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ERGODIC COMPLEX PLANE AND CYLINDER TRANSFORMATION ON A PERIODIC TIME SCALE

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Abstract. We introduce the ergodic complex plane, a global analogue of Hilger's local complex plane on time scales, which simultaneously encodes exponential growth and frequency. Averaging the cylinder transformation on a periodic time scale leads to the notions of ergodic growth rate and ergodic frequency, unifying local and global stability perspectives. This yields the ergodic cylinder transformation, a univalent map inducing an orthogonal curvilinear coordinate system on the regressive complex plane. Within this framework, we develop a decomposition analogous to Hilger's real and imaginary parts, and define the box plus operation, extending the circle plus operation globally.

1. Hilger's complex plane and exponential stability

We begin with a brief summary of the Hilger complex plane [3] on a time scale $\mathbb{T}$. To match the narrative of later sections, we frame the Hilger complex plane entirely in terms of the cylinder transformation, which is a different perspective from [3].

Let $\mathbb{T}$ be a time scale and let $t \in \mathbb{T}$. Define the *Hilger complex numbers* at $t \in \mathbb{T}$, denoted $\mathbb{C}_{\mu(t)}$, by $\mathbb{C}_{\mu(t)} = \mathbb{C} \setminus \{-1/\mu(t)\}$. Defining $\Omega(\mu(t)) := \pi/\mu(t)$, let the *cylinder strip* at $t \in \mathbb{T}$ be defined as

$$\mathbb{C}^{\Omega(\mu(t))} := \{z \in \mathbb{C} \mid -\Omega(\mu(t)) < \operatorname{Im}(z) \le \Omega(\mu(t))\}.$$

Definition 1.1. Let $\mathbb{T}$ be a time scale and let $t \in \mathbb{T}$. Define the *cylinder transformation* $\xi_{\mu(t)} : \mathbb{C}_{\mu(t)} \to \mathbb{C}^{\Omega(\mu(t))}$ by

$$\xi_{\mu(t)}(z) := \begin{cases} \frac{\operatorname{Log}(1+z\mu(t))}{\mu(t)}, & \mu(t) > 0, \\ z, & \mu(t) = 0. \end{cases} \tag{1.1}$$

Note that $\xi_{\mu(t)}$ is one-to-one, and so $\xi^{-1}_{\mu(t)}$ exists. In fact, it has the explicit form

$$\xi^{-1}_{\mu(t)}(z) := \begin{cases} \frac{e^{\mu(t)z}-1}{\mu(t)}, & \mu(t) > 0, \\ z, & \mu(t) = 0. \end{cases} \tag{1.2}$$

The *Hilger real part* at $t \in \mathbb{T}$ of $z \in \mathbb{C}_{\mu(t)}$ can be defined as

$$\operatorname{Re}_{\mu(t)}(z) = \xi^{-1}_{\mu(t)}(\operatorname{Re}(\xi_{\mu(t)}(z))).$$

The *Hilger imaginary part* at $t \in \mathbb{T}$ of z is given by

$$\operatorname{Im}_{\mu(t)}(z) = \operatorname{Im}(\xi_{\mu(t)}(z)),$$

while the *Hilger pure imaginary part* of z at $t \in \mathbb{T}$ is given by

$$\overset{o}{\imath} \operatorname{Im}_{\mu(t)}(z) := \xi^{-1}_{\mu(t)}(i \operatorname{Im}(\xi_{\mu(t)}(z))).$$

It is straightforward to establish the following identities (which are usually taken as definitions).

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Proposition 1.2. *Let $\mathbb{T}$ be a time scale and let $t \in \mathbb{T}$. Then*

$$\begin{aligned} \mathrm{Re}_{\mu(t)}(z) &= \lim_{\tau \searrow \mu(t)} \frac{|1+\tau z| - 1}{\tau}, \\ \mathrm{Im}_{\mu(t)}(z) &= \lim_{\tau \searrow \mu(t)} \frac{\mathrm{Arg}(z\tau + 1)}{\tau}, \\ \overset{o}{\iota}\,\mathrm{Im}_{\mu(t)}(z) &= \lim_{\tau \searrow \mu(t)} \frac{e^{i\,\mathrm{Im}_{\mu(t)}(z)\tau} - 1}{\tau}, \end{aligned}$$

where Arg *is the principal argument satisfying* $-\pi < \mathrm{Arg}(z) \le \pi$.

