

COMBINED EFFECTS OF CRITICAL HARDY-SOBOLEV EXPONENT AND SINGULAR NONLINEARITIES IN NONLOCAL PROBLEMS WITH VARIABLE WEIGHTS

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ABSTRACT. This article studies a fractional elliptic equation that involves a critical Hardy-Sobolev nonlinearity along with a singular term,

$$\begin{aligned} (-\Delta_p)^s u &= \lambda f(x)u^{-\gamma} \pm \frac{g(x)|u|^{p_s^*(t)-2}u}{|x|^t} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $0 \in \Omega$. The dimension N satisfies $N > sp$, $s \in (0, 1)$, $\lambda > 0$, $0 < \gamma < 1$, and $p_s^*(t) = \frac{p(N-t)}{N-sp}$ represent the critical Hardy-Sobolev exponent. The weight functions f and g are elements of $L^\infty(\Omega)$ and satisfy specific positivity conditions, and $(-\Delta_p)^s u$ is the fractional p -Laplacian operator. We use the method of sub- and super-solutions combined with monotonicity arguments, to establish the existence and nonexistence of solutions. Furthermore, we prove that any weak solution is locally Hölder continuous.

1. INTRODUCTION

The purpose of this article is to study a fractional equation characterized by a critical Sobolev exponent and singular nonlinearity,

$$\begin{aligned} (-\Delta_p)^s u &= \lambda f(x)u^{-\gamma} \pm \frac{g(x)|u|^{p_s^*(t)-2}u}{|x|^t} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1.1}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and $N > sp$ with $s \in (0, 1)$. The parameters satisfy $\lambda > 0$ and $0 < \gamma < 1$, and the term $p_s^*(t) = \frac{p(N-t)}{N-sp}$ represents the critical Sobolev-Hardy exponent. The variable weight functions f and g belong to $L^\infty(\Omega)$ and satisfy:

$$\operatorname{ess\,inf}_{x \in \Omega} f(x) > 0 \quad \text{and} \quad \operatorname{ess\,inf}_{x \in \Omega} g(x) > 0. \tag{1.2}$$

The fractional p -Laplacian $(-\Delta_p)^s u$ is defined for smooth functions as

$$(-\Delta_p)^s u(x) = 2 \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy,$$

where $x \in \mathbb{R}^N$ and B_ϵ is a ball of radius ϵ centered at x .

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For clarity, we state the two versions of problem (1.1) as follows

$$\begin{aligned} (-\Delta_p)^s u &= \lambda f(x)u^{-\gamma} + \frac{g(x)|u|^{p_s^*(t)-2}u}{|x|^t} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} (-\Delta_p)^s u &= \lambda f(x)u^{-\gamma} - \frac{g(x)|u|^{p_s^*(t)-2}u}{|x|^t} \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{1.4}$$

Before presenting the key results of this work, we will outline several related problems to problem (1.1). Numerous studies have investigated solutions for elliptic partial differential equations (PDEs) with singularity. Many authors have studied the nonlocal problem

$$\begin{aligned} (-\Delta_p)^s u &= \frac{\lambda a(x)}{u^\gamma} + Mf(x, u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \\ u &> 0 \quad \text{in } \Omega, \end{aligned} \tag{1.5}$$

where $N > ps$, $M \geq 0$, and $a : \Omega \rightarrow \mathbb{R}$ is a nonnegative bounded function. In the case where $M = 0$, the purely singular problem was investigated by Canino et al. [6]. The authors proved that when $0 < \gamma < 1$, problem (1.5) possesses a unique solution u in the space $W_0^{s,p}(\Omega)$. Additionally, they proved that the essential infimum of u over K is greater than zero, i.e., $\text{ess inf}_K u > 0$. In [18], variational techniques were used to establish multiple results for $f(x, u) = u^{2_s^*-1}$. Using the Nehari method, Ghanmi et al. [11] obtained multiple results for $0 < \gamma < 1$ and $\lambda = 1$. Additionally, Alomair et al. [1] considered $M = b(x)$ and $f(x, u) = u^{2_s^*-1}$, employing sub and supersolution methods along with monotonicity arguments to derive both existence and non-existence results. Lastly, in Daouas et al. [8], variational methods were combined with a perturbation approach to establish multiple results.

The study of the existence and multiplicity of solutions, as well as their regularity, for regular PDEs involving nonlocal operators and Sobolev-Hardy nonlinearities has indeed attracted considerable interest in recent years. In particular, works such as those by Chen et al. [7], Yan [24], as well as Alotaibi et al. [2], along with the references therein, have contributed significantly to this field. These studies have explored various aspects of nonlocal problems, particularly focusing on the critical growth of nonlinearities and the implications of Sobolev embeddings.

It is important to highlight that research on singular PDEs with Hardy-Sobolev nonlinearities in a fractional setting has not been as extensively developed as its classical counterpart. Although there are foundational results for regular PDEs, such as the existence, multiplicity, and regularity of solutions, the analysis of singular problems involving fractional operators remains comparatively limited. For instance, although some studies have addressed the existence of solutions for singular fractional PDEs, key questions regarding their qualitative properties (e.g. regularity, asymptotic behavior, or stability) and the optimality of functional frameworks (e.g. weighted fractional Sobolev spaces) are far from being fully resolved. This gap presents an opportunity for further investigation into singular nonlocal problems, particularly those combining fractional diffusion operators (e.g. the fractional Laplacian), Hardy-type singular potentials, and critical nonlinearities tied to fractional Sobolev embeddings.

In this article, we will explore both the existence and nonexistence of solutions to the nonlocal problem (1.1), as well as investigate its regularity. It is essential to note that including weight functions $f(x)$ and $g(x)$ in problem (1.1) is significant, as they enhance the mathematical richness of our model and broaden its applicability to real-world scenarios. For example, these weight functions can represent varying material properties in inhomogeneous media, account for spatial differences in resource availability in ecological contexts, and play a crucial role in optimization

problems. By incorporating weight functions, we address new mathematical challenges and ensure that our work remains relevant to practical applications across various fields.

Now, we outline the main results that will be established, related to problem (1.1).

Theorem 1.1. *Suppose $0 < \gamma < 1$ and that the weight functions $f, g \in L^\infty(\Omega)$ satisfy condition (1.2). Then, there exists a positive constant Λ_1 with the following properties:*

- (1) *For every $\lambda \in (0, \Lambda_1)$, problem (1.3) admits a minimal solution u_λ .*
- (2) *Problem (1.3) has a solution when $\lambda = \Lambda_1$.*
- (3) *Problem (1.3) has no solutions for $\lambda > \Lambda_1$.*

Theorem 1.2. *Consider $0 < \gamma < 1$ and the weight functions $f, g \in L^\infty(\Omega)$ that satisfy condition (1.2). Then, there exists a positive constant Λ_2 such that*

- (1) *For $\lambda > \Lambda_2$, problem (1.4) has at least one solution.*
- (2) *For $\lambda < \Lambda_2$, problem (1.4) has no solutions.*

The third result concerns the regularity of weak solutions of problem (1.1).

