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MULTIPLICITY OF SOLUTIONS FOR BIHARMONIC EQUATIONS WITH CRITICAL EXPONENT AND PRESCRIBED SINGULARITY

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Abstract. In this article, we study the singular critical biharmonic problem

$$\Delta^2 u - \mu V(x)u = |u|^{2^*-2}u + \lambda f(x) \quad \text{in } \Omega,$$
$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where Δ^2 is the biharmonic operator, Ω is an open bounded domain in $\mathbb{R}^N$ $(N \geq 5)$ with smooth boundary $\partial\Omega, 2^* = \frac{2N}{N-4}$, $0 < \mu < \bar{\mu} := \big(\frac{N(N-4)}{4}\big)^2$, $f(x)$ and $V(x)$ are given functions. By using variational method and Nehari-type constraint, we establish the existence of multiple solutions for this problem when $0 < \lambda < \lambda^*$, for some $\lambda^* > 0$.

1. Introduction

This article concerns the biharmonic problem

$$\begin{gathered} \Delta^2 u - \mu V(x)u = |u|^{2^*-2}u + \lambda f(x) \quad \text{in } \Omega, \\ u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Δ^2 denotes the biharmonic operator, $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, $2^* = \frac{2N}{N-4}$ is the critical Sobolev exponent and $0 < \mu < \bar{\mu}$, where $\bar{\mu}$ is the best constant for the Rellich inequality

$$\int_\Omega \frac{u^2}{|x-a|^4}\,dx \leq \frac{1}{\bar{\mu}} \int_\Omega |\Delta u|^2\,dx, \quad \forall a \in \Omega,\ u \in H_0^2(\Omega).$$

Here, $H_0^2(\Omega)$ denotes the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \Big(\int_\Omega |\Delta u|^2\,dx\Big)^{1/2}.$$

In elasticity theory, biharmonic equations with multipolar singular potentials effectively model thin elastic plates that include k localized defects or concentrated loads situated at finitely many points $a_1, a_2, \dots, a_k$. These singularities strongly affect deformation and stability and can be used to eliminate unwanted frequencies or localize vibrational energy (see Lindsay et al. [9]). Similar models appear in composite and metamaterials, where point inclusions generate stress multipoles influencing effective properties and cloaking effects (see Mao-Huang [12]). In addition, biharmonic singular models can also arise in thin-film physics, surface growth, and nanomechanical resonators (see [3], [4], and [5]).

In recent years, many authors have studied biharmonic problems, for instance, in the regular case $(\mu = 0)$, Qian-Wang in [14] proved the existence of at least two distinct solutions for (1.1) with $\mu = 0$, $\lambda = 1$ and $f(x)$ small. In the singular case $(\mu \neq 0)$, we recall that the existence of multiple

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solutions of (1.1) has been studied under different hypotheses on $V(x)$. For the homogeneous case, D'Ambrossio-Jannelli [2] considered the Hardy potentiel $V(x) = |x|^{-4}$ and Kang-Xiong in [7] established the existence and nonexistence of ground state solutions of (1.1) where $V(x)$ has prescribed singularities of the form $|x-a_j|^{-4}$ with $a_j \in \Omega$ and $j = 1, \dots, k$, by using a complicated asymptotic analysis and variational arguments. For the nonhomogenous case, Li et al. [11] studied (1.1) with $V(x) = |x|^{-s}$, $0 < s \le 4$. Very recently, the authors in [13] have proved the existence of at least $2k$ solutions of the problem

$$\Delta^2 u - \sum_{j=1}^{k} \frac{\mu_j}{|x-a_j|^4} u = |u|^{2^*-2}u + \sum_{j=1}^{k} \frac{\lambda_j}{|x-a_j|^{4-\alpha_j}} u + f(x) \quad \text{in } \Omega$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

where Ω is an open bounded domain of $\mathbb{R}^N$ $(N \ge 5)$ with smooth boundary $\partial\Omega$, for all $j = 1, \dots, k$, $a_j \in \Omega$ denote the singularity points, $\lambda_j > 0$ are parameters, $0 < \alpha_j < 4$ and $\mu_j > 0$ are real constants satisfying $\sum_{j=1}^{k} \mu_j < \bar{\mu}$. We recall that the problem (1.1) with the Laplacian operator was studied by Chen [1]. He proved the existence of at least k positive solutions by the argument developed in [15]. The main goal of this paper is to generalize the result in [1] to the biharmonic operator.

From the Rellich inequality, it follows that the best constant

$$\mathcal{A}_\mu(\Omega) = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega \Big(|\Delta u|^2 - \mu \frac{u^2}{|x-a|^4}\Big)\,dx}{\Big(\int_\Omega |u|^{2^*}\,dx\Big)^{2/2^*}}, \quad \forall a \in \Omega,\ \mu < \bar{\mu}, \tag{1.2}$$

is well defined. Moreover, as shown in [2, 8], $\mathcal{A}_\mu(\Omega)$ is independent of the domain Ω and is attained in $\mathbb{R}^N$ by the family of translated extremals

$$\{y_\varepsilon(x-a) = \varepsilon^{\frac{4-N}{2}} U_\mu(\varepsilon^{-1}(x-a)),\ \varepsilon > 0\},$$

where U_μ is a positive, radially symmetric, and radially decreasing solution of

$$\Delta^2 u - \mu \frac{u}{|x|^4} = u^{2^*-1} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad u > 0.$$

The normalized translated profiles satisfy

$$\int_{\mathbb{R}^N} \Big(|\Delta y_\varepsilon(x-a)|^2 - \mu \frac{|y_\varepsilon(x-a)|^2}{|x-a|^4}\Big)\,dx = \int_{\mathbb{R}^N} |y_\varepsilon(x-a)|^{2^*}\,dx = \mathcal{A}_\mu^{N/4}.$$

By setting $\rho = |x|$, the profile U_μ has the following sharp asymptotics:

$$U_\mu(\rho) = O_1(\rho^{-a(\mu)}) \quad \text{as } \rho \to 0,$$

$$U_\mu(\rho) = O_1(\rho^{-b(\mu)}), \quad U_\mu'(\rho) = O_1(\rho^{-b(\mu)-1}) \quad \text{as } \rho \to \infty,$$

where $\delta = \frac{N-4}{2}$ and

$$a(\mu) = \delta\,\varphi(\mu), \quad b(\mu) = \delta\big(2 - \varphi(\mu)\big),$$

with

$$\varphi(\mu) = 1 - \frac{\sqrt{N^2 - 4N + 8 - 4\sqrt{(N-2)^2 + \mu}}}{N-4}, \quad \mu \in [0, \bar{\mu}].$$

In particular, for $\mu \in [0, \bar{\mu})$ one has $0 \le a(\mu) \le \delta \le b(\mu) \le 2\delta$. There exist positive constants $\mathcal{C}_1(\mu), \mathcal{C}_2(\mu)$ such that

$$0 < \mathcal{C}_1(\mu) \le U_\mu(x)\big(|x|^{a(\mu)/\delta} + |x|^{b(\mu)/\delta}\big)^\delta \le \mathcal{C}_2(\mu) \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Before stating our main assumptions, we briefly fix some notations. We denote by $\mathcal{D}^{2,2}(\mathbb{R}^N)$ the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |\Delta u|^2\,dx\Big)^{1/2}.$$

For $1 \le p < \infty$, $L^p(\Omega)$ denotes the Lebesgue space endowed with the norm

$$\|u\|_p = \Big(\int_\Omega |u|^p \, dx \Big)^{1/p}.$$

We write $\|\cdot\|_-$ for the norm in $H^{-2}(\Omega)$, the dual space of $H_0^2(\Omega)$. The ball of center $a \in \mathbb{R}^N$ and radius $r > 0$ is denoted by $B(a,r)$. l_i, ν_i, and C_i denote positive constants whose values are unimportant. For all $\varepsilon > 0, t > 0, O(\varepsilon^t)$ denotes the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \le C_1$, $O_1(\varepsilon^t)$ denotes $C_2\varepsilon^t \leqslant O_1(\varepsilon^t) \leqslant C_3\varepsilon^t$ and $o(\varepsilon^t)$ means $o(\varepsilon^t)/\varepsilon^t \to 0$ as $\varepsilon \to 0$ and $o(1)$ a generic infinitesimal value, $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. Finally, we note

$$\mathcal{A}_0 = \inf \Big\{ \int_\Omega |\Delta u|^2; \; u \in H_0^2(\Omega), \; \int_\Omega |u|^{2^*} = 1 \Big\}.$$

We now state the structural assumptions used throughout the paper:

(A1) $f \in H^{-2}(\Omega)$ and $f(x) > 0$ a.e. in Ω.

