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NORMALIZED SOLUTIONS FOR A FRACTIONAL KIRCHHOFF-SCHRÖDINGER-POISSON SYSTEMS WITH CRITICAL GROWTH

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Abstract. In this article, we study the fractional Kirchhoff-Schrödinger-Poisson system with Sobolev critical growth

$$\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u+\phi u=\lambda u+\mu|u|^{p-2}u+|u|^{2_s^*-2}u,\quad \text{in } \mathbb{R}^3,$$
$$(-\Delta)^s\phi=u^2,\quad \text{in } \mathbb{R}^3,$$

where $a,b>0$, $s\in(\frac{3}{4},1)$, $p\in(2,2_s^*)$, and $\mu>0$ is a parameter, $\lambda\in\mathbb{R}$ is an undermined parameter. For this problem, under the L^2-subcritical, $p\in(2,\frac{4s}{3}+2)$, we obtain the multiplicity of the normalized solutions by means of the truncation technique, concentration-compactness principle, and genus theory. In the L^2-supercritical, $p\in(\frac{8s}{3}+2,2_s^*)$, we prove a couple of normalized solutions by developing a fiber map and using the concentration-compactness principle.

1. Introduction

In this article, we study the nonlinear Kirchhoff-Schrödinger-Poisson system with Sobolev critical growth

$$\begin{gathered}(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx)(-\Delta)^s u+\phi u=\lambda u+\mu|u|^{p-2}u+|u|^{2_s^*-2}u,\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^s\phi=u^2,\quad \text{in } \mathbb{R}^3,\end{gathered}\tag{1.1}$$

where $a,b>0$, $s\in(\frac{3}{4},1)$, and $p\in(2,2_s^*)$, $\mu>0$ and $\lambda\in\mathbb{R}$ are parameters. Here $(-\Delta)^s$ $(s\in(0,1))$ is the fractional Laplacian operator which is defined by

$$(-\Delta)^s u(x)=C_s\text{P.V.}\int_{\mathbb{R}^3}\frac{u(x)-u(y)}{|x-y|^{3+2s}}dy=C_s\lim_{\epsilon\to 0}\int_{\mathbb{R}^3\setminus B_\epsilon(x)}\frac{u(x)-u(y)}{|x-y|^{3+2s}}dy,$$

for $u\in\mathcal{S}(\mathbb{R}^3)$, where $\mathcal{S}(\mathbb{R}^3)$ is the Schwartz space of rapidly decaying C^∞ functions, $B_\epsilon(x)$ denote an open ball of radius ϵ at x and C_s is a normalization constant.

System (1.1) has been motivated by the time-dependent fractional Schrödinger-Poisson system

$$\begin{gathered}i\frac{\partial\Psi}{\partial\tau}=(-\Delta)^s\Psi+\lambda\phi\Psi-f(x,|\Psi|),\quad x\in\mathbb{R}^3,\\ (-\Delta)^t\phi=|\Psi|^2,\quad x\in\mathbb{R}^3,\end{gathered}\tag{1.2}$$

where $\Psi:\mathbb{R}\times\mathbb{R}^3\to\mathbb{C}$, $s,t\in(0,1)$, $\lambda\in\mathbb{R}$. It is well-known that, the first equation in system (1.2) was used by Laskin (see [20, 21]) to extend the Feynman path integral, from Brownian-like to Lévy-like quantum mechanical paths. This class of fractional Schrödinger equations with a repulsive nonlocal Coulombic potential can be approximated by the Hartree-Fock equations to describe a quantum mechanical system of many particles. For more application backgrounds on the fractional Laplacian see [5, 7, 23].

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When looking for solutions to system (1.1), there are two distinct options regarding the frequency parameter λ. One is to regard the frequency λ as a given constant. Xiang and Wang [32] first investigated the fractional Kirchhoff-Schrödinger-Poisson system, and obtained the existence, multiplicity and asymptotic behavior of nonnegative solutions. In recent years, researchers have shown growing interest in the fractional Kirchhoff-Schrödinger-Poisson system

$$\begin{gathered}(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx)(-\Delta)^s u+V(x)u+\mu\phi u=f(x,u),\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=\mu u^2,\quad \text{in } \mathbb{R}^3,\end{gathered}\tag{1.3}$$

where $a>0$, $b\geq 0$. When $V(x)=0$, $f(x,u)=f(u)$, by utilizing minimax argument, Ambrosio [1] obtained the existence of a nontrivial solutions for system (1.3) with Berestycki-Lions type nonlinearities. When $V(x)\neq 0$, Wang et.al [30] studied the existence of ground solutions for system (1.3) with $V(x)=1$ and $f(x,u)=(|x|^{-\theta}*F(u))f(u)$, $\theta\in(0,3-2t)$, and used the Pohozǎev type manifold; Then, under some assumptions on V and f, by using constraint variational approach and a quantitative deformation lemma, Meng et.al [24] proved the existence of the least energy sign-changing solutions for system (1.3). Feng et al. [12] applied a similar method studied the least energy sign-changing solutions to the following critical fractional Kirchhoff-Schrödinger-Poisson system with steep potential well

$$\begin{gathered}\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u+V_\lambda(x)u+\phi u=|u|^{p-2}u+|u|^{2_s^*-2}u,\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=u^2,\quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $s\in(\frac{3}{4},1)$, $t\in(0,1)$, $V_\lambda(x)=\lambda V(x)+1$, $\lambda>0$ and $p>4$. Jian et al. [18], deal with the fractional Kirchhoff-Schrödinger-Poisson system with steep potential well

$$\begin{gathered}(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx)(-\Delta)^s u+\lambda V(x)u+\mu\phi u=|u|^{p-2}u,\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=u^2,\quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $s\in[\frac{3}{4},1)$, $t\in(0,1)$, $2<p<4$, $a>0$ is a constant, and b,λ,μ are positive parameters. By applying the truncation technique and the parameter-dependent compactness lemma, they first proved the existence of positive solutions. Furthermore, they investigated the asymptotic behavior as $b\to 0$, $\lambda\to\infty$ and $\mu\to 0$, respectively. For other existence results, we refer to [11, 28, 29] and the references therein.

When $a=1$, $b=0$, system (1.3) reduces to the following fractional Schrödinger-Poisson system

$$\begin{gathered}(-\Delta)^s u+V(x)u+\mu\phi u=f(x,u),\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=\mu u^2,\quad \text{in } \mathbb{R}^3.\end{gathered}\tag{1.4}$$

Recently, under various potentials and nonlinear terms, most scholars have investigated the existence and multiplicity of ground state solutions, sign-changing solutions, and nontrivial solutions for system (1.4). For further details, we refer the interesting readers to see [6, 10, 13, 15, 26] and so on.

Alternatively, the other one is to regard the frequency λ as an unknown quantity. In such point of view, it is natural to prescribe the mass, i.e., the L^2-norm, so that λ can be interpreted as a Lagrange multiplier. Solutions of this type are often referred to as normalized solutions. Nowadays, from a physical point of view, some physicists are very interested in the normalized solutions, see for example [4, 22, 33]. There a few results are related to the study of normalized solutions for the fractional Kirchhoff-Schrödinger-Poisson system except Wang et al. [31]. They considered the fractional Kirchhoff-Schrödinger-Poisson system

$$\begin{gathered}\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)(-\Delta)^s u+\phi u=f(u)+\lambda u\quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t\phi=u^2\quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $a, b > 0$, $s, t \in (0,1)$, $2s + 2t > 3$, $\lambda \in \mathbb{R}$, $f \in C(\mathbb{R}, \mathbb{R})$ satisfies some general conditions which contain the case $f(u) \sim |u|^{p-2}u$ with $p \in (\frac{8s+4t-3}{s+t}, \frac{8s}{3} + 2) \cup (\frac{8s}{3} + 2, 2_s^*)$, $2_s^* = \frac{6}{3-2s}$. They obtained the existence of normalized solutions by using the Pohozǎev manifold and variational method.

Recently, He and Meng [14] studied the existence and multiplicity of the normalized solutions for the nonlinear fractional Schrödinger-Poisson system with Sobolev critical exponent

$$\begin{gathered}(-\Delta)^s u + \alpha\phi u = \mu|u|^{p-2}u + |u|^{2_s^*-2}u + \lambda u \quad \text{in } \mathbb{R}^3,\\ (-\Delta)^t \phi = u^2 \quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $s, t \in (0,1)$, $2s + 2t > 3$, $p \in (2, 2_s^*)$, $\alpha, \mu > 0$ are parameters and $\lambda \in \mathbb{R}$.

Motivated by [14] and [31], a natural question is whether the fractional Kirchhoff-Schrödinger-Poisson system with the Sobolev critical growth can be applied to obtain the existence and multiplicity of normalized solutions for p in distinct ranges. Therefore, we study system (1.1) and give an affirmative answer. In addition, to recover the compactness, we will take $H_r^s(\mathbb{R}^3)$ as a working space.

By using the Lax-Milgram theorem, for any $u \in H_r^s(\mathbb{R}^3)$, a unique $\phi_u^s(x) \in D^{s,2}(\mathbb{R}^3)$ is given by

$$\phi_u^s(x) = |x|^{2s-3} * |u|^2 = \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2s}} dy,$$

such that $(-\Delta)^s\phi = u^2$ and that inserting it into the first equation of system (1.1), then system (1.1) can be transformed into the following single equation

$$\Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx\Big)(-\Delta)^s u + \phi_u^s u = \lambda u + \mu|u|^{p-2}u + |u|^{2_s^*-2}u, \quad \text{in } \mathbb{R}^3. \tag{1.5}$$

It can be proved that to find the normalized solutions of system (1.1) is to seek the critical points of the functional

$$\begin{aligned} I_\mu(u) = &\frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx\Big)^2 \\ &+ \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx, \end{aligned}$$

under the constraint

$$S_r(c) = \Big\{u \in H_r^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = c^2, c > 0\Big\}.$$

It is well known that $I_\mu \in C^1(H_r^s(\mathbb{R}^3), \mathbb{R})$. Here are our main results.

Theorem 1.1. *Assume $a, b > 0$, $s \in (\frac{3}{4}, 1)$ and $2 < p < \frac{4s}{3} + 2$, for given $k \in \mathbb{N}$, then there exist $\beta > 0$, $\Lambda = \big(\frac{1}{2} - \frac{1}{2_s^*}\big)a^{\frac{3}{2s}}S^{\frac{3}{2s}} > 0$, $D = \big(\frac{1}{p} - \frac{1}{2_s^*}\big)C(p,s)R_1^{p\delta_{p,s}} > 0$ independent of k and $\mu_k^* > 0$ large, such that system (1.1) possesses at least k couples $(u_j, \lambda_j) \in H_r^s(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions for $\mu > \mu_k^*$ and*

$$c \in \Big(0, \min\Big\{\Big(\frac{\beta}{\mu}\Big)^{\frac{1}{p(1-\delta_{p,s})}}, \Big(\frac{\Lambda}{D\mu}\Big)^{\frac{1}{p(1-\delta_{p,s})}}\Big\}\Big)$$

with $\int_{\mathbb{R}^3} |u_j|^2 dx = c^2$, $\lambda_j < 0$ for all $j = 1, 2, ..., k$, $\delta_{p,s} = \frac{3(p-2)}{2ps}$.

Theorem 1.2. *Assume $a, b > 0$, $s \in (\frac{3}{4}, 1)$ and $2 + \frac{8s}{3} < p < 2_s^*$, then there exists $\mu^* = \mu^*(c) > 0$ large, such that as $\mu > \mu^*$, system (1.1) possesses a couple $(u_c, \lambda) \in H_r^s(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions with $\int_{\mathbb{R}^3} |u_c|^2 dx = c^2$, $\lambda < 0$.*

Remark 1.3. To our best knowledge, our results are up to date. On the one hand, the Sobolev critical exponent leads to the lack of compactness. Even the embedding of the radially symmetric space of $H_r^s(\mathbb{R}^3)$ into $L^{2_s^*}(\mathbb{R}^3)$ is not compact. Furthermore, $H_r^s(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3)$ is also not compact. Then, the weak limit of Palais-Smale sequences could leave the constrained manifold $S_r(c)$. Hence, we need to estimate finely the Lagrange multipliers, which is vital in obtaining compactness. We shall employ the concentration-compactness principle, mountain pass theorem and the truncation method to overcome the loss of compactness caused by the critical growth.

On the other hand, no matter $2 < p < \frac{4s}{3} + 2$ or $\frac{8s}{3} + 2 < p < 2_s^*$, I_μ on the constrained manifold $S_r(c)$ is all unbounded from below. Hence, it is unlikely to obtain a solution to system (1.1) by minimizing method. We adopt some ideas from [2] to overcome the difficulty.

This article is structured as follows: in section 2, we presents some preliminary results that will be used frequently in the sequel. Theorem 1.1 is proved in section 3, which presents the multiplicity of normalized solutions for system (1.1) when $p \in (2, \frac{4s}{3} + 2)$. In this section, we address three main challenges. First, we employ the truncation technique to establish the boundedness of the (PS) sequence. Subsequently, we apply the concentration compactness principle to restore the compactness lost of the (PS) sequence due to the critical growth. Finally, we use genus theory to prove the multiplicity of normalized solutions for system (1.1). In section 4, when the parameter $\mu > 0$ is large, we give another existence result for system (1.1) with $p \in (\frac{8s}{3} + 2, 2_s^*)$ by using the fiber map and concentration-compactness principle.

Notation Throughout this paper, we denote $\|\cdot\|_q$ the usual norm of the space $L^q(\mathbb{R}^3)$, $1 \le q < \infty$, $B_r(x)$ denotes the open ball with center at x and radius r, C or $C_i (i = 1, 2, ...)$ denote various positive constants whose exact values are irrelevant. $\rightharpoonup$ and $\to$ mean the weak and strong convergence.

2. Preliminaries

In this section, we first introduce some notations. For any $s \in (0, 1)$, the homogeneous Sobolev space $D^{s,2}(\mathbb{R}^3)$ is defined by $D^{s,2}(\mathbb{R}^3) = \{u \in L^{2_s^*}(\mathbb{R}^3) : \|u\|_{D^{s,2}} < \infty\}$, where

$$\|u\|_{D^{s,2}}^2 = \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}}\, dx\, dy.$$

The fractional space $H^s(\mathbb{R}^3)$ is defined by

$$H^s(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \|u\|_{D^{s,2}} < \infty\},$$

endowed with the norm

$$\|u\|_{H^s}^2 = \|u\|_2^2 + \|u\|_{D^{s,2}}^2.$$

The best fractional Sobolev constant S is defined as

$$S = \inf_{u \in D^{s,2}, u \neq 0} \frac{\|(-\Delta)^{s/2} u\|_2^2}{(\int_{\mathbb{R}^3} |u|^{2_s^*} dx)^{2/2_s^*}}. \tag{2.1}$$

The work space is

$$H_r^s(\mathbb{R}^3) = \{u \in H^s(\mathbb{R}^3) |\ u(x) = u(|x|)\}.$$

Let $\mathbb{H} = H_r^s(\mathbb{R}^3) \times \mathbb{R}$ with the scalar product $\langle \cdot, \cdot \rangle_{H_r^s} + \langle \cdot, \cdot \rangle_{\mathbb{R}}$, and the corresponding norm $\|(\cdot, \cdot)\|_{\mathbb{H}}^2 = \|\cdot\|_{H_r^s}^2 + |\cdot|_{\mathbb{R}}^2$.