Theorem 1.3. *Suppose u is a weak positive solution to problem (1.1). Then, there exists a value $\alpha \in (0, s]$ such that u belongs to the Hölder space $C_{\text{loc}}^\alpha(\Omega')$ for every compactly embedded subset $\Omega' \Subset \Omega$.*

We highlight that the analysis of problem (1.1) has yielded valuable insights into fractional elliptic equations with weight functions. It demonstrates how these weight functions can alter the properties of solutions, affecting their existence, uniqueness, and regularity, which is crucial for modeling situations where material properties vary. We have established new results showing that solutions can still exhibit desirable behavior, even in the presence of singularities. Additionally, we have refined methods for identifying sub- and super-solutions, enhancing our understanding of how weight functions influence solutions in nonlinear contexts. The interaction between weight functions and critical nonlinear terms has also been clarified, revealing how they can lead to multiple solutions or impact the stability of a solution. These findings are relevant in various fields, including materials science, biology, and finance. Looking to the future, research can explore more complex systems with multiple weight functions, develop specific numerical methods for these equations, investigate higher-order fractional models, and examine applications in control theory. Overall, this work opens many new avenues for research and practical applications in different scientific disciplines.

This manuscript is structured as follows: Section 2 provides essential foundational results that will be utilized in subsequent sections. The proofs of Theorems 1.1 and 1.2 can be found in Sections 3 and 4. Section 5 addresses the regularity of weak solutions, as detailed in Theorem 1.3.

2. FUNCTIONAL FRAMEWORK AND PRELIMINARY RESULTS

In this section, we introduce key definitions, notation, and function spaces that will be used throughout this work. We define the fractional Sobolev space as

$$W_0^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : u \text{ is measurable, } |u|_{s,p} < +\infty, u \equiv 0 \text{ a.e. on } \Omega^c\}.$$

In addition, the homogeneous fractional Sobolev space is defined as

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : u \text{ is measurable, } |u|_{s,p} < +\infty\}.$$

The associated Gagliardo norm for these spaces is defined as

$$\|u\|_{s,p} := \left(\|u\|_p^p + |u|_{s,p}^p \right)^{1/p}.$$

For a comprehensive overview of the properties of the space $W^{s,p}(\mathbb{R}^N)$ we refer to [19] and references therein.

Now, let $\Omega \subset \mathbb{R}^N$ and define $Q = \mathbb{R}^{2N} \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega))$. The space $(X, \|\cdot\|_X)$ is defined as:

$$X = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ is measurable, } u|_\Omega \in L^p(\Omega) \text{ and } \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+ps}{p}}} \in L^p(Q) \right\}$$

equipped with the Gagliardo norm

$$\|u\|_X = \|u\|_p + \left(\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p}.$$

Here, $\|u\|_p$ denotes the L^p -norm of u . We also define the subspace

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

which is equipped with the norm

$$\|u\| = \left(\int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p}.$$

Lastly, for $u \in L^{p_s^*(t)}(\mathbb{R}^N)$, we define the norm

$$|u|_{p_s^*(t)} = \left(\int_{\mathbb{R}^N} \frac{|u|^{p_s^*(t)}}{|x|^t} dx \right)^{1/p_s^*(t)}.$$

3. STUDY OF PROBLEM (1.3)

Our goal is to demonstrate the existence of solutions for specific values of the parameter λ . To start, we introduce the concept of a weak solution of problem (1.3).

Definition 3.1. A function $u_\lambda \in X_0$ is called a weak solution of problem (1.3) if, for every test function $\phi \in X_0$, the following conditions hold

- (i) $u_\lambda > 0$, $u_\lambda^{-\gamma} \phi \in L^1(\Omega)$,
- (ii)

$$\begin{aligned} & \int_Q \frac{|u_\lambda(x) - u_\lambda(y)|^{p-2} (u_\lambda(x) - u_\lambda(y)) (\phi(x) - \phi(y))}{|x - y|^{N+ps}} dx dy \\ &= \lambda \int_\Omega f(x) u_\lambda^{-\gamma} \phi dx + \int_\Omega \frac{g(x) |u_\lambda|^{p_s^*(t)-2} u_\lambda}{|x|^t} \phi dx \end{aligned} \quad (3.1)$$

We now present the concepts of a subsolution and a supersolution for problem (1.3).

Definition 3.2. A function $\underline{u}_\lambda$ is defined as a weak subsolution of problem (1.3) if it satisfies

- (1) The function is strictly positive on the domain. Additionally, the term $\underline{u}_\lambda^{-\gamma} \phi$ is integrable on Ω ;
- (2) For every nonnegative test function $\phi \in X_0$,

$$\begin{aligned} & \int_Q \frac{|\underline{u}_\lambda(x) - \underline{u}_\lambda(y)|^{p-2} (\underline{u}_\lambda(x) - \underline{u}_\lambda(y)) (\phi(x) - \phi(y))}{|x - y|^{N+ps}} dx dy \\ & - \int_\Omega \frac{\lambda f(x)}{\underline{u}_\lambda^\gamma} \phi dx - \int_\Omega \frac{g(x) |\underline{u}_\lambda|^{p_s^*(t)-2} \underline{u}_\lambda}{|x|^t} \phi dx \leq 0. \end{aligned}$$

Definition 3.3. A function $\bar{u}_\lambda$ is defined as a weak supersolution of problem (1.3) if it satisfies

- (1) The function is strictly positive on the domain. Additionally, the term $\bar{u}_\lambda^{-\gamma} \phi$ is integrable on Ω ;
- (2) For every nonnegative test function $\phi \in X_0$,

$$\begin{aligned} & \int_Q \frac{|\bar{u}_\lambda(x) - \bar{u}_\lambda(y)|^{p-2} (\bar{u}_\lambda(x) - \bar{u}_\lambda(y)) (\phi(x) - \phi(y))}{|x - y|^{N+ps}} dx dy \\ & - \int_\Omega \frac{\lambda f(x)}{\bar{u}_\lambda^\gamma} \phi dx - \int_\Omega \frac{g(x) |\bar{u}_\lambda|^{p_s^*(t)-2} \bar{u}_\lambda}{|x|^t} \phi dx \geq 0. \end{aligned}$$

Now, we define the functional $I_\lambda : X_0 \rightarrow \mathbb{R}$ as

$$I_\lambda(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{1-\gamma} \int_\Omega f(x) |u|^{1-\gamma} dx - \frac{1}{p_s^*(t)} \int_\Omega \frac{g(x) |u|^{p_s^*(t)}}{|x|^t} dx. \quad (3.2)$$

Next, we introduce the critical value

$$\Lambda_1 := \inf \{ \lambda > 0 : \text{problem (1.3) admits no weak solution} \}.$$

We now establish the following key result.