(A2) There exist k different points $a_1, a_2, \dots, a_k \in \Omega$ such that

$$V(x) \in L^\infty_{\rm loc}\big(\Omega \setminus \{a_1, a_2, \dots, a_k\}\big), \quad \lim_{x \to a_j} V(x)\,|x - a_j|^4 = 1.$$

Moreover, there exist $\delta_0 > 0$ and $\alpha, \beta > 2\big(b(\mu) - \delta\big) > 0$ such that, for all $x \in B(a_j, \delta_0)$ and $j \in \{1, \dots, k\}$,

$$1 - |x - a_j|^\beta \;\le\; |x - a_j|^4 V(x) \;\le\; 1 - |x - a_j|^\alpha. \tag{1.3}$$

Here, δ_0 is chosen so that $|a_i - a_j| \ge 4\delta_0$ for $i \ne j$ and $B(a_j, \delta_0) \subset \Omega$.

(A3) There exists a constant $0 < C < 1$ such that

$$\mu \int_\Omega V(x) u^2 \, dx \;\le\; C \int_\Omega |\Delta u|^2 \, dx, \quad \forall u \in H_0^2(\Omega).$$

We are ready to state our main result.

Theorem 1.1. *Assume that* (A1)–(A3) *are satisfied and* $0 < \mu < \bar{\mu}$. *Then there exists* $\lambda^* > 0$ *such that for all* $0 < \lambda < \lambda^*$, *problem* (1.1) *has at least* k *solutions on* $H_0^2(\Omega)$.

Remark 1.2. It is worth noting that, unlike [13], which studied nonhomogeneous biharmonic problem with Rellich-type singularities $V(x) = \sum_{j=1}^k |x - a_j|^{-4}$ with a classical perturbation $\sum_{j=1}^k |x - a_j|^{\alpha_j - 4}$, our work considers a more rigid class of multiple singular potentials $V(x)$. Specifically, we impose sharper two-sided bounds near each pole and uniform separation between the singularities, making the problem (1.1) more interesting and delicate.

This article is organized as follows. In Section 2, we give some preliminaries. In Section 3, we present the proofs of several technical lemmas and propositions. Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminary results

To prove Theorem 1.1, we will use critical point theory. On $H_0^2(\Omega)$, we define the energy functional associated with the problem (1.1) by

$$J_\mu(u) = \frac{1}{2} \int_\Omega (|\Delta u|^2 - \mu V(x) u^2) \, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*} \, dx - \lambda \int_\Omega f u \, dx.$$

We say that u is a weak solution of (1.1) if $u \in H_0^2(\Omega)$ and for all $\varphi \in H_0^2(\Omega)$, we have

$$\langle J_\mu'(u), \varphi \rangle = \int_\Omega (\Delta u \Delta \varphi - \mu V(x) u \varphi) \, dx - \int_\Omega |u|^{2^*-2} u \varphi \, dx - \lambda \int_\Omega f \varphi \, dx = 0.$$

We define

$$\mathcal{N}_\mu = \{u \in H_0^2(\Omega) : u \ne 0, \; \langle J_\mu'(u), u \rangle = 0\}.$$

First, we give some energy estimates.

Lemma 2.1. *Let u be a solution of* (1.1)*, then for any $\lambda > 0$,*
$$J_\mu(u) \geq -\chi\lambda^2,$$
where
$$\chi = \frac{(N+4)^2}{32N(1-C)}\|f\|_-^2.$$

Proof. Let u be a solution of (1.1), then we have
$$\begin{aligned}
J_\mu(u) &= \frac{1}{2}\int_\Omega (|\Delta u|^2 - \mu V(x)u^2)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx - \lambda\int_\Omega fu\,dx \\
&= \frac{2}{N}\int_\Omega (|\Delta u|^2 - \mu V(x)u^2)\,dx - \lambda\frac{N+4}{2N}\int_\Omega fu\,dx \\
&\geq \frac{2}{N}(1-C)\|u\|^2 - \lambda\frac{N+4}{2N}\|u\|\|f\|_-.
\end{aligned}$$
For $t > 0$, we set
$$\bar{h}(t) = \frac{2}{N}(1-C)t^2 - \lambda\frac{N+4}{2N}\|f\|_- t,$$
then, we obtain
$$\bar{h}(t) \geq \bar{h}(\bar{t}) = -\lambda^2\frac{(N+4)^2}{32N(1-C)}\|f\|_-,$$
where $\bar{t} = \lambda\frac{N+4}{8(1-C)}\|f\|_-$. This completes the proof. □

Next, we define $\psi_\mu : (0,+\infty) \to \mathbb{R}$ by $\psi_\mu(t) = \langle J'_\mu(tu), tu\rangle$, that is
$$\psi_\mu(t) = t^2\int_\Omega (|\Delta u|^2 - \mu V(x)u^2)\,dx - t^{2^*}\int_\Omega |u|^{2^*}\,dx - \lambda t\int_\Omega fu\,dx,$$
for all $u \in \mathcal{N}_\mu$. So
$$\begin{aligned}
\psi'_\mu(1) &= 2\int_\Omega \left(|\Delta u|^2 - \mu V(x)u^2\right)\,dx - 2^*\int_\Omega |u|^{2^*}\,dx - \lambda\int_\Omega fu\,dx, \\
&= \int_\Omega \left(|\Delta u|^2 - \mu V(x)u^2\right)\,dx - (2^*-1)\int_\Omega |u|^{2^*}\,dx.
\end{aligned}$$
We split $\mathcal{N}_\mu$ into three parts
$$\begin{aligned}
\mathcal{N}_\mu^+ &= \{u \in N_\mu : \psi'_\mu(1) > 0\}, \\
\mathcal{N}_\mu^0 &= \{u \in N_\mu : \psi'_\mu(1) = 0\}, \\
\mathcal{N}_\mu^- &= \{u \in N_\mu : \psi'_\mu(1) < 0\}.
\end{aligned}$$
We now derive some basic properties of $\mathcal{N}_\mu^+, \mathcal{N}_\mu^0$ and $\mathcal{N}_\mu^-$.

Lemma 2.2. *Assume that* (A1)–(A3) *are satisfied and $0 < \mu < \bar{\mu}$. Then there exists $\lambda_1 > 0$ such that for any $\lambda \in (0,\lambda_1)$, $\mathcal{N}_\mu^0 = \emptyset$ and $\mathcal{N}_\mu^- \neq \emptyset$.*

Proof. Arguing by contradiction, we assume that there are $\lambda_n \to 0$ such that $\mathcal{N}_\mu^0 \neq \emptyset$, then
$$\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx = (2^*-1)\int_\Omega |u_n|^{2^*}\,dx, \tag{2.1}$$
and
$$\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx = \int_\Omega |u_n|^{2^*}\,dx + \lambda_n\int_\Omega fu_n\,dx, \tag{2.2}$$
by Sobolev inequality, (A3) and (2.1), we obtain
$$\mathcal{A}_0^{-2^*/2}\Big(\int_\Omega |\Delta u_n|^2\,dx\Big)^{2^*/2} \geq \int_\Omega |u_n|^{2^*}\,dx \geq \frac{1-C}{2^*-1}\int_\Omega |\Delta u_n|^2\,dx,$$
and so
$$\int_\Omega |\Delta u_n|^2\,dx \geq \tilde{C}\mathcal{A}_0^{N/4}, \tag{2.3}$$

with $\tilde{C}=\big(\frac{1-C}{2^*-1}\big)^{\frac{N-4}{4}}$. Combining (2.1), (2.2) with (2.3), we obtain

$$
\begin{aligned}
0 &= \Big(1-\frac{1}{2^*-1}\Big)\int_\Omega (|\Delta u_n|^2-\mu V(x)u_n^2)\,dx-\lambda_n\int_\Omega f u_n\,dx\\
&\geq \Big(1-\frac{1}{2^*-1}\Big)(1-C)\int_\Omega |\Delta u_n|^2\,dx-\lambda_n\|u_n\|\|f\|_- > 0.
\end{aligned}
$$

Since $\lambda_n\to 0$. This contradiction implies that there is $\bar\lambda>0$ such that $\mathcal{N}_\mu^0=\emptyset$ for any $\lambda\in(0,\bar\lambda)$. Now, for $u\in H_0^2(\Omega)\setminus\{0\}, t>0$, we consider the function

$$
g(t)=t\int_\Omega(|\Delta u|^2-\mu V(x)u^2)\,dx-t^{2^*-1}\int_\Omega |u|^{2^*}\,dx-\lambda\int_\Omega fu\,dx.
$$