Proposition 2.1 (Hardy-Littlewood-Sobolev inequality)**.** *Let $r, l > 1$ and $0 < \lambda < N$ with $\frac{1}{r} + \frac{1}{l} + \frac{\lambda}{N} = 2$. Let $f \in L^r(\mathbb{R}^N)$ and $g \in L^l(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, \lambda, l, r) > 0$, independent of f and g, such that*

$$\Big| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(x) g(y) |x - y|^{-\lambda}\, dx\, dy \Big| \le C(N, \lambda, l, r) \|f\|_r \|g\|_l.$$

Moreover, if $r = l = \frac{2N}{2N - \lambda}$, then

$$C(N, \lambda, l, r) = \pi^{\lambda/2} \frac{\Gamma(\frac{N}{2} - \frac{\lambda}{2})}{\Gamma(N - \frac{\lambda}{2})} \Big(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \Big)^{1 - \frac{\lambda}{N}}.$$

From Proposition 2.1, with $r = l = \frac{6}{3+2s}$, the Hardy-Littlewood-Sobolev inequality implies that

$$\int_{\mathbb{R}^3} \phi_u^s u^2 dx = \int_{\mathbb{R}^3} \big(|x|^{2s-3} * u^2 \big) u^2 dx \le \Gamma_s \|u\|_{\frac{12}{3+2s}}^4, \tag{2.2}$$

where

$$\Gamma_s = C\big(3, 3 - 2s, \frac{6}{3+2s}, \frac{6}{3+2s}\big) = \pi^{\frac{3-2s}{2}} \frac{\Gamma(s)}{\Gamma(\frac{3}{2} + s)} \Big(\frac{\Gamma(3)}{\Gamma(\frac{3}{2})} \Big)^{\frac{2s}{3}},$$

where $\Gamma(t)$ is the Gamma function with $t>0$. Now, we introduce the Pohozăev manifold associated to equation (1.5), which can be derived from [25].

Proposition 2.2. *Let $u\in H_r^s(\mathbb{R}^3)$ be a weak solution of equation (1.5), then u satisfies*

$$\begin{aligned}&\frac{3-2s}{2}\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\frac{3+2s}{4}\int_{\mathbb{R}^3}\phi_u^su^2dx\\&=\frac{3}{2}\lambda\int_{\mathbb{R}^3}|u|^2dx+\frac{3}{p}\mu\int_{\mathbb{R}^3}|u|^pdx+\frac{3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.\end{aligned}$$

Lemma 2.3 ([9])**.** *The embedding $H_r^s(\mathbb{R}^3)\hookrightarrow L^q(\mathbb{R}^3)$ is compact for any $2<q<2_s^*$.*

Lemma 2.4 (Fractional Gagliardo-Nirenberg inequality)**.** *Let $0<s<1$, and $p\in(2,2_s^*)$. Then there exists a constant $C(p,s)=S^{-\frac{\delta_{p,s}}{2}}>0$ such that*

$$\|u\|_p^p\le C(p,s)\|(-\Delta)^{s/2}u\|_2^{\frac{3}{s}(\frac{p}{2}-1)}\|u\|_2^{\frac{3}{s}(1-\frac{3-2s}{6}p)},\quad \forall u\in H^s(\mathbb{R}^3),\tag{2.3}$$

where $\delta_{p,s}=\frac{3(p-2)}{2ps}$.

Lemma 2.5 ([8])**.** *If $u_n\rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3}\phi_{u_n}^su_n^2dx\to\int_{\mathbb{R}^3}\phi_u^su^2dx,\quad \int_{\mathbb{R}^3}\phi_{u_n}^su_n\varphi dx\to\int_{\mathbb{R}^3}\phi_u^su\varphi dx,\forall\varphi\in H_r^s(\mathbb{R}^3).$$

Lemma 2.6. *Let $u\in H_r^s(\mathbb{R}^3)$ be a weak solution of (1.5), then we can construct the Pohozăev manifold*

$$\mathcal{P}_c=\{u\in S_r(c):P_\mu(u)=0\},$$

where

$$\begin{aligned}P_\mu(u)=&\,s\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\frac{3-2s}{4}\int_{\mathbb{R}^3}\phi_u^su^2dx\\&-s\mu\delta_{p,s}\int_{\mathbb{R}^3}|u|^pdx-s\int_{\mathbb{R}^3}|u|^{2_s^*}dx.\end{aligned}$$

Proof. From Proposition 2.2, we know that u satisfies the Pohozăev identity

$$\begin{aligned}&\frac{3-2s}{2}\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\frac{3+2s}{4}\int_{\mathbb{R}^3}\phi_u^su^2dx\\&=\frac{3}{2}\lambda\int_{\mathbb{R}^3}|u|^2dx+\frac{3}{p}\mu\int_{\mathbb{R}^3}|u|^pdx+\frac{3}{2_s^*}\int_{\mathbb{R}^3}|u|^{2_s^*}dx.\end{aligned}\tag{2.4}$$

Moreover, since u is the weak solution of equation (1.5), we have

$$\begin{aligned}&\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\int_{\mathbb{R}^3}\phi_u^su^2dx\\&=\lambda\int_{\mathbb{R}^3}|u|^2dx+\mu\int_{\mathbb{R}^3}|u|^pdx+\int_{\mathbb{R}^3}|u|^{2_s^*}dx.\end{aligned}\tag{2.5}$$

Combining this with (2.4) and (2.5), we obtain

$$\begin{aligned}&s\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\frac{3-2s}{4}\int_{\mathbb{R}^3}\phi_u^su^2dx\\&-s\mu\delta_{p,s}\int_{\mathbb{R}^3}|u|^pdx-s\int_{\mathbb{R}^3}|u|^{2_s^*}dx=0,\end{aligned}$$

which completes the proof. □

3. Proof of Theorem 1.1

In this section, we show the multiplicity of normalized solutions to system (1.1). To begin by recalling the definition of genus. Let X be a Banach space and let A be a subset of X. The set A is said to be symmetric if $u \in A$ implies that $-u \in A$. We denote the set

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, define

$$\gamma(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ \inf\{k \in \mathbb{N} : \exists \text{ an odd } \varphi \in C(A, \mathbb{R}^k \setminus \{0\})\}, & \\ +\infty, & \text{if no such odd map exists,} \end{cases}$$

and that $\Sigma_k = \{A \in \Sigma : \gamma(A) \geq k\}$.

Lemma 3.1 ([34]). *Let $\{u_n\}$ be a bounded sequence in $D^{s,2}(\mathbb{R}^3)$ converging weakly and a.e. to some $u \in D^{s,2}(\mathbb{R}^3)$. We have that $|(-\Delta)^{s/2}u_n|^2 \rightharpoonup \omega$ and $|u_n|^{2_s^*} \rightharpoonup \zeta$ in the sense of measures. Then, there exist some at most a countable set J, a family of points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$, and families of positive numbers $\{\zeta_j\}_{j\in J}$ and $\{\omega_j\}_{j\in J}$ such that*

$$\omega \geq |(-\Delta)^{s/2}u|^2 + \sum_{j\in J} \omega_j \delta_{x_j}, \tag{3.1}$$

$$\zeta = |u|^{2_s^*} + \sum_{j\in J} \zeta_j \delta_{x_j}, \tag{3.2}$$

$$\omega(\{x_j\}) \geq S\zeta_j^{2/2_s^*}, \tag{3.3}$$

where δ_{x_j} is the Dirac-mass of mass 1 concentrated at $x_j \in \mathbb{R}^3$.

Lemma 3.2 ([34]). *Let $\{u_n\} \subset D^{s,2}(\mathbb{R}^3)$ be a sequence in Lemama 3.1 and define*

$$\omega_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |(-\Delta)^{s/2}u_n|^2 dx, \quad \zeta_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geq R} |u_n|^{2_s^*} dx.$$

Then it follows that

$$\omega_\infty \geq S\zeta_\infty^{2/2_s^*},$$
$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx = \int_{\mathbb{R}^3} d\omega + \omega_\infty,$$
$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx = \int_{\mathbb{R}^3} d\zeta + \zeta_\infty.$$

For $u \in S_r(c)$, in view of Lemma 2.4, and the Sobolev inequality, one has

$$\begin{aligned} I_\mu(u) &= \frac{a}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{\frac{s}{2}}u|^2 dx\Big)^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx \\ &\quad - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &\geq \frac{a}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{b}{4}\|(-\Delta)^{s/2}u\|_2^4 - \frac{\mu}{p} c^{p(1-\delta_{p,s})} C(p,s) \|(-\Delta)^{s/2}u\|_2^{p\delta_{p,s}} \\ &\quad - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}} \|(-\Delta)^{s/2}u\|_2^{2_s^*} \\ &:= h(\|(-\Delta)^{s/2}u\|_2), \end{aligned}$$

where

$$h(r) = \frac{a}{2}r^2 + \frac{b}{4}r^4 - \frac{\mu}{p} c^{p(1-\delta_{p,s})} C(p,s) r^{p\delta_{p,s}} - \frac{1}{2_s^*} S^{-\frac{2_s^*}{2}} r^{2_s^*}.$$

Recalling that $p \in (2, \frac{4s}{3}+2)$, we obtain that $p\delta_{p,s} < 2$, and there exists $\beta > 0$ such that, if $\mu c^{p(1-\delta_{p,s})} < \beta$, the function $h(\cdot)$ attains its positive local maximum. More precisely, there exist two constants $0 < R_1 < R_2 < +\infty$, such that

$$h(r) > 0, \forall r \in (R_1, R_2), \quad h(r) < 0, \quad \forall r \in (0, R_1) \cup (R_2, +\infty).$$

Let $\tau : \mathbb{R}^+ \to [0,1]$ be a non-increasing and C^∞ function satisfying

$$\tau(r) = \begin{cases} 1, & \text{if } r \in [0, R_1], \\ 0, & \text{if } r \in [R_2, +\infty). \end{cases}$$

In the sequel, we consider the truncated functional

$$\begin{aligned} I_{\mu,\tau}(u) = \frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx\Big)^2 + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx \\ - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p dx - \frac{\tau(\|(-\Delta)^{s/2}u\|_2)}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned} \tag{3.4}$$

For $u \in S_r(c)$, again by Lemma 2.4, and the Sobolev inequality, it is easy to see that

$$\begin{aligned} I_{\mu,\tau}(u) &\geq \frac{a}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{b}{4}\|(-\Delta)^{s/2}u\|_2^4 - \frac{\mu}{p}c^{p(1-\delta_{p,s})}C(p,s)\|(-\Delta)^{s/2}u\|_2^{p\delta_{p,s}} \\ &\quad - \frac{\tau(\|(-\Delta)^{s/2}u\|_2)}{2_s^*}\|(-\Delta)^{s/2}u\|_2^{2_s^*} \\ &:= \tilde{h}(\|(-\Delta)^{s/2}u\|_2), \end{aligned}$$

where

$$\tilde{h}(r) = \frac{a}{2}r^2 + \frac{b}{4}r^4 - \frac{\mu}{p}c^{p(1-\delta_{p,s})}C(p,s)r^{p\delta_{p,s}} - \frac{\tau(r)}{2_s^*}S^{-\frac{2_s^*}{2}}r^{2_s^*}.$$

Then, by the definition of $\tau(\cdot)$, when $c \in \big(0, (\frac{\beta}{\mu})^{\frac{1}{p(1-\delta_{p,s})}}\big)$, we have

$$\tilde{h}(r) < 0, \forall r \in (0, R_1), \quad \tilde{h}(r) > 0, \forall r \in (R_1, +\infty).$$

In what follows, we assume that $c \in \big(0, (\frac{\beta}{\mu})^{\frac{1}{p(1-\delta_{p,s})}}\big)$. Without loss of generality, in the sequel, we assume that

$$\frac{a}{2}r^2 + \frac{b}{4}r^4 - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}r^{2_s^*} \geq 0, \quad \forall\, r \in [0, R_1] \tag{3.5}$$

Lemma 3.3. *The functional $I_{\mu,\tau}$ has the following characteristics:*

(i) $I_{\mu,\tau} \in C^1(H_r^s(\mathbb{R}^3), \mathbb{R})$;

(ii) *$I_{\mu,\tau}$ is coercive and bounded from below on $S_r(c)$. Moreover, if $I_{\mu,\tau}(u) \leq 0$, then $\|(-\Delta)^{s/2}u\|_2 \leq R_1$ and $I_{\mu,\tau}(u) = I_\mu(u)$;*

(iii) *$I_{\mu,\tau}|_{S_r(c)}$ satisfies the $(PS)_d$ condition for all $d < min\{0, \Lambda - \mu c^{p(1-\delta_{p,s})}D\}$, provided that $\mu > \mu_1^* > 0$ large, where $\Lambda = (\frac{1}{2} - \frac{1}{2_s^*})a^{\frac{3}{2s}}S^{\frac{3}{2s}}$, $D = (\frac{1}{p} - \frac{1}{2_s^*})C(p,s)R_1^{p\delta_{p,s}}$.*

Proof. The proofs of (i) and (ii) are easy. To prove item (iii), Let $\{u_n\}$ be a $(PS)_d$ sequence of $I_{\mu,\tau}$ restricted to $S_r(c)$ with $d < min\{0, \Lambda - \mu c^{q(1-\delta_{p,s})}D\}$. By (ii), we see that $\|(-\Delta)^{\frac{s}{2}}u_n\|_2 \leq R_1$ for large n, and thus $\{u_n\}$ is a $(PS)_d$ sequence of $I_\mu|_{S_r(c)}$ with $d < min\{0, \Lambda - \mu c^{p(1-\delta_{p,s})}D\}$; i.e., $I_\mu(u_n) \to d$ and $\|I_\mu|'_{S_r(c)}(u_n)\| \to 0$ as $n \to \infty$. Then, $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$. Therefore, up to a subsequence, there exists $u \in H_r^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$ and $u_n \to u$ in $L^p(\mathbb{R}^3)$ for $2 < p < 2_s^*$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^3$. From $2 < p < \frac{4s}{3} + 2 < 2_s^*$ and Lemma 2.5, we infer to

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n|^p dx = \int_{\mathbb{R}^3} |u|^p dx, \quad \int_{\mathbb{R}^3} \phi_{u_n}^s {u_n}^2 dx \to \int_{\mathbb{R}^3} \phi_u^s u^2 dx.$$

