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VARIATIONAL APPROACH FOR THE n-DIMENSIONAL STATIONARY NAVIER-STOKES EQUATIONS WITH A DAMPING TERM

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Abstract. harvesting effort We study the n-dimensional stationary Navier-Stokes equations with a damping term by developing a new general minimax principle. This principle is sufficiently broad to be applied in various contexts, and here it is used to establish the existence of weak solutions for both linear and nonlinear damping, without restrictions on the damping constant. The damping term, which models physical effects such as porous media flow, drag, friction, and dissipation, also provides a mathematical advantage by improving the regularity of solutions compared to the classical Navier-Stokes system.

Our results cover the cases of positive and negative damping constants and yield existence theorems under different ranges of p and spatial dimensions. In particular, we prove solvability even in borderline situations, such as when $\mu = -\lambda_1$, where coercivity is lost and traditional minimax arguments typically fail. The general minimax framework we introduce is flexible and can be adapted to other nonlinear PDEs, especially when symmetry or structural properties are involved.

1. Introduction

In this article we study the n-dimensional stationary Navier-Stokes equations with a damping term,

$$\begin{gathered} -\Delta u + (u\cdot\nabla)u + \mu|u|^{p-2}u = f(x) + \nabla P \quad \forall x\in\Omega \\ \nabla u = 0 \quad \forall x\in\Omega \\ u = 0 \quad \forall x\in\partial\Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $p \geq 1$ and $\mu \in \mathbb{R}$. We address both linear and nonlinear dampings and we are allowing μ to take both positive and negative values. Here $u = (u_1, u_2, \dots, u_n)$ is the velocity, P stand for scalar pressure and f is the external force.

The analysis of the Navier-Stokes equations is a central theme in mathematical fluid mechanics. For the classical evolutionary system without damping, the existence of global weak solutions was established by Leray [11] and Hopf [6], but the uniqueness of weak solutions and the global existence of strong solutions remain open problems. These longstanding challenges have motivated researchers to study modified models where additional terms improve the mathematical structure of the equations.

One such modification is the inclusion of a damping term of the form $\mu|u|^{p-2}u$. From the physical viewpoint, this term models resistance to motion and arises naturally in contexts such as porous media flows, drag or friction effects, and other dissipative mechanisms (see [7, 23]). From the mathematical viewpoint, damping often leads to better control of solutions, sometimes yielding results that are out of reach for the standard Navier-Stokes equations. This has stimulated extensive research on the evolutionary Navier-Stokes system with damping, leading to results on global existence, regularity, decay rates, attractors, and stability (see, e.g., [2, 8, 9, 10, 20, 25, 24, 26]).

In contrast, stationary Navier-Stokes equations with damping have received comparatively less attention. Some existence and uniqueness results are available for the case $\mu > 0$ [13], and there is

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growing literature on numerical approaches (see [12, 14, 18, 19, 27]). However, general variational approaches capable of handling both positive and negative damping, as well as borderline cases where coercivity is lost, are still lacking.

In this work, we develop a new minimax principle on convex subsets of Banach spaces and apply it to the n-dimensional stationary Navier-Stokes equations with damping. Our framework is broad and flexible: depending on the choice of convex set, it allows us to obtain solutions with additional structural properties, such as symmetry. With this method we prove existence results for both linear ($p = 2$) and nonlinear damping, without restriction on the damping constant, and even in the critical case $\mu = -\lambda_1$. This general minimax principle is of independent interest and may find applications to other nonlinear PDEs beyond the Navier-Stokes system.

In this work, we consider the Banach space

$$V = \{u \in H_0^1(\Omega) \cap L^p(\Omega), \nabla.u = 0\},$$

equipped with the norm

$$\|u\| := \|u\|_{H_0^1(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Let Λu be the operator $\Lambda u := (u.\nabla)u$, and K be a convex and weakly closed subset of V. We shall define $M : K \times K \to \mathbb{R}$ as follows,

$$M(u, v) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \int_\Omega (\Lambda u - f(x) - \frac{1}{p}|u|^{p-2}u)(u - v)\,dx, \tag{1.2}$$

where $f \in L^2(\Omega)$. The following variational principle on general convex sets K is a key component in our arguments. It is also broad enough to deal with various other cases by choosing a convex set K accordingly.

Theorem 1.1. *Let K be a convex and weakly closed subset of V. Assume that the following two assertions hold:*

(i) *There exists $\bar{u} \in K$ for such that*

$$M(\bar{u}, v) \le 0, \quad \forall v \in K,$$

where M is defined in (1.2).

(ii) *There exists $\bar{v} \in K$ such that*

$$-\Delta\bar{v} + \nabla P = f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda\bar{u},$$

in the weak sense, i.e.,

$$\int_\Omega \nabla\bar{v}.\nabla\eta\,dx = \int_\Omega (f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda\bar{u})\eta\,dx, \quad \forall \eta \in V.$$

Then $\bar{u} \in K$ is a weak solution of the equation

$$-\Delta u + \Lambda u = f(x) + \nabla P + |u|^{p-2}u\,.$$

It is worth noting that the primary consequence of this theorem centres on the choice of K, i.e., by choosing an appropriate K, one is able to establish the existence of a solution enjoying all the properties induced by the set K (see Remark 1.5 for an application where the problem (1.1) has some symmetry properties). Also, Condition (i) in Theorem 1.1 is most of the time guarantied because of the well-known Ky Fan's min-max principle by Brezis-Nirenberg-Stampacchia [1]. We provide more details of how to apply the above theorem in the sequel. As an application of the above Theorem we first prove the following result.