Lemma 3.4. *For $\lambda \in (0, \Lambda_1)$, problem (1.3) has at least one weak solution.*

Proof. To demonstrate the existence of a solution for problem (1.3), we will construct a well-ordered pair of functions: a weak subsolution $\underline{u}_\lambda$ and a weak supersolution $\bar{u}_\lambda$ such that the inequality $\underline{u}_\lambda \leq \bar{u}_\lambda$ holds throughout the domain.

First, let λ_1 denote the principal eigenvalue of the fractional p -Laplacian operator $(-\Delta_p)^s$ in the domain Ω . Specifically, λ_1 and its corresponding eigenfunction ϕ_1 satisfy the eigenvalue problem

$$\begin{aligned} (-\Delta_p)^s \phi_1 &= \lambda_1 h(x) |\phi_1|^{p-2} \phi_1 \quad \text{in } \Omega, \\ \phi_1 &> 0 \quad \text{in } \Omega, \\ \phi_1 &= 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned}$$

where $h(x) = \min\{f(x), g(x)\}$. Here, the eigenfunction ϕ_1 is positive and belongs to $L^\infty(\Omega)$. For further details, we refer the reader to [15].

Assume $\lambda_1 \leq \lambda$. Define the subsolution $\underline{u}_\lambda = \epsilon \phi_1$, where $\epsilon > 0$ is a small constant. Then

$$(-\Delta_p)^s(\underline{u}_\lambda) = \lambda_1 \epsilon^{p-1} h(x) \phi_1^{p-1} \leq \lambda \epsilon^{p-1} h(x) \phi_1^{p-1} \leq \lambda h(x) \epsilon^\gamma \phi_1^{-\gamma}.$$

Since $\underline{u}_\lambda = \epsilon \phi_1$, we have

$$\begin{aligned} \frac{\lambda h(x) \epsilon^\gamma}{\phi_1^{-\gamma}} &\leq h(x) \left(\frac{\lambda \epsilon^\gamma}{\phi_1^{-\gamma}} + \frac{\epsilon^{p_s^*(t)-1} |\phi_1|^{p_s^*(t)-2} \phi_1}{|x|^t} \right) \\ &\leq \lambda f(x) \underline{u}_\lambda^{-\gamma} + \frac{g(x) |\underline{u}_\lambda|^{p_s^*(t)-2} \underline{u}_\lambda}{|x|^t}. \end{aligned}$$

Thus, $\underline{u}_\lambda$ satisfies

$$(-\Delta_p)^s(\underline{u}_\lambda) \leq \lambda f(x) \underline{u}_\lambda^{-\gamma} + \frac{g(x) |\underline{u}_\lambda|^{p_s^*(t)-2} \underline{u}_\lambda}{|x|^t},$$

which shows that $\underline{u}_\lambda$ is a subsolution of problem (1.3).

Now, let $h(x) = \max\{f(x), g(x)\}$ and assume $\lambda_1 \geq \lambda$. Define the supersolution $\bar{u}_\lambda = C \phi_1$, where $C > 0$ is a sufficiently large constant. Then

$$(-\Delta_p)^s(\bar{u}_\lambda) = \lambda_1 C^{p-1} h(x) \phi_1^{p-1} \geq \lambda C^{p-1} h(x) \phi_1^{p-1} \geq \lambda h(x) C^\gamma \phi_1^{-\gamma}.$$

Since $\bar{u}_\lambda = C \phi_1$, we have

$$\frac{\lambda h(x) C^\gamma}{\phi_1^{-\gamma}} \geq h(x) \left(\frac{\lambda C^\gamma}{\phi_1^{-\gamma}} + \frac{C^{p_s^*(t)-1} |\phi_1|^{p_s^*(t)-2} \phi_1}{|x|^t} \right) \geq h(x) \left(\frac{\lambda}{\bar{u}_\lambda^{-\gamma}} + \frac{|\bar{u}_\lambda|^{p_s^*(t)-2} \bar{u}_\lambda}{|x|^t} \right).$$

Thus, $\bar{u}_\lambda$ satisfies

$$(-\Delta_p)^s(\bar{u}_\lambda) \geq \lambda f(x) \bar{u}_\lambda^{-\gamma} + \frac{g(x) |\bar{u}_\lambda|^{p_s^*(t)-2} \bar{u}_\lambda}{|x|^t},$$

which proves that $\bar{u}_\lambda$ is a supersolution of problem (1.3).

To ensure $\underline{u}_\lambda \leq \bar{u}_\lambda$, we choose $\epsilon > 0$ small enough and $C > 0$ large enough such that

$$\epsilon \phi_1 \leq C \phi_1 \quad \text{for all } x \in \Omega.$$

This holds trivially since $\epsilon \leq C$.

By the weak comparison principle (see Lemma 3.1 in Saoudi et al. [20]), we have

$$\underline{u}_\lambda \leq \bar{u}_\lambda.$$

Therefore, using the classical iteration method, we conclude that problem (1.3) admits a solution u_λ that satisfies $\underline{u}_\lambda \leq u_\lambda \leq \bar{u}_\lambda$ for all $\lambda < \Lambda_1$. $\square$

Lemma 3.5. *For all $\lambda \in (0, \Lambda_1)$, problem (1.3) has a minimal solution u_λ .*

Proof. To prove the existence of a minimal solution, we introduce the monotone iterative scheme

$$\begin{aligned} (-\Delta_p)^s u_n - \lambda f(x) u_n^{-\gamma} &= \frac{g(x) |u_{n-1}|^{p_s^*(t)-2} u_{n-1}}{|x|^t} \quad \text{in } \Omega, \\ u_n|_{\partial\Omega} &= 0. \end{aligned}$$

We initialize the scheme with $u_0 = \underline{u}_\lambda$, where $\underline{u}_\lambda$ is the unique solution to the purely singular problem, as established in the work of Canino et al. [6],

$$\begin{aligned} (-\Delta_p)^s u &= \lambda f(x) u^{-\gamma} \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0, \quad u > 0 \quad \text{in } \Omega. \end{aligned}$$

According to the weak comparison principle (as detailed in Saoudi et al. [20, Lemma 3.1]), it follows that u_0 is a weak subsolution to $(P)_+$ and fulfills condition $u_0 \leq v$, where v denotes any weak solution of $(P)_+$. Furthermore, applying the weak comparison principle once more, we can demonstrate that the sequence $\{u_n\}_{n=1}^\infty$ is not monotonically decreasing, which means $u_0 \leq u_1 \leq u_2 \leq \dots$. Furthermore, since $u_n \leq v$ for every n , the sequence $\{u_n\}$ remains uniformly bounded in X_0 .

From the monotonicity and uniform boundedness of the sequence $\{u_n\}$, it follows that $\{u_n\}$ converges weakly in X_0 and pointwise to a function u_λ , which serves as a weak solution to (1.3).