Let

$$
t_{\max}=\Big(\frac{\int_\Omega(|\Delta u|^2-\mu V(x)u^2)\,dx}{(2^*-1)\int_\Omega|u|^{2^*}\,dx}\Big)^{\frac{1}{2^*-2}}.
$$

It is clear that $g(t)$ achieves its maximum at $t_{\max}$ and we have

$$
\begin{aligned}
g(t_{\max}) &= \Big(\Big(\frac{1}{2^*-1}\Big)^{\frac{1}{2^*-2}}-\Big(\frac{1}{2^*-1}\Big)^{\frac{2^*-1}{2^*-2}}\Big)\frac{\big(\int_\Omega(|\Delta u|^2-\mu V(x)u^2)\,dx\big)^{\frac{2^*-1}{2^*-2}}}{\|u\|_{2^*}^{\frac{2^*}{2^*-2}}}-\lambda\int_\Omega fu\,dx\\
&= C_N\frac{\big(\int_\Omega(|\Delta u|^2-\mu V(x)u^2)\,dx\big)^{\frac{N+4}{8}}}{\|u\|_{2^*}^{N/4}}-\lambda\int_\Omega fu\,dx\\
&\geq C_N\frac{\big((1-C)\int_\Omega|\Delta u|^2\,dx\big)^{\frac{N+4}{8}}}{\Big(\frac{\|u\|}{\sqrt{\mathcal{A}_0}}\Big)^{N/4}}-\lambda\int_\Omega fu\,dx\\
&\geq C_N(1-C)^{\frac{N+4}{8}}\|u\|(\sqrt{\mathcal{A}_0})^{N/4}-\lambda\|u\|\|f\|_-\,,
\end{aligned}
$$

where $C_N=\big(\frac{1}{2^*-1}\big)^{\frac{1}{2^*-2}}-\Big(\frac{1}{2^*-1}\Big)^{\frac{2^*-1}{2^*-2}}$.
Let

$$
\bar{\bar\lambda}=\frac{C_N(1-C)^{\frac{N+4}{8}}\big(\sqrt{\mathcal{A}_0}\big)^{N/4}}{\|f\|_-}>0,
$$

then for $\lambda\in(0,\bar{\bar\lambda})$, one has that $g(t_{\max})>0$. Then there exists $t^+=t^+(u)$ such that $0<t_{\max}<t^+, g(t^+)=0$ and $g'(t^+)<0$, it follows that $t^+u\in\mathcal{N}_\mu^-$. In conclusion, for $\lambda_1=\min\{\bar\lambda,\bar{\bar\lambda}\}$ we have

$$
\mathcal{N}_\mu^0=\emptyset\quad\text{and}\quad \mathcal{N}_\mu^-\neq\emptyset. \qquad\square
$$

3. Localization of constraints

We minimize the functional J_μ on some subsets of constraints $\mathcal{N}_\mu$. For this similar to [10], we define a map of "Barycenter type" $\beta_j: H_0^2(\Omega)\setminus\{0\}\to\mathbb{R}^N$, as

$$
\beta_j(u)=\frac{\int_\Omega\psi_j(x)|\Delta u|^2\,dx}{\int_\Omega|\Delta u|^2\,dx},\quad \text{for } j\in\{1,2,\dots,k\}
$$

where $\psi_j(x)=\min\{\delta_0,|x-a_j|\}$. For $r_0=\frac{\delta_0}{3}$, we set

$$
\mathcal{N}_j^-=\{u\in\mathcal{N}_\mu^-,\beta_j(u)<r_0\},\quad \Gamma_j^-=\{u\in\mathcal{N}_\mu^-,\beta_j(u)=r_0\}.
$$

From [1] and [6], we derive the following result.

Proposition 3.1. *Let r_0 be as above, if $\beta_j(u)<r_0$ then*

$$
\int_\Omega|\Delta u|^2\,dx\geq 3\int_{\Omega\setminus B(a_j,r_0)}|\Delta u|^2\,dx.
$$

Remark 3.2. We deduce from Proposition 3.1, that if $u \in H_0^2(\Omega)\setminus\{0\}, \beta_j(u) \le r_0$ and $\beta_i(u) \le r_0$, then $i = j$. Indeed, notice that if $\beta_j(u) \le r_0,\ \beta_i(u) \le r_0$ and $j \ne i$, then by Proposition 3.1, we have

$$2\int_\Omega |\Delta u|^2\,dx \ge 3\Big(\int_{\Omega\setminus B(a_j,r_0)} |\Delta u|^2\,dx + \int_{\Omega\setminus B(a_i,r_0)} |\Delta u|^2\,dx\Big) \ge 3\int_\Omega |\Delta u|^2\,dx,$$

which is a contradiction.

Now, for $j \in \{1,2,\dots,k\}$, we consider the following variational problems

$$c_{\lambda,j} = \inf\{J_\mu(u), u \in \mathcal{N}_j^-\}, \quad \hat{c}_{\lambda,j} = \inf\{J_\mu(u), u \in \Gamma_j^-\}.$$

For $j \in \{1,2,\dots,k\}$ fixed, choose the radial cut-off function $\phi(x) = \phi(|x|) \in \mathcal{C}_0^\infty(B(0,2\delta_0))$ such that $0 \le \phi(x) \le 1$ in $B(0,2\delta_0)$ and $\phi(x) = 1$ in $B(0,\delta_0)$.

We set $u_{\varepsilon,j}(x) = \phi(x-a_j)y_\varepsilon(x-a_j)$. The following asymptotic properties hold.

Lemma 3.3. *Assume that $N \ge 5$ and $0 < \mu < \bar{\mu}$. Then, as $\varepsilon \to 0$, we have the following estimates*

$$\int_\Omega \Big(|\Delta u_{\varepsilon,j}|^2 - \mu\frac{|u_{\varepsilon,j}|^2}{|x-a_j|^4}\Big)\,dx = \mathcal{A}_\mu^{N/4} + O(\varepsilon^{2(b(\mu)-\delta)}), \tag{3.1}$$

$$\int_\Omega |u_{\varepsilon,j}|^{2^*}\,dx = \mathcal{A}_\mu^{N/4} + O(\varepsilon^{2^*(b(\mu)-\delta)}), \tag{3.2}$$

$$\int_\Omega |x-a_j|^{\alpha-4}|u_{\varepsilon,j}|^2\,dx = O(\varepsilon^{2(b(\mu)-\delta)}), \quad \textit{since } \alpha > 2(b(\mu)-\delta). \tag{3.3}$$

Proof. The proof is similar to [2] and [8]. □

Proposition 3.4. *Let $(A1),(A2)$ and $(A3)$ be verified, and $0 < \mu < \bar{\mu}$. Then, there exists $\lambda_2 > 0$, such that for all $\lambda \in (0,\lambda_2)$ there holds*

$$c_{\lambda,j} < \frac{2}{N}\mathcal{A}_\mu^{N/4} - \chi\lambda^2. \tag{3.4}$$

Proof. As $u_{\varepsilon,j}(x) \ne 0$, from Lemma 2.2 we can find $t_{\varepsilon,j}^- = t^-(u_{\varepsilon,j}) > 0$, such that for any $\lambda \in (0,\lambda_1)$ we have $t_{\varepsilon,j}^- u_{\varepsilon,j} \in \mathcal{N}_j^-$. Since

$$\beta_j(t_{\varepsilon,j}^- u_{\varepsilon,j}) = \frac{\int_\Omega \psi_j(x)|\Delta u_{\varepsilon,j}|^2\,dx}{\int_\Omega |\Delta u_{\varepsilon,j}|^2\,dx} \to 0, \quad \text{as} \quad \varepsilon \to 0,$$

it follows that there exists $\varepsilon_1 > 0$ such that $\beta_j(t_{\varepsilon,j}^- u_{\varepsilon,j}) < r_0$ for any $\varepsilon \in (0,\varepsilon_1)$, that is $t_{\varepsilon,j}^- u_{\varepsilon,j} \in \mathcal{N}_j^-$. Hence, we have

$$\sup_{t>t_{\max}} J_\mu(tu_{\varepsilon,j}) \le \sup_{t>0}\Big\{\frac{t^2}{2}\int_\Omega \big(|\Delta u_{\varepsilon,j}|^2 - \mu V(x)u_{\varepsilon,j}^2\big)\,dx - \frac{t^{2^*}}{2^*}\int_\Omega |u_{\varepsilon,j}|^{2^*}\,dx\Big\} - t_{\max}\lambda\int_\Omega f u_{\varepsilon,j}\,dx,$$

for $t > 0$, we consider the function

$$\bar{g}(t) = \frac{t^2}{2}\int_\Omega \big(|\Delta u_{\varepsilon,j}|^2 - \mu V(x)u_{\varepsilon,j}^2\big)\,dx - \frac{t^{2^*}}{2^*}\int_\Omega |u_{\varepsilon,j}|^{2^*}\,dx.$$