Theorem 1.2. *Let Ω be a bounded C^2 domain in $\mathbb{R}^n$ and $\mu < 0$. Then for $f \in L^2(\Omega)$ small enough, the following statements hold:*

(i) *For $n \le 4$ and $p > 2$, the Navier-Stokes equation* (1.1) *has a solution $u \in W^{2,2}(\Omega)$.*

(ii) *For $5 \le n \le 7$ and $2 < p \le \frac{2n-4}{n-4}$, the Navier-Stokes equation* (1.1) *has a solution in $W^{2,2}(\Omega)$.*

In both cases there exists a scalar function $P:\Omega\to\mathbb{R}$ and a constant $C>0$ such that

$$\|\Delta u\|_{L^2(\Omega)}+\|\nabla P\|_{L^2(\Omega)}\le C\big(\|f\|_{L^2(\Omega)}+\|u\|^{p-1}_{L^{2(p-1)}}+\|u\|^2_{W^{1,2^*}(\Omega)}\|u\|^2_{L^{\frac{n}{2}}(\Omega)}\big). \tag{1.3}$$

where $2^*=2n/(n-2)$.

When the constant μ in the damping term is non-negative we can cover higher values for p as shown in the following theorem.

Theorem 1.3. *Let $\Omega\subset\mathbb{R}^n$ be a smooth bounded domain and $\mu>0$. Suppose that $p\ge 1$ and $f\in L^2(\Omega)$. Then there exists $u\in V$ such that the following holds:*

(i) *If $n\ge 2$, then*

$$\int_\Omega \nabla u.\nabla\eta\,dx+\int_\Omega |u|^{p-2}u\,\eta+\int_\Omega \Lambda u\,\eta\,dx=\int_\Omega f(x)\eta\,dx,\quad \forall\eta\in C^1_c(\Omega),\ \text{with } \nabla.\eta=0.$$

(ii) *If $n\le 4$ or $p\ge 4$, then*

$$\int_\Omega \nabla u.\nabla\eta\,dx+\int_\Omega |u|^{p-2}u\,\eta\,dx+\int_\Omega \Lambda u\,\eta\,dx=\int_\Omega f(x)\eta\,dx,\quad \forall\eta\in V.$$

We would like to remark that the solution we are getting in part (i) of the above theorem is weaker than the one we are getting in part (ii). This is so because all the test functions η in part (i) are coming from $C^1_c(\Omega)$ on contrary to part (ii) where the test functions η live in a less regular space $H^1_0(\Omega)\cap L^p(\Omega)$.

We shall also deal with the linear damping term where $p=2$ for positive and negative values of μ. To state our result we first recall the following standard fact about the first eigenfunction of the Laplacian on bounded domains. Recall that

$$\lambda_1=\min_{\psi\in H^1_0(\Omega)\setminus\{0\}}\frac{\int_\Omega|\nabla\psi|^2\,dx}{\int_\Omega|\psi|^2\,dx}.$$

where the minimum is taken over all $\psi: H^1_0(\Omega)\to\mathbb{R}$. Note that in Theorem 1.3 we have already covered the case $\mu>0$. Here is our result for the linear case where we are allowing negative values for μ.

Theorem 1.4. *Let Ω be smooth bounded domain in $\mathbb{R}^n$ and $p=2$. Assume that $-\lambda_1\le\mu<0$, and $f\in L^2(\Omega)$. Then there exists $u\in V$ such that the following assertions hold:*

(i) *If $n\ge 2$, then*

$$\int_\Omega \nabla u.\nabla\eta\,dx+\mu\int_\Omega u\eta\,dx+\int_\Omega \Lambda u\,\eta\,dx=\int_\Omega f(x)\eta\,dx,\quad \forall\eta\in C^1_c(\Omega),\quad \text{with } \nabla.\eta=0.$$

(ii) *If $n\le 4$, then*

$$\int_\Omega \nabla u.\nabla\eta\,dx+\mu\int_\Omega u\,\eta\,dx+\int_\Omega \Lambda u\,\eta\,dx=\int_\Omega f(x)\eta\,dx,\quad \forall\eta\in V.$$

The highlight of the above theorem is the case where $\mu=-\lambda_1$ in which case one losses the coercivity required in most minimax arguments.

Remark 1.5. Even though our main objective in this paper is to prove existence results having a damping term in mind, we would like emphasize that the applications of Theorem 1.1 goes well beyond this goal. In light of this remark, let us define the maps $\pi_1,\pi_2,\pi_3:\Omega\subset\mathbb{R}^3\to\Omega$ as follow:

$$\begin{aligned}\pi_1(x_1,x_2,x_3)&=(-x_1,x_2,x_3),\\ \pi_2(x_1,x_2,x_3)&=(x_1,-x_2,x_3),\\ \pi_3(x_1,x_2,x_3)&=(x_1,x_2,-x_3).\end{aligned}$$

We consider the 3D case of the stationary Navier-Stokes equations with damping presented in equation (1.1). Assume that Ω is invariant under the maps $\pi_1,\pi_2,\pi_3:\Omega\to\Omega$. Moreover, assume that K_S is a subset of V containing all $u \in V$ with the following properties:

$$\begin{aligned} u_1(x_1, x_2, x_3) &= -u_1(-x_1, x_2, x_3), \\ u_2(x_1, x_2, x_3) &= u_2(-x_1, x_2, x_3), \\ u_3(x_1, x_2, x_3) &= u_3(-x_1, x_2, x_3). \end{aligned} \tag{1.4}$$

Furthermore, assume that $f(x) \in L^2(\Omega)$ also holds the same properties; i.e.,

$$\begin{aligned} f_1(x_1, x_2, x_3) &= -f_1(-x_1, x_2, x_3), \\ f_2(x_1, x_2, x_3) &= f_2(-x_1, x_2, x_3), \\ f_3(x_1, x_2, x_3) &= f_3(-x_1, x_2, x_3). \end{aligned}$$

Then, the solution $u = (u_1, u_2, u_3)$ obtained in Theorems 1.2, 1.3 and 1.4 is symmetric in the sense (1.4). Indeed, the symmetry of solutions follows from the uniqueness of the solution to the corresponding linear problem.