The *circle plus operation* $\oplus_{\mu(t)}$ is defined by $a \oplus_{\mu(t)} b := a + b + \mu(t)ab$. The *circle minus operation* $\ominus_{\mu(t)}$ is defined to be the additive inverse of $\oplus_{\mu(t)}$. Bohner and Peterson [3, Theorem 2.24] establish that $(\mathbb{C}_{\mu(t)}, \oplus_{\mu(t)})$ is homomorphic to $(\mathbb{C}^{\Omega(\mu(t))}, +(\mathrm{mod}\ 2\pi i/\mu(t)))$ with group homomorphism $\xi_{\mu(t)}$. Although not explicitly stated in [3], it follows from the fact that $\xi_{\mu(t)}$ is a bijection that $(\mathbb{C}_{\mu(t)}, \oplus_{\mu(t)})$ is isomorphic to $(\mathbb{C}^{\Omega(\mu(t))}, +(\mathrm{mod}\ 2\pi i/\mu))$.

The group isomorphism $\xi_{\mu(t)}$ implies we can understand the circle plus addition via, for $w, z \in \mathbb{C}_{\mu(t)}$,

$$z \oplus_{\mu(t)} w = \xi_{\mu(t)}^{-1}(\xi_{\mu(t)}(z) + \xi_{\mu(t)}(w) \pmod{2\pi i/\mu(t)}). \tag{1.3}$$

Using circle plus, for fixed $t \in \mathbb{T}$, every $z \neq -\frac{1}{\mu(t)}$ has the *Hilger decomposition at time* t given by

$$z = \mathrm{Re}_{\mu(t)}(z) \oplus_{\mu(t)} \overset{o}{\iota}\,\mathrm{Im}_{\mu(t)}(z).$$

The Hilger complex plane, $\mathbb{C}_{\mu(t)}$, is shown in Figure 1 for $\mu(t) \neq 0$. Important components of $\mathbb{C}_{\mu(t)}$ are the *Hilger disk* at $t \in \mathbb{T}$, given by

$$\mathcal{H}_{\mu(t)} := \{z \in \mathbb{C}_{\mu(t)} \mid \mathrm{Re}_{\mu(t)}(z) < 0\},$$

which can be thought of as the preimage of the left half-plane of $\mathbb{C}^{\Omega(\mu(t))}$ under $\xi_{\mu(t)}$. The *Hilger circle* at $t \in \mathbb{T}$ is the boundary of $\mathcal{H}_{\mu(t)}$, which we denote by $\mathbb{I}_{\mu(t)}$. The Hilger circle can be thought of as the preimage of the imaginary axis of $\mathbb{C}^{\Omega(\mu(t))}$ under $\xi_{\mu(t)}$.

The cylinder transformation maps contours of equal Hilger imaginary part (which are rays emanating from $-1/\mu(t)$) to horizontal lines in $\mathbb{C}^{\Omega(\mu(t))}$. Also, contours of equal Hilger real part (which are circles centered at $-1/\mu(t)$) are mapped to vertical lines in $\mathbb{C}^{\Omega(\mu(t))}$. In particular, these contours form an orthogonal curvilinear coordinate system on $\mathbb{C}_{\mu(t)}$.

Remark 1.3. The cylinder transformation is strongly related to the concepts of growth rate and frequency on time scales with constant graininess. On the time scale $\mathbb{T} = h\mathbb{Z}$, the contours of equal Hilger imaginary part correspond to frequency given by

$$\omega_h(z) := \mathrm{Im}_h(z) = \mathrm{Im}(\xi_h(z)) = \frac{\mathrm{Arg}(1+zh)}{h}. \tag{1.4}$$

This is the same quantity that represents frequency in Hilger's definition of the time scale sinusoids [5]. Moreover, the contours of equal Hilger real part all have the same exponential growth rate corresponding to the Lyapunov exponent

$$\gamma_h(z) := \frac{\ln(1 + h\,\mathrm{Re}_h(z))}{h} = \xi_h(\mathrm{Re}_h(z)) = \mathrm{Re}(\xi_h(z)) = \frac{\ln|1+hz|}{h}. \tag{1.5}$$

Similarly, on the time scale $\mathbb{T} = \mathbb{R}$, $\xi_0(z) = z$, and the contours of equal exponential growth rate are vertical lines determined by the Lyapunov exponent $\gamma_0(z) := \mathrm{Re}(z)$, while the contours of equal frequency are horizontal lines given by $\omega_0(z) = \mathrm{Im}(z)$.