Moreover, we have that $u \not\equiv 0$. Indeed, assume by contradiction that, $u \equiv 0$, then $\lim_{n\to\infty}\int_{\mathbb{R}^3} |u_n|^p dx = 0$. From (3.5) and the definition of $I_{\mu,\tau}$, we infer that

$$0 > d = \lim_{n\to\infty} I_{\mu,\tau}(u_n) = \lim_{n\to\infty} I_\mu(u_n)$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big[\frac{a}{2}\|(-\Delta)^{s/2}u_n\|_2^2 + \frac{b}{4}\|(-\Delta)^{s/2}u_n\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}^s {u_n}^2 dx \\
&\quad - \frac{\mu}{p}\int_{\mathbb{R}^3}|u_n|^p dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\Big] \\
&\geq \lim_{n\to\infty} \Big[\frac{a}{2}\|(-\Delta)^{s/2}u_n\|_2^2 + \frac{b}{4}\|(-\Delta)^{s/2}u_n\|_2^4 \\
&\quad - \frac{\mu}{p}\int_{\mathbb{R}^3}|u_n|^p dx - \frac{1}{2_s^*}S^{-\frac{2_s^*}{2}}\|(-\Delta)^{s/2}u_n\|_2^{2_s^*}\Big] \\
&\geq -\frac{\mu}{p}\int_{\mathbb{R}^3}|u|^p dx = 0,
\end{aligned}
$$

which is absurd. On the other hand, setting the function $\Theta(v) : H_r^s(\mathbb{R}^3) \to \mathbb{R}$ by

$$
\Theta(v) = \frac{1}{2}\int_{\mathbb{R}^3}|v|^2 dx,
$$

it follows that $S_r(c) = \Theta^{-1}(\{\frac{c^2}{2}\})$. Then, by [27, Proposition 5.12], there exists $\lambda_n \in \mathbb{R}$ such that

$$
\|I'_\mu(u_n) - \lambda_n\Theta'(u_n)\| \to 0, \ as\ n \to \infty.
$$

Hence, in $H_r^{-s}(\mathbb{R}^3)$, we have

$$
\Big(a + b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx\Big)(-\Delta)^s u_n + \phi_{u_n}^s u_n - \mu|u_n|^{p-2}u_n - |u_n|^{2_s^*-2}u_n = \lambda_n u_n + o(1),
$$

where $H_r^{-s}(\mathbb{R}^3)$ is the dual space of $H_r^s(\mathbb{R}^3)$. Thus, we have for $\varphi \in H_r^s(\mathbb{R}^3)$, that

$$
\begin{aligned}
&\Big(a + b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx\Big)\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}\varphi dx + \int_{\mathbb{R}^3}\phi_{u_n}^s u_n\varphi dx \\
&\quad - \mu\int_{\mathbb{R}^3}|u_n|^{p-2}u_n\varphi dx - \int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n\varphi dx \\
&= \lambda_n\int_{\mathbb{R}^3}u_n\varphi dx + o(1),
\end{aligned} \tag{3.6}
$$

and if we choose $\varphi = u_n$, we obtain

$$
\begin{aligned}
&\Big(a + b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx + \int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx - \mu\int_{\mathbb{R}^3}|u_n|^p dx - \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \\
&= \lambda_n\int_{\mathbb{R}^3}u_n^2 dx + o(1),
\end{aligned} \tag{3.7}
$$

from (3.7), and the boundedness of $\{u_n\}$ in $D^{s,2}(\mathbb{R}^3)$, we can deduce that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Then we can assume that, up to subsequence, $\lambda_n \to \lambda$ for some $\lambda \in \mathbb{R}$.

Next, we shall prove $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^3)$ by using the concentration-compactness principle due to Lions [19]. Since $\|(-\Delta)^{s/2}u_n\|_2 \leq R_1$, for n large enough, by Lemma 3.1, there exist two positive measures, ζ, $\omega \in \mathcal{M}(\mathbb{R}^3)$, such that

$$
|(-\Delta)^{s/2}u_n|^2 \rightharpoonup \omega, |u_n|^{2_s^*} \rightharpoonup \zeta \quad \text{in } \mathcal{M}(\mathbb{R}^3) \tag{3.8}
$$

as $n \to \infty$. Then, by Lemma 3.1, either $u_n \to u$ in $L_{loc}^{2_s^*}(\mathbb{R}^3)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$ and positive numbers $\{\zeta_j\}_{j\in J}$ such that

$$
\zeta = |u|^{2_s^*} + \Sigma_{j\in J}\zeta_j\delta_{x_j}.
$$

Moreover, there exist some at most a countable set $J \subset \mathbb{N}$, a corresponding set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$, and two sets of positive numbers $\{\zeta_j\}_{j\in J} \subset \mathbb{R}^3$ and $\{\omega_j\}_{j\in J} \subset \mathbb{R}^3$ such that items (3.1)-(3.3) hold. Now, assume that $J \neq \emptyset$. We split the proof into three steps.

Step 1. We prove that $a\omega(\{x_j\}) \le \zeta_j$, where $\omega(\{x_j\})$, and ζ_j come from Lemma 3.1. We define $\varphi \in C_0^\infty(\mathbb{R}^3)$ as a cut-off function with $\varphi \in [0,1]$, $\varphi = 1$ in $B_{\frac{1}{2}}(0)$, $\varphi = 0$ in $\mathbb{R}^3 \setminus B_1(0)$. For any $\rho > 0$, define

$$\varphi_\rho(x) := \varphi\Big(\frac{x-x_j}{\rho}\Big) = \begin{cases} 1, & |x-x_j| \le \frac{\rho}{2}, \\ 0, & |x-x_j| \ge \rho. \end{cases}$$

By the boundedness of $\{u_n\}$ in $H_r^s(\mathbb{R}^3)$, we have that $\{u_n\varphi_\rho\}$ is also bounded in $H_r^s(\mathbb{R}^3)$. Thus, one has

$$\begin{aligned} o_n(1) &= \langle I'_\mu(u_n), u_n\varphi_\rho\rangle \\ &= \Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx\Big)\int_{\mathbb{R}^3} (-\Delta)^{s/2}u_n(-\Delta)^{s/2}(u_n\varphi_\rho)dx + \int_{\mathbb{R}^3} \phi_{u_n}^s u_n\varphi_\rho dx \\ &\quad - \mu\int_{\mathbb{R}^3} |u_n|^p\varphi_\rho dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*}\varphi_\rho dx. \end{aligned} \tag{3.9}$$

It is easy to check that

$$\begin{aligned} \int_{\mathbb{R}^3} (-\Delta)^{s/2}u_n(-\Delta)^{s/2}(u_n\varphi_\rho)dx &= \int\int_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(u_n(x)\varphi_\rho(x)-u_n(y)\varphi_\rho(y))}{|x-y|^{3+2s}}\,dx\,dy \\ &= \int\int_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2\varphi_\rho(y)}{|x-y|^{3+2s}}\,dx\,dy \\ &\quad + \int\int_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(\varphi_\rho(x)-\varphi_\rho(y))u_n(x)}{|x-y|^{3+2s}}\,dx\,dy \\ &= T_1 + T_2\,. \end{aligned}$$

For T_1, by (3.8), we obtain

$$\begin{aligned} \lim_{\rho\to 0}\lim_{n\to\infty} T_1 &= \lim_{\rho\to 0}\lim_{n\to\infty} \int\int_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2\varphi_\rho(y)}{|x-y|^{3+2s}}\,dx\,dy \\ &= \lim_{\rho\to 0}\int_{\mathbb{R}^3} \varphi_\rho d\omega = \omega(\{x_j\}). \end{aligned}$$

From Hölder's inequality, we have

$$\begin{aligned} |T_2| &= \Big|\int\int_{\mathbb{R}^6} \frac{(u_n(x)-u_n(y))(\varphi_\rho(x)-\varphi_\rho(x))u_n(x)}{|x-y|^{3+2s}}\,dx\,dy\Big| \\ &\le \int\int_{\mathbb{R}^6} \Big|\frac{(u_n(x)-u_n(y))(\varphi_\rho(x)-\varphi_\rho(x))u_n(x)}{|x-y|^{3+2s}}\Big|\,dx\,dy \\ &\le \Big(\int\int_{\mathbb{R}^6} \frac{|\varphi_\rho(x)-\varphi_\rho(x)|^2u_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2}\Big(\int\int_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2} \\ &\le C_1\Big(\int\int_{\mathbb{R}^6} \frac{|\varphi_\rho(x)-\varphi_\rho(x)|^2u_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2}. \end{aligned}$$

Analogously to the proof of [34, Lemma 3.4], one has

$$\begin{gathered} \lim_{\rho\to 0}\lim_{n\to\infty} \int\int_{\mathbb{R}^6} \frac{|\varphi_\rho(x)-\varphi_\rho(y)|^2u_n^2(x)}{|x-y|^{3+2s}}\,dx\,dy = 0, \\ \lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^3} (-\Delta)^{s/2}u_n(-\Delta)^{s/2}(u_n\varphi_\rho)dx = \omega(\{x_j\}). \end{gathered} \tag{3.10}$$

Again by (3.8), we have

$$\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^{2_s^*}\varphi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3} \varphi_\rho d\zeta = \zeta(\{x_j\}) = \zeta_j. \tag{3.11}$$

By the definition of φ_ρ, and the absolute continuity of the Lebesgue integral, one has

$$\lim_{\rho\to 0}\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^p\varphi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3} |u|^p\varphi_\rho dx = \lim_{\rho\to 0}\int_{|x-x_j|\le\rho} |u|^p\varphi_\rho dx = 0. \tag{3.12}$$

Thus, by Proposition 2.1 and Lemma 2.3, we have

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2\varphi_\rho dx &\le C\left(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2s}}dx\right)^{\frac{3+2s}{6}}\left(\int_{\mathbb{R}^3}|u_n^2\varphi_\rho|^{\frac{6}{3+2s}}dx\right)^{\frac{3+2s}{6}}\\ &\le C_2\|u_n\|_{H_r^s}^2\Big(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2s}}|\varphi_\rho|^{\frac{6}{3+2s}}dx\Big)^{\frac{3+2s}{6}}\\ &\le C_3\Big(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2s}}\varphi_\rho dx\Big)^{\frac{3+2s}{6}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{\rho\to0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2\varphi_\rho dx &\le \lim_{\rho\to0}\lim_{n\to\infty}C_3\Big(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2s}}\varphi_\rho dx\Big)^{\frac{3+2s}{6}}\\ &=\lim_{\rho\to0}C_3\Big(\int_{\mathbb{R}^3}|u|^{\frac{12}{3+2s}}\varphi_\rho dx\Big)^{\frac{3+2s}{6}}\\ &=\lim_{\rho\to0}C_3\Big(\int_{|x-x_j|\le\rho}|u|^{\frac{12}{3+2s}}\varphi_\rho dx\Big)^{\frac{3+2s}{6}}=0.\end{aligned}\tag{3.13}$$

Summing (3.9)-(3.13), taking the limit as $n\to\infty$, and then the limit as $\rho\to0$, we arrive at

$$\zeta_j\ge a\omega(\{x_j\}).$$

Step 2. We show that $a\omega_\infty\le\zeta_\infty$, where ω_∞ and ζ_∞ are given in Lemma 3.2. Let $\psi\in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\psi\in[0,1]$, $\psi=0$ in $B_{\frac12}(0)$, $\psi=1$ in $\mathbb{R}^3\setminus B_1(0)$. For any $R>0$, define

$$\psi_R(x):=\psi\left(\frac{x}{R}\right)=\begin{cases}0, & |x|\le\frac{R}{2},\\ 1, & |x|\ge R.\end{cases}$$

Using again the boundedness of $\{u_n\}$ and $\{u_n\psi_R\}$ in $H_r^s(\mathbb{R}^3)$, we have

$$\begin{aligned}o_n(1)&=\langle I_\mu'(u_n),u_n\psi_R\rangle\\ &=\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx\Big)\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}(u_n\psi_R)dx\\ &\quad+\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2\psi_R dx-\mu\int_{\mathbb{R}^3}|u_n|^p\psi_R dx-\int_{\mathbb{R}^3}|u_n|^{2_s^*}\psi_R dx.\end{aligned}\tag{3.14}$$

It is easy to derive that

$$\begin{aligned}&\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}(u_n\psi_R)dx\\ &=\int\int_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y))(u_n(x)\psi_R(x)-u_n(y)\psi_R(y))}{|x-y|^{3+2s}}\,dx\,dy\\ &=\int\int_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\psi_R(y)}{|x-y|^{3+2s}}\,dx\,dy\\ &\quad+\int\int_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))u_n(x)}{|x-y|^{3+2s}}\,dx\,dy=T_3+T_4\,.\end{aligned}$$

For T_3, by (3.8) and Lemma 3.2, we infer that

$$\lim_{R\to\infty}\lim_{n\to\infty}T_3=\lim_{R\to\infty}\lim_{n\to\infty}\int\int_{\mathbb{R}^6}\frac{|u_n(x)-u_n(y)|^2\psi_R(y)}{|x-y|^{3+2s}}\,dx\,dy=\omega_\infty.$$

From Hölder's inequality, we have

$$\begin{aligned}|T_4|&=\Big|\int\int_{\mathbb{R}^6}\frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))u_n(x)}{|x-y|^{3+2s}}\,dx\,dy\Big|\\ &\le\int\int_{\mathbb{R}^6}\Big|\frac{(u_n(x)-u_n(y))(\psi_R(x)-\psi_R(y))u_n(x)}{|x-y|^{3+2s}}\Big|\,dx\,dy\end{aligned}$$

$$
\begin{aligned}
&\le \Big(\int\int_{\mathbb{R}^6} \frac{|\psi_R(x)-\psi_R(x)|^2|u_n(x)|^2}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2}\Big(\int\int_{\mathbb{R}^6} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2}\\
&\le C\Big(\int\int_{\mathbb{R}^6} \frac{|\psi_R(x)-\psi_R(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}\,dx\,dy\Big)^{1/2}.
\end{aligned}
$$

Combining the above, we conclude that

$$
\begin{aligned}
&\lim_{R\to\infty}\lim_{n\to\infty}\int\int_{\mathbb{R}^6} \frac{|\psi_R(x)-\psi_R(y)|^2|u_n(x)|^2}{|x-y|^{3+2s}}\,dx\,dy\\
&=\lim_{R\to\infty}\lim_{n\to\infty}\int\int_{\mathbb{R}^6} \frac{|[1-\psi_R(x)]-[1-\psi_R(y)]|^2|u_n(x)|^2}{|x-y|^{3+2s}}\,dx\,dy=0.
\end{aligned}
$$

Hence,

$$
\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}(u_n\psi_R)dx=\omega_\infty. \tag{3.15}
$$

By Lemma 3.2, one has

$$
\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\psi_R dx=\zeta_\infty. \tag{3.16}
$$

Analogous the proof of [34, Lemma 3.3], we infer that

$$
\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^p\psi_R dx=\lim_{R\to\infty}\int_{\mathbb{R}^3}|u|^p\psi_R dx=\lim_{R\to\infty}\int_{|x|\ge\frac{R}{2}}|u|^p\psi_R dx=0. \tag{3.17}
$$

Moreover, we can obtain

$$
\begin{aligned}
\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2\psi_R dx &\le \lim_{R\to\infty}\lim_{n\to\infty}C_3\Big(\int_{\mathbb{R}^3}|u_n|^{\frac{12}{3+2s}}\psi_R dx\Big)^{\frac{3+2s}{6}}\\
&=\lim_{R\to\infty}C_3\Big(\int_{\mathbb{R}^3}|u|^{\frac{12}{3+2s}}\psi_R dx\Big)^{\frac{3+2s}{6}}\\
&=\lim_{R\to\infty}C_3\Big(\int_{|x|\ge\frac{R}{2}}|u|^{\frac{12}{3+2s}}\psi_R dx\Big)^{\frac{3+2s}{6}}=0.
\end{aligned} \tag{3.18}
$$

Summing up, from (3.14)-(3.18), taking the limit as $n\to\infty$, and then the limit as $R\to\infty$, we have

$$
\zeta_\infty\ge a\omega_\infty.
$$

Step 3. We claim that $\zeta_j=0$ for any $j\in J$ and $\zeta_\infty=0$. Suppose by contradiction that, there exists $j_0\in J$ such that $\zeta_{j_0}>0$ or $\zeta_\infty>0$. Then step 1, step 2, Lemma 3.1 and Lemma 3.2 imply that

$$
\zeta_{j_0}\le (S^{-1}\omega(\{x_{j_0}\}))^{\frac{2_s^*}{2}}\le (S^{-1}a^{-1}\zeta_{j_0})^{\frac{2_s^*}{2}},\quad \zeta_\infty\le (S^{-1}\omega_\infty)^{\frac{2_s^*}{2}}\le (S^{-1}a^{-1}\zeta_\infty)^{\frac{2_s^*}{2}}.
$$