The article is organized as follows. In section 2, we prove Theorems 1.1 and 1.2 through a minimax principle. Section 3 is devoted to the proof of our results in Theorems 1.3 and 1.4.

2. A minimax principle and the proof of Theorem 1.2

In this section, we first prove an adapted version of variational principle presented in Theorem 1.1 which is applicable specifically to our problem when $\mu < 0$, and $p > 2$. Afterwards, we proceed with the proof of Theorem 1.2.

We consider the Banach space $V = \{u \in H_0^1(\Omega) \cap L^p(\Omega), \nabla.u = 0\}$ equipped with the norm

$$\|u\| := \|u\|_{H_0^1(\Omega)} + \|u\|_{L^p(\Omega)}.$$

Let Λu be the operator $\Lambda u := (u.\nabla)u$, that is

$$\langle \Lambda u, v\rangle = \int_\Omega (\Lambda u)v = \int_\Omega \sum_{j,k=1}^n u_k \frac{\partial u_j}{\partial x_k} v_j.$$

Let K be a convex and weakly closed subset of V. As stated in Theorem 1.1 we shall consider the functional $M : K \times K \to \mathbb{R}$ given in (1.2).

Proof of Theorem 1.1. It follows from condition (i) in the theorem that there exists $\bar{u} \in K$ such that

$$\frac{1}{2}\int_\Omega |\nabla \bar{u}|^2\,dx - \frac{1}{2}\int_\Omega |\nabla v|^2\,dx \le \int_\Omega (f(x) + \frac{1}{p}|\bar{u}|^p - \Lambda \bar{u})(\bar{u} - v)\,dx, \quad \forall v \in K. \tag{2.1}$$

It also follows from (ii) that there exists $\bar{v} \in K$ such that

$$\int_\Omega \nabla \bar{v}.\nabla \eta\,dx = \int_\Omega (f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda \bar{u})\eta\,dx, \quad \forall \eta \in V. \tag{2.2}$$

Substituting $\eta = \bar{u} - \bar{v}$ in the latter equality gives

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\,dx = \int_\Omega (f(x) + |\bar{u}|^{p-2}\bar{u} - \Lambda \bar{u})(\bar{u} - \bar{v})\,dx, \quad \forall \eta \in V. \tag{2.3}$$

Setting $v = \bar{v}$ in (2.1) and taking into account the equality (2.3) we obtain that

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\,dx \ge \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2\,dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2\,dx. \tag{2.4}$$

On the other hand, it follows from the convexity of $g(t) = \frac{1}{2}t^2$ that

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\,dx \le \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2\,dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2\,dx. \tag{2.5}$$

Inequalities (2.4) and (2.5) together imply that

$$\int_\Omega \nabla \bar{v}.\nabla(\bar{u} - \bar{v})\,dx = \frac{1}{2}\int_\Omega |\nabla \bar{u}|^2\,dx - \frac{1}{2}\int_\Omega |\nabla \bar{v}|^2\,dx.$$

Therefore,

$$\int_\Omega |\nabla\bar{u} - \nabla\bar{v}|^2 = 0,$$

from which it follows that $\bar{v} = \bar{u}$ for a.e. $x \in \Omega$. Hence, the equality (2.2) proves the desired result. $\square$

We shall apply Theorem 1.1 to prove the existence of solution in Theorem 1.2. The convex subset K of V required in Theorem 1.1 is defined by

$$K(r) = \{u \in V : \|u\|_{W^{2,2}(\Omega)} \le r\}, \tag{2.6}$$

for some $r > 0$ to be determined. To see that $K(r)$ is weakly closed, we present the proof of this statement in the following lemma.

Lemma 2.1. *Let $r > 0$ be fixed. The set*

$$K(r) = \{u \in V : \|u\|_{W^{2,2}(\Omega)} \le r\}$$

is weakly closed in V.

Proof. Let $\{u_m\}$ be a sequence in $K(r)$ such that $u_m \rightharpoonup u$ weakly in V. Then there exists a subsequence of u_m, denoted by u_m again such that $u_m \to u$ a.e in Ω. On the other hand, $\|u_m\|_{W^{2,2}(\Omega)} \le r$ for all $m \in \mathbb{N}$ and so $\{u_m\}$ is bounded in $W^{2,2}(\Omega)$. Going if necessary to a subsequence, there exists $\bar{u} \in W^{2,2}(\Omega)$ such that $u_m \rightharpoonup \bar{u}$ weakly in $W^{2,2}(\Omega)$ and $u_m(x) \to \bar{u}(x)$ for a.e. $x \in \Omega$. It follows then $u(x) = \bar{u}(x)$ for a.e. $x \in \Omega$. Thus $u_m \rightharpoonup u$ weakly in $W^{2,2}(\Omega)$. Now from the weak lower semi-continuity of the norm in $W^{2,2}(\Omega)$ follows that

$$\|u\|_{W^{2,2}(\Omega)} \le \liminf_{m\to\infty} \|u_m\|_{W^{2,2}(\Omega)} \le r,$$

which means that $u \in K(r)$. $\square$

To apply Theorem 1.1, we need to verify both conditions (i) and (ii) in this Theorem. To verify condition (i) we use the following version of the well-known Ky Fan's min-max principle [1]. We refer to [5, Lemma 12.1] for a proof.

Lemma 2.2. *Let E be a closed convex subset of a reflexive Banach space H, and consider $M : E \times E \to \bar{\mathbb{R}}$ to be a functional such that:*

1. *For each $y \in E$, the map $x \to M(x, y)$ is weakly lower semi- continuous on E.*
2. *For each $x \in E$, the map $y \to M(x, y)$ is concave on E.*
3. *There exists $\gamma \in \mathbb{R}$ such that $M(x, x) \le \gamma$ for every $x \in E$.*
4. *There exists a $y_0 \in E$ such that $E_0 = \{x \in E : M(x, y_0) \le \gamma\}$ is bounded.*

Then there exits $\bar{x} \in E$ such that $M(\bar{x}, y) \le \gamma$ for all $y \in E$.