Using (1.3), we obtain

$$\bar{g}(t) \le \frac{t^2}{2}\Big[\int_\Omega \Big(|\Delta u_{\varepsilon,j}|^2 - \mu\frac{u_{\varepsilon,j}^2}{|x-a_j|^4}\Big)\,dx + \mu\int_\Omega |x-a_j|^{\beta-4}u_{\varepsilon,j}^2\,dx\Big] - \frac{t^{2^*}}{2^*}\int_\Omega |u_{\varepsilon,j}|^{2^*}\,dx.$$

On the other hand, from

$$\sup_{t\ge0}\Big(\frac{t^2}{2}B_1 - \frac{t^{2^*}}{2^*}B_2\Big) = \frac{2}{N}B_1^{N/4}B_2^{\frac{4-N}{4}}; \quad B_1, B_2 > 0,$$

and (3.1), (3.2) and (3.3), we obtain

$$\sup_{t\ge0}\bar{g}(t) \le \frac{2}{N}\mathcal{A}_\mu^{N/4} + o(\varepsilon^{2(b(\mu)-\delta)}).$$

Hence, for all $0 < \varepsilon < \varepsilon_1$, we have

$$\begin{aligned}\sup_{t>t_{\max}} J_\mu(tu_{\varepsilon,j}) &\le \frac{2}{N}\mathcal{A}_\mu^{N/4} + o(\varepsilon^{2(b(\mu)-\delta)}) - t_{\max}\lambda\int_\Omega f u_{\varepsilon,j}\,dx \\ &\le \frac{2}{N}\mathcal{A}_\mu^{N/4} + o(\varepsilon^{2(b(\mu)-\delta)}) - C_4\lambda\varepsilon^{b(\mu)-\delta}\end{aligned}$$

for some positive constant C_4 independent of ε and j. We write

$$o(\varepsilon^{2(b(\mu)-\delta)}) = k(\varepsilon)\varepsilon^{2(b(\mu)-\delta)},$$

where $k(\varepsilon) \to 0$ as $\varepsilon \to 0$. Taking

$$\varepsilon^{2(b(\mu)-\delta)} = (\frac{2\chi}{C_1})^2\lambda^2, \quad k(\varepsilon) < \chi(\frac{C_1}{2\chi})^2, \tag{3.5}$$

and choosing λ_2 small enough, we have taht $\varepsilon < \varepsilon_1$ for all $\lambda \in (0, \lambda_2)$. Substituting ε as above to (3.5) gives (3.4). □

Proposition 3.5. *For $j \in \{1, \dots, k\}$. Let* (A1)–(A3) *be satisfied and $0 < \mu < \bar{\mu}$. Then there exists $\lambda_3 > 0$, such that for all $\lambda \in (0, \lambda_3)$*

$$\hat{c}_{\lambda,j} > \frac{2}{N}\mathcal{A}_\mu^{N/4}.$$

Proof. We argue by contradiction. Suppose that there exists a sequence $\lambda_n \to 0$ such that

$$\hat{c}_{\lambda_n,j} \to c \le \frac{2}{N}\mathcal{A}_\mu^{N/4}$$

for some $1 \le j \le k$. Hence, we can find a sequence $\{u_n\}_n \subset H_0^2(\Omega)$ with $u_n \in \mathcal{N}_j^-$ satisfying

$$J_\mu(u_n) \to c \le \frac{2}{N}\mathcal{A}_\mu^{N/4}, \quad \text{as } n \to \infty. \tag{3.6}$$

From $\{u_n\}_n \subset \mathcal{N}_j^-$, we have

$$\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx - \int_\Omega |u_n|^{2^*}\,dx = \lambda_n \int_\Omega f u_n\,dx = o(1). \tag{3.7}$$

This means that

$$\begin{aligned} c + o(1) &= J_\mu(u_n) \\ &= \frac{1}{2}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx - \frac{1}{2^*}\int_\Omega |u_n|^{2^*}\,dx + o(1) \\ &= \frac{2}{N}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx + o(1),\end{aligned}$$

and

$$\begin{aligned}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx &= \int_\Omega |u_n|^{2^*}\,dx + o(1) \\ &\le \Big(\frac{\int_\Omega |\Delta u_n|^2\,dx}{\mathcal{A}_0}\Big)^{2*/2} + o(1) \\ &\le \Big(\frac{\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx}{(1-C)\mathcal{A}_0}\Big)^{2^*/2} + o(1),\end{aligned}$$

which together imply that

$$Nc + 1 \ge \int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx \ge ((1-C)\mathcal{A}_0)^{N/4} > 0$$

holds for n large. From (A3), we obtain

$$(1-C)\|u_n\|^2 \le \int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx \le C'\|u_n\|^2,$$

which provides that

$$0 < \nu_1 \le \|u_n\| \le \nu_2 \tag{3.8}$$

for some positive constants ν_1 and ν_2. From (3.7), we set

$$\lim_{n\to\infty}\int_\Omega (|\Delta u_n|^2 - \mu V(x)u_n^2)\,dx = \lim_{n\to\infty}\int_\Omega |u_n|^{2^*}\,dx = l_1 > 0.$$

We define the Levy concentration functions

$$\varrho_n(r) = \sup_{y\in\Omega}\int_{B(y,r)} |u_n|^{2^*}\,dx,$$

since, for every n we have $r_n > 0$ such that $\varrho_n(r_n) = \frac{l_1}{2}$, then there exists $y_n \in \Omega$ such that

$$\int_{B(y_n,r_n)} |u_n|^{2^*}\,dx = \varrho_n(r_n) = \frac{l_1}{2}.$$

Let us define the rescaled functions by

$$w_n(\tilde{x}) = r_n^{\frac{N-4}{2}} u_n(r_n\tilde{x} + y_n),$$

then $w_n(\tilde{x}) \in H_0^2(\Omega_n)$ with $\Omega_n = (\Omega - y_n)/r_n$. By extending $w_n(\tilde{x})$ to be zero for $\tilde{x}$ outside Ω_n, we obtain $w_n(\tilde{x}) \in \mathcal{D}^{2,2}(\mathbb{R}^N)$, with

$$\begin{aligned}
\int_{\mathbb{R}^N} |\Delta w_n(\tilde{x})|^2\,d\tilde{x} &= \int_{\Omega_n} |r_n^{\frac{N-4}{2}} \Delta u_n(r_n\tilde{x} + y_n)|^2\,d\tilde{x} \\
&= \int_{\Omega_n} r_n^N |\Delta u_n(r_n\tilde{x} + y_n)|^2\,d\tilde{x} \\
&= \int_\Omega |\Delta u_n(x)|^2\,dx.
\end{aligned}$$

Similarly, we obtain

$$\int_{\mathbb{R}^N} r_n^4 V(r_n\tilde{x} + y_n) w_n^2(\tilde{x})\,d\tilde{x} = \int_\Omega V(x)u_n^2(x)\,dx,$$

$$\int_{\mathbb{R}^N} |w_n(\tilde{x})|^{2^*}\,d\tilde{x} = \int_\Omega |u_n(x)|^{2^*}\,dx.$$

Then by (3.8), $\{w_n\}_n$ is bounded in $\mathcal{D}^{2,2}(\mathbb{R}^N)$, thus we can assume that

$$w_n(\tilde{x}) \rightharpoonup w_0(\tilde{x}) \text{ in } \mathcal{D}^{2,2}(\mathbb{R}^N), \quad w_n(\tilde{x}) \to w_0(\tilde{x}) \text{ a.e in } \mathbb{R}^N.$$