Consequently, we obtain $\zeta_{j_0}\ge (aS)^{\frac{3}{2s}}$ or $\zeta_\infty\ge (aS)^{\frac{3}{2s}}$. If $\zeta_{j_0}\ge (aS)^{\frac{3}{2s}}$, one has

$$
\begin{aligned}
d&=\lim_{n\to\infty}\Big[I_\mu(u_n)-\frac{1}{2_s^*}\langle I_\mu'(u_n),u_n\rangle\Big]\\
&=\lim_{n\to\infty}\Big[\Big(\frac12-\frac{1}{2_s^*}\Big)a\|(-\Delta)^{s/2}u_n\|_2^2+\Big(\frac14-\frac{1}{2_s^*}\Big)b\|(-\Delta)^{s/2}u_n\|_2^4\\
&\quad+\Big(\frac14-\frac{1}{2_s^*}\Big)\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx-\Big(\frac1p-\frac{1}{2_s^*}\Big)\mu\int_{\mathbb{R}^3}|u_n|^pdx\Big]\\
&\ge \lim_{n\to\infty}\Big[\Big(\frac12-\frac{1}{2_s^*}\Big)a\|(-\Delta)^{s/2}u_n\|_2^2-\Big(\frac1p-\frac{1}{2_s^*}\Big)\mu\int_{\mathbb{R}^3}|u_n|^pdx\Big]\\
&\ge \lim_{n\to\infty}\Big(\frac12-\frac{1}{2_s^*}\Big)aS\|u_n\|_{2_s^*}^2-\lim_{n\to\infty}\Big(\frac1p-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)\|(-\Delta)^{s/2}u_n\|_2^{p\delta_{p,s}}\\
&\ge \Big(\frac12-\frac{1}{2_s^*}\Big)aS\lim_{\rho\to0}\lim_{n\to\infty}\Big(\int_{\mathbb{R}^3}|u_n|^{2_s^*}\varphi_\rho dx\Big)^{2/2_s^*}-\Big(\frac1p-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)R_1^{p\delta_{p,s}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)aS(\zeta_{j_0})^{\frac{2}{2_s^*}} - \Big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)R_1^{p\delta_{p,s}}\\
&\geq \Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)a^{\frac{3}{2s}}S^{\frac{3}{2s}} - \big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)R_1^{p\delta_{p,s}}\\
&= \Lambda - \mu c^{p(1-\delta_{p,s})}D,
\end{aligned}
$$

which contradicts $d < \min\{0, \Lambda - \mu c^{p(1-\delta_{p,s})}D\}$. If $\zeta_\infty \geq (aS)^{\frac{3}{2s}}$, we have

$$
\begin{aligned}
d &= \lim_{n\to\infty}\Big[I_\mu(u_n) - \frac{1}{2_s^*}\langle I'_\mu(u_n), u_n\rangle\Big]\\
&= \lim_{n\to\infty}\Big[\Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)a\|(-\Delta)^{s/2}u_n\|_2^2 + \Big(\frac{1}{4}-\frac{1}{2_s^*}\Big)b\|(-\Delta)^{s/2}u_n\|_2^4\\
&+ \Big(\frac{1}{4}-\frac{1}{2_s^*}\Big)\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx - \Big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu\int_{\mathbb{R}^3}|u_n|^pdx\Big]\\
&\geq \lim_{n\to\infty}\Big[\Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)a\|(-\Delta)^{s/2}u_n\|_2^2 - \Big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu\int_{\mathbb{R}^3}|u_n|^pdx\Big]\\
&\geq \lim_{n\to\infty}\Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)aS\|u_n\|_{2_s^*}^2 - \lim_{n\to\infty}\Big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)\|(-\Delta)^{s/2}u_n\|_2^{p\delta_{p,s}}\\
&\geq \Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)aS\lim_{R\to\infty}\lim_{n\to\infty}\Big(\int_{\mathbb{R}^3}|u_n|^{2_s^*}\psi_Rdx\Big)^{2/2_s^*} - \Big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)R_1^{p\delta_{p,s}}\\
&= \Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)aS(\zeta_\infty)^{2/2_s^*} - \Big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)R_1^{p\delta_{p,s}}\\
&\geq \Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)a^{\frac{3}{2s}}S^{\frac{3}{2s}} - \Big(\frac{1}{p}-\frac{1}{2_s^*}\Big)\mu c^{p(1-\delta_{p,s})}C(p,s)R_1^{p\delta_{p,s}}\\
&= \Lambda - \mu c^{p(1-\delta_{p,s})}D,
\end{aligned}
$$

which also contradicts $d < min\{0, \Lambda - \mu c^{p(1-\delta_{p,s})}D\}$. Therefore, $\zeta_j = 0$ for any $j \in J$ and $\zeta_\infty = 0$. As a result, by Lemma 3.1, we obtain that $u_n \to u$ in $L_{loc}^{2_s^*}(\mathbb{R}^3)$. Combining with Lemma 3.2, we know that $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^3)$.

Now, we prove there exists $\mu_1^* > 0$ independently on $n \in \mathbb{N}$ such that if $\mu > \mu_1^*$, the Lagrange multiplier $\lambda < 0$. Indeed, note that $\{u_n\} \subset S_r(c)$ and $\|(-\Delta)^{s/2}u_n\|_2 \leq R_1$ for large n, as can be seen from the previous proof of this Lemma, and (2.2)-(2.3) that, there exists $Q_1 > 0$ independently on n, such that for large n

$$
\begin{aligned}
Q_1 &\leq \int_{\mathbb{R}^3}|u_n|^pdx\\
&\leq C(p,s)\|(-\Delta)^{s/2}u_n\|_2^{p\delta_{p,s}}\|u_n\|_2^{p(1-\delta_{p,s})}\\
&\leq C(p,s){R_1}^{p\delta_{p,s}}c^{p(1-\delta_{p,s})},
\end{aligned}
\tag{3.19}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx \leq \Gamma_s\|u_n\|_{\frac{12}{3+2s}}^4 &\leq \Gamma_sC(p_s,s)^{\frac{3+2s}{3}}\|(-\Delta)^{s/2}u_n\|_2^{\frac{3-2s}{s}}\|u_n\|_2^{\frac{6s-3}{s}}\\
&\leq \Gamma_sC(p_s,s)^{\frac{3+2s}{3}}R_1^{\frac{3-2s}{s}}c^{\frac{6s-3}{s}}\\
&:= Q_2,
\end{aligned}
\tag{3.20}
$$

where $p_s := \frac{12}{3+2s}$ and $Q_2 = Q_2(s, R_1, c) > 0$. We define the constant

$$
\mu_1^* = \frac{p(6s-3)Q_2}{2[6-p(3-2s)]Q_1}. \tag{3.21}
$$

By (3.19)-(3.21), we have

$$
\mu_1^* > \lim_{n\to+\infty}\frac{p(6s-3)\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx}{2[6-p(3-2s)]\int_{\mathbb{R}^3}|u_n|^pdx} = \frac{p(6s-3)\int_{\mathbb{R}^3}\phi_u^s u^2dx}{2[6-p(3-2s)]\int_{\mathbb{R}^3}|u|^pdx} > 0. \tag{3.22}
$$

Set $B := \lim_{n\to\infty} \|(-\Delta)^{s/2}u_n\|_2^2 \geq 0$, then, $0 < \|(-\Delta)^{s/2}u\|_2^2 \leq B$. For any $\varphi \in H_r^s(\mathbb{R}^3)$, it follows by $u_n \rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$ and $\lambda_n \to \lambda$, that

$$\int_{\mathbb{R}^3} (-\Delta)^{s/2}u_n(-\Delta)^{s/2}\varphi dx \to \int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi dx \quad \text{and} \quad \lambda_n \int_{\mathbb{R}^3} u_n\varphi dx \to \lambda \int_{\mathbb{R}^3} u\varphi dx$$

as $n \to \infty$. Since $\{|u_n|^{2_s^*-2}u_n\}$ is bounded in $L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3)$, $\{|u_n|^{p-2}u_n\}$ is bounded in $L^{\frac{2_s^*}{p-1}}(\mathbb{R}^3)$, and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^3$, we obtain that

$$|u_n|^{2_s^*-2}u_n \rightharpoonup |u|^{2_s^*-2}u \quad \text{in } L^{\frac{2_s^*}{2_s^*-1}}(\mathbb{R}^3) \quad \text{and} \quad |u_n|^{p-2}u_n \rightharpoonup |u|^{p-2}u \quad \text{in } L^{\frac{2_s^*}{p-1}}(\mathbb{R}^3),$$

and so

$$\int_{\mathbb{R}^3} |u_n|^{2_s^*-2}u_n\varphi dx \to \int_{\mathbb{R}^3} |u|^{2_s^*-2}u\varphi dx \quad \text{and} \quad \int_{\mathbb{R}^3} |u_n|^{p-2}u_n\varphi dx \to \int_{\mathbb{R}^3} |u|^{p-2}u\varphi dx,$$

as $n \to \infty$. Recall from Lemma 2.5 that

$$\int_{\mathbb{R}^3} \phi_{u_n}^s u_n\varphi dx \to \int_{\mathbb{R}^3} \phi_u^s u\varphi dx, \quad \forall \varphi \in H_r^s(\mathbb{R}^3).$$

Thus, by (3.6), for all $\varphi \in H_r^s(\mathbb{R}^3)$, we have

$$\begin{aligned} &(a+bB)\int_{\mathbb{R}^3} (-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi dx + \int_{\mathbb{R}^3} \phi_u^s u\varphi dx \\ &- \mu\int_{\mathbb{R}^3} |u|^{p-2}u\varphi dx - \int_{\mathbb{R}^3} |u|^{2_s^*-2}u\varphi dx \\ &= \lambda\int_{\mathbb{R}^3} u\varphi dx, \end{aligned} \tag{3.23}$$

we can derive that u solves the equation

$$(a+bB)(-\Delta)^s u + \phi_u^s u - \mu|u|^{q-2}u - |u|^{2_s^*-2}u = \lambda u. \tag{3.24}$$

Moreover, by Lemma 2.6, u satisfies

$$s(a+bB)\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx + \frac{3-2s}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx - s\mu\delta_{p,s}\int_{\mathbb{R}^3} |u|^p dx - s\int_{\mathbb{R}^3} |u|^{2_s^*} dx = 0. \tag{3.25}$$

Combining (3.24) and (3.25), one has

$$s\lambda\|u\|_2^2 = \frac{6s-3}{4}\int_{\mathbb{R}^3} \phi_u^s u^2 dx + s\mu\frac{(3-2s)p-6}{2p}\int_{\mathbb{R}^3} |u|^p dx. \tag{3.26}$$

Now, if $\mu > \mu_1^*$, we conclude from (3.22), that

$$\mu > \frac{p(6s-3)\int_{\mathbb{R}^3} \phi_u^s u^2 dx}{2[6-p(3-2s)]\int_{\mathbb{R}^3} |u|^p dx}.$$

Thus, from (3.26), we infer to $\lim_{n\to+\infty} \lambda_n = \lambda < 0$. By (3.7) and (3.23), we derive

$$\begin{aligned} &\lim_{n\to\infty} \Big[\Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx\Big)\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u_n|^2 dx + \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx - \lambda_n\int_{\mathbb{R}^3} u_n^2 dx\Big] \\ &= \lim_{n\to\infty} \Big[\mu\int_{\mathbb{R}^3} |u_n|^p dx + \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx\Big] \\ &= \mu\int_{\mathbb{R}^3} |u|^p dx + \int_{\mathbb{R}^3} |u|^{2_s^*} dx \\ &= (a+bB)\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 dx + \int_{\mathbb{R}^3} \phi_u^s u^2 dx - \lambda\int_{\mathbb{R}^3} u^2 dx. \end{aligned} \tag{3.27}$$

Since $\lambda < 0$ for $\mu > \mu_1^*$ large, by Fatou's Lemma we obtain,

$$\begin{aligned}
&\lim_{n\to\infty}\Big[\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx+\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx-\lambda_n\int_{\mathbb{R}^3}u_n^2dx\Big]\\
&=\lim_{n\to\infty}\Big[(a+bB)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx+\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx-\lambda\int_{\mathbb{R}^3}u_n^2dx\Big]\\
&\geq (a+bB)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\int_{\mathbb{R}^3}\phi_u^s u^2dx-\liminf_{n\to\infty}\lambda\int_{\mathbb{R}^3}u_n^2dx,
\end{aligned}\tag{3.28}$$

and from (3.27)-(3.28), one has

$$-\lambda\int_{\mathbb{R}^3}u^2dx\geq\liminf_{n\to\infty}\Big(-\lambda\int_{\mathbb{R}^3}u_n^2dx\Big).\tag{3.29}$$

But by Fatou's Lemma, we have

$$\liminf_{n\to\infty}\Big(-\lambda\int_{\mathbb{R}^3}u_n^2dx\Big)\geq-\lambda\int_{\mathbb{R}^3}u^2dx.\tag{3.30}$$

Combining (3.29) with (3.30), we obtain

$$\lim_{n\to\infty}\Big(-\lambda\int_{\mathbb{R}^3}u_n^2dx\Big)=-\lambda\int_{\mathbb{R}^3}u^2dx;$$

that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}u_n^2dx=\int_{\mathbb{R}^3}u^2dx.$$

Thus, by (3.27), one gets

$$\lim_{n\to\infty}\|(-\Delta)^{s/2}u_n\|_2^2=\|(-\Delta)^{s/2}u\|_2^2.$$

Therefore, $u_n\to u$ in $H_r^s(\mathbb{R}^3)$ and $\|u\|_2=c$. This completes the proof. □

For $\epsilon>0$, we introduce the set

$$I_{\mu,\tau}^{-\epsilon}=\{u\in S_r(c):I_{\mu,\tau}(u)\leq-\epsilon\}\subset H_r^s(\mathbb{R}^3).$$

Because $I_{\mu,\tau}$ is continuous and even on $H_r^s(\mathbb{R}^3)$, $I_{\mu,\tau}^{-\epsilon}$ is closed and symmetric.

Lemma 3.4. *For any fixed $k\in\mathbb{N}$, there exist $\epsilon_k=\epsilon(k)>0$ and $\mu:=\mu(k)>0$ such that, for any $0<\epsilon\leq\epsilon_k$ and $\mu\geq\mu_k$, one has that $\gamma(I_{\mu,\tau}^{-\epsilon})\geq k$.*

The proof of Lemma 3.4 is similar to [2, Lemma 3.2], so we omit it here. We define the set

$$\Sigma_k:=\{\Omega\subset S_r(c):\Omega\text{ is closed and symmetric, }\gamma(\Omega)\geq k\},$$

and by Lemma 3.3-(ii), we know that

$$d_k:=\inf_{\Omega\in\Sigma_k}\sup_{u\in\Omega}I_{\mu,\tau}>-\infty$$

for all $k\in\mathbb{N}$. To prove Theorem 1.1, we introduce the critical value, we define

$$K_d=\{u\in S_r(c):I'_{\mu,\tau}(u)=0,I_{\mu,\tau}(u)=d\}.$$

Then, we can derive the following conclusion.