One of the requirements in Lemma 2.2 is the lower semi-continuity of $M(. , v)$ for a fixed v. To verify that, we begin with the following Lemma.

Lemma 2.3. *For each $v \in K$, the map $u \to \langle \Lambda u, v\rangle$ is weakly continuous on K for the values of n, p in Theorem 1.2.*

Proof. Fix $v \in K$, and let $u^m \rightharpoonup u$ weakly in K. We have

$$\begin{aligned}
\big|\langle \Lambda u^m, v\rangle - \langle \Lambda u, v\rangle\big| &= \Big|\sum_{j,k=1}^n \int_\Omega \big(u_k^m \frac{\partial u_j^m}{\partial x_k} v_j - u_k \frac{\partial u_j}{\partial x_k} v_j\big)\,dx\Big| \\
&= \Big|\sum_{j,k=1}^n \int_\Omega \big((u_k^m - u_k)\frac{\partial u_j^m}{\partial x_k} v_j + u_k \frac{\partial (u_j^m - u_j)}{\partial x_k} v_j\big)\,dx\Big| \\
&\le \sum_{j,k=1}^n \int_\Omega \Big|(u_k^m - u_k)\frac{\partial u_j^m}{\partial x_k} v_j\Big| + \sum_{j,k=1}^n \int_\Omega \Big|u_k \frac{\partial (u_j^m - u_j)}{\partial x_k} v_j\Big|\,dx.
\end{aligned}$$

On the other hand, by Hölder inequality we conclude that

$$\sum_{j,k=1}^{n}\int_\Omega\Big|(u_k^m-u_k)\frac{\partial u_j^m}{\partial x_k}v_j\Big| \le\|(u^m-u)v\|_{L^2}\|\nabla u^m\|_{L^2} \le\|u^m-u\|_{L^4}\|v\|_{L^4}\|\nabla u^m\|_{L^2}.$$

Therefore,

$$\big|\langle\Lambda u^m,v\rangle-\langle\Lambda u,v\rangle\big| \le\|u^m-u\|_{L^4}\|v\|_{L^4}\|\nabla u^m\|_{L^2}+\sum_{j,k=1}^{n}\int_\Omega\Big|u_k\frac{\partial(u_j^m-u_j)}{\partial x_k}v_j\Big|\,dx.$$

Moreover, since the space $W^{2,2}(\Omega)$ is compactly imbedded into $L^4(\Omega)$, for all $n\le 7$, it follows that $u^m\to u$ strongly in $L^4(\Omega)$. Furthermore, since u,v are in $W^{2,2}(\Omega)$, we deduce from Hölder's inequality that $u_kv_j\in L^2(\Omega)$. Finally, since $\nabla u^m\rightharpoonup\nabla u$ weakly in K, by definition of weak convergence the result follows. □

Lemma 2.4. *Fro each $v\in K$, the map $u\to M(u,v)$ is weakly lower semi-continuous on K for the values of n,p in Theorem 1.2.*

Proof. Let $v\in K$ be fixed and $u_m\rightharpoonup u$ weakly in K. Since $<\Lambda u,u>=0$ resulting from (1.2) we have

$$\begin{aligned} M(u,v)=\frac12\int_\Omega|\nabla u|^2\,dx-\langle\Lambda u,v\rangle-\int_\Omega f(x)u\,dx-\frac1p\int_\Omega|u|^p\,dx\\ +\frac1p\int_\Omega|u|^{p-2}uv\,dx+\frac12\int_\Omega|\nabla v|^2\,dx+\int_\Omega f(x)v\,dx,\end{aligned} \tag{2.7}$$

Now we shall verify lower semi-continuity of every single part in (2.7) separately. Note that the last two terms in (2.7) are constant with respect to u.

- Since the function $g(u)=|u|^2$ is convex, it can easily be shown that
$$\int_\Omega|\nabla u|^2\,dx\le\liminf_{m\to\infty}\int_\Omega|\nabla u^m|^2\,dx.$$
that implies the map $u\to\int_\Omega|\nabla u|^2\,dx$ is weakly lower semi-continuous.
- The map $u\to-\int_\Omega\Lambda u.v\,dx$ is weakly lower semi-continuous by Lemma 2.3.
- Since $f\in L^2(\Omega)$, applying the definition of weak convergence leads to
$$\int_\Omega f(x)\ u\,dx=\liminf_{n\to\infty}\int_\Omega f(x)u^m\,dx.$$
- The map $u\to\int_\Omega|u|^p\,dx$ is weakly lower semi-continuous for n,p in Theorem 1.2 because
  (i) if $n\le 4$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^p(\Omega)$ for all $p>2$, and
  (ii) if $5\le n\le 7$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^p(\Omega)$ for all $2<p<\frac{2n}{n-4}$.
  It then follows for both of cases that
$$\lim_{n\to\infty}\int_\Omega|u_m|^P=\int_\Omega|u|^P\,dx,$$
- The map $u\to\int_\Omega|u|^{p-2}uv\,dx$ is weakly lower semi-continuous for n,p in Theorem 1.2 because
  (i) if $n\le 4$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^{2(p-1)}(\Omega)$ for all $p>2$, and
  (ii) if $5\le n\le 7$, then $W^{2,2}(\Omega)$ is compactly imbedded into $L^{2(p-1)}(\Omega)$ for all $2<p<\frac{2n-4}{n-4}$.
  In both cases, we have $|u|^{p-2}u\in L^2(\Omega)$ from which we deduce that the map $u\to\int_\Omega|u|^{p-2}uv\,dx$ is continuous functional and
$$\lim_{n\to\infty}\int_\Omega|u_m|^{p-2}uv\,dx=\int_\Omega|u|^{p-2}uv\,dx.$$

This completes the proof. □

We are now in a position to state the following result addressing condition (i) in Theorem 1.2.