Set $\tilde{w}_n(\tilde{x}) = w_n(\tilde{x}) - w_0(\tilde{x})$, we have from (3.7)

$$\int_{\mathbb{R}^N} \left(|\Delta w_n(\tilde{x})|^2 - r_n^4 V(r_n\tilde{x} + y_n) w_n^2(\tilde{x})\right)\,d\tilde{x} - \int_{\mathbb{R}^N} |w_n(\tilde{x})|^{2^*}\,d\tilde{x} = o(1), \tag{3.9}$$

by using Brezis-Lieb lemma, we obtain

$$\int_{\mathbb{R}^N} |\Delta \tilde{w}(\tilde{x})|^2\,d\tilde{x} = \int_{\mathbb{R}^N} |\Delta w_n(\tilde{x})|^2\,d\tilde{x} - \int_{\mathbb{R}^N} |\Delta w_0(\tilde{x})|^2\,d\tilde{x} + o(1)$$

$$\int_{\mathbb{R}^N} |\tilde{w}_n(\tilde{x})|^{2^*}\,d\tilde{x} = \int_{\mathbb{R}^N} |w_n(\tilde{x})|^{2^*}\,d\tilde{x} - \int_{\mathbb{R}^N} |w_0(\tilde{x})|^{2^*}\,d\tilde{x} + o(1)$$

$$\int_{\mathbb{R}^N} r_n^4 V(r_n\tilde{x} + y_n)\tilde{w}_n^2(\tilde{x})\,d\tilde{x} = \int_{\mathbb{R}^N} r_n^4 V(r_n\tilde{x} + y_n) w_n^2(\tilde{x})\,d\tilde{x} - \int_{\mathbb{R}^N} r_n^4 V(r_n\tilde{x} + y_n) w_0^2(\tilde{x})\,d\tilde{x} + o(1).$$

Recalling that $r_n\tilde{x} + y_n \in \Omega$ and Ω is bounded, we can assume up to a subsequence, that $r_n \to \bar{r} \geq 0$ and $y_n \to \bar{y} \in \overline{\Omega}$, so we will distinguish two cases. Before doing so, we need to clarify one notation. Whenever writing $r_n\tilde{x} + y_n \in (or \notin) B(a_j, \delta_0)$, always mean that there is a natural number N_1 such that for all $n > N_1$, there holds $r_n\tilde{x} + y_n \in (or \notin) B(a_j, \delta_0)$, for any given $\tilde{x}$.

Case (I): If $r_n\tilde{x} + y_n \notin B(a_j, \delta_0)$, then by taking into account the definition of $\psi_j(x)$, we obtain

$$r_0 = \beta_j(u_n) = \frac{\int_\Omega \psi_j(x)|\Delta u_n(x)|^2\,dx}{\int_\Omega |\Delta u_n(x)|^2\,dx} = \frac{\int_{\mathbb{R}^N} \psi_j(r_n\tilde{x} + y_n)|\Delta w_n(\tilde{x})|^2\,d\tilde{x}}{\int_{\mathbb{R}^N} |\Delta w_n(\tilde{x})|^2\,d\tilde{x}} \to \delta_0,$$

which is a contradiction to the choice of r_0.

Case (II): When $r_n\tilde{x} + y_n \in B(a_j, \delta_0)$, we distinguish three steps.

Step 1. If $r_n \to \bar r = 0$ and $\bar y = a_j$, then we have

$$r_0 = \beta_j(u_n) = \frac{\int_{\mathbb{R}^N} \psi_j(r_n\tilde x + y_n)|\Delta w_n(\tilde x)|^2\,d\tilde x}{\int_{\mathbb{R}^N} |\Delta w_n(\tilde x)|^2\,d\tilde x} \to \psi_j(a_j) = 0, \quad \text{as } n \to \infty,$$

which is a contradiction.

Step 2. If $r_n \to \bar r > 0$, we consider the following two sub-steps:

Sub-step 2.1. If $\|\tilde w_n\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$, from (1.2) and (3.9), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde x)|^2 - \mu \frac{r_n^4}{|r_n\tilde x + y_n - a_j|^4} w_n^2(\tilde x)\Big)\,d\tilde x \\
&\ge \mathcal{A}_\mu\Big(\int_{\mathbb{R}^N} |w_n(\tilde x)|^{2^*}\,d\tilde x\Big)^{2/2^*} \\
&= \mathcal{A}_\mu\Big(\int_{\mathbb{R}^N} \big(|\Delta w_n(\tilde x)|^2 - \mu r_n^4 V(r_n\tilde x + y_n) w_n^2(\tilde x)\big)\,d\tilde x\Big)^{2/2^*} \\
&\ge \mathcal{A}_\mu\Big(\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde x)|^2 - \mu \frac{r_n^4}{|r_n\tilde x + y_n - a_j|^4} w_n^2(\tilde x)\Big)\,d\tilde x\Big)^{2/2^*}.
\end{aligned}$$

Hence, we obtain

$$\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde x)|^2 - \mu \frac{r_n^4}{|r_n\tilde x + y_n - a_j|^4} w_n^2(\tilde x)\Big)\,d\tilde x \ge \mathcal{A}_\mu^{N/4},$$

and so

$$\int_{\mathbb{R}^N} \Big(|\Delta w_0(\tilde x)|^2 - \mu \frac{r_n^4}{|r_n\tilde x + y_n - a_j|^4} w_0^2(\tilde x)\Big)\,d\tilde x \ge \mathcal{A}_\mu^{N/4}.$$

Consequently,

$$\begin{aligned}
J_\mu(u_n) &= \frac{1}{2}\int_\Omega (|\Delta u_n|^2 - \mu V(x) u_n^2)\,dx - \frac{1}{2^*}\int_\Omega |u_n|^{2^*}\,dx + o(1) \\
&= \frac{2}{N}\int_{\mathbb{R}^N} \big(|\Delta w_n(\tilde x)|^2 - \mu r_n^4 V(r_n\tilde x + y_n) w_n^2(\tilde x)\big)\,d\tilde x + o(1) \\
&\ge \frac{2}{N}\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde x)|^2 - \mu r_n^4 \frac{w_n^2(\tilde x)}{|r_n\tilde x + y_n - a_j|^4}\Big)\,d\tilde x \\
&\quad + \frac{2}{N}\mu \int_{\mathbb{R}^N} r_n^4 |r_n\tilde x + y_n - a_j|^{\alpha-4} w_n^2(\tilde x)\,d\tilde x + o(1) \\
&= \frac{2}{N}\int_{\mathbb{R}^N} \Big(|\Delta w_0(\tilde x)|^2 - \mu r_n^4 \frac{w_0^2(\tilde x)}{|r_n\tilde x + y_n - a_j|^4}\Big)\,d\tilde x \\
&\quad + \frac{2}{N}\mu \int_{\mathbb{R}^N} \bar r^2 |\bar r\tilde x + \bar y - a_j|^{\alpha-4} w_0^2(\tilde x)\,d\tilde x \\
&> \frac{2}{N}\mathcal{A}_\mu^{N/4},
\end{aligned}$$

which contradicts the assumption that $J_\mu(u_n) \le \frac{2}{N}\mathcal{A}_\mu^{N/4}$.

Sub-step 2-2. If $\|\tilde w_n\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to L > 0$, set

$$A + o(1) = \int_{\mathbb{R}^N} \big(|\Delta w_0(\tilde x)|^2 - \mu r_n^4 V(r_n\tilde x + y_n) w_0^2(\tilde x)\big)\,d\tilde x - \int_{\mathbb{R}^N} |w_0(\tilde x)|^{2^*}\,d\tilde x,$$

then, by (3.9) we obtain

$$-A + o(1) = \int_{\mathbb{R}^N} \big(|\Delta w_n(\tilde x)|^2 - \mu r_n^4 V(r_n\tilde x + y_n) w_n^2(\tilde x)\big)\,d\tilde x - \int_{\mathbb{R}^N} |w_n(\tilde x)|^{2^*}\,d\tilde x.$$

Suppose that $A > 0$ ($A < 0$ can be considered similarly). We can find $t_n \to 1$, $s_n \to 1$ such that $\bar w_n = t_n w_0$ and $\bar v_n = s_n \tilde w_n$ satisfy

$$A = \int_{\mathbb{R}^N} \big(|\Delta \bar w_n|^2 - \mu r_n^4 V(r_n\tilde x + y_n) \bar w_n^2\big)\,d\tilde x - \int_{\mathbb{R}^N} |\bar w_n|^{2^*}\,d\tilde x, \tag{3.10}$$

$$-A=\int_{\mathbb{R}^N}\left(|\Delta\bar v_n|^2-\mu r_n^4V(r_n\tilde x+y_n)\bar v_n^2\right)\,d\tilde x-\int_{\mathbb{R}^N}|\bar v_n|^{2^*}\,d\tilde x. \tag{3.11}$$