Lemma 3.5. *If $d=d_k=d_{k+1}=\cdots=d_{k+l}$,*

$$c\in\Big(0,\min\Big\{\big(\frac{\beta}{\mu}\big)^{\frac{1}{p(1-\delta_{p,s})}},\big(\frac{\Lambda}{D\mu}\big)^{\frac{1}{p(1-\delta_{p,s})}}\Big\}\Big),$$

$\mu>\mu_k^*=\max\{\mu_1^*,\mu_k\}$, then one has $\gamma(K_d)\geq\ell+1$. Especially, $I_{\mu,\tau}(u)$ admits at least $\ell+1$ nontrivial critical points.

Proof. For $\epsilon > 0$, it is easy to check that $I_{\mu,\tau}^{-\epsilon} \in \Sigma$. For any fixed $k \in \mathbb{N}$, by Lemma 3.4, there exists $\epsilon_k := \epsilon(k) > 0$ and $\mu_k := \mu(k) > 0$ such that, if $0 < \epsilon < \epsilon_k$ and $\mu \geq \mu_k$, we have $\gamma(I_{\mu,\tau}^{-\epsilon_k}) \geq k$. Thus, $I_{\mu,\tau}^{-\epsilon_k} \in \Sigma_k$, and moreover,

$$d_k \leq \sup_{u \in I_{\mu,\tau}^{-\epsilon_k}} I_{\mu,\tau}(u) = -\epsilon_k < 0.$$

Assume that $0 > d = d_k = d_{k+1} = \cdots = d_{k+l}$, since $c \in \Big(0, \min\Big\{\big(\frac{\beta}{\mu}\big)^{\frac{1}{p(1-\delta_{p,s})}}, \big(\frac{\Lambda}{D\mu}\big)^{\frac{1}{p(1-\delta_{p,s})}}\Big\}\Big)$, by Lemma 3.3-(iii), when $\mu > \mu_1^* > 0$ large, $I_{\mu,\tau}(u)$ satisfies the $(PS)_d$ condition at the level $d < 0$. So, K_d is a compact set. By [17, Theorem 2.1], we know that the restricted function $I_{\mu,\tau}|_{S_r(c)}$ possesses at least $\ell + 1$ nontrivial critical points. The proof is complete. □

Proof of Theorem 1.1. Let

$$c \in \Big(0, \min\Big\{\big(\frac{\beta}{\mu}\big)^{\frac{1}{p(1-\delta_{p,s})}}, \big(\frac{\Lambda}{D\mu}\big)^{\frac{1}{p(1-\delta_{p,s})}}\Big\}\Big),$$

$\mu \geq \mu_k^* = \max\{\mu_1^*, \mu_k\}$. From Lemma 3.3-(ii), we see that the critical points of $I_{\mu,\tau}$ found in Lemma 3.5 are the critical points of I_μ, which completes the proof. □

4. Proof of Theorem 1.2

From Lemma 2.6, we see that any critical point of $I_\mu|_{S_r(c)}$ belongs to $\mathcal{P}_c$. Consequently, the properties of the manifold $\mathcal{P}_c$ have relation to the mini-max structure of $I_\mu|_{S_r(c)}$. For $u \in S_r(c)$ and $\theta \in \mathbb{R}$, we introduce the transformation:

$$(\theta \star u)(x) := e^{\frac{3\theta}{2}} u(e^\theta x), \quad x \in \mathbb{R}^3,\ \theta \in \mathbb{R}.$$

It is easy to check that the dilations preserve the L^2-norm such that $\theta \star u \in S_r(c)$, by direct calculation, one has

$$\begin{aligned} I(u,\theta) &= I_\mu((\theta \star u)) \\ &:= \frac{a}{2} e^{2s\theta} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx + \frac{b}{4} e^{4s\theta} \Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx\Big)^2 \\ &\quad + \frac{1}{4} e^{(3-2s)\theta} \int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{\mu}{p} e^{\frac{3(p-2)}{2}\theta} \int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*} e^{\frac{3(2_s^*-2)}{2}\theta} \int_{\mathbb{R}^3} |u|^{2_s^*} dx. \end{aligned}$$

Lemma 4.1. *Let $u \in S_r(c)$, then*

(i) $\int_{\mathbb{R}^3} |(-\Delta)^{s/2} (\theta \star u)|^2 dx \to 0$ *and* $I_\mu((\theta \star u)) \to 0$ *as* $\theta \to -\infty$;
(ii) $\int_{\mathbb{R}^3} |(-\Delta)^{s/2} (\theta \star u)|^2 dx \to +\infty$ *and* $I_\mu((\theta \star u)) \to -\infty$ *as* $\theta \to +\infty$.

Proof. A direct computation shows that

$$\begin{gathered} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} (\theta \star u)|^2 dx = e^{2s\theta} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx, \\ \int_{\mathbb{R}^3} |(-\Delta)^{s/2} (\theta \star u)|^2 dx \to 0 \quad \text{as } \theta \to -\infty, \\ \int_{\mathbb{R}^3} |(-\Delta)^{s/2} (\theta \star u)|^2 dx \to +\infty \quad \text{as } \theta \to +\infty. \end{gathered}$$

Notice that

$$\begin{aligned} I_\mu((\theta \star u)) &:= \frac{a}{2} e^{2s\theta} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx + \frac{b}{4} e^{4s\theta} \Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx\Big)^2 \\ &\quad + \frac{1}{4} e^{(3-2s)\theta} \int_{\mathbb{R}^3} \phi_u^s u^2 dx - \frac{\mu}{p} e^{\frac{3(p-2)}{2}\theta} \int_{\mathbb{R}^3} |u|^p dx - \frac{1}{2_s^*} e^{\frac{3(2_s^*-2)}{2}\theta} \int_{\mathbb{R}^3} |u|^{2_s^*} dx, \end{aligned}$$

by $\frac{3(2_s^*-2)}{2} > \frac{3(p-2)}{2} > 4s > 2s > 3 - 2s$, we infer that

$$I_\mu((\theta \star u)) \to 0 \quad \text{as } \theta \to -\infty, \quad I_\mu((\theta \star u)) \to -\infty \ as\ \theta \to +\infty.$$

This completes the proof. □

Lemma 4.2. *There exist $K = K_c > 0$ and $\tilde{c} > 0$ such that for all $0 < c < \tilde{c}$,*

$$0 < \sup_{u\in\mathcal{A}_c} I_\mu(u) < \inf_{u\in\mathcal{B}_c} I_\mu(u), \tag{4.1}$$

where $\mathcal{A}_c = \{u \in S_r(c) : \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx \le K_c\}$, $\mathcal{B}_c = \{u \in S_r(c) : \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx = 2K_c\}$.

Proof. By Lemma 2.4, for any $p \in (2, 2_s^*)$, we have

$$\|u\|_p^p \le C(p, s)\|(-\Delta)^{s/2}u\|_2^{\frac{3}{s}(\frac{p}{2}-1)}\|u\|_2^{\frac{3}{s}(1-\frac{3-2s}{6}p)}, \quad \forall u \in H^s(\mathbb{R}^3). \tag{4.2}$$

By the Sobolev inequality (2.1) and (4.2), for $u, v \in S_r(c)$, one has

$$\begin{aligned}
I_\mu(v) - I_\mu(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}v|^2dx - \frac{a}{2}\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx \\
&\quad + \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}v|^2dx\Big)^2 - \frac{b}{4}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx\Big)^2 \\
&\quad + \frac{1}{4}\int_{\mathbb{R}^3} \phi_v^s v^2dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u^s u^2dx + \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^pdx \\
&\quad - \frac{\mu}{p}\int_{\mathbb{R}^3} |v|^pdx + \frac{1}{2_s^*}\int_{\mathbb{R}^3} |u|^{2_s^*}dx - \frac{1}{2_s^*}\int_{\mathbb{R}^3} |v|^{2_s^*}dx \\
&\ge \frac{a}{2}\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}v|^2dx - \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx\Big) \\
&\quad + \frac{b}{4}\Big[\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}v|^2dx\Big)^2 - \Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2dx\Big)^2\Big] \\
&\quad - \frac{1}{4}\Gamma_s C(p_s, s)^{\frac{3+2s}{3}}\|(-\Delta)^{s/2}u\|_2^{\frac{3-2s}{s}} c^{\frac{6s-3}{s}} \\
&\quad - \frac{\mu}{p}C(p, s)\|(-\Delta)^{s/2}v\|_2^{p\delta_{p,s}} c^{\frac{6-p(3-2s)}{2s}} - \frac{S^{-\frac{2_s^*}{2}}}{2_s^*}\|(-\Delta)^{s/2}v\|_2^{2_s^*}.
\end{aligned}$$

Let $\|(-\Delta)^{s/2}u\|_2^2 \le K_c$ and $\|(-\Delta)^{s/2}v\|_2^2 = 2K_c$, here K_c will be determined later. Set

$$\tilde{c} = \Big(\frac{K_c^{\frac{4s-3}{2s}}}{4\Gamma_s C(p_s, s)^{\frac{3+2s}{3}}}a\Big)^{\frac{s}{6s-3}},$$

by a direct computation, we obtain

$$\begin{aligned}
&I_\mu(v) - I_\mu(u) \\
&\ge \frac{a}{2}K_c + \frac{3b}{4}K_c^2 - \frac{1}{4}\Gamma_s C(p_s, s)^{\frac{3+2s}{3}} K_c^{\frac{3-2s}{2s}}\Big(\frac{K_c^{\frac{4s-3}{2s}}}{4\Gamma_s C(p_s, s)^{\frac{3+2s}{3}}}a\Big)^{\frac{s}{6s-3}\cdot\frac{6s-3}{s}} \\
&\quad - \frac{\mu}{p}C(p, s)(2K_c)^{\frac{p\delta_{p,s}}{2}}\Big(\frac{K_c^{\frac{4s-3}{2s}}}{4\Gamma_s C(p_s, s)^{\frac{3+2s}{3}}}a\Big)^{\frac{s}{6s-3}\cdot\frac{6-p(3-2s)}{2s}} - \frac{S^{-\frac{2_s^*}{2}}}{2_s^*}(2K_c)^{\frac{2_s^*}{2}} \\
&\ge \frac{a}{2}K_c - \frac{a}{16}K_c - \frac{\mu}{p}2^{\frac{p\delta_{p,s}}{2}}C(p, s)\Big(\frac{1}{4\Gamma_s C(p_s, s)^{\frac{3+2s}{3}}}a\Big)^{\frac{6-p(3-2s)}{2(6s-3)}} K_c^{\frac{(4s-3)(6-p(3-2s))}{4s(6s-3)}} K_c^{\frac{3(p-2)}{4s}} \\
&\quad - \frac{S^{-\frac{2_s^*}{2}}}{2_s^*}2^{\frac{2_s^*}{2}}(K_c)^{\frac{2_s^*}{2}} \\
&= \frac{7}{16}aK_c - \frac{\mu 2^{\frac{3(p-2)}{4s}}C(p, s)a^{\frac{6-p(3-2s)}{2(6s-3)}}}{p(4\Gamma_s C(p_s, s)^{\frac{3+2s}{3}})^{\frac{6-p(3-2s)}{2(6s-3)}}}K_c^{\gamma_1}K_c - \frac{2^{\frac{2_s^*}{2}}}{2_s^* S^{\frac{2_s^*}{2}}}K_c^{\frac{2_s^*-2}{2}}K_c \\
&\ge \frac{5}{16}aK_c > 0,
\end{aligned} \tag{4.3}$$

where

$$\gamma_1 = \frac{[6 - p(3 - 2s)][4s - 3] + [3(p - 2) - 4s][6s - 3]}{4s(6s - 3)}.$$

If we take

$$K_c = \min\Big\{\Big(\frac{p(4\Gamma_s C(p_s,s)^{\frac{3+2s}{3}})^{\frac{6-p(3-2s)}{2(6s-3)}}}{16\mu 2^{\frac{3(p-2)}{4s}} C(p,s) a^{\frac{6-p(3-2s)}{2(6s-3)}}} a\Big)^{\gamma_2}, \Big(\frac{2_s^* S^{\frac{2_s^*}{2}}}{2^{\frac{2_s^*}{2}} 16} a\Big)^{\frac{2}{2_s^*-2}}\Big\},$$

with

$$\gamma_2 = \frac{4s(6s-3)}{[6-p(3-2s)][4s-3]+[3(p-2)-4s][6s-3]},$$

then, by (4.3) we deduce that (4.1) holds. The proof is complete. □

From Lemma 4.2 we can deduce the following conclusion.

Corollary 4.3. *Let K_c and $\tilde{c}$ be given in Lemma 4.2, and $u \in S_r(c)$ with $\|(-\Delta)^{s/2}u\|_2^2 \le K_c$, then $I_\mu(u) > 0$.*

Proof. A direct computation shows that

$$\begin{aligned} I_\mu(u) \ge \frac{a}{2}\|(-\Delta)^{s/2}u\|_2^2 + \frac{b}{4}\|(-\Delta)^{s/2}u\|_2^4 - \frac{\mu}{p}C(p,s)c^{\frac{6-p(3-2s)}{2s}}\|(-\Delta)^{s/2}u\|_2^{\frac{3(p-2)}{2s}} \\ - \frac{S^{-\frac{2_s^*}{2}}}{2_s^*}\|(-\Delta)^{s/2}u\|_2^{2_s^*} > 0, \end{aligned}$$

if $\|(-\Delta)^{s/2}u\|_2^2 \le K_c$, and the conclusion holds. □

Next, we study the characterizations of the mountain pass levels for $I(u,\theta)$ and $I_\mu(u)$. We denote the closed set $I_\mu^d := \{u \in S_r(c) : I_\mu(u) \le d\}$.

Proposition 4.4. *Assuming that $\frac{8s}{3} + 2 < p < 2_s^*$, we define*

$$\tilde{c}_\mu(c) := \inf_{\tilde{\gamma}\in\tilde{\Gamma}} \max_{t\in[0,1]} I(\tilde{\gamma}(t)),\ c_\mu(c) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\mu(\gamma(t)),$$

where

$$\tilde{\Gamma}_c = \{\tilde{\gamma} \in C([0,1], S_r(c)\times\mathbb{R}) : \tilde{\gamma}(0) \in (\mathcal{A}_c, 0), \tilde{\gamma}(1) \in (I_\mu^0, 0)\},$$
$$\Gamma_c = \{\gamma \in C([0,1], S_r(c)) : \gamma(0) \in \mathcal{A}_c, \gamma(1) \in I_\mu^0\}.$$

Then we have $\tilde{c}_\mu(c) = c_\mu(c) > 0$.

Proof. On the one hand, for any $\tilde{\gamma} \in \tilde{\Gamma}_c$, we can write it into

$$\tilde{\gamma}(t) = (\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)) \in S_r(c)\times\mathbb{R}.$$

We set $\gamma(t) = \tilde{\gamma}_2(t) \star \tilde{\gamma}_1(t)$, then $\gamma \in \Gamma_c$, and

$$\max_{t\in[0,1]} I(\tilde{\gamma}(t)) = \max_{t\in[0,1]} I_\mu(\tilde{\gamma}_2(t)\star\tilde{\gamma}_1(t)) = \max_{t\in[0,1]} I_\mu(\gamma(t)),$$

which implies $\tilde{c}_\mu(c) \ge c_\mu(c) > 0$, using Corollary 4.3. On the other hand, for any $\gamma \in \Gamma_c$, if we set $\tilde{\gamma}(t) = (\gamma(t), 0)$, then we obtain $\tilde{\gamma} \in \tilde{\Gamma}_c$ and

$$\max_{t\in[0,1]} I(\tilde{\gamma}(t)) = \max_{t\in[0,1]} I_\mu(\gamma(t)).$$

This infers that $\tilde{c}_\mu(c) \le c_\mu(c)$. So, $\tilde{c}_\mu(c) = c_\mu(c) > 0$. □

Next, we show the existence of the $(PS)_{c_\mu(c)}$-sequence for $I(u,\theta)$ on $S_r(c)\times\mathbb{R} \subset \mathbb{H}$. It is obtained by a standard argument using Ekeland's variational principle and constructing pseudo-gradient flow, see [16, Proposition 2.2].