Lemma 2.5. *Let $K = K(r)$ be a convex and weakly closed subset of V defined in (2.6). Let $M : K \times K \to \mathbb{R}$ be defined as (1.2) and n, p as in Theorem 1.2. Then there exists $\bar{u} \in K$ such that*

$$M(\bar{u}, v) \leq 0 \quad \forall v \in K.$$

Proof. We shall show that the function M satisfies all the conditions of the Ky Fan's Min-Max Principle presented in Lemma 2.2. The condition (1) is provided by Lemma 2.4. For each $u \in K$, the map $v \to M(u, v)$ is concave on K since $M(u, v)$ is a linear functional with respect to v except $-\frac{1}{2} \int_\Omega |\nabla v|^2\,dx$, which is in fact concave. Also we have $M(u, u) = 0 = \gamma$ for every $u \in K$. Finally, since $u \in K$ we have that $\|u\|_{W^{2,2}(\Omega)} \leq r$. Thus, we can conclude that $\{u \in K : M(u, v) \leq 0\}$ is bounded. It now follows by Lemma 2.2 that there exists $\bar{u} \in K$ such that

$$M(\bar{u}, v) \leq 0 \quad \forall v \in K,$$

as desired. □

Our next task consists of verifying condition (ii) in Theorem 1.2. To do this, we start with the following two lemmas, which provide us the required estimates. Hereafter C will denote a positive constant, not necessarily the same one.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p$. Then for any $u \in K(r)$ we have*

$$\big\|f + |u|^{p-1}u - \Lambda u\big\|_{L^2(\Omega)} \leq \big(\|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}} + \|u\|^2_{W^{1,2^*}(\Omega)}\|u\|^2_{L^{\frac{n}{2}}(\Omega)}\big).$$

Proof. Let $u \in K(r)$. By Hölder's inequality we have

$$\begin{aligned}
\big\|f + |u|^{p-2}u - \Lambda u\big\|_{L^2(\Omega)} &\leq \|f\|_{L^2(\Omega)} + \|u^{p-1}\|_{L^2(\Omega)} + \|\Lambda u\|_{L^2(\Omega)} \\
&\leq \|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|\nabla u\|^2_{L^{2^*}(\Omega)}\|u\|^2_{L^{\frac{n}{2}}(\Omega)}, \quad (\text{where } 2^* = \frac{2n}{n-2}) \\
&\leq \|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}(\Omega)} + \|u\|^2_{W^{1,2^*}(\Omega)}\|u\|^2_{L^{\frac{n}{2}}(\Omega)}.
\end{aligned}$$

as desired. □

Lemma 2.7. *Let $p > 2$ and $C > 0$ be given. Then there exists $0 < r \in \mathbb{R}$ which satisfies*

$$C(\|f\|_{L^2(\Omega)} + r^{p-1} + r^4) \leq r,$$

where $\|f\|_{L^2(\Omega)}$ be small enough.

Proof. Since $p > 2$, we can choose r such that

$$C(r^{p-1} + r^4) \leq \frac{r}{2}.$$

Now if $C\|f\|_{L^2(\Omega)} \leq \frac{r}{2}$ then we have

$$C(\|f\|_{L^2(\Omega)} + r^{p-1} + r^4) \leq r,$$

as desired. □

Here is another useful results that we shall use in the sequel. See [3, Theorem 1.2] for a more general version of the following result.

Lemma 2.8. *If $g \in L^2(\Omega)$, then there exists $u \in W^{2,2}(\Omega) \cap H^1_0(\Omega)$, a scalar function $P : \Omega \to \mathbb{R}$ and a constant C such that*

$$\Delta u + \nabla P = g, \quad \nabla \cdot u = 0, \quad u|_{\partial\Omega} = 0,$$

and

$$rVert\Delta u\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \leq C\|g\|_{L^2(\Omega)}.$$

The following inequality is proved in [4, Lemma 9.17].

Lemma 2.9. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ and let the operator $Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$ be strictly Elliptic in Ω with coefficients $a_{ij} \in C(\Omega)$, $b_i, c \in L^\infty(\Omega)$, with $i, j = 1, \dots, n$ and $c \le 0$. Then there exists a positive constant C (independent of u) such that*

$$rVertu\|_{W^{2,p}(\Omega)} \le C\|Lu\|_{L^p(\Omega)},$$

for all $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $1 < p < \infty$.

Here comes a direct consequence of Lemma 2.9.

Corollary 2.10. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$. Then there exists a constant C such that*

$$\|u\|_{W^{2,2}(\Omega)} \le C\|\Delta u\|_{L^2(\Omega)},$$

for all $u \in W^{2,2}(\Omega) \cap H_0^1(\Omega)$.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Without loss of generality we may suppose that $\mu = -1$. We define $K = K(r)$ for $r > 0$ to be determined presently. By Lemma 2.5 we have the existence of a non-trivial $\bar{u} \in K$ such that

$$M(\bar{u}, v) \le 0 \quad \forall v \in K.$$

Now we shall show the existence of $\bar{v}$ that satisfy condition (ii) in the theorem. Consider

$$g(x) = f + |\bar{u}|^{p-1}\bar{u} - \Lambda\bar{u}.$$

Thus we have to show there exists $\bar{v} \in K$ that the following equation holds in the weak sense,

$$-\Delta v + \nabla P = g(x). \tag{2.8}$$

By Lemma 2.8 there exists $\bar{v} \in V$ which satisfies (2.8) and

$$\|\Delta\bar{v}\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Omega)}. \tag{2.9}$$

It is sufficient to show that $\bar{v} \in K$. The estimate (2.9) together with Lemma 2.6 imply that

$$\begin{aligned} \|\Delta\bar{v}\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} &\le C\|f + |\bar{u}|^{p-1}\bar{u} - \Lambda\bar{u}\| \\ &\le C\big(\|f\|_{L^2(\Omega)} + \|\bar{u}\|^{p-1}_{L^{2(p-1)}} + \|\bar{u}\|^2_{W^{1,2^*}(\Omega)}\|\bar{u}\|^2_{L^{\frac{n}{2}}(\Omega)}\big). \end{aligned} \tag{2.10}$$