We say that $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. For regressive p, the *time scale exponential function* $e_p(t,s)$ is defined by

$$e_p(t,s) := \exp\Big(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta\tau\Big).$$

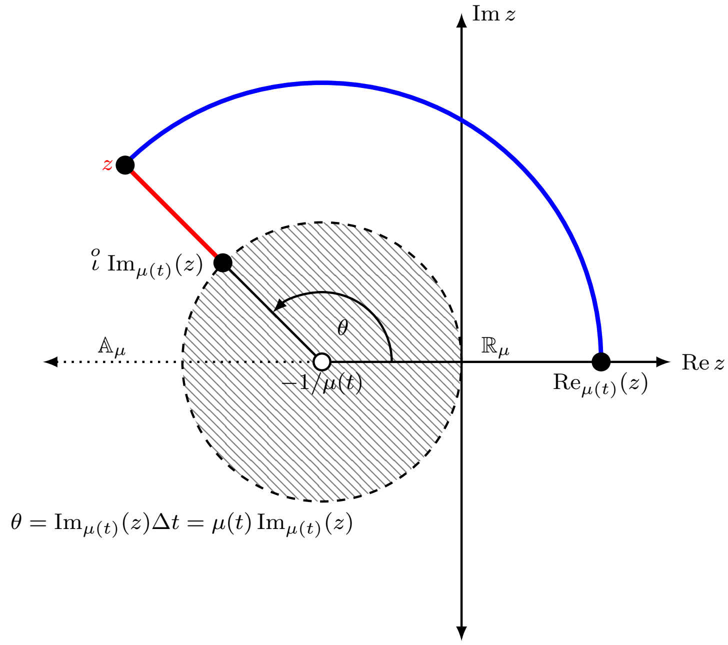

FIGURE 1. Hilger's complex plane, $\mathbb{C}_{\mathcal{H}}$. The z inside the circle have negative Hilger real part, the z on the circle have zero Hilger real part, and the z outside the circle have positive Hilger real part. Points z on the Hilger real axis $\mathbb{R}_{\mu(t)}$ (the solid ray on the real axis) are such that $e_z(t,t_0) > 0$ for all t, while points on the Hilger alternating axis $\mathbb{A}_{\mu(t)}$ (the dotted ray on the real axis) are such that $e_z(t,t_0)$ changes sign at each $t \in \mathbb{T}$.

Following Karpuz [8], denote the set of all *regressive points* in the complex plane by

$$\mathbb{C}_{\mu(\mathbb{T})} := \{z \in \mathbb{C} \mid 1 + \mu(t)z \neq 0 \text{ for all } t \in \mathbb{T}\}.$$

The definitions related to Hilger's complex plane are *local* and *dynamic* in nature; that is, they depend on and change in t. In 2003, Pötzsche, Siegmund, and Wirth introduced the concept of a *global* and *static* exponential growth rate for the time scale exponential function.

Theorem 1.4 (Pötzsche, Siegmund, and Wirth [9])**.** *Let $\mathbb{T}$ be a time scale which is unbounded above and let $\lambda \in \mathbb{C}$. Then the scalar equation*

$$x^{\Delta}(t) = \lambda x(t), \quad x(t_0) = x_0,$$

is exponentially stable if and only if one of the following conditions is satisfied for arbitrary $t_0 \in \mathbb{T}$:

(C1) $\gamma(\lambda) := \limsup_{T\to\infty} \frac{1}{T-t_0}\int_{t_0}^{T} \lim_{s\searrow\mu(t)} \frac{\ln|1+\lambda s|}{s}\Delta t < 0$,

(C2) *For every $T \in \mathbb{T}$, there exists a $t \in \mathbb{T}$ with $t > T$ such that $1 + \mu(t)\lambda = 0$,*

where we use the convention $\ln 0 = -\infty$ in (C1)*.*

Note that condition (C1) can be restated using (1.5) as

(C1*) $\limsup_{T\to\infty} \frac{1}{T-t_0}\int_{t_0}^{T} \lim_{s\searrow\mu(t)} \gamma_s(\lambda)\Delta t < 0$.

This theorem naturally leads to the following definition.