To establish that u_λ is the minimal solution, let $\tilde{u}_\lambda$ represent any weak solution to (1.3) for $0 < \lambda < \Lambda_1$. Since $u_0 = \underline{u}_\lambda \leq \tilde{u}_\lambda$, the weak comparison principle allows us to assert that $u_n \leq \tilde{u}_\lambda$ for all $n \geq 0$. Taking the limit as $n \rightarrow \infty$, we derive $u_\lambda \leq \tilde{u}_\lambda$. This establishes that u_λ is in fact the minimal solution to (1.3). $\square$

Lemma 3.6. *Problem (1.3) admits a solution when $\lambda = \Lambda_1$.*

Proof. Consider a sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ such that $\lambda_n \uparrow \Lambda_1$ as $n \rightarrow \infty$. According to Lemma 3.4, we know that there exists a weak positive solution $u_n = u_{\lambda_n} \geq \underline{u}_{\lambda_n}$ to problem (1.3) for the corresponding parameter. Consequently, for any $\phi \in C_c^\infty(\Omega)$, we can derive that

$$\begin{aligned} &\int_Q \frac{|u_n(x) - u_n(y)|^{p-2} (u_n(x) - u_n(y)) (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \\ &= \lambda_n \int_\Omega f(x) u_n^{-\gamma} \phi dx + \int_\Omega \frac{g(x) |u_n|^{p_s^*(t)-2} u_n}{|x|^t} \phi dx. \end{aligned} \quad (3.3)$$

Since $u_n \in X_0$ and $u_n \geq \underline{u}_{\lambda_n}$, it is evident that (3.3) holds for $\phi \in X_0$ as well. Furthermore, we have

$$\int_Q \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+sp}} dx dy - \lambda_n \int_\Omega f(x) u_n^{1-\gamma} dx - \int_\Omega \frac{g(x) |u_n|^{p_s^*(t)}}{|x|^t} dx = 0. \quad (3.4)$$

Again, using Lemma 3.4, we deduce that

$$I_{\lambda_n}(u_n) \leq I_{\lambda_n}(\underline{u}_{\lambda_n}) = \frac{1}{p} \int_Q \frac{|\underline{u}_{\lambda_n}(x) - \underline{u}_{\lambda_n}(y)|^p}{|x - y|^{N+sp}} dx dy - \frac{\lambda_n}{1-\gamma} \int_\Omega f(x) \underline{u}_{\lambda_n}^{1-\gamma} dx < 0, \quad (3.5)$$

because u_n is a minimizer of I_{λ_n} over $\{u \geq \underline{u}_{\lambda_n}\}$, and the subsolution $\underline{u}_{\lambda_n}$ is constructed to have negative energy because of the singular term.

So, substituting Eq. (3.4) into Eq. (3.5), we obtain

$$\frac{1}{p} \left(\lambda_n \int_\Omega f(x) u_n^{1-\gamma} dx + \int_\Omega \frac{g(x) |u_n|^{p_s^*(t)}}{|x|^t} dx \right) - \frac{\lambda_n}{1-\gamma} \int_\Omega f(x) u_n^{1-\gamma} dx - \frac{1}{p_s^*(t)} \int_\Omega \frac{g(x) |u_n|^{p_s^*(t)}}{|x|^t} dx < 0.$$

Therefore,

$$\left(\frac{1}{p} - \frac{1}{p_s^*(t)} \right) \int_\Omega \frac{g(x) |u_n|^{p_s^*(t)}}{|x|^t} dx < \lambda_n \left(\frac{1}{1-\gamma} - \frac{1}{p} \right) \int_\Omega f(x) u_n^{1-\gamma} dx. \quad (3.6)$$

Insetting (3.6) into (3.4), and using the Sobolev embedding along with the fact that $f \in L^\infty(\Omega)$, we derive the inequality

$$\|u_n\|_{X_0}^{p+\gamma-1} \leq C_1 \quad (3.7)$$

where C_1 is a positive constant. This leads us to easily conclude that

$$\sup_{n \in \mathbb{N}} \|u_n\|_{X_0} < \infty.$$

Thus, given that the space X_0 is reflexive, we can assert the existence of a subsequence, still denoted by $\{u_n\}$, such that $u_n \rightharpoonup u_{\Lambda_1}$ in X_0 as n approaches ∞ . By taking the limit of (3.3) as $n \rightarrow \infty$, we obtain

$$\begin{aligned} & \int_Q \frac{|u_{\Lambda_1}(x) - u_{\Lambda_1}(y)|^{p-2} (u_{\Lambda_1}(x) - u_{\Lambda_1}(y)) (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \\ &= \Lambda_1 \int_{\Omega} f(x) u_{\Lambda_1}^{-\gamma} \phi dx + \int_{\Omega} \frac{g(x) |u_{\Lambda_1}|^{p_s^*(t)-2} u_{\Lambda_1}}{|x|^t} \phi dx. \end{aligned}$$

We deduce that u_{Λ_1} is a weak solution to problem (1.3). $\square$

We shall now prove that Λ_1 is finite.

Lemma 3.7. *Problem (1.3) has no solution if $\lambda > \Lambda_1$.*

Proof. We proceed by contradiction. Assume that there exists a sequence $\lambda_n \rightarrow \infty$ for which problem (1.3) has a solution u_n . Setting

$$h := \min \left\{ \operatorname{ess\,inf}_{x \in \Omega} f(x), \operatorname{ess\,inf}_{x \in \Omega} g(x) \right\} > 0. \quad (3.8)$$

So, we derive the existence of $\Lambda_1 > 0$ satisfying

$$h \left(\lambda r^{-\gamma} + \frac{|r|^{p_s^*(t)-2} r}{|x|^t} \right) \geq (\lambda_1 + \delta) r^{p-1} \quad \text{for all } r > 0, \delta \in (0, 1) \text{ and } \lambda > \Lambda_1 \quad (3.9)$$

where λ_1 is the principal eigenvalue of $(-\Delta_p)^s$ in Ω .

Fix $\lambda_n > \Lambda_1$. It is simple to see that u_n is a supersolution of problem

$$\begin{aligned} (-\Delta)_p^s u &= (\lambda_1 + \delta) u^{p-1} \quad \text{in } \Omega, \\ u &> 0, \quad u|_{\partial\Omega} = 0, \end{aligned} \quad (3.10)$$

for all $\delta \in (0, 1)$. According to the results presented in [6, Theorem 1.2], we can select $\epsilon < \lambda_1 + \delta$ sufficiently small such that $\epsilon\phi_1(x) < u_n(x)$ ensures that $\epsilon\phi_1$ is a valid subsolution to problem (3.10). Through a monotone iteration process, we can derive a solution to problem (3.10) for any $\delta \in (0, 1)$, which contradicts the assertion that $\lambda_1(s, p)$ is a positive, simple and isolated point in the spectrum of $(-\Delta)_p^s$ in X_0 (refer, e.g. [10, Theorems 4.9 and 4.11]). $\square$