Now for $v=t\bar v_n=t(s_n\tilde w_n),\ t\in(0,1)$, we have

$$\int_{\mathbb{R}^N}\left(|\Delta v|^2-\mu r_n^4V(r_n\tilde x+y_n)v^2\right)\,d\tilde x=\int_{\mathbb{R}^N}|v|^{2^*}\,d\tilde x.$$

We denote

$$\tilde J_\mu(w)=\frac12\int_{\mathbb{R}^N}\left(|\Delta w|^2-\mu r_n^4V(r_n\tilde x+y_n)w^2\right)\,d\tilde x-\frac{1}{2^*}\int_{\mathbb{R}^N}|w|^{2^*}\,d\tilde x.$$

Then, we have

$$\begin{aligned}
\tilde J_\mu(v)&=\frac2N\int_{\mathbb{R}^N}\left(|\Delta v|^2-\mu r_n^4V(r_n\tilde x+y_n)v^2\right)\,d\tilde x\\
&=\frac2N t^2\int_{\mathbb{R}^N}\left(|\Delta\bar v_n|^2-\mu r_n^4V(r_n\tilde x+y_n)\bar v_n^2\right)\,d\tilde x\\
&<\frac2N\int_{\mathbb{R}^N}\left(|\Delta\bar v_n|^2-\mu r_n^4V(r_n\tilde x+y_n)\bar v_n^2\right)\,d\tilde x\\
&=\frac12\int_{\mathbb{R}^N}\left(|\Delta\bar v_n|^2-\mu r_n^4V(r_n\tilde x+y_n)\bar v_n^2\right)\,d\tilde x-\frac{1}{2^*}\Big(\int_{\mathbb{R}^N}|\bar v_n|^{2^*}\,d\tilde x-A\Big)\\
&=\tilde J_\mu(\bar v_n)+\frac{1}{2^*}A\\
&=\tilde J_\mu(\bar v_n)+\frac12A+\Big(\frac{1}{2^*}-\frac12\Big)A\\
&<\tilde J_\mu(w_0)+\tilde J_\mu(\tilde w_n)+\Big(\frac{1}{2^*}-\frac12\Big)A+o(1).
\end{aligned}$$

Thus, we obtain

$$\tilde J_\mu(w_0)+\tilde J_\mu(\tilde w_n)>\tilde J_\mu(v)+\Big(\frac12-\frac{1}{2^*}\Big)A.$$

Using an argument as in sub-step 2-1, we obtain

$$\tilde J_\mu(v)\ge\frac2N\mathcal{A}_\mu^{N/4}.$$

It follows that

$$\tilde J_\mu(w_0)+\tilde J_\mu(\tilde w_n)+o(1)\ge\frac2N\mathcal{A}_\mu^{N/4}+\Big(\frac12-\frac{1}{2^*}\Big)A>\frac2N\mathcal{A}_\mu^{N/4}. \tag{3.12}$$

On the other hand, we have

$$\begin{aligned}
J_\mu(u_n)&=\frac12\int_{\mathbb{R}^N}\left(|\Delta w_0|^2-\mu r_n^4V(r_n\tilde x+y_n)w_0^2\right)\,d\tilde x-\frac{1}{2^*}\int_{\mathbb{R}^N}|w_0|^{2^*}\,d\tilde x+o(1)\\
&\quad+\frac12\int_{\mathbb{R}^N}\left(|\Delta\tilde w_n|^2-\mu r_n^4V(r_n\tilde x+y_n)\tilde w_n^2\right)\,d\tilde x-\frac{1}{2^*}\int_{\mathbb{R}^N}|\tilde w_n|^{2^*}\,d\tilde x\\
&=\tilde J_\mu(w_0)+\tilde J_\mu(\tilde w_n)+o(1).
\end{aligned}$$

Using (3.12), we obtain

$$J_\mu(u_n)>\frac2N\mathcal{A}_\mu^{N/4}.$$

That is absurd in contrast to (3.6). When $A=0$, the proof is similar. Using (3.10), (3.11) with $A=0$, we obtain from the same argument as above that

$$J_\mu(u_n)=\tilde J_\mu(w_0)+\tilde J_\mu(\tilde w_n)+o(1)\ge\frac2N\mathcal{A}_\mu^{N/4}+\frac2N\mathcal{A}_\mu^{N/4}>\frac2N\mathcal{A}_\mu^{N/4}.$$

Again, we obtain a contradiction.

Step 3. If $r_n\to\bar r=0$ and $\bar y\ne a_j$, then

$$r_n^4V(r_n\tilde x+y_n)\le\frac{r_n^4}{|r_n\tilde x+y_n-a_j|^4}-r_n^4\,|r_n\tilde x+y_n-a_j|^{\alpha-4}\to0.$$

Thus, we obtain

$$J_\mu(u_n) = \frac{1}{2}\int_{\mathbb{R}^N} |\Delta w_n|^2 d\tilde{x} - \frac{1}{2^*}\int_{\mathbb{R}^N} |w_n|^{2^*} d\tilde{x} + o(1). \tag{3.13}$$

We also divide this step into two sub-steps.

Sub-step 3.1. When $\|\tilde{w}_n\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to 0$, we have

$$\int_{\mathbb{R}^N} |\Delta w_n|^2 \, d\tilde{x} \geq \mathcal{A}_0 \Big(\int_{\mathbb{R}^N} |w_n|^{2^*} \, d\tilde{x} \Big)^{2/2^*} = \mathcal{A}_0 \Big(\int_{\mathbb{R}^N} |\Delta w_n|^2 \, d\tilde{x} \Big)^{2/2^*},$$

hence, we obtain

$$\int_{\mathbb{R}^N} |\Delta w_n|^2 \, d\tilde{x} \geq \mathcal{A}_0^{N/4}.$$

It follows from (3.13) that

$$\begin{aligned} J_\mu(u_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\Delta w_n|^2 \, d\tilde{x} - \frac{1}{2^*}\int_{\mathbb{R}^N} |w_n|^{2^*} \, d\tilde{x} + o(1) \\ &= \frac{2}{N}\int_{\mathbb{R}^N} |\Delta w_n|^2 \, d\tilde{x} + o(1) \geq \frac{2}{N}\mathcal{A}_0^{N/4} > \frac{2}{N}\mathcal{A}_\mu^{N/4}, \end{aligned}$$

which also contradicts (3.6).

Sub-step 3.2. If $\|\tilde{w}_n\|_{\mathcal{D}^{2,2}(\mathbb{R}^N)} \to L > 0$, then the proof is similar. Using (3.12), we obtain from the same arguments as above that

$$\begin{aligned} J_\mu(u_n) &= \tilde{J}_\mu(w_0) + \tilde{J}_\mu(\tilde{w}_n) + o(1) \\ &\geq \begin{cases} \frac{2}{N}\mathcal{A}_0^{N/4} + \left(\frac{1}{2} - \frac{1}{2^*}\right) A, & \text{if } A \neq 0, \\ \frac{2}{N}\mathcal{A}_0^{N/4} + \frac{2}{N}\mathcal{A}_0^{N/4}, & \text{if } A = 0. \end{cases} \\ &> \frac{2}{N}\mathcal{A}_\mu^{N/4}. \end{aligned}$$

Again, we obtain a contradiction.

This completes the proof of Proposition 3.5. □

4. Proof of Theorem 1.1

To obtain the existence of multiple solutions for the problem (1.1), we need several lemmas.