Definition 1.5 ([9])**.** Given a time scale $\mathbb{T}$ which is unbounded above, for arbitrary $t_0 \in \mathbb{T}$, define the sets

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) := \{\lambda \in \mathbb{C} : \text{ (C1) holds}\},$$
$$\mathcal{S}_{\mathbb{R}}(\mathbb{T}) := \{\lambda \in \mathbb{R} : \text{ (C2) holds}\}.$$

Then the *set of exponential stability* for $\mathbb{T}$ is given by

$$\mathcal{S}(\mathbb{T}) := \mathcal{S}_{\mathbb{C}}(\mathbb{T}) \cup \mathcal{S}_{\mathbb{R}}(\mathbb{T}).$$

The calculation of the set of exponential stability is simplified for periodic time scales [[9, Lemma 3.4(b)], , which will be a focus of this paper.

Definition 1.6. A time scale $\mathbb{T}$ is *periodic* if there exists $L > 0$ such that $t \in \mathbb{T}$ implies $t + L \in \mathbb{T}$ and $\mu(t + L) = \mu(t)$ for all $t \in \mathbb{T}$. We call L a *period* of the time scale.

Remark 1.7. Throughout this paper, we will illustrate various results on the following prototypical time scales.

(1) For $\mu_1, \mu_2, \dots, \mu_n > 0$, set $L = \mu_1 + \mu_2 + \cdots + \mu_n$. Then $\mathbb{T}_{\mu_1,\mu_2,\dots,\mu_n}$ is the time scale starting at 0 whose graininesses form the periodic sequence $\{\mu_1, \mu_2, \dots, \mu_n\}$, i.e.,

$$\mathbb{T}_{\mu_1,\mu_2,\dots,\mu_n} = \Big\{0, \mu_1, \mu_1 + \mu_2, \dots, \underbrace{\mu_1 + \cdots + \mu_n}_{L}, L + \mu_1, L + \mu_1 + \mu_2, \dots, 2L, \dots\Big\}.$$

(2) $\mathbb{P}_{a,b}$ is the periodic time scale starting at 0 with an interval of length a followed by a gap of length b, i.e.,

$$\mathbb{P}_{a,b} = [0, a] \cup [a + b, 2a + b] \cup [2a + 2b, 3a + 2b] \cup \cdots$$

A key insight for this paper is that since the cylinder transformation is linked to both growth rate (via the Hilger real part) and frequency (via the Hilger imaginary part) on constant graininess time scales, and since the average of the real part of the cylinder transformation is related to global exponential growth rate, it seems reasonable that the average of the imaginary part of the cylinder transformation is related to the global frequency.

2. Ergodic growth rate and frequency

To solidify the connection between the average of the cylinder transformation and the growth rate and frequency of the time scale exponential, notice for $\lambda \in \mathbb{C}_h$ that the time scale exponential function decomposes as

$$\begin{aligned}
e_\lambda(t, t_0) &= \exp\Big(\int_{t_0}^{t} \xi_{\mu(\tau)}(\lambda)\Delta\tau\Big) \\
&= \exp\Big(\frac{\int_{t_0}^{t} \xi_{\mu(\tau)}(\lambda)\Delta\tau}{t - t_0}(t - t_0)\Big) \\
&= \exp\Big(\frac{\int_{t_0}^{t} \mathrm{Re}(\xi_{\mu(\tau)}(\lambda))\Delta\tau}{t - t_0}(t - t_0)\Big)\exp\Big(i\frac{\int_{t_0}^{t} \mathrm{Im}(\xi_{\mu(\tau)}(\lambda))\Delta\tau}{t - t_0}(t - t_0)\Big) \\
&\overset{(1.4),(1.5)}{=} \exp\Big(\frac{\int_{t_0}^{t} \gamma_{\mu(\tau)}(\lambda)\Delta\tau}{t - t_0}(t - t_0)\Big)\exp\Big(i\frac{\int_{t_0}^{t} \omega_{\mu(\tau)}(\lambda)\Delta\tau}{t - t_0}(t - t_0)\Big) \\
&= \exp\Big(\frac{\int_{t_0}^{t} \gamma_{\mu(\tau)}(\lambda)\Delta\tau}{t - t_0}(t - t_0)\Big) \\
&\quad \times \Big[\cos\Big(\frac{\int_{t_0}^{t} \omega_{\mu(\tau)}(\lambda)\Delta\tau}{t - t_0}(t - t_0)\Big) + i\sin\Big(\frac{\int_{t_0}^{t} \omega_{\mu(\tau)}(\lambda)\Delta\tau}{t - t_0}(t - t_0)\Big)\Big].
\end{aligned} \tag{2.1}$$

Therefore, over an interval $[t, t_0]_{\mathbb{T}}$, the growth rate and frequency of the time scale exponential function are averages (in the integral sense) of $\gamma_{\mu(\tau)}$ and $\omega_{\mu(\tau)}$, respectively.