Proof of Theorem 1.1. Its assertion can be derived from the results established in Lemmas 3.5, 3.6, and 3.7. $\square$

4. STUDY OF PROBLEM (1.4)

Our main objective is to investigate the existence of solutions by analyzing the critical points of the associated functional energy. To this end, we define the functional $E_\lambda : X_0 \rightarrow \mathbb{R}$, as

$$E_\lambda(u) = \frac{1}{p} \|u\|^p - \frac{\lambda}{1-\gamma} \int_{\Omega} f(x) |u|^{1-\gamma} dx + \frac{1}{p_s^*(t)} \int_{\Omega} \frac{g(x) |u|^{p_s^*(t)}}{|x|^t} dx.$$

Here, the functional E_λ is considered within the Sobolev space X_0 , which serves as the appropriate functional setting for this variational approach. The existence of solutions is determined by identifying the critical points of E_λ , which correspond to weak solutions to problem (1.4). To establish the main result, namely Theorem 1.2, we proceed through a sequence of intermediate steps, each formulated as a lemma. The first key step in our analysis is to establish the coercivity of the functional energy E_λ . This property plays a crucial role in ensuring the boundedness of minimizing sequences and, consequently, in proving the existence of solutions via variational methods. Now, we shall prove the following lemma.

Lemma 4.1. *The energy functional E_λ is coercive.*

Proof. We begin by applying Hölder's inequality with the exponents

$$q = \frac{p_s^*(t)}{1-\gamma} \quad \text{and} \quad q' = \frac{p_s^*(t)}{p_s^*(t) - (1-\gamma)}.$$

This yields the estimate

$$\int_{\Omega} f(x)|u|^{1-\gamma} dx \leq \|f\|_{L^\infty} \left(\int_{\Omega} |u|^{p_s^*(t)} dx \right)^{\frac{1-\gamma}{p_s^*(t)}} |\Omega|^{\frac{p_s^*(t) - (1-\gamma)}{p_s^*(t)}},$$

where $|\Omega|$ denotes the measure of the domain Ω .

Next, we bound the term $\int_{\Omega} |u|^{p_s^*(t)} dx$ using the fractional Sobolev-Hardy inequality,

$$\|u\|_{L^{p_s^*(t)}} \leq S\|u\|,$$

where $S > 0$ is the best Sobolev constant associated with the fractional Sobolev space X_0 and the dimension N . Applying this inequality, we obtain

$$\left(\int_{\Omega} |u|^{p_s^*(t)} dx \right)^{\frac{1-\gamma}{p_s^*(t)}} \leq S^{1-\gamma} \|u\|^{1-\gamma}.$$

Substituting this estimate into the singular term, we obtain

$$\lambda \int_{\Omega} \frac{f(x)|u|^{1-\gamma}}{1-\gamma} dx \leq C_1 \|u\|^{1-\gamma},$$

where

$$C_1 = \lambda \frac{\|f\|_{L^\infty}}{1-\gamma} S^{1-\gamma} |\Omega|^{\frac{p_s^*(t) - (1-\gamma)}{p_s^*(t)}}.$$

Therefore, using the fractional Sobolev-Hardy inequality once again, we have

$$\int_{\Omega} \frac{g(x)|u|^{p_s^*(t)}}{|x|^t} dx \geq \left(\operatorname{ess\,inf}_{x \in \Omega} g(x) \right) \|u\|_{L^{p_s^*(t)}}^{p_s^*(t)}.$$

Since $\|u\|_{L^{p_s^*(t)}} \geq 0$, it follows that

$$E_\lambda(u) \geq \frac{1}{p} \|u\|^p - C_1 \|u\|^{1-\gamma}.$$

Noting that $1-\gamma < p$, we conclude that $E_\lambda(u) \rightarrow +\infty$ as $\|u\| \rightarrow \infty$. Therefore, E_λ is coercive. $\square$

Lemma 4.2. *The energy functional E_λ has a global minimizer.*

Proof. Let $\{u_k\}_k$ be a minimizing sequence for the functional E_λ in the space X_0 . By Lemma 4.1, the sequence $\{u_k\}_k$ is bounded in X_0 . Moreover, since $E_\lambda(u) = E_\lambda(|u|)$, we may assume without loss of generality that $\{u_k\}_k$ is nonnegative, that is, $u_k \geq 0$ for all k .

By the reflexivity of X_0 , there exists a subsequence, still denoted by $\{u_k\}$, and a function $u \in X_0$ such that

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{weakly in } X_0, \\ u_k &\rightharpoonup u \quad \text{weakly in } L^{p_s^*(t)}(\Omega), \\ u_k &\rightarrow u \quad \text{strongly in } L^r(\Omega, \frac{dx}{|x|^t}) \text{ for } r \in [1, p_s^*(t)), \\ u_k &\rightarrow u \quad \text{almost everywhere in } \Omega. \end{aligned}$$

These convergence properties are a consequence of compact embeddings $X_0 \hookrightarrow L^{1-\gamma}(\Omega)$ and $X_0 \hookrightarrow L^{p_s^*(t)}(\Omega)$, as well as the weak lower semi-continuity of the norm $\|\cdot\|$.

From the weak lower semicontinuity of the norm $\|\cdot\|$, we have

$$\|u\|^p \leq \liminf_{k \rightarrow \infty} \|u_k\|^p.$$

Additionally, the strong convergence in $L^r(\Omega, \frac{dx}{|x|^t})$ and the convergence almost everywhere imply

$$\int_{\Omega} f(x)u^{1-\gamma} dx = \lim_{k \rightarrow \infty} \int_{\Omega} f(x)u_k^{1-\gamma} dx,$$

$$\int_{\Omega} \frac{g(x)|u|^{p_s^*(t)}}{|x|^t} dx = \lim_{k \rightarrow \infty} \int_{\Omega} \frac{g(x)|u_k|^{p_s^*(t)}}{|x|^t} dx.$$

Combining these results, we obtain

$$E_{\lambda}(u) \leq \liminf_{k \rightarrow \infty} E_{\lambda}(u_k).$$

Since $\{u_k\}$ is a minimizing sequence for E_{λ} , it follows that u is a global minimizer of E_{λ} in X_0 . $\square$

The next Lemma guaranties the existence of at least one solution for sufficiently large values of the parameter λ .