On the other hand, Corollary 2.10 together with (2.10) yield that

$$\begin{aligned} \|\bar{v}\|_{W^{2,2}(\Omega)} \le C\|\Delta\bar{v}\|_{L^2(\Omega)} &\le C\big(\|\Delta\bar{v}\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \\ &\le C\big(\|f\|_{L^2(\Omega)} + \|\bar{u}\|^{p-1}_{L^{2(p-1)}} + \|\bar{u}\|^2_{W^{1,2^*}(\Omega)}\|\bar{u}\|^2_{L^{\frac{n}{2}}(\Omega)}\big). \end{aligned} \tag{2.11}$$

From the imbeddings of $W^{2,2}(\Omega) \hookrightarrow L^{2(p-1)}$ and $W^{2,2}(\Omega) \hookrightarrow W^{1,2^*}(\Omega)$ we obtain from (2.11) that

$$\|\bar{v}\|_{W^{2,2}(\Omega)} \le C\big(\|f\|_{L^2(\Omega)} + \|\bar{u}\|^{p-1}_{w^{2,2}(\Omega)} + \|\bar{u}\|^2_{W^{2,2}(\Omega)}\|\bar{u}\|^2_{W^{2,2}(\Omega)}\big). \tag{2.12}$$

Let r be as in Lemma 2.7 for C given in the last inequality above. The inequality (2.12) and Lemmas 2.7 yield that

$$\|\bar{v}\|_{W^{2,2}(\Omega)} \le C(\|f\|_{L^2(\Omega)} + r^{p-1} + r^4) \le r,$$

where $\|f\|_{L^2(\Omega)}$ is small enough. That means $\bar{v} \in K$ and so $\bar{v} = \bar{u}$. This completes the proof of (i) and (ii). Now the inequality (2.10) concludes that

$$\|\Delta u\|_{L^2(\Omega)} + \|\nabla P\|_{L^2(\Omega)} \le C\big(\|f\|_{L^2(\Omega)} + \|u\|^{p-1}_{L^{2(p-1)}} + \|u\|^2_{W^{1,2^*}(\Omega)}\|u\|^2_{L^{\frac{n}{2}}(\Omega)}\big). \tag{2.13}$$

□

3. Proof of Theorems 1.3 and 1.4

We need some preliminary results before proving the theorems in this section. We consider the same notation for the Banach space $V = \{u \in H_0^1(\Omega) \cap L^p(\Omega), \nabla.u = 0\}$ with norm $\|u\| = \|u\|_{H_0^1(\Omega)} + \|u\|_{L^p(\Omega)}$. Where Ω is a bounded domain of $\mathbb{R}^n$, the operator $\Lambda u = (u.\nabla)u$ may not be defined on whole space $H_0^1(\Omega)$. Although, there exist constant C such that

$$|\langle \Lambda u, v\rangle| = \Big|\int_\Omega \sum_{j,k=1}^n u_k \frac{\partial u_j}{\partial x_k} v_j\Big| \le C\|u\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\|v\|_{C^1(\Omega)}\,.$$

which means that for the dense linear subspace

$$E = \{u \in C_c^1(\Omega), \nabla.u = 0\}$$

of V, we have that Λ is well defined. We shall define $\Phi : V \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{p}\int_\Omega |u|^p\,dx - \int_\Omega f\ u\,dx.$$

We also define $H : V \times V \to \mathbb{R}$ by

$$H(v, u) = \Phi(u) - \Phi(v).$$

For $r > 1$, we set

$$K(r) = \{u \in V; \|u\| \le r\},$$

that is convex and weakly closed in V by similar arguments as in the proof of Lemma 2.1. Let

$$K_0(r) = K(r) \cap E,$$

and define $M : K(r) \times K_0(r) \to \mathbb{R}$ by

$$M(u, v) = H(v, u) - \langle \Lambda u, v\rangle. \tag{3.1}$$

When $\mu > 0$ in the damping term, we shall use a different version of Ky-Fan minimax theorem (See [5, Lemma 12.1]) for a proof). This version is more practical when one expects less regularity of the solution. For a subset set D, we denote its convex hull by $conv(D)$.

Lemma 3.1. *Let $\emptyset \ne D \subset E \subset H$ where E is a weakly compact convex set in a Banach space H, and consider $M : E \times conv(D) \to \mathbb{R}$ to be a function such that:*

1. *For each $y \in D$, the map $x \to M(x, y)$ is weakly lower semi- continuous on E.*
2. *For each $x \in E$, the map $y \to M(x, y)$ is concave on $conv(D)$.*
3. *$M(x, x) \le 0$ for every $x \in conv(D)$.*

Then there exits $\bar{x} \in E$ such that $M(\bar{x}, y) \le 0$ for all $y \in D$.

Proof of Theorem 1.3. Without loss of generality we may suppose that $\mu = 1$. By similar arguments as in Lemma 2.4 for M defined in (3.1) we obtain that

- For each $v \in K_0(r)$ the function $u \to M(v, u)$ is weakly lower semi-continuous.
- For each $u \in K(r)$ the function $v \to M(v, u)$ is concave.
- $M(u, u) = 0, \forall u \in K_0(r)$

Now we can apply Ky-Fan minimax principle (Lemma 3.1), which yields there exits $\bar{u}_r \in K(r)$ such that

$$M(\bar{u}_r, v) = H(v, \bar{u}_r) - \langle \Lambda \bar{u}_r, v\rangle \le 0, \qquad \forall v \in K_0(r) \tag{3.2}$$