Since we are averaging the cylinder transformation, which is a logarithm for $\mu(t) > 0$, we want to be careful about the branch cut. We begin by defining a subset of the complex plane which avoids the branch cuts of (1.1) for all $\tau \in \mathbb{T}$. Assuming (P), $\mu_{\max} := \max\{\mu(t) \mid t \in \mathbb{T}\}$ exists, so define

$$B := (-\infty, -1/\mu_{\max}], \quad B^c = \mathbb{C} \setminus B.$$

Motivated by the form (2.1), we now formalize the averaged quantities introduced above.

Definition 2.1. Let $\mathbb{T}$ be a time scale and $x, y \in \mathbb{R}$ such that $x + iy \in B^c$. The *ergodic growth rate of* $e_{x+iy}(t, t_0)$ *on* $[t_0, t]_{\mathbb{T}}$ *off the branch cut* is given by

$$\gamma_{B^c}(x, y, t_0, t) := \frac{1}{t - t_0} \int_{t_0}^{t} \gamma_{\mu(\tau)}(x + iy)\Delta\tau \tag{2.2}$$

Similarly, the *ergodic frequency of* $e_{x+iy}(t, t_0)$ *on* $[t_0, t]_{\mathbb{T}}$ *off the branch cut* is given by

$$\omega_{B^c}(x, y, t_0, t) := \frac{1}{t - t_0} \int_{t_0}^{t} \omega_{\mu(\tau)}(x + iy)\, \Delta\tau. \tag{2.3}$$

Remark 2.2. Several remarks are in order regarding $\gamma_{B^c}(x, y, t_0, t)$ and $\omega_{B^c}(x, y, t_0, t)$.

(1) We use the term *ergodic* to describe the growth rate and frequency here because, if one considers the space $\mathbb{X} = \mu^*(\mathbb{T})$ of all extended graininesses of $\mathbb{T}$ [6] in the order that they appear in $\mathbb{T}$, then the global (spatial) averages of the growth rates and frequencies derived from $\mathbb{X}$ are the pointwise limit of the local (time) averages of the local growth rates and frequencies derived from $\mathbb{X}$ as defined above. We may treat the collection of asymptotic extended graininesses as an irreducible Markov chain, which guarantees the ergodicity of the associated stochastic process. See [10].

(2) By (1.5), $\gamma_h(x+iy)$ has a few useful forms. The choice of form gives different interpretations to the equation. Writing $\gamma_h(x + iy) = \xi_h(\mathrm{Re}_h(x + iy))$ is useful when a characterization of growth rates in terms of the Hilger real part of $x + iy$ is advantageous. By contrast, writing $\gamma_h(x + iy) = \mathrm{Re}(\xi_h(x + iy))$ is in line with Pötzsche, Siegmund, and Wirth's condition.

(3) Using properties of the logarithm, we get

$$\gamma_h(x + iy) = \ln(1 + h\,\mathrm{Re}_h(x + iy))^{1/h}.$$

This implies $\gamma_{B^c}(x, y, t_0, t)$ is the natural log of the geometric mean (in the time scales sense) on $[t_0, t]_{\mathbb{T}}$ of

$$\lim_{s\searrow\mu(t)} (1 + \mathrm{Re}_s(\lambda)s)^{1/s} = \lim_{s\searrow\mu(t)} |1 + \lambda s|^{1/s}. \tag{2.4}$$

Thus, (2.4) can be interpreted as a type of local growth rate of $e_{x+iy}(t, t_0)$ at time t. Since the geometric mean is the best measure of average local growth rates, expressing $\gamma_h(x + iy)$ in this way shows that (2.2) is an effective measure of the average growth rate of the time scale exponential function over $[t_0, t]_{\mathbb{T}}$. This observation was made earlier in [4, 6] –albeit from the geometric perspective– by geometrically averaging Hilger circles.