Lemma 4.5. *Let $\{\tilde{h}_n\} \subset \tilde{\Gamma}_c$ satisfy that*

$$\max_{t\in[0,1]} I(\tilde{h}_n(t)) \le \tilde{c}_\mu(c) + \frac{1}{n},$$

then there exists a sequence $\{(v_n,\theta_n)\} \subset S_r(c)\times\mathbb{R}$ such that

(i) $I(v_n,\theta_n) \in [\tilde{c}_\mu(c) - \frac{1}{n}, \tilde{c}_\mu(c) + \frac{1}{n}]$,
(ii) $\min_{t\in[0,1]} \|(v_n,\theta_n) - \tilde{h}_n(t)\|_{\mathbb{H}} \le \frac{1}{\sqrt{n}}$; *and*

(iii) $\|(I|_{S_r(c)\times\mathbb{R}})'(v_n,\theta_n)\| \le \frac{2}{\sqrt{n}}$, *that is,*

$$|\langle I'(v_n,\theta_n), z\rangle|_{\mathbb{H}^{-1}\times\mathbb{H}} \le \frac{2}{\sqrt{n}}\|z\|_{\mathbb{H}},$$

for all

$$z\in \tilde{T}_{(v_n,\theta_n)} = \{(z_1,z_2)\in\mathbb{H} : \langle v_n, z_1\rangle_{L^2}=0\}.$$

It follows from the above proposition, we can obtain a special $(PS)_{c_\mu(c)}$-sequence for $I_\mu(u)$ on $S_r(c)\subset H_r^s(\mathbb{R}^3)$.

Lemma 4.6. *Under the assumption $2+\frac{8s}{3}<p<2_s^*$, there exists a sequence $\{u_n\}\subset S_r(c)$ such that*

1. $I_\mu(u_n)\to c_\mu(c)$ *as* $n\to\infty$;
2. $P_\mu(u_n)\to 0$ *as* $n\to\infty$;
3. $(I_\mu|_{S_r(c)})'(u_n)\to 0$ *as* $n\to\infty$, *i.e.,* $\langle I_\mu'(u_n), z\rangle_{H_r^{-s}\times H_r^s}\to 0$, *uniformly for all z satisfying*

$$\|z\|_{H_r^s}\le 1,\quad \text{where } z\in T_{u_n} := \{z\in H_r^s(\mathbb{R}^3):\ \langle u_n, z\rangle_{L^2}=0\}.$$

Proof. Let $\{h_n\}\subset\Gamma_c$ satisfy

$$\max_{t\in[0.1]} I_\mu(h_n(t)) \le c_\mu(c)+\frac{1}{n}, \tag{4.4}$$

we define $\tilde{h}_n(t)=(h_n(t),0), \forall t\in[0,1]$. It is easy to see that $\tilde{h}_n\in\tilde{\Gamma}_c$ and $I_\mu(h_n(t))=I(\tilde{h}_n(t))$. By Proposition 4.4, we have $\tilde{c}_\mu(c)=c_\mu(c)$, then it follows from (4.4) that

$$\max_{t\in[0.1]} I(\tilde{h}_n(t)) \le \tilde{c}_\mu(c)+\frac{1}{n}.$$

It follows from Lemma 4.5 that, there exists a sequence $\{(v_n,\theta_n)\}\subset S_r(c)\times\mathbb{R}$ such that as $n\to\infty$, one has

$$I(v_n,\theta_n)\to c_\mu(c),\ \theta_n\to 0, \tag{4.5}$$

$$(I|_{S_r(c)\times\mathbb{R}})'(v_n,\theta_n)\to 0. \tag{4.6}$$

Set $u_n=\theta_n\star v_n$. Then, $I_\mu(u_n)=I(v_n,\theta_n)$, and by (4.5), item (1) holds. To prove conclusion (2), we utilize

$$\begin{aligned}
\partial_\theta I(v_n,\theta_n) &= ase^{2s\theta_n}\|(-\Delta)^{s/2}v_n\|_2^2 + bse^{4s\theta_n}\|(-\Delta)^{s/2}v_n\|_2^4 + \frac{3-2s}{4}e^{(3-2s)\theta_n}\int_{\mathbb{R}^3}\phi_{v_n}^s v_n^2dx\\
&\quad -\frac{\mu}{p}\frac{3(p-2)}{2}e^{\frac{3(p-2)}{2}\theta n}\int_{\mathbb{R}^3}|v_n|^pdx - \frac{3(2_s^*-2)}{22_s^*}e^{\frac{3(2_s^*-2)}{2}\theta n}\int_{\mathbb{R}^3}|v_n|^{2_s^*}dx\\
&= s(a+b\|(-\Delta)^{s/2}u_n\|_2^2)\|(-\Delta)^{s/2}u_n\|_2^2 + \frac{3-2s}{4}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx\\
&\quad -\frac{3(p-2)}{2p}\mu\int_{\mathbb{R}^3}|u_n|^pdx - s\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\\
&= P_\mu(u_n),
\end{aligned}$$

which implies item (2) by (4.6). To show item (3), we set $z_n\in T_{u_n}$. Then,

$$\begin{aligned}
\langle I_\mu'(u_n), z_n\rangle &= a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_n(x)-u_n(y))(z_n(x)-z_n(y))}{|x-y|^{3+2s}}\,dx\,dy\\
&\quad + b\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_n(x)-u_n(y))^2}{|x-y|^{3+2s}}\,dx\,dy\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(u_n(x)-u_n(y))(z_n(x)-z_n(y))}{|x-y|^{3+2s}}\,dx\,dy\\
&\quad + \int_{\mathbb{R}^3}\phi_{u_n}^s u_n z_ndx - \mu\int_{\mathbb{R}^3}|u_n|^{p-2}u_nz_ndx - \int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_nz_ndx\\
&= a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(e^{\frac{3\theta_n}{2}}v_n(e^{\theta_n}x) - e^{\frac{3\theta_n}{2}}v_n(e^{\theta_n}y))(z_n(x)-z_n(y))}{|x-y|^{3+2s}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&+ b\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(e^{\frac{3\theta_n}{2}}v_n(e^{\theta_n}x) - e^{\frac{3\theta_n}{2}}v_n(e^{\theta_n}y))^2}{|x-y|^{3+2s}}\,dx\,dy \\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(e^{\frac{3\theta_n}{2}}v_n(e^{\theta_n}x) - e^{\frac{3\theta_n}{2}}v_n(e^{\theta_n}y))(z_n(x)-z_n(y))}{|x-y|^{3+2s}}\,dx\,dy \\
&+ e^{\frac{3-4s}{2}\theta_n}\int_{\mathbb{R}^3}\phi^s_{v_n}v_n z_n(e^{-\theta_n}x)dx - \mu e^{\frac{3(p-3)}{2}\theta_n}\int_{\mathbb{R}^3}|v_n|^{p-2}v_n z_n(e^{-\theta_n}x)dx \\
&- e^{\frac{3(2^*_s-3)}{2}\theta_n}\int_{\mathbb{R}^3}|v_n|^{2^*_s-2}v_n z_n(e^{-\theta_n}x)dx \\
= {}& e^{2s\theta_n}a\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(v_n(x)-v_n(y))e^{-\frac{3}{2}\theta_n}(z_n(e^{-\theta_n}x)-z_n(e^{-\theta_n}y))}{|x-y|^{3+2s}}\,dx\,dy \\
&+ e^{4s\theta_n}b\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(v_n(x)-v_n(y))^2}{|x-y|^{3+2s}}\,dx\,dy \\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{(v_n(x)-v_n(y))e^{-\frac{3}{2}\theta_n}(z_n(e^{-\theta_n}x)-z_n(e^{-\theta_n}y))}{|x-y|^{3+2s}}\,dx\,dy \\
&+ e^{\frac{3-4s}{2}\theta_n}\int_{\mathbb{R}^3}\phi^s_{v_n}v_n z_n(e^{-\theta_n}x)dx - \mu e^{\frac{3(p-3)}{2}\theta_n}\int_{\mathbb{R}^3}|v_n|^{p-2}v_n z_n(e^{-\theta_n}x)dx \\
&- e^{\frac{3(2^*_s-3)}{2}\theta_n}\int_{\mathbb{R}^3}|v_n|^{2^*_s-2}v_n z_n(e^{-\theta_n}x)dx.
\end{aligned}
$$

Denoting $\tilde{z}_n(x) = e^{-\frac{3\theta_n}{2}}z_n(e^{-\theta_n}x)$, we obtain

$$
\langle I'_\mu(u_n), z_n\rangle_{H_r^{-s}\times H_r^s} = \langle I'(v_n,\theta_n),(\tilde{z}_n,0)\rangle_{\mathbb{H}^{-1}\times\mathbb{H}}.
$$

It is easy to check that

$$
\langle v_n,\tilde{z}_n\rangle_{L^2} = \int_{\mathbb{R}^3}v_n(x)e^{-\frac{3\theta_n}{2}}z_n(e^{-\theta_n}x)dx = \int_{\mathbb{R}^3}v_n(e^{\theta_n}x)e^{\frac{3\theta_n}{2}}z_n(x)dx = \int_{\mathbb{R}^3}u_n(x)z_n(x)dx = 0.
$$

Therefore, $(\tilde{z}_n,0)\in\tilde{T}_{(v_n,\theta_n)}$. On the other hand,

$$
\|(\tilde{z}_n,0)\|^2_{\mathbb{H}} = \|\tilde{z}_n\|^2_{H^s_r} = \|z_n\|_2^2 + e^{-2s\theta_n}\|z_n\|^2_{D^{s,2}} \le C\|z_n\|^2_{H^s_r},
$$

where the last inequality follows by $\theta_n\to 0$. Consequently, we conclude item (3). The proof is complete. □

Lemma 4.7. *The (PS) sequence $\{u_n\}\subset S_r(c)$ for $I_\mu(u)$ with the level $c_\mu(c)$ mentioned in Lemma 4.6 is bounded in $H^s_r(\mathbb{R}^3)$.*

Proof. From Lemma 4.6 (1), we see that $I_\mu(u)$ is bounded. In fact, by $P_\mu(u_n)\to 0$ as $n\to\infty$, one obtains

$$
|(1+2s)I_\mu(u_n) + P_\mu(u_n)| \le 3c_\mu(c),
$$

which implies that

$$
\begin{aligned}
-3c_\mu(c) \le{}& \frac{1+4s}{2}a\|(-\Delta)^{s/2}u_n\|_2^2 + \frac{1+6s}{4}b\|(-\Delta)^{s/2}u_n\|_2^4 + \int_{\mathbb{R}^3}\phi^s_{u_n}u_n^2dx \\
&- \mu\Big(\frac{1+2s}{p}+s\delta_{p,s}\Big)\int_{\mathbb{R}^3}|u_n|^pdx - \Big(\frac{1+2s}{2^*_s}+s\Big)\int_{\mathbb{R}^3}|u_n|^{2^*_s}dx.
\end{aligned} \tag{4.7}
$$

In view of the boundedness of $I_\mu(u)$, we have

$$
\begin{aligned}
&a\|(-\Delta)^{s/2}u_n\|_2^2 + \frac{b}{2}\|(-\Delta)^{s/2}u_n\|_2^4 + \frac{1}{2}\int_{\mathbb{R}^3}\phi^s_{u_n}u_n^2dx \\
&\le 6c_\mu(c) + \frac{2\mu}{p}\int_{\mathbb{R}^3}|u_n|^pdx + \frac{2}{2^*_s}\int_{\mathbb{R}^3}|u_n|^{2^*_s}dx.
\end{aligned} \tag{4.8}
$$

By (4.7) and (4.8), we obtain

$$
\mu\frac{(p\delta_{p,s}-4)s}{p}\int_{\mathbb{R}^3}|u_n|^pdx + \frac{(2^*_s-4)s}{2^*_s}\int_{\mathbb{R}^3}|u_n|^{2^*_s}dx + \frac{(6s-3)}{4}\int_{\mathbb{R}^3}\phi^s_{u_n}u_n^2dx \le 3(2+6s)c_\mu(c).
$$

Note that $s\in(\frac{3}{4},1)$, $p>\frac{8s}{3}+2$, we have that $p\delta_{p,s}-4>0$, and so

$$\int_{\mathbb{R}^3}|u_n|^p dx, \int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \quad \text{and} \quad \int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx$$

are all bounded. Thus, $\|(-\Delta)^{s/2}u_n\|_2\le R_3$ for some $R_3>0$ independently on $n\in\mathbb{N}$. Since $\{u_n\}\subset S_r(c)$, we see that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$. This completes the proof. □

Now, we set the functional $\Phi: H_r^s(\mathbb{R}^3)\to\mathbb{R}$ as

$$\Phi(u)=\frac{1}{2}\int_{\mathbb{R}^3}|u|^2dx,$$

then $S_r(c)=\Phi^{-1}(\frac{c^2}{2})$. As a result, it can be derived from [27, Proposition 5.12] that there is a sequence $\{\lambda_n\}\subset\mathbb{R}$ such that

$$I'_\mu(u_n)-\lambda_n\Phi'(u_n)\to 0, \ in\ H_r^{-s}(\mathbb{R}^3)\ as\ n\to\infty.$$

That is, in $H_r^{-s}(\mathbb{R}^3)$, we have

$$\Big(a+b\|(-\Delta)^{s/2}u_n\|_2^2\Big)(-\Delta)^s u_n+\phi_{u_n}^s u_n-\mu|u_n|^{p-2}u_n-|u_n|^{2_s^*-2}u_n=\lambda_n u_n+o_n(1). \tag{4.9}$$

Therefore, for any $\varphi\in H_r^s(\mathbb{R}^3)$, one has

$$\begin{aligned}&\Big(a+b\|(-\Delta)^{s/2}u_n\|_2^2\Big)\int_{\mathbb{R}^3}(-\Delta)^{s/2}u_n(-\Delta)^{s/2}\varphi dx+\int_{\mathbb{R}^3}\phi_{u_n}^s u_n\varphi dx\\&-\int_{\mathbb{R}^3}\mu|u_n|^{p-2}u_n\varphi dx-\int_{\mathbb{R}^3}|u_n|^{2_s^*-2}u_n\varphi dx\\&=\lambda_n\int_{\mathbb{R}^3}u_n\varphi dx+o_n(1).\end{aligned} \tag{4.10}$$

Next, we study the asymptotical behavior of the mountain pass level value $c_\mu(c)$ as $\mu\to+\infty$, and the properties of the $(PS)_{c_\mu(c)}$-sequence $\{u_n\}\subset S_r(c)$ as $n\to+\infty$.