Lemma 4.3. *There exists a positive number Λ_2 such that for all $\lambda > \Lambda_2$, problem (1.4) admits at least one solution.*

Proof. We begin by introducing the constrained variational problem

$$\lambda_* := \inf \left\{ \frac{1}{p} \|w\|^p + \frac{1}{p_s^*(t)} \int_{\Omega} \frac{g(x)|w|^{p_s^*(t)}}{|x|^t} dx : w \in X_0 \quad \text{and} \quad \frac{1}{1-\gamma} \int_{\Omega} f(x)|w|^{1-\gamma} dx = 1 \right\}. \quad (4.1)$$

Additionally, we define

$$\Lambda_2 := \inf \{ \lambda > 0 : (1.4) \text{ admits a nontrivial weak solution} \}. \quad (4.2)$$

Consider a sequence $\{v_k\}_k$ to be a minimizing sequence for λ_* . Employing an argument similar to the one in Lemma 4.2, we may assume that $\{v_k\}_k$ converges weakly to some $v \in X_0$, satisfying

$$\begin{aligned} \lambda_* &= \frac{1}{p} \|v\|^p + \frac{1}{p_s^*(t)} \int_{\Omega} \frac{g(x)|v|^{p_s^*(t)}}{|x|^t} dx, \\ \frac{1}{1-\gamma} \int_{\Omega} f(x)|v|^{1-\gamma} dx &= 1. \end{aligned}$$

Consequently, for any $\lambda > \lambda_*$, we have $E_{\lambda}(v) = \lambda_* - \lambda < 0$.

Now, fix $\lambda > \Lambda_2$. By the definition of Λ_2 , there exists $\beta \in (\Lambda_2, \lambda)$ such that the functional E_{β} has a non-trivial critical point $u_{\beta} \in X_0$. Since $\beta < \lambda$, u_{β} serves as a sub-solution to problem (1.4).

To construct a super-solution, consider the minimization problem

$$\inf \left\{ \frac{1}{p} \|w\|^p - \frac{\lambda}{1-\gamma} \int_{\Omega} f(x)|w|^{1-\gamma} dx + \frac{1}{p_s^*(t)} \int_{\Omega} \frac{g(x)|w|^{p_s^*(t)}}{|x|^t} dx : w \in X_0, w \geq u_{\beta} \right\}. \quad (4.3)$$

Then, using arguments analogous to those for problem (4.1), we deduce that problem (4.3) admits a solution w_{λ} satisfying $w_{\lambda} > u_{\beta}$. Furthermore, for all $\lambda > \Lambda_2$, the function w_{λ} remains also a weak solution to problem (1.4). Using the reasoning described in [20], we infer that a solution exists for (1.4) when $\lambda = \Lambda_2$.

Finally, the positivity of u_{λ} follows from the strong maximum principle (see [16, Lemma 2.3]), as u_{λ} is a weak C^1 positive solution of the differential inequality

$$(-\Delta_p)^s u + \frac{g(x)|u|^{p_s^*(t)-2}u}{|x|^t} \geq 0 \quad \text{in } \Omega.$$

We derive that u_{λ} is nonnegative everywhere in Ω . This completes the proof. $\square$

Lemma 4.4. *There exists a positive number Λ_2 such that if $\lambda < \Lambda_2$, problem (1.1) does not have solution.*

Proof. Fix $\delta \in (0, \lambda_1)$, where λ_1 is the principal eigenvalue. Set $\Lambda_2 = \min\{1, \delta\} > 0$. We now claim that for all $\lambda \in (0, \Lambda_2)$ and for all $r > 0$, the following inequality holds

$$\lambda f(x)r^{-\gamma} - \frac{g(x)r^{p_s^*(t)-2}r}{|x|^t} < \delta r^{p-1}. \quad (4.4)$$

Indeed, first observe that for $\lambda < \Lambda_2 \leq 1$ and $0 < \gamma < 1 < p_s^*(t)$, we derive for all $r \in [0, 1]$ that

$$\lambda f(x)r^{-\gamma} - \frac{g(x)r^{p_s^*(t)-2}r}{|x|^t} < 0.$$

Furthermore, for $r > 1$ and since $1 - \gamma < p$, we deduce that

$$\lambda f(x)r^{-\gamma} - \frac{g(x)r^{p_s^*(t)-2}r}{|x|^t} < \lambda f(x)r^{-\gamma} < \delta r^{p-1}$$

for all $\Lambda_2 \in [\lambda, \delta]$. Which proves (4.4). We now consider $\lambda \in (0, \Lambda_2)$ and let u be a critical point to the energy functional E_λ . Therefore,

$$\begin{aligned} \|u\|_{W_0^{s,p}(\Omega)}^p &= \langle (-\Delta_p)^s u, u \rangle \\ &= \lambda \int_{\Omega} f(x)|u|^{1-\gamma} dx - \int_{\Omega} \frac{g(x)|u|^{p_s^*(t)}}{|x|^t} dx \\ &\leq \delta \|u\|^p. \end{aligned} \quad (4.5)$$

On the other hand, using the variational description of λ_1 , we have

$$\lambda_1 = \inf_{u \in W_0^{s,p}(\Omega)} \frac{\|u\|_{W_0^{s,p}(\Omega)}^p}{\|u\|_{L^p(\Omega)}^p}.$$

This implies

$$\|u\|_{W_0^{s,p}(\Omega)}^p \geq \lambda_1 \|u\|_{L^p(\Omega)}^p. \quad (4.6)$$

Combining the two inequalities (4.5) and (4.6), we obtain

$$\lambda_1 \|u\|^p \leq \|u\|_{W_0^{s,p}(\Omega)}^p \leq \delta \|u\|^p.$$

Since $\delta < \lambda_1$, this implies

$$\lambda_1 \|u\|^p \leq \delta \|u\|^p.$$

If $\|u\|^p \neq 0$, then dividing both sides by $\|u\|^p$ gives $\lambda_1 \leq \delta$. But this contradicts the assumption that $\delta < \lambda_1$. Therefore, the only possibility is $\|u\|^p = 0$, which implies $u = 0$ almost everywhere in Ω . Consequently (1.4) has no solution if $\lambda < \Lambda_2$. $\square$

Proof of Theorem 1.2. The proof follows directly from Lemma 4.3 and Lemma 4.4. $\square$

5. REGULARITY RESULTS

In this section, we study the regularity of weak solutions to problem (1.1). We begin by recalling the following elementary inequalities, which are proved in [4, 5]. These inequalities serve as fundamental tools in our analysis.

Lemma 5.1 ([4, Lemma C.2]). *Let $1 < p < \infty$ and $r \geq 1$. For any non-negative real numbers a , b , and M , it holds*

$$|a - b|^{p-2}(a - b)(a_M^r - b_M^r) \geq \frac{rp^p}{(r + p - 1)^p} \left| a_M^{\frac{r+p-1}{p}} - b_M^{\frac{r+p-1}{p}} \right|,$$

with $a_M = \min\{a, M\}$ and $b_M = \min\{b, M\}$.

Lemma 5.2 (cite[Lemma C.3]BrPa). *Let $1 < p < \infty$ and an increasing function $g : \mathbb{R} \rightarrow \mathbb{R}$. For each $a, b \in \mathbb{R}$, it holds*

$$|G(a) - G(b)|^p \leq |a - b|^{p-2}(a - b)(g(a) - g(b)),$$

with $G(t) = \int_0^t g'(\tau)^{1/p} d\tau$ for $t \in \mathbb{R}$.

Now, we establish the boundedness of weak solutions to problem (1.1). The proof relies on iterative techniques and careful estimates involving fractional Sobolev spaces.