(4) The formulation in (2.1) is similar to the one provided by Karpuz [8]:

$$e_z(t, t_0) = e_{\mathrm{Re}_{\mu(t)}(z)}(t, t_0)\Big[\cos\Big(\int_{t_0}^{t} \mathrm{Im}_{\mu(\eta)}(z)\Delta\eta\Big) + i\sin\Big(\int_{t_0}^{t} \mathrm{Im}_{\mu(\eta)}(z)\Delta\eta\Big)\Big].$$

To use the terminology of [7], Karpuz gave a type II (time scale) exponential representation whereas (2.1) is a type I (continuous) exponential representation.

(5) In this new notation, condition (C1*) becomes

$$\limsup_{T\to\infty} \gamma_{B^c}(x, y, t_0, T) < 0, \tag{2.5}$$

and thus (C1*) is indeed a requirement that the largest cluster point of the average exponential growth rate over the tail of the time scale is negative. Therefore, (2.5) is a condition on the *global* exponential growth rate. Moreover, the formulation here ties $\gamma_{B^c}(x, y, t_0, t)$ to the local Lyapunov exponent $\gamma_{\mu(t)}$, in the spirit of Pötzsche, Siegmund, and Wirth's (C1) condition.

(6) Note that the dynamic equation

$$z^{\Delta}(t) = \lambda z(t), \quad z(t_0) = z_0, \tag{2.6}$$

for $\lambda \in B^c$ has a growth rate of $\gamma_{B^c}(x, y, t_0, \infty)$ rather than a growth rate of λ. Furthermore, unlike on $\mathbb{R}$ where the terms *growth rate* and *exponential order* are often used synonymously, these two quantities will be distinct in general. Indeed, if λ in (2.6) is real and positive, then λ is best described as the *exponential order* of the equation.

We now see that we can understand the *global* growth rate and frequency of the time scale exponential function by understanding the asymptotic behavior of the ergodic growth rate and ergodic frequency as $t \to \infty$. Since this asymptotic behavior is complicated in general (indeed, the limit as $t \to \infty$ may not exist), we begin by studying the asymptotic behavior with the following simplifying assumption:

$$\mathbb{T} \text{ is a periodic time scale with a period of } L > 0 \text{ and } 0 \in \mathbb{T}. \tag{P}$$

Under this assumption, we have

$$\lim_{t\to\infty} \gamma_{B^c}(x, y, t, 0) = \gamma_{B^c}(x, y, L, 0),$$

and similarly

$$\lim_{t\to\infty} \omega_{B^c}(x, y, t, 0) = \omega_{B^c}(x, y, L, 0).$$

Therefore, we can simplify notation by rewriting (2.2), (2.3), for $x + iy \in B^c$, as

$$\gamma_{B^c}(x, y) := \frac{1}{L} \int_0^L \lim_{s\searrow\mu(\tau)} \gamma_s(x + iy)\, \Delta\tau, \tag{2.7}$$

$$\omega_{B^c}(x, y) := \frac{1}{L} \int_0^L \lim_{s\searrow\mu(\tau)} \omega_s(x + iy)\, \Delta\tau. \tag{2.8}$$

Remark 2.3. While the assumption of periodicity may appear to be limiting, a larger class of aperiodic time scales can be analyzed using periodic techniques. We will call a time scale *simple* if there is a representative periodic time scale which has the same limiting functions γ_{B^c} and ω_{B^c}.

To illustrate this, consider the symmetric time scale $\mathbb{T}_{\text{PTM}} = \{0, t_1, t_2, \dots\}$ with the graininess function $\mu(t_n)$ equal to the n^{th} term of the celebrated Prouhet-Thue-Morse sequence on the symbols one and two [1]. Since the Prouhet-Thue-Morse sequence is aperiodic, $\mathbb{T}_{\text{PTM}}$ is also aperiodic. Additionally, since the symbols of the sequence appear with equal weight in the limit as the sequence length increases,

$$\begin{aligned}\limsup_{T\to\infty} \gamma_{B^c}(x, y, t_0, T) &= \limsup_{T\to\infty} \frac{1}{T - t_0} \int_{t_0}^T \frac{\ln|1 + (x + iy)s|}{s} \Delta\tau \\ &= \frac{1}{3} \left[\ln|1 + (x + iy)| + \ln|1 + 2(x + iy)|\right].\end{aligned}$$

This last expression is equal to $\gamma_{B^c}(x, y)$ on the periodic time scale $\mathbb{T}_{1,2}$. Similar results hold for calculating the limiting value of ω_{B^c}.