Lemma 4.8. *The limit* $\lim_{\mu\to+\infty}c_\mu(c)=0$ *holds.*

Proof. Recall Lemma 4.1 and Corollary 4.3, we see that for fixed $u_0\in S_r(c)$, there exists two constants θ_1,θ_2 satisfying $\theta_1<0<\theta_2$ such that $u_1=\theta_1\star u_0\in\mathcal{A}_c$ and $I_\mu(u_2)=I_\mu((\theta_2\star u_0))<0$. Then, we can define a path

$$\eta_0: t\in[0,1]\to((1-t)\theta_1+t\theta_2)\star u_0\in\Gamma_c.$$

Therefore,

$$\begin{aligned}0<c_\mu(c)&\le\max_{t\in[0,1]}I_\mu(\eta_0(t))\\&\le\max_{r\ge0}\Big\{\frac{a}{2}r^{2s}\|(-\Delta)^{s/2}u_0\|_2^2+\frac{b}{4}r^{4s}\|(-\Delta)^{s/2}u_0\|_2^4\\&\quad+\frac{1}{4}r^{3-2s}\int_{\mathbb{R}^3}\phi_{u_0}^s u_0^2dx-\frac{\mu}{p}r^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\}\\&:=\max_{r\ge0}g(r).\end{aligned}$$

Note that $\frac{3(p-2)}{2}>4s>2s>3-2s$, we have that $\lim_{r\to0^+}g(r)=0^+$, $\lim_{r\to+\infty}g(r)=-\infty$. Then, there exists a unique maximum point $r_0>0$ such that $\max_{r\ge0}g(r)=g(r_0)>0$. Hence, we distinguish two cases: $r_0\ge1$ and $0\le r_0<1$.

If $r_0\ge1$, then by $s\in(\frac{3}{4},1)$, we have

$$\begin{aligned}&\max_{t\in[0,1]}I_\mu(\eta_0(t))\le g(r_0)\\&\le\Big\{\frac{a}{2}r_0^{4s}\|(-\Delta)^{s/2}u_0\|_2^2+\frac{b}{4}r_0^{4s}\|(-\Delta)^{s/2}u_0\|_2^4\\&\quad+\frac{1}{4}r_0^{4s}\int_{\mathbb{R}^3}\phi_{u_0}^s u_0^2dx-\frac{\mu}{p}r_0^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\}\end{aligned}$$

$$
\begin{aligned}
&\le \max_{r\ge 0}\Big\{3\max\Big\{\frac{a}{2}\|(-\Delta)^{s/2}u_0\|_2^2, \frac{b}{4}\|(-\Delta)^{s/2}u_0\|_2^4, \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_0}^s u_0^2dx\Big\}r^{4s} \\
&\quad -\frac{\mu}{p}r^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\} \\
&= 3m(r_{\max})^{4s}-\frac{\mu}{p}n(r_{\max})^{\frac{3(p-2)}{2}} \\
&= \frac{m(3p-6-8s)}{p-2}\Big[\frac{8psm}{(p-2)\mu n}\Big]^{\frac{8s}{3p-6-8s}},
\end{aligned}
$$

where $r_{\max}=\Big[\frac{8psm}{(p-2)\mu n}\Big]^{\frac{2}{3p-6-8s}}$, $m=\max\big\{\frac{a}{2}\|(-\Delta)^{s/2}u_0\|_2^2, \frac{b}{4}\|(-\Delta)^{s/2}u_0\|_2^4, \frac14\int_{\mathbb{R}^3}\phi_{u_0}^s u_0^2dx\big\}$, $n=\int_{\mathbb{R}^3}|u_0|^pdx$. Therefore, for $\frac{8s}{3}+2<p<2_s^*$, we have a positive constant $\tilde{C}$ independent of μ such that

$$
c_\mu(c)\le \tilde{C}\mu^{-\frac{8s}{3p-6-8s}}\to 0,\ as\ \mu\to+\infty.
$$

If $0\le r_0<1$, we infer to

$$
\begin{aligned}
\max_{t\in[0,1]} I_\mu(\eta_0(t)) &\le g(r_0) \\
&\le \Big\{\frac{a}{2}r_0^{3-2s}\|(-\Delta)^{s/2}u_0\|_2^2+\frac{b}{4}r_0^{3-2s}\|(-\Delta)^{s/2}u_0\|_2^4 \\
&\quad +\frac{1}{4}r_0^{3-2s}\int_{\mathbb{R}^3}\phi_{u_0}^s u_0^2dx-\frac{\mu}{p}r_0^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\} \\
&\le \max_{r\ge 0}\Big\{3\max\Big\{\frac{a}{2}\|(-\Delta)^{s/2}u_0\|_2^2, \frac{b}{4}\|(-\Delta)^{s/2}u_0\|_2^4, \frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_0}^s u_0^2dx\Big\}r^{3-2s} \\
&\quad -\frac{\mu}{p}r^{\frac{3(p-2)}{2}}\int_{\mathbb{R}^3}|u_0|^pdx\Big\} \\
&= 3m(r_{\max})^{3-2s}-\frac{\mu}{p}n(r_{\max})^{\frac{3(p-2)}{2}} \\
&= \frac{m(3p+4s-12)}{p-2}\Big[\frac{2pm(3-2s)}{(p-2)\mu n}\Big]^{\frac{6-4s}{3p+4s-12}},
\end{aligned}
$$

where $r_{\max}=[\frac{2pm(3-2s)}{(p-2)\mu n}]^{\frac{2}{3p+4s-12}}$. Therefore, for $2+\frac{8s}{3}<p<2_s^*$, and $s\in(\frac34,1)$, we can deduce that $3p+4s-12>0$, then there exists a positive constant $\bar{C}$ independent of μ such that

$$
c_\mu(c)\le \bar{C}\mu^{-\frac{6-4s}{3p+4s-12}}\to 0,\ as\ \mu\to+\infty.
$$

This completes the proof. □

Lemma 4.9. *There exists a constant $C=C(p,s)>0$ such that*

$$
\limsup_{n\to\infty}\int_{\mathbb{R}^3}F(u_n)dx\le Cc_\mu(c),\quad \limsup_{n\to\infty}\int_{\mathbb{R}^3}f(u_n)u_ndx\le Cc_\mu(c),
$$
$$
\limsup_{n\to\infty}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx\le Cc_\mu(c),\quad \limsup_{n\to\infty}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx\le Cc_\mu(c),
$$
$$
\limsup_{n\to\infty}\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx\Big)^2\le Cc_\mu(c),
$$

where $f(u)=\mu|u|^{p-2}u+|u|^{2_s^*-2}u$, $F(u)=\int_0^u f(s)ds$.

Proof. Since $I_\mu(u_n) \to c_\mu(c)$ and $P_\mu(u_n) \to 0$ as $n \to \infty$, one obtains

$$\begin{aligned}
&3c_\mu(c) + o_n(1) = 3I_\mu(u_n) + P_\mu(u_n)\\
&= \frac{3+2s}{2} a \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx + \frac{3+4s}{4} b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx\Big)^2\\
&\quad + \frac{3-s}{2}\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} f(u_n)u_n dx\\
&\le \frac{3+4s}{2} a \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx + \frac{3+4s}{4} b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 dx\Big)^2\\
&\quad + \frac{3-s}{2}\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} f(u_n)u_n dx\\
&= (3+4s)\Big(-\frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx + \int_{\mathbb{R}^3} F(u_n)dx + c_\mu(c) + o_n(1)\Big)\\
&\quad + \frac{3-s}{2}\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} f(u_n)u_n dx\\
&= (3+4s)\Big(\int_{\mathbb{R}^3} F(u_n)dx + c_\mu(c) + o_n(1)\Big)\\
&\quad - \frac{6s-3}{4}\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx - \frac{3}{2}\int_{\mathbb{R}^3} f(u_n)u_n dx\\
&\le (3+4s)\Big(\int_{\mathbb{R}^3} F(u_n)dx + c_\mu(c) + o_n(1)\Big) - \frac{3}{2}\int_{\mathbb{R}^3} f(u_n)u_n dx\\
&\le (3+4s)(c_\mu(c) + o_n(1)) - \frac{3}{2}p\int_{\mathbb{R}^3} F(u_n)dx + (3+4s)\int_{\mathbb{R}^3} F(u_n)dx.
\end{aligned} \tag{4.11}$$

Hence,

$$4sc_\mu(c) + o_n(1) \ge \frac{3p-6-8s}{2}\int_{\mathbb{R}^3} F(u_n)dx,$$

which implies that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)dx \le \frac{8s}{3p-6-8s} c_\mu(c) \le Cc_\mu(c) \tag{4.12}$$

and then

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} f(u_n)u_n dx \le Cc_\mu(c). \tag{4.13}$$

Then, from (4.11)-(4.13), one has

$$\begin{aligned}
&\limsup_{n\to\infty} \Big\{\frac{3+2s}{2} a\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx + \frac{3+4s}{4} b(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx)^2 + \frac{3-s}{2}\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx\Big\}\\
&= \limsup_{n\to\infty} \Big\{\frac{3}{2}\int_{\mathbb{R}^3} f(u_n)u_n dx + 3c_\mu(c)\Big\}\\
&\le Cc_\mu(c).
\end{aligned}$$

Consequently, the proof is complete. □

Lemma 4.10. *Let $\{u_n\} \subset S_r(c)$ be the (PS) sequence for the constrained functional $I_\mu|_{S_r(c)}$ at level $c_\mu(c) \in \Big(0, \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)(aS)^{\frac{3}{2s}}\Big)$ with $P_\mu(u_n) \to 0$ as $n \to \infty$. Then*

(i) *$\{\lambda_n\}$ is bounded in $\mathbb{R}$, and $\limsup_{n\to\infty} |\lambda_n| \le \frac{C}{c^2} c_\mu(c)$ has the estimation*

$$\lambda_n = \frac{1}{c^2}\Big[\frac{6s-3}{4s}\int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx + \mu\frac{p(3-2s)-6}{2ps}\int_{\mathbb{R}^3} |u_n|^p dx\Big] + o_n(1).$$

Moreover, there exists some $\mu_2^* := \mu_2^*(c) > 0$ such that $\lim_{n\to+\infty} \lambda_n = \lambda < 0$, if $\mu > \mu_2^*$ large;

(ii) *there exist $u \in S_r(c)$ such that, up to a subsequence, $u_n \to u$ strongly in $H_r^s(\mathbb{R}^3)$ as $\mu > \mu_2^*$ large and u is a solution of system (1.1) for some $\lambda < 0$.*

Proof. We split the proof into three steps.

Step 1. We assert that $u \not\equiv 0$. From Lemma 4.7, we know that $\{u_n\}$ is a bounded (PS) sequence for I_μ in $H_r^s(\mathbb{R}^3)$, and by Lemma 2.3, up to a subsequence, there exists $u \in H_r^s(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H_r^s(\mathbb{R}^3)$, $u_n \to u$ strongly in $L^p(\mathbb{R}^3)$, for $p \in (2, 2_s^*)$, $u_n(x) \rightharpoonup u(x)$ a.e. on $\mathbb{R}^3$. In view of $2 + \frac{8s}{3} < p < 2_s^*$, and Lemma 2.3 and Lemma 2.5, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^p dx = \int_{\mathbb{R}^3} |u|^p dx, \ \lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx = \int_{\mathbb{R}^3} \phi_u^s u^2 dx. \tag{4.14}$$

Suppose by contradiction that, $u \equiv 0$. Then, by (4.14) and $P_\mu(u_n) = o_n(1)$, we deduce that

$$\begin{aligned} o_n(1) &= \Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx\Big) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx + \frac{3-2s}{4s} \int_{\mathbb{R}^3} \phi_u^s u^2 dx \\ &\quad - \mu\delta_{p,s} \int_{\mathbb{R}^3} |u_n|^p dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \\ &= \Big(a + b\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx\Big) \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx - \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx + o_n(1). \end{aligned}$$

Without loss of generality, we assume that

$$a\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx + b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx\Big)^2 \to l \geq 0, \quad \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \to l,$$

as $n \to \infty$. If $l = 0$, then we can deduce from the expression of $I_\mu(u_n)$ that $c_\mu(c) = 0$, which is absurd since $c_\mu(c) > 0$. Hence, $l > 0$. By the definition of S, we have

$$S \leq \frac{\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx}{\|u_n\|_{2_s^*}^2} \leq \frac{1}{a} \frac{a\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx + b\Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx\Big)^2}{\big(\int_{\mathbb{R}^3} |u_n|^{2_s^*} dx\big)^{2/2_s^*}} \to \frac{1}{a} l^{\frac{2s}{3}}$$

as $n \to \infty$. It follows that $l \geq (aS)^{\frac{3}{2s}}$. Consequently, by (4.14) we have

$$\begin{aligned} c_\mu(c) &= \lim_{n\to\infty} I_\mu(u_n) \\ &= \lim_{n\to\infty} \Big\{ \frac{a}{2} \int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx + \frac{b}{4} \Big(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u_n|^2 dx \Big)^2 \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n}^s u_n^2 dx - \frac{\mu}{p} \int_{\mathbb{R}^3} |u_n|^p dx - \frac{1}{2_s^*} \int_{\mathbb{R}^3} |u_n|^{2_s^*} dx \Big\} \\ &\geq \Big(\frac{1}{4} - \frac{1}{2_s^*} \Big) l \\ &\geq \Big(\frac{1}{4} - \frac{1}{2_s^*} \Big) (aS)^{\frac{3}{2s}}, \end{aligned}$$

which contradicts $I_\mu(u_n) \to c_\mu(c) < \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)(aS)^{\frac{3}{2s}}$. Therefore, $u \not\equiv 0$.

Step 2. We prove that $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^3)$. Again by Lemma 4.7, we can obtain $\{\|(-\Delta)^{s/2} u_n\|_2\}$ is bounded in $\mathbb{R}$, by Prohorov's theorem [3], there exist two positive measures, ζ, $\omega \in \mathcal{M}(\mathbb{R}^3)$, such that

$$|(-\Delta)^{s/2} u_n|^2 \rightharpoonup \omega, |u_n|^{2_s^*} \rightharpoonup \zeta \quad \text{in } \mathcal{M}(\mathbb{R}^3)$$

as $n \to \infty$. Then, by Lemma 3.1, either $u_n \to u$ in $L_{loc}^{2_s^*}(\mathbb{R}^3)$ or there exists a (at most countable) set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$ and positive numbers $\{\zeta_j\}_{j\in J}$ such that

$$\zeta = |u|^{2_s^*} + \Sigma_{j\in J} \zeta_j \delta_{x_j}.$$

Moreover, there exist some at most a countable set $J \subset \mathbb{N}$, a corresponding set of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^3$, and two sets of positive numbers $\{\zeta_j\}_{j\in J} \subset \mathbb{R}^3$ and $\{\omega_j\}_{j\in J} \subset \mathbb{R}^3$ such that items (3.1)-(3.3) hold. Now, assume that $J \neq \emptyset$.

Similar to the proof in step 1 and step 2 of Lemma 3.3, we can obtain

$$a\omega(\{x_j\}) \leq \zeta_j, \quad a\omega_\infty \leq \zeta_\infty. \tag{4.15}$$

Next, we claim that $\zeta_j = 0$ for any $j \in J$ and $\zeta_\infty = 0$.