Substituting $v = 0$ in the latter inequality implies that $\Phi(\bar{u_r}) \le 0$. Now the coercivity of the functional Φ follows that $\{\bar{u}_r\}_r$ is bounded in V and so there exists a sequence $r_n \to \infty$ and $\bar{u} \in V$ such that $\bar{u}_{r_n} \to \bar{u}$ weakly in V. If $v \in E$ is fixed, then from (3.2) and the weak lower semi-continuity of the functions involved, we obtain

$$H(v, \bar{u}) - \langle \Lambda \bar{u}, v\rangle \le 0 \tag{3.3}$$

This indeed implies that

$$\sup_{v \in E, \|v\|_E \le 1} \langle \Lambda \bar{u}, v\rangle + \inf_{\|z\| \le 1} H(z, \bar{u}) \le 0 \tag{3.4}$$

Therefore,

$$\sup_{v\in E,\|v\|_E\le 1}\langle \Lambda\bar u, v\rangle \le -\inf_{\|z\|\le 1} H(z,\bar u)\le \infty. \tag{3.5}$$

This implies that the linear functional $l: E\to\mathbb{R}$ defined by $l(v) =< \Lambda\bar u, v >$ is continuous. It now follows from the bounded linear extension theorem that l can be extended to a bounded linear operator $L: V\to\mathbb{R}$ with the same operator norm as l. It then follows that there exists $\hat\Lambda\bar u\in V^*$ such that

$$\langle\hat\Lambda\bar u, v\rangle =< \Lambda\bar u, v >,\quad \forall v\in E. \tag{3.6}$$

This together with (3.3) yield that

$$H(v,\bar u)-\langle\hat\Lambda\bar u, v\rangle\le 0,\quad \forall v\in E. \tag{3.7}$$

But since E is dense in V and expression (3.7) is continuous with respect to v, we can conclude that

$$H(v,\bar u)-\langle\hat\Lambda\bar u, v\rangle\le 0,\quad \forall v\in V. \tag{3.8}$$

Now by substituting $v=\bar u+t\eta,\ \eta\in V$, into (3.8) we obtain that

$$H(\bar u+t\eta,\bar u)-\langle\hat\Lambda\bar u,\bar u+t\eta\rangle\le 0,\quad \forall t\in\mathbb{R}. \tag{3.9}$$

Dividing (3.9) by $t>0$ and letting t converge to zero yields that

$$\int_\Omega \nabla\bar u.\nabla\eta\,dx+\int_\Omega |\bar u|^{p-2}\bar u\eta\,dx-\int_\Omega f\eta\,dx+\int_\Omega\hat\Lambda\bar u\;\eta\ge 0,\quad \forall\eta\in V. \tag{3.10}$$

Now substituting η by $-\eta$ in (3.10) we deduce the opposite inequality and thus

$$\int_\Omega \nabla\bar u.\nabla\eta\,dx+\int_\Omega |\bar u|^{p-2}\bar u\eta\,dx-\int_\Omega f\eta\,dx+\int_\Omega\hat\Lambda\bar u\;\eta= 0,\quad \forall\eta\in V. \tag{3.11}$$

This together with (3.6) follow that

$$\int_\Omega \nabla\bar u.\nabla\eta\,dx+\int_\Omega |\bar u|^{p-2}\bar u\;\eta\,dx-\int_\Omega f\eta\,dx+\int_\Omega\Lambda\bar u\;\eta= 0,\quad \forall\eta\in E.$$

This completes the proof of part (i).

For the proof of the second part we consider two cases $n\le 4$ and $p\ge 4$ separately.

Case 1: $(n\le 4)$**.** Let $v\in V$. If $n\le 4$, then $4\le\frac{2n}{n-2}$. From continuous imbedding of Sobolev space $H^1_0(\Omega)$ into $L^4(\Omega)$, and by the Hölder inequality we obtain for operator Λu

$$\begin{aligned}|\langle\Lambda u, v\rangle| = \Big|\int_\Omega\sum_{j,k=1}^n u_k\frac{\partial u_j}{\partial x_k}v_j\Big| &\le C\|uv\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\\ &\le C\|u\|_{L^4(\Omega)}\|v\|_{L^4(\Omega)}\|u\|_{H^1_0(\Omega)}<\infty.\end{aligned}$$

This means, the operator Λu is well defined on V. Since E is a dense subspace of V, from uniqueness of the bounded linear extension theorem we have $\langle\hat\Lambda\bar u, v\rangle =< \Lambda\bar u, v >,\ \forall v\in V$. Now the result follows from (3.11).

Case 2: $p\ge 4$**.** For $v\in V$, since $V\subset L^p(\Omega)$ we can deduce that

$$|\langle\Lambda u, v\rangle|\le C\|uv\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}\le C\|u\|_{H^1_0(\Omega)}\|u\|_{L^p(\Omega)}\|v\|_{L^{\frac{2p}{p-2}}(\Omega)}<\infty$$

where the last inequality follows from $\frac{2p}{p-2}\le p$. Thus operator Λu is well defined on whole V and this completes the proof. □

We would like to remark that in the last part of the proof we make extensive use of $\frac{2p}{p-2}\le p$. Note that if $2<p<4$ and $n\ge 5$, then $\frac{2p}{p-2}>p$, and there is no guarantee for $\|v\|_{L^{\frac{2p}{p-2}}(\Omega)}<\infty$.