Lemma 4.1. *For each $u \in \mathcal{N}_j^-$ there exist a number $\rho_u > 0$ and a continuous function $h > 0$ defined on $\{w \in H_0^2(\Omega) : \|w\| < \rho_u\}$, satisfying*

$$h(0) = 1, \quad h(w)(u - w) \in \mathcal{N}_j^-, \ \textit{for } \|w\| < \rho_u,$$

and

$$\langle h'(0), w\rangle = \frac{2\int_\Omega (\Delta u \Delta w - \mu V(x) u w) \, dx - 2^* \int_\Omega |u|^{2^*} u w \, dx - \lambda \int_\Omega f u \, dx}{\int_\Omega (|\Delta u|^2 - \mu V(x) u^2) \, dx - (2^* - 1)\int_\Omega |u|^{2^*} \, dx}.$$

Proof. Define $H : \mathbb{R} \times H_0^2(\Omega) \to \mathbb{R}$ as follows

$$\begin{aligned} H(t, w) = t\Big(&\int_\Omega \left(|\Delta(u - w)|^2 - \mu V(x)(u - w)^2\right) \, dx\Big) \\ &- t^{2^*-1}\int_\Omega |u - w|^{2^*} dx - \lambda \int_\Omega f(u - w) \, dx. \end{aligned}$$

Since $u \in \mathcal{N}_j^-$, we have

$$H(1, 0) = \int_\Omega \left(|\Delta u|^2 - \mu V(x) u^2\right) \, dx - \int_\Omega |u|^{2^*} \, dx - \lambda \int_\Omega f u \, dx = 0.$$

and

$$\frac{\partial H}{\partial t}(1, 0) = \int_\Omega \left(|\Delta u|^2 - \mu V(x)(u)^2\right) \, dx - (2^* - 1)\int_\Omega |u|^{2^*} \, dx \neq 0.$$

We obtain the result by applying the implicit function Theorem at the point $(1, 0)$. □

Lemma 4.2. *Assume that* (A1)–(A3) *are verified, and* $0 < \mu < \bar{\mu}$. *Then, for any sequence* $\{u_n^j\} \subset \mathcal{N}_j^-$, *satisfying*

$$J_\mu(u_n^j) \to m < \frac{2}{N}\mathcal{A}_\mu^{N/4} - \chi\lambda^2,$$

and

$$J'_\mu(u_n^j) \to 0, \text{ in } H^{-2}(\Omega), \tag{4.1}$$

$\{u_n^j\}$ *is relatively compact in* $H_0^2(\Omega)$.

Proof. Since $\{u_n^j\} \subset \mathcal{N}_j^-$ and $J_\mu(u_n^j) \to m$ as $n \to \infty$, we can assume from a similar argument to Proposition 3.5 that

$$0 < \nu_3 \leq \|u_n^j\| \leq \nu_4$$

for some positive constants ν_3 and ν_4. Going if necessary to a subsequence, we can assume that $u_n^j \rightharpoonup u^j$ in $H_0^2(\Omega)$ and a.e. in Ω. Using (4.1) then for any $\varphi \in H_0^2(\Omega)$,

$$\begin{aligned} o(1) &= \langle J'_\mu(u_n^j), \varphi\rangle \\ &= \int_\Omega \left(\Delta u_n^j \Delta\varphi - \mu V(x) u_n^j \varphi\right)\, dx - \int_\Omega |u_n^j|^{2^*-2} u_n^j \varphi\, dx - \lambda \int_\Omega f\varphi\, dx \\ &= \int_\Omega \left(\Delta u^j \Delta\varphi - \mu V(x) u^j \varphi\right)\, dx - \int_\Omega |u^j|^{2^*-2} u^j \varphi\, dx - \lambda \int_\Omega f\varphi\, dx + o(1) \\ &= \langle J'_\mu(u^j), \varphi\rangle + o(1), \end{aligned}$$

that is, u^j is a weak solution of (1.1), and

$$\int_\Omega \left(|\Delta u^j|^2 - \mu V(x)(u^j)^2\right)\, dx - \int_\Omega |u^j|^{2^*}\, dx - \lambda \int_\Omega f u^j\, dx = 0. \tag{4.2}$$

Now, we claim that $u^j \neq 0$. Arguing by contradiction, we assume that $u^j \equiv 0$, then $\lim_{n\to\infty} \int_\Omega f u_n^j dx = \int_\Omega f u^j dx = 0$ and therefore, from $\{u_n^j\} \subset \mathcal{N}_j^-$ we have

$$\int_\Omega \left(|\Delta u_n^j|^2 - \mu V(x)(u_n^j)^2\right)\, dx - \int_\Omega |u_n^j|^{2^*}\, dx = o(1),$$

thus, we obtain

$$\lim_{n\to\infty} \int_\Omega (|\Delta u_n^j|^2 - \mu V(x)(u_n^j)^2)\, dx = \lim_{n\to\infty} \int_\Omega |u_n^j|^{2^*}\, dx = l_2 > 0,$$

and we find $y_n \in \Omega, r_n \geq 0$ such that

$$\int_{B(y_n, r_n)} |u_n^j|^{2^*}\, dx = \frac{l_2}{2}.$$

Again denoting $w_n(\tilde{x}) = r_n^{\frac{N-4}{2}} u_n^j(r_n\tilde{x} + y_n)$, by extending $w_n(\tilde{x})$ to be zero for $\tilde{x}$ outside Ω_n, we obtain $w_n(\tilde{x}) \in \mathcal{D}^{2,2}(\mathbb{R}^N)$ with

$$\int_{\mathbb{R}^N} \left(|\Delta w_n(\tilde{x})|^2 - r_n^4 V(r_n\tilde{x} + y_n) w_n^2(\tilde{x})\right)\, d\tilde{x} - \int_{\mathbb{R}^N} |w_n(\tilde{x})|^{2^*}\, d\tilde{x} = o(1).$$

We divide the discussion into two cases.

Case III. If $r_n\tilde{x} + y_n \notin B(a_j, \delta_0)$, it follows from the definition of $\beta_j(u)$, ψ_j and r_0 that

$$\frac{\delta_0}{3} = r_0 = \beta_j(u_n^j) = \frac{\int_\Omega \psi_j(x)|\Delta u_n^j(x)|^2\, dx}{\int_\Omega |\Delta u_n^j(x)|^2\, dx} = \frac{\int_{\mathbb{R}^N} \psi_j(r_n\tilde{x} + y_n)|\Delta w_n(\tilde{x})|^2\, d\tilde{x}}{\int_{\mathbb{R}^N} |\Delta w_n(\tilde{x})|^2\, d\tilde{x}} \to \delta_0,$$

which is a contradiction.

Case IV. If $r_n\tilde{x} + y_n \in B(a_j, \delta_0)$, then

$$\begin{aligned} J_\mu(u_n^j) &= \frac{1}{2}\int_\Omega (|\Delta u_n^j|^2 - \mu V(x)(u_n^j)^2)\, dx - \frac{1}{2^*}\int_\Omega |u_n^j|^{2^*}\, dx + o(1) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta w_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x} + y_n) w_n^2(\tilde{x})\right)\, d\tilde{x} - \frac{1}{2^*}\int_{\mathbb{R}^N} |w_n(\tilde{x})|^{2^*}\, d\tilde{x} + o(1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{N}\int_{\mathbb{R}^N} \big(|\Delta w_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n) w_n^2(\tilde{x})\big)\, d\tilde{x} + o(1)\\
&= \tilde{J}_\mu(w_n) + o(1),
\end{aligned}
$$

The proof is divided into several steps: step 1: $r_n \to \bar{r} = 0$ and $\bar{y} = a_j$; step 2: $r_n \to \bar{r} > 0$; step 3: $r_n \to \bar{r} = 0$ and $\bar{y} \neq a_j$.

We use the same arguments as those in the corresponding steps in the proof of Proposition 3.5 to get that $\tilde{J}_\mu(w_n) \geq \frac{2}{N}\mathcal{A}_\mu^{N/4}$. Here, we only sketch the argument for step 2. It follows from (1.2), that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde{x})|^2 - \mu \frac{r_n^4}{|r_n\tilde{x}+y_n-a_j|^4} w_n^2(\tilde{x})\Big)\, d\tilde{x}\\
&\geq \mathcal{A}_\mu \Big(\int_{\mathbb{R}^N} |w_n(\tilde{x})|^{2^*}\, d\tilde{x}\Big)^{2/2^*}\\
&= \mathcal{A}_\mu \Big(\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n) w_n^2(\tilde{x})\Big)\, d\tilde{x}\Big)^{2/2^*}\\
&\geq \mathcal{A}_\mu \Big(\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde{x})|^2 - \mu \frac{r_n^4}{|r_n\tilde{x}+y_n-a_j|^4} w_n^2(\tilde{x})\Big)\, d\tilde{x}\Big)^{2/2^*}.
\end{aligned}
$$

So, that

$$
\int_{\mathbb{R}^N} \Big(|\Delta w_n(\tilde{x})|^2 - \mu \frac{r_n^4}{|r_n\tilde{x}+y_n-a_j|^4} w_n^2(\tilde{x})\Big)\, d\tilde{x} \geq \mathcal{A}_\mu^{N/4}.
$$

Thus, from (A2) it follows that

$$
\int_{\mathbb{R}^N} \big(|\Delta w_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n) w_n^2(\tilde{x})\big)\, d\tilde{x} \geq \mathcal{A}_\mu^{N/4}.
$$

Then, we have

$$
\frac{2}{N}\mathcal{A}_\mu^{N/4} - \chi\lambda^2 > m = \lim_{n\to\infty} J_\mu(u_n^j) = \lim_{n\to\infty} \tilde{J}_\mu(w_n) \geq \frac{2}{N}\mathcal{A}_\mu^{N/4},
$$

which is a contradiction. In sum, we have proved that $u^j \neq 0$.