Lemma 5.3. *If u is a weak solution to problem (1.1), then u belongs to $L^\infty(\Omega)$.*

Proof. The proof adapts methods from [2], which are inspired by the work in [5]. This method has been further incorporated and applied in the current context. Setting $u_l = \min\{(u-1)^+, l\}$ with $l > 0$. Also, let us define the function φ by $\varphi = (u_l + \theta)^r - \theta^r$ with $\theta > 0$ and $r \geq 1$. Combining lemma 5.1 with the triangle inequality and applying it to problem (1.1) with $\varphi \in X_0$, we obtain the key inequality

$$\begin{aligned} & \frac{rp^p}{(r+p-1)^p} \int_{\mathbb{R}^{2N}} |((u_l(x) + \theta))^{\frac{r+p-1}{p}}(x) - ((u_l(y) + \theta))^{\frac{r+p-1}{p}}(y)| \, dx \, dy \\ & \leq \int_{\mathbb{R}^{2N}} \frac{||u(x)| - |u(y)||^{p-2} (|u(x)| - |u(y)|) ((u_l(x) + \theta)^r - (u_l(y) + \theta)^r)}{|x - y|^{N+sp}} \, dx \, dy \\ & \leq \int_{\Omega} \lambda u^{-\gamma} ((u_l + \theta)^r - \theta^r) \, dx + \int_{\Omega} \frac{|u|^{p_s^*(t)-2} u}{|x|^t} ((u_l + \theta)^r - \theta^r) \, dx. \end{aligned} \quad (5.1)$$

Then with the help of the Hölder inequality combine with the support of u_l , we derive

$$\begin{aligned} & \int_{\Omega} \lambda u^{-\gamma} ((u_l + \theta)^r - \theta^r) \, dx + \int_{\Omega} \frac{|u|^{p_s^*(t)-2} u}{|x|^t} ((u_l + \theta)^r - \theta^r) \, dx \\ & = \int_{\{u \geq 1\}} \lambda u^{-\gamma} ((u_l + \theta)^r - \theta^r) \, dx + \int_{\{u \geq 1\}} \frac{|u|^{p_s^*(t)-2} u}{|x|^t} ((u_l + \theta)^r - \theta^r) \, dx \\ & \leq K_1 \int_{\{u \geq 1\}} \left(1 + \frac{|u|^{p_s^*(t)-2} u}{|x|^t}\right) ((u_l + \theta)^r - \theta^r) \, dx \\ & \leq 2K_1 \int_{\{u \geq 1\}} \frac{|u|^{p_s^*(t)-2} u}{|x|^t} ((u_l + \theta)^r - \theta^r) \, dx \\ & \leq 2K_1 |u|_{p_s^*(t)-1}^{p_s^*(t)-1} |(u_l + \theta)^r|_q \end{aligned} \quad (5.2)$$

with $K_1 = \max\{\lambda, 1\}$ and $q = p_s^*/(p_s^* + 1 - p_s^*(t))$. Furthermore, applying [19, Theorem 1] with inequality $(u_l + \theta)^r \leq (u_l + \theta)^{r+p-1} \theta^{1-p}$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^{2N}} \left| (u_l(x) + \theta)^{\frac{r+p-1}{p}} - (u_l(y) + \theta)^{\frac{r+p-1}{p}} \right|^p \, dx \, dy \\ & \geq C_{N,p,s} \left(\int_{\mathbb{R}^N} \left((u_l(x) + \theta)^{\frac{r+p-1}{p}} - \theta^{\frac{r+p-1}{p}} \right)^{p_s^*} \, dx \right)^{\frac{p}{p_s^*}} \\ & \geq \left(\frac{\theta}{2}\right)^{p-1} C_{N,p,s} \left(\int_{\mathbb{R}^N} ((u_l(x) + \theta)^{\frac{r+p-1}{p}})^{p_s^*} \, dx \right)^{\frac{p}{p_s^*}} - \theta^{r+p-1} |\Omega|^{\frac{p}{p_s^*}} \end{aligned} \quad (5.3)$$

with $C_{N,p,s}$ being a positive constant. From (5.2) and (5.3), we derive

$$\left(\int_{\mathbb{R}^N} ((u_l(x) + \theta)^{\frac{r+p-1}{p}})^{p_s^*} \, dx \right)^{\frac{p}{p_s^*}} \leq C_{N,p,s} \frac{|u|_{p_s^*(t)-1}^{p_s^*(t)-1}}{r} \left(\frac{r+p-1}{p\theta^{\frac{p-1}{p}}} \right)^{p-1} |(u_l + \theta)^r|_q + \theta^r |\Omega|^{\frac{p}{p_s^*}}. \quad (5.4)$$

Now, noting that for $r \geq 1$, we have

$$\frac{rp^p}{(r+p-1)^p} \geq \left(\frac{p}{r+p-1} \right)^{p-1}. \quad (5.5)$$

Applying (5.5), it is easy to see that

$$\theta^r |\Omega|^{\frac{p}{p_s^*}} \leq \frac{1}{r} \left(\frac{r+p-1}{p} \right)^p |\Omega|^{1-\frac{1}{q}-\frac{sp}{N}} |(u_l + \theta)^r|_q. \quad (5.6)$$

Therefore, using (5.6) in (5.4), we obtain

$$\begin{aligned} & \left(\int_{\mathbb{R}^N} ((u_l(x) + \theta)^{\frac{r+p-1}{p}})^{p_s^*} \, dx \right)^{\frac{p}{p_s^*}} \\ & \leq C_{N,p,s} \left(\frac{r+p-1}{p\theta^{\frac{p-1}{p}}} \right)^{p-1} |(u_l + \theta)^r|_q \times \left(\frac{|u|_{p_s^*(t)-1}^{p_s^*(t)-1}}{\theta^{p-1}} + |\Omega|^{1-\frac{1}{q}-\frac{sp}{N}} \right). \end{aligned} \quad (5.7)$$

We now fix $\theta > 0$ as

$$\theta = \left(|u|_{p_s^*}^{\frac{p_s^*(t)-1}{p-1}} + |\Omega|^{\frac{1}{p-1}(1-\frac{1}{q}-\frac{sp}{N})} \right)$$

Setting $w = rq$ and $\sigma = \frac{p_s^*}{pq} > 1$. Therefore, (5.7) can be formulated as

$$\left(\int_{\Omega} (u_l(x) + \theta)^{w\sigma} dx \right)^{\frac{1}{w\sigma}} \leq \left(C_{N,p,s} |\Omega|^{1-\frac{1}{q}-\frac{sp}{N}} \right)^{\frac{q}{w}} \left(\frac{q}{w} \right)^{\frac{q}{w}} \left(\frac{w+qp-q}{q'p} \right)^{\frac{pq}{w}} |(u_l + \theta)|_w. \quad (5.8)$$