As we have just seen, the case of $\mu>0$ with the linear damping term was covered in the theorem 1.3. But for $\mu<0$, due to an essential role of Lemma 2.7 in the proof of theorem 1.2 we were not be able to deal with the linear damping term in this theorem. However, with a similar argument to theorem 1.3, we would be in a position to manage it separately. Note that when $p = 2$ we have that $V = \{u \in H_0^1(\Omega), \nabla u = 0\}$, and

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\mu}{2}\int_\Omega |u|^2\,dx - \int_\Omega f\,u\,dx.$$

Proof of Theorem 1.4. In the same way as in proof of Theorem 1.3, it follows from the Ky-Fan minimax principle (Lemma 3.1) that there exits $\bar{u}_r \in K(r)$ with

$$M(\bar{u}_r, v) = H(v, \bar{u}_r) - \langle \Lambda \bar{u}_r, v\rangle \le 0, \quad \forall v \in K_0(r). \tag{3.12}$$

Now we claim that $\{\bar{u}_r\}_r$ is bounded in V and so there exists a sequence $r_n \to \infty$ and $\bar{u} \in V$ such that $\bar{u}_{r_n} \rightharpoonup \bar{u}$ weakly in V. Thus, by similar arguments as in proof of Theorem 1.3 we obtain the result.

Now to complete the proof we have to show the claim. Assume, by contradiction, that $\{\bar{u}_r\}_r$ is unbounded. So there exists a sequence $r_m \to \infty$ such that $\{\bar{u}_{r_m}\}_m$ is unbounded. By substituting $v = 0$ in (3.12) we obtain that

$$\Phi(\bar{u}_{r_m}) = \frac{1}{2}\int_\Omega |\nabla \bar{u}_{r_m}|^2\,dx + \frac{\mu}{2}\int_\Omega |\bar{u}_{r_m}|^2\,dx - \int_\Omega f\,\bar{u}_{r_m}\,dx \le 0. \tag{3.13}$$

Let $t_m^2 = \int_\Omega |\nabla \bar{u}_{r_m}|^2\,dx$, and $w_m = \frac{\bar{u}_{r_m}}{t_m}$. Note that $\|w_m\|_{H_0^1(\Omega)} = 1$. Thus, there exists a $w = (z_1, \dots, z_n) \in V$ such that $w_m \rightharpoonup w$ weakly in V. It follows that $w \neq 0$, because dividing (3.13) by t_m^2 we obtain

$$\frac{1}{2} + \frac{\mu}{2}\int_\Omega |w_m|^2\,dx \le \frac{1}{t_m}\int_\Omega f\,w_m\,dx, \tag{3.14}$$

and letting $m \to \infty$, due to the compact imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ we obtain that

$$\frac{1}{2} + \frac{\mu}{2}\int_\Omega |w|^2\,dx \le 0, \tag{3.15}$$

which implies $w \neq 0$. Also, we have

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \frac{\mu}{2}\int_\Omega |w|^2\,dx &\le \liminf_{m\to\infty}\big(\frac{1}{2}\int_\Omega |\nabla w_m|^2\,dx + \frac{\mu}{2}\int_\Omega |w_m|^2\,dx\big)\\ &\le \frac{1}{2} + \frac{\mu}{2}\int_\Omega |w|^2\,dx.\end{aligned}$$

This estimate together with (3.15) yield that

$$\int_\Omega |\nabla w|^2\,dx + \mu\int_\Omega |w|^2\,dx \le 0.$$

Therefore,

$$\frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx} \le -\mu, \tag{3.16}$$

from which with hypothesis $-\lambda_1 \le \mu$ of theorem we obtain that

$$\frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx} \le \lambda_1. \tag{3.17}$$

On the other hand, for first eigenvalue λ_1 of $-\Delta$ we have

$$\frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx} = \frac{\sum_{i=1}^n \int_\Omega |\nabla z_i|^2\,dx}{\sum_{i=1}^n \int_\Omega z_i^2\,dx} \ge \frac{\sum_{i=1}^n \lambda_1 \int_\Omega z_i^2\,dx}{\sum_{i=1}^n \int_\Omega z_i^2\,dx} = \lambda_1, \tag{3.18}$$

where $w = (z_1, \dots, z_n)$. It then follows from (3.17) and (3.18) that

$$\lambda_1 = \frac{\int_\Omega |\nabla w|^2\,dx}{\int_\Omega |w|^2\,dx}, \tag{3.19}$$

from which we obtain that

$$\int_\Omega |\nabla z_i|^2\,dx = \lambda_1 \int_\Omega z_i^2\,dx, \quad (i = 1, .., n).$$

Therefore,

$$-\Delta z_i = \lambda_1 z_i, \quad i = 1, \dots, n. \tag{3.20}$$

Since the first eigenvalue of the $-\Delta$ is simple it follows that there exists $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that

$$z_i = \alpha_i \psi_1, \quad i = 1, \dots, n, \tag{3.21}$$

where $\psi_1 > 0$ is the unique eigenfunction of $-\Delta$ corresponding to λ_1 with $\|\psi_1\|_{L^2(\Omega)} = 1$, i.e.

$$-\Delta \psi_1 = \lambda_1 \psi_1, \quad \psi_1|_{\partial\Omega} = 0.$$

Since $\nabla . w = 0$, it follows from (3.21) that

$$0 = \sum_{i=1}^{n} \frac{\partial z_i}{\partial x_i} = \sum_{i=1}^{n} \alpha_i \frac{\partial \psi_1}{\partial x_i} = \alpha . \nabla \psi_1. \tag{3.22}$$

Now let x be an interior point of Ω and $\bar{x}$ the closest point on $\partial\Omega$ to x such that $\bar{x} - x = C\alpha$ for some constant $C \in \mathbb{R}$, and the line joining x to $\bar{x}$ lies in $\bar{\Omega}$. Define $g : [0, 1] \to \mathbb{R}$ by

$$g(t) = \psi_1(tx + (1 - t)\bar{x}).$$

It can be easily deduced from (3.22) that

$$g'(t) = (x - \bar{x}).\nabla\psi_1(tx + (1 - t)\bar{x}) = C\alpha.\nabla\psi_1(tx + (1 - t)\bar{x}) = 0.$$

Thus, g is a constant function and since $\psi_1|_{\partial\Omega} = 0$ we have

$$g(t) = g(0) = \psi_1(\bar{x}) = 0, \quad \forall t \in [0, 1],$$

which implies that $\psi_1(x) = 0$. This is the contradiction we wanted. □

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