Next, we prove that the limit u^j is indeed strong. Suppose by contradiction that $\|u_n^j - u^j\| \to \nu_5 > 0$ and denote $v_n = u_n^j - u^j$, then we have $v_n \rightharpoonup 0$ in $H_0^2(\Omega)$ and $\|v_n\| \to \nu_5 > 0$. It follows from Brezis-Lieb Lemma and (4.2), that

$$
\begin{aligned}
&\int_\Omega \big(|\Delta v_n(x)|^2 - V(x) v_n^2(x)\big)\, dx - \int_\Omega |v_n(x)|^{2^*}\, dx\\
&= \int_\Omega \big(|\Delta u_n^j(x)|^2 - V(x)(u_n^j)^2(x)\big)\, dx - \int_\Omega |u_n^j(x)|^{2^*}\, dx\\
&\quad - \int_\Omega \big(|\Delta u^j(x)|^2 - r_n^4 V(x)(u^j)^2(x)\big)\, dx + \int_\Omega |u^j(x)|^{2^*}\, dx + o(1)\\
&= \lambda \int_\Omega f u_n^j\, dx - \lambda \int_\Omega f u^j\, dx + o(1)\\
&= o(1).
\end{aligned}
$$

Similarly to the previous proof, we can find $y_n \in \Omega$, $r_n > 0$ such that

$$
\int_{B(y_n,r_n)} |u_n^j|^{2^*}\, dx = \frac{l_3}{2} > 0.
$$

Letting $z_n(\tilde{x}) = r_n^{\frac{N-4}{2}} v_n(r_n\tilde{x}+y_n)$, and extending z_n to be zero for $\tilde{x}$ outside $\Omega_n = r_n^{-1}(\Omega - y_n)$, we can write $z_n(\tilde{x}) \in D^{2,2}(\mathbb{R}^N)$ and we have that

$$
\int_{\mathbb{R}^N} \big(|\Delta z_n(\tilde{x})|^2 - r_n^4 V(r_n\tilde{x}+y_n) z_n^2(\tilde{x})\big)\, d\tilde{x} - \int_{\mathbb{R}^N} |z_n(\tilde{x})|^{2^*}\, d\tilde{x} = o(1). \tag{4.3}
$$

We will distinguish two cases.

Case V. If $r_n\tilde{x}+y_n \notin B(a_j,\delta_0)$, we can use similar arguments as those in case III to obtain a contradiction, so we omit the details here.

Case VI. If $r_n\tilde{x}+y_n \in B(a_j,\delta_0)$, then

$$\begin{aligned}
J_\mu(u_n^j) &= \frac{1}{2}\int_\Omega (|\Delta v_n|^2 - \mu V(x) v_n^2)\,dx - \frac{1}{2^*}\int_\Omega |v_n|^{2^*}\,dx + J_\mu(u^j) + o(1)\\
&= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta z_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n) z_n^2(\tilde{x})\right)\,d\tilde{x} - \frac{1}{2^*}\int_{\mathbb{R}^N} |z_n(\tilde{x})|^{2^*}\,d\tilde{x} + J_\mu(u^j) + o(1)\\
&= \tilde{J}_\mu(z_n) + J_\mu(u^j) + o(1).
\end{aligned}$$

Now, we need to distinguish three steps.

Step 1. If $r_n \to \bar{r} > 0$, it follows from (1.2) and (4.3), that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(|\Delta z_n(\tilde{x})|^2 - \mu \frac{r_n^4}{|r_n\tilde{x}+y_n-a_j|^4} z_n^2(\tilde{x})\Big)\,d\tilde{x}\\
&\geq \mathcal{A}_\mu \Big(\int_{\mathbb{R}^N} |z_n(\tilde{x})|^{2^*}\Big)^{2/2^*}\,d\tilde{x}\\
&\geq \mathcal{A}_\mu \Big(\int_{\mathbb{R}^N} |\Delta z_n(\tilde{x})|^2 - \mu \frac{r_n^4}{|r_n\tilde{x}+y_n-a_j|^4} z_n^2(\tilde{x})\Big)^{2/2^*}\,d\tilde{x}.
\end{aligned}$$

Thus, we obtain

$$\tilde{J}_\mu(z_n) = \frac{2}{N}\int_{\mathbb{R}^N} \left(|\Delta z_n(\tilde{x})|^2 - \mu r_n^4 V(r_n\tilde{x}+y_n) z_n^2(\tilde{x})\right)\,d\tilde{x} \geq \frac{2}{N}\mathcal{A}_\mu^{N/4}. \tag{4.4}$$

Step 2. If $r_n \to \bar{r} = 0$ and $\bar{y} \neq a_j$ then we have

$$r_n^4 V(r_n\tilde{x}+y_n) \leq \frac{r_n^4}{|r_n\tilde{x}+y_n-a_j|^4} - r_n^4\,|r_n\tilde{x}+y_n-a_j|^{\alpha-4} \to 0.$$

Thus, we obtain $\tilde{J}_\mu(z_n) = \frac{2}{N}\int_{\mathbb{R}^N} |\Delta z_n(\tilde{x})|^2\,d\tilde{x}$. Then, we obtain

$$\int_{\mathbb{R}^N} |\Delta z_n|^2\,d\tilde{x} \geq \mathcal{A}_0\Big(\int_{\mathbb{R}^N} |z_n|^{2^*}\,d\tilde{x}\Big)^{2/2^*} = \mathcal{A}_0\Big(\int_{\mathbb{R}^N} |\Delta z_n|^2\,d\tilde{x}\Big)^{2/2^*},$$

hence

$$\tilde{J}_\mu(z_n) \geq \frac{2}{N}\mathcal{A}_0^{N/4} \geq \frac{2}{N}\mathcal{A}_\mu^{N/4}. \tag{4.5}$$

Step 3. If $r_n \to \bar{r} = 0$ and $\bar{y} = a_j$, then similarly, we can obtain

$$\tilde{J}_\mu(z_n) \geq \frac{2}{N}\mathcal{A}_0^{N/4} \geq \frac{2}{N}\mathcal{A}_\mu^{N/4}. \tag{4.6}$$

It follows from (4.4), (4.5) and (4.6) that

$$\tilde{J}_\mu(z_n) \geq \frac{2}{N}\mathcal{A}_\mu^{N/4}.$$

Since u^j is a solution of (1.1) then from Lemma 2.1, we obtain

$$J_\mu(u^j) \geq -\chi\lambda^2.$$

Therefore,

$$J_\mu(u_n^j) = \tilde{J}_\mu(z_n) + J_\mu(u^j) + o(1) \geq \frac{2}{N}\mathcal{A}_\mu^{N/4} - \chi\lambda^2,$$

which is a contradiction with the choice of $\{u_n^j\}$.

This completes the proof of Lemma 4.2. □

Proof of Theorem 1.1. Fix $\lambda^* = \min\{\lambda_1,\lambda_2,\lambda_3\}$. For any $j \in \{1,\dots,k\}$, we obtain $\overline{\mathcal{N}_j^-} = \mathcal{N}_j^- \cup \Gamma_j^-$. Propositions 3.4 and 3.5 imply that

$$c_{\lambda,j} < \frac{2}{N}\mathcal{A}_\mu^{N/4} - \chi\lambda^2 < \hat{c}_{\lambda,j}.$$

Thus, $c_{\lambda,j} = \inf\{J_\mu(u), u \in \overline{\mathcal{N}_j^-}\}$. By the Ekeland variational principle, we can obtain the minimizing sequence $\{u_n^j\} \subset \overline{\mathcal{N}_j^-}$ satisfying

$$c_{\lambda,j} < J_\mu(u_n^j) < c_{\lambda,j} + \frac{1}{n} \quad \text{and} \quad J'_\mu(u_n^j) \to 0, \ in \ H^{-2}(\Omega).$$

Therefore, by Lemma 4.2 up to a subsequence we have $u_n^j \to u^j$ and $u^j \neq 0$ in $H_0^2(\Omega)$ and so u^j is a solution of (1.1). Moreover, from Remark 3.2, we know that u^i and u^j are distinct if $i \neq j$. This implies that problem (1.1) has at least k solutions $u^j \in \mathcal{N}_j^-$. □

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