So, we now iterate (5.8) using

$$w_0 = 1 \quad w_{k+1} = w_k \sigma = \sigma^{k+1}.$$

By initiating from $k = 0$ at step k , we can express the inequality (5.8) as

$$|(u_l + \theta)|_{w_{k+1}} \leq \left(C_{N,p,s} |\Omega|^{1-\frac{1}{q}-\frac{sp}{N}} \right)^{\sum_{i=0}^k \frac{q}{w_i}} \prod_{i=0}^k \left(\frac{q}{w_i} \right)^{q/w_i} \left(\frac{w_i + qp - q}{qp} \right)^{pq/w_i} |(u_l + \theta)|_w. \quad (5.9)$$

We now notice that w_k diverges at infinity, and furthermore

$$\sum_{k=0}^{\infty} \frac{1}{w_k} = \sum_{k=0}^{\infty} \frac{1}{\sigma^k} = \frac{\sigma}{\sigma-1} \quad \text{and} \quad \prod_{k=0}^{\infty} \left(\frac{q}{w_k} \right)^{q/w_k} \left(\frac{w + qp - q}{qp} \right)^{pq/w} < +\infty.$$

Through an infinite iteration of inequality (5.9), we arrive at

$$|u_l|_{\infty} \leq \left(C_{N,p,s} |\Omega|^{1-\frac{1}{q}-\frac{sp}{N}} \right)^{q \frac{\sigma}{\sigma-1}} |(u_l + r)|_q.$$

Given that $u_l \leq (u-1)^+$, we obtain

$$|u_l|_{\infty} \leq \left(C_{N,p,s} |\Omega|^{1-\frac{1}{q}-\frac{sp}{N}} \right)^{q \frac{\sigma}{\sigma-1}} \left(|(u-1)^+|_q + \theta |\Omega|^{1/q} \right).$$

Taking $l \rightarrow \infty$, it follows that

$$|(u-1)^+|_{\infty} \leq \left(C_{N,p,s} |\Omega|^{1-\frac{1}{q}-\frac{sp}{N}} \right)^{q \frac{\sigma}{\sigma-1}} \left(|(u-1)^+|_q + \theta |\Omega|^{1/q} \right).$$

Thus, in particular, we conclude that $u \in L^{\infty}(\Omega)$. $\square$

Proof of Theorem 1.3. The proof follows from Lemma 5.3 and [17, Theorem 6.5]. $\square$

REFERENCES

- [1] K. Kefi, M. Kratou, K. Saoudi; *Combined effects of critical and singular nonlinearities in fractional problems*, submitted (2025).
- [2] S. Rashed Mohamed Alotaibi, K. Saoudi; *Regularity and multiplicity of solutions for a nonlocal problem with critical Sobolev-Hardy nonlinearities*, Journal of the Korean Mathematical Society, **57** (3) (2020), 747-775.
- [3] B. Barrios, I. De Bonis, M. Maria, I. Peral; *Semilinear problems for the fractional laplacian with a singular nonlinearity*, Open Math., **13** (2015), 390-407.
- [4] L. Brasco, E. Lindgren, E. Parini; *The fractional Cheeger problem*, Interfaces and Free Boundaries, **16** (2) (2014), 419-458.
- [5] L. Brasco, E. Parini; *The second eigenvalue of the fractional p-laplacian*, Adv. Calc. Var., **9** (4) (2016), 323-355.
- [6] A. Canino, L. Montoro, B. Sciunzi, M. Squassina; *Nonlocal problems with singular nonlinearity*, Bulletin des Sciences Mathématiques, **141** (3) (2017), 223-250.
- [7] W. Chen, S. Mosconi, M. Squassina; *Nonlocal problems with critical Hardy non-linearity*, Journal of Functional Analysis, **275** (11) (2018), 3065-3114.
- [8] A. Daoues, A. Hammami, K. Saoudi; *Multiplicity results of nonlocal singular PDEs with critical Sobolev-Hardy exponent*, Electronic Journal of Differential Equations, **2023** (10) (2023), 1-19.
- [9] A. Daoues, A. Hammami, K. Saoudi; *Existence and multiplicity of solutions for a nonlocal problem with critical Sobolev-Hardy nonlinearities*, Mediterranean Journal of Mathematics, **17** (167) (2020), 166-188.
- [10] L. Del Pezzo, A. Quaas; *Global bifurcation for fractional p-Laplacian and application*, Z. Anal. Anwend., **354** (2016), 411-447.
- [11] A. Ghanmi, K. Saoudi; *The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator*, Fractional Differential Calculus, **6** (2) (2016), 201-217.
- [12] A. Ghanmi, K. Saoudi; *A multiplicity results for a singular problem involving the fractional p-Laplacian operator*, Complex. Var. Elliptic Equ., **61** (2016), 1199-1216.
- [13] N. Ghoussoub, S. Shakerian; *Borderline variational problems involving fractional Laplacians and critical singularities*, Adv. Nonlinear Stud. **15** (3) (2015), 527-555.

- [14] A. Iannizzotto, S. Mosconi, M. Squassina; *Global Hölder regularity for the fractional p -Laplacian*, Revista Matematica Iberoamericana, **32** (4) (2016), 1353-1392.
- [15] E. Lindgren, P. Lindqvist; *Fractional eigenvalues*, Calc. Var. Partial Differential Equations, **49** (2014), 795-826.
- [16] S. Mosconi, M. Squassina; *Nonlocal problems at nearly critical growth*, Nonlinear Analysis: Theory, Methods & Applications, **136** (2016), 84-101.
- [17] T. Mukherjee, K. Sreenadh; *On Dirichlet problem for fractional p -Laplacian with singular non-linearity*, Advances in Nonlinear Analysis, **8** (2019), 52-72.
- [18] T. Mukherjee, K. Sreenadh; *Fractional elliptic equations with critical growth and singular nonlinearities*, Electronic Journal of Differential Equations, **2016** (54) (2016), 1-23.
- [19] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., **136** (2012), 521-573.
- [20] K. Saoudi, S. Ghosh, D. Choudhuri; *Multiplicity and Hölder regularity of solutions for a nonlocal elliptic PDE involving singularity*, Journal of Mathematical Physics, **60** (10) (2019), 101509, 28 pp.
- [21] R. Servadei, E. Valdinoci; *Mountain pass solutions for non-local elliptic operators*, Journal of Mathematical Analysis and Applications, **389** (2) (2012), 887-898.
- [22] R. Servadei, E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discrete and Continuous Dynamical Systems, **33** (5) (2013), 2105-2137.
- [23] M. Q. Xiang, B.L. Zhang, V. Radulescu; *Existence of solutions for perturbed fractional p -Laplacian equations*, J. Differ. Equ., **260** (2) (2016), 1392-1413.
- [24] Y. Yan; *The Brezis Nirenberg problem for the fractional p -Laplacian involving critical Hardy Sobolev exponents*, Ithaca: Cornell University Library, (2017) <https://arxiv.org/abs/1710.04654>.

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