Suppose by contradiction that, there exists $j_1 \in J$ such that $\zeta_{j_1} > 0$ or $\zeta_\infty > 0$. By (4.15), Lemma 3.1 and Lemma 3.2, we obtain

$$\zeta_{j_1} \geq (aS)^{\frac{3}{2s}} \quad \text{or} \quad \zeta_\infty \geq (aS)^{\frac{3}{2s}}.$$

If the former case occurs, one has

$$\begin{aligned}
\Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)(aS)^{\frac{3}{2s}} > c_\mu(c) &= \lim_{n\to\infty}\Big[I_\mu(u_n) - \frac{1}{4s}P_\mu(u_n)\Big] \\
&= \lim_{n\to\infty}\Big[\frac{1}{4}a\|(-\Delta)^{s/2}u_n\|_2^2 + \Big(\frac{1}{4} - \frac{3-2s}{16s}\Big)\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx \\
&\quad + \Big(\frac{3(p-2)}{8ps} - \frac{1}{p}\Big)\mu\int_{\mathbb{R}^3}|u_n|^p dx + \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\Big] \\
&= \lim_{n\to\infty}\Big[\frac{1}{4}a\|(-\Delta)^{s/2}u_n\|_2^2 + \frac{6s-3}{16s}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx \\
&\quad + \frac{3(p-2)-8s}{8ps}\mu\int_{\mathbb{R}^3}|u_n|^p dx + \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\Big] \\
&\geq \lim_{n\to\infty}\Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \\
&\geq \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\varphi_\rho dx \\
&= \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\zeta_{j_1} \\
&\geq \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)(aS)^{\frac{3}{2s}},
\end{aligned}$$

which is a contradiction. If the latter case happens, we have

$$\begin{aligned}
\Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)(aS)^{\frac{3}{2s}} > c_\mu(c) &\geq \lim_{n\to\infty}\Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx \\
&\geq \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\lim_{R\to\infty}\lim_{n\to\infty}\int_{\mathbb{R}^3}|u_n|^{2_s^*}\psi_R dx \\
&= \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)\zeta_\infty \\
&\geq \Big(\frac{1}{4} - \frac{1}{2_s^*}\Big)(aS)^{\frac{3}{2s}}.
\end{aligned}$$

This is also a contradiction. Therefore, $\zeta_j = 0$ for any $j \in J$ and $\zeta_\infty = 0$. As a result, by Lemma 3.1, we obtain that $u_n \to u$ in $L_{loc}^{2_s^*}(\mathbb{R}^3)$. Combining this with Lemma 3.2, we know that $u_n \to u$ in $L^{2_s^*}(\mathbb{R}^3)$.

Step 3. We prove that there exists some $\mu_2^* := \mu_2^*(c) > 0$ such that $\lim_{n\to+\infty}\lambda_n = \lambda < 0$, if $\mu > \mu_2^*$ large.

By (4.9) and the fact that $u_n \in S_r(c)$, one has

$$\begin{aligned}
&a\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx + b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx\Big)^2 + \int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2 dx - \int_{\mathbb{R}^3}f(u_n)u_n dx \\
&= \lambda_n\int_{\mathbb{R}^3}|u_n|^2 dx + o_n(1) \\
&= \lambda_n c^2 + o_n(1).
\end{aligned}$$

It indicates that

$$\lambda_n = \frac{1}{c^2}\Big[a\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx + b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2 dx\Big)^2$$

$$+\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx-\int_{\mathbb{R}^3}f(u_n)u_ndx\Big]+o_n(1).$$

By Lemma 4.7, we know that $\{u_n\}$ is bounded in $H_r^s(\mathbb{R}^3)$, and so, $\{\lambda_n\}$ is bounded in $\mathbb{R}$. By Lemma 4.9 we know that $\limsup_{n\to\infty}|\lambda_n|\le\frac{C}{c^2}c_\mu(c)$. From this and $P_\mu(u_n)\to 0$ as $n\to\infty$, we derive that

$$\begin{aligned}\lambda_n&=\frac{1}{c^2}\Big[a\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx+b\Big(\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx\Big)^2\\&\quad+\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx-\int_{\mathbb{R}^3}f(u_n)u_ndx-\frac{1}{s}P_\mu(u_n)\Big]+o_n(1)\\&=\frac{1}{c^2}\Big[\frac{6s-3}{4s}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx+\mu\frac{p(3-2s)-6}{2ps}\int_{\mathbb{R}^3}|u_n|^pdx\Big]+o_n(1).\end{aligned}$$

By (4.14) and similar arguments to that of (3.19)-(3.22), we see that there exists $\mu_2^*:=\mu_2^*(c)>0$, such that

$$\begin{aligned}\lambda&=\lim_{n\to\infty}\lambda_n\\&=\lim_{n\to\infty}\frac{1}{c^2}\Big\{\frac{6s-3}{4s}\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx+\mu\frac{p(3-2s)-6}{2ps}\int_{\mathbb{R}^3}|u_n|^pdx\Big\}\\&=\frac{1}{c^2}\Big\{\frac{6s-3}{4s}\int_{\mathbb{R}^3}\phi_u^s u^2dx+\mu\frac{p(3-2s)-6}{2ps}\int_{\mathbb{R}^3}|u|^pdx\Big\}<0,\end{aligned}\tag{4.16}$$

for $\mu>\mu_2^*$ large.

With the help of the above step 1, step 2, step 3, we can prove $u_n\to u$ strongly in $H_r^s(\mathbb{R}^3)$. Let $\mu>\mu_2^*$, set $B:=\lim_{n\to\infty}\|(-\Delta)^{s/2}u_n\|_2^2\ge 0$, by the weak convergence of $u_n\rightharpoonup u$ in $H_r^s(\mathbb{R}^3)$ and (4.10), one obtains

$$\begin{aligned}&(a+bB)\int_{\mathbb{R}^3}(-\Delta)^{s/2}u(-\Delta)^{s/2}\varphi dx+\int_{\mathbb{R}^3}\phi_u^s u\varphi dx\\&-\mu\int_{\mathbb{R}^3}|u|^{q-2}u\varphi dx-\int_{\mathbb{R}^3}|u|^{2_s^*-2}u\varphi dx\\&=\lambda\int_{\mathbb{R}^3}u\varphi dx.\end{aligned}\tag{4.17}$$

Therefore, from (4.14), (4.16), and (4.17) and $u_n\to u$ in $L^{2_s^*}(\mathbb{R}^3)$, it follows that

$$\begin{aligned}&\lim_{n\to\infty}\Big[\Big(a+b\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx\Big)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u_n|^2dx+\int_{\mathbb{R}^3}\phi_{u_n}^s u_n^2dx-\lambda\int_{\mathbb{R}^3}u_n^2dx\Big]\\&=\lim_{n\to\infty}\Big[\mu\int_{\mathbb{R}^3}|u_n|^pdx+\int_{\mathbb{R}^3}|u_n|^{2_s^*}dx\Big]\\&=\mu\int_{\mathbb{R}^3}|u|^pdx+\int_{\mathbb{R}^3}|u|^{2_s^*}dx\\&=(a+bB)\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2dx+\int_{\mathbb{R}^3}\phi_u^s u^2dx-\lambda\int_{\mathbb{R}^3}u^2dx.\end{aligned}$$

Since $\lambda<0$, as in the proof of Lemma 3.3, we can derive that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}u_n^2dx=\int_{\mathbb{R}^3}u^2dx,\quad \lim_{n\to\infty}\|(-\Delta)^{s/2}u_n\|_2^2=\|(-\Delta)^{s/2}u\|_2^2.$$

Therefore, $u_n\to u$ in $H_r^s(\mathbb{R}^3)$ and $\|u\|_2=c$. The proof is complete. □

With the help of the above technical lemmas, we can prove Theorem 1.2 as follows.

Proof of Theorem 1.2. From Lemmas 4.1 and 4.2, the functional I_μ satisfies the Mountain pass geometry. By Lemmas 4.5 and 4.6, there exist a $(PS)_{c_\mu(c)}$-sequence $\{u_n\}\subset S_r(c)$ satisfying $P_\mu(u_n)\to 0$ as $n\to\infty$, (4.9), and (4.10), which is bounded in $H_r^s(\mathbb{R}^3)$. Furthermore, by Lemma 4.8, there exists $\mu_3^*:=\mu_3^*(c)$ large enough such that $0<c_\mu(c)<\Big(\frac14-\frac{1}{2_s^*}\Big)(aS)^{\frac{3}{2s}}$ for $\mu>\mu_3^*$.

Then, by Lemma 4.10, there exist $u \in S_r(c)$ and $\lambda < 0$ such that passing to a subsequence $u_n \to u$ in $H_r^s(\mathbb{R}^3)$ if $\mu > \mu^*(c) := max\{\mu_2^*, \mu_3^*\}$. This completes the proof. □

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References

[1] Ambrosio, V.; *An existence result for a fractional Kirchhoff-Schrödinger-Poisson system*, Z. Angew. Math. Phys., 2018, 69(2), 30.

[2] Alves, C. O.; Ji, C,; Miyagaki, O. H.; *Normalized solutions for a Schrödinger equation with critical growth in* $\mathbb{R}^N$, Calc. Var. Partial Differential Equations, 2022, 61(1), 18.

[3] Bogachev. V. I.; *Measure Theory*, vol. II, Springer-Verlag, Berlin, 2007.

[4] Bartsch, T.; De Valeriola. S.; *Normalized solutions of nonlinear Schrödinger equations*, Arch. Math. (Basel), 2013, 100(1), 75–83.

[5] Cho, Y.; Hwang, G.; Kwon, S.; Lee, S.; *On finite time blow-up for the mass-critical Hartree equations*, Proc. Roy. Soc. Edinburgh Sect. A, 2015, 14(3), 467–479.

[6] Che, G. F.; Chen. H. B.; Shi, H. X.; Wang, Z. W.; *Existence of nontrial solutions for fractional Schrödinger-Poisson system with sign-changing potentials*, Math. Methods Appl. Sci., 2018, 41(13), 5050–5064.

[7] Di Nezza, E.; Palatucci, G.; Valdinoci, E.; *Hitchhiker's guide to the fractional Sobolev spaces*. Bull. Sci. Math., 2012, 136(5), 521-573.

[8] d'Avenia, P.; Siciliano, G.; *Nonlinear Schrödinger equation in the Bopp-Podolsky electrodynamics: solutions in the electrostatic case*, J. Differ. Equa., 2019, 267(2), 1025–1065.

[9] Dinh, V. D.; *Existence, non-existence and blow-up behaviour of minimizers for the mass-critical fractional non-linear Schrödinger equations with periodic potentials*, Proc. Roy. Soc. Edinburgh Sect. A, 2020, 150(6), 3252–3292.

[10] Duan, X. L.; Wei, G. M.; Yang, H. T.; *Ground states for a fractional Schrödinger-Poisson system involving Hardy potentials*, Appl. Anal., 2022, 101(4), 1225–1242.

[11] Feng, S. H.; Wang, L.; Huang, L.; *Least energy sign-changing solutions of fractional Kirchhoff-Schrödinger-Poisson system with critical and logarithmic nonlinearity*, Complex Var. Elliptic Equ., 2023, 68(1), 81–106.

[12] Feng, S. H.; Chen, J. H.; Sun, J. J.; Huang, X. J.; *Least energy sign-changing solutions for fractional critical Kirchhoff-Schrödinger-Poisson with steep potential well*, Fract. Calc. Appl. Anal., 2024, 27(1), 124–156.

[13] Guo, L.; *Sign-changing solutions for fractional Schrödinger-Poisson system in* $\mathbb{R}^3$, Appl. Anal., 2019, 98(11), 2085–2104.

[14] He, X. M.; Meng, Y. X.; Squassina, M.; *Normalized solutions for a fractional Schrödinger-Poisson system with critical growth*, Calc. Var. Partial Differential Equations, 2024, 63(6), 142.

[15] Huang, X. Q.; Liao, J. F.; *Existence and asymptotic behavior for ground state sign-changing solutions of fractional Schrödinger-Poisson system with steep potential well*, Commun. Anal. Mech., 2024, 16(2), 307–333.

[16] Jeanjean, L.; *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Anal., 1997, 28(10), 1633–1659.

[17] Jeanjean, L.; Lu, S. S.; *Nonradial normalized solutions for nonlinear scalar field equations*, Nonlinearity, 2019, 32(12), 4942–4966.

[18] Jian, H.; Zhong, Q. C.; Wang, L.; *Positive solutions for fractional Kirchhoff-Schrödinger-Poisson system with steep potential well*, Rev. Math. Phys., 2023, 35(07), 2350024.

[19] Lions, P. L.; *The concentration-compactness principle in the calculus of variations: The locally compact case, II*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1984, 1(4), 223–283.

[20] Laskin, N.; *Fractional quantum mechanics and Lévy path integrals*, Phys. Lett. A, 2000, 268(4–6), 298–305.

[21] Laskin, N.; *Fractional Schrödinger equation*, Phys. Rev. E, 2002, 66(5), 56–108.

[22] Luo, X.; Wang, Q. F.; *Existence and asymptotic behavior of high energy normalized solutions for the Kirchhoff type equations in* $\mathbb{R}^3$, Nonlinear Anal. Real World Appl., 2017, 33, 19–32.

[23] Molica B., G.; Radulescu, V. D.; Servadei, R.; *Variational methods for nonlocal fractional problems. Encyclopedia of Mathematics and its Applications*, 162. Cambrige University Press, Cambrige, 2016.

[24] Meng, Y. X.; Zhang, X. R.; He, X. M.; *Least energy sign-changing solutions for a class of fractional Kirchhoff-Poisson system*, J. Math. Phys., 2021, 62(9), 091508.

[25] Teng, K. M.; *Existence of ground state solutions for the nonlinear fractional Schrödinger-Poisson system with critical Sobolev exponent*, J. Differential Equations, 2016, 261(6), 3061–3106.

[26] Teng, K. M.; *Ground state solutions for the non-linear fractional Schrödinger-Poisson system*, Appl. Anal., 2019, 98(11), 1959–1996.

[27] Willem, M.; *Minimax Theorems*, Birkhauser, Boston 1996.

[28] Wang, L.; Rădulescu, V. D.; Zhang, B. L.; *Infinitely many solutions for fractional Kirchhoff-Schrödinger-Poisson systems*, J. Math. Phys., 2019, 60(1), 011506.

[29] Wang, D. B.; Zhang, J. L.; *Least energy sign-changing solutions of fractional Kirchhoff-Schrödinger-Poisson system with critical growth*, Appl. Math. Lett., 2020, 106, 106372.

[30] Wang, L.; Han, T.; Cheng, K.; Wang, J. X.; *Ground state solutions for nonlinear fractional Kirchhoff-Schrödinger-Poisson systems*, Int. J. Nonlinear Sci. Numer. Simul., 2021, 22(5), 531–542.

[31] Wang, L.; Tang, L. Q.; Wang, J.; Wang, J. X.; *Existence of normalized solutions for fractional Kirchhoff-Schrödinger-Poisson equations with general nonlinearities*, Math. Methods Appl. Sci., 2024, 47(11), 8786–8807.

[32] Xiang, M. Q.; Wang, F. L.; *Fractional Schrödinger-Poisson-Kirchhoff type systems involving critical nonlinearities*, Nonlinear Anal., 2017, 164, 1–26.

[33] Yang, J. F.; Guo, W.; Li, W. M.; Zhang, J. F.; *Existence of normalized solutions for a class of Kirchhoff-Schrödinger-Poisson equations in* $\mathbb{R}^3$, Ann. Func. Anal., 2023, 14(1), 13.

[34] Zhang, X.; Zhang, B. L.; Repovš, D.; *Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials*, Nonlinear Anal., 2016, 142, 48–68.

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