In this paper, any result which assumes (P) can be extended to a simple time scale via its representative (asymptotically equivalent) periodic time scale since our arguments rely only on the limiting values of γ_{B^c} and ω_{B^c} rather than the precise nature of $\mathbb{T}$ itself.

In light of the definitions above, the level curves of $\gamma_{B^c}(x, y)$ are the curves along which $e_{x+iy}(t, t_0)$ has a constant global exponential growth rate. The level curves of $\omega_{B^c}(x, y)$ are the curves along which $e_{x+iy}(t, t_0)$ has a constant global (signal) frequency. As we will see, these two families of level curves induce a natural coordinate system on $\mathbb{C}_{\mu(\mathbb{T})}$.

Using these definitions, we can reformulate $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ as

$$\mathcal{S}_{\mathbb{C}}(\mathbb{T}) = \{x + iy \in \mathbb{C} \mid \gamma_{B^c}(x, y) < 0\}. \tag{2.9}$$

Equation (2.9) is reminiscent of condition (C1) and $\mathcal{S}_{\mathbb{C}}(\mathbb{T})$ has been thought of as an average of Hilger circles [4].

Finally, since γ_{B^c} and ω_{B^c} are the average value of $\text{Re}(\xi_{\mu(\tau)})$ and $\text{Im}(\xi_{\mu(\tau)})$, respectively, it makes sense to study directly the complex function given by the average of $\xi_{\mu(\tau)}$. In the next section, we will define and study the properties of a map $\overline{\xi}$, called the *ergodic cylinder transformation*, from the xy plane to the $\gamma\omega$ plane, i.e., a mapping from the pair (x, y) to the pair $(\gamma(x, y), \omega(x, y))$ for periodic time scales.

3. The ergodic cylinder transformation

We begin by defining the codomain of the ergodic cylinder transformation. We define

$$\Omega := \sup_{x+iy\in B^c} \omega_{B^c}(x,y),$$
$$\mathbb{C}^\Omega := \{z \in \mathbb{C} \mid -\Omega < \operatorname{Im}(z) \le \Omega\}.$$

If $\Omega = \infty$, then $\mathbb{C}^\Omega = \mathbb{C}$.

Definition 3.1. Assume (P). Suppose $x+iy \in B^c$. We define the *ergodic cylinder transformation off the branch cut*, $\overline{\xi}_{B^c} : B^c \to \mathbb{C}^\Omega$, by

$$\overline{\xi}_{B^c}(x+iy) := \frac{1}{L}\int_0^L \xi_{\mu(t)}(x+iy)\Delta t.$$

It is straightforward to show from (2.1), for $x+iy \in B^c$,

$$\overline{\xi}_{B^c}(x+iy) = \gamma_{B^c}(x,y) + i\,\omega_{B^c}(x,y). \tag{3.1}$$

Our goal is to extend $\overline{\xi}_{B^c}$ to a map $\overline{\xi} : \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}^\Omega$ that induces an orthogonal curvilinear coordinate system on $\mathbb{C}_{\mu(\mathbb{T})}$ for which the contours are given by the level curves of γ and ω. We will first show that this is the case for $\overline{\xi}_{B^c}$, and then we will define the extension.

Analytic functions that are globally injective, or univalent, induce an orthogonal coordinate system as the preimage of the rectangular coordinate system in the complex plane via the Cauchy-Riemann equations. Therefore, we begin by showing $\overline{\xi}_{B^c} : B^c \to \mathbb{C}^\Omega$ is analytic and univalent. To do so, we need the following result, in the form first presented in [2, Theorem 8].

Theorem 3.2 (Noshiro-Warschawski criterion)**.** *If f is analytic and nonconstant in a convex domain D, and*

$$\operatorname{Re}(e^{i\alpha} f'(z)) > 0$$

for all $z \in D$ and for $\alpha \in \mathbb{R}$ fixed, then f is univalent in D.

Theorem 3.3. *Suppose* (P) *holds. Then*

1. $\overline{\xi}_{B^c} : B^c \to \mathbb{C}^\Omega$ *is analytic, and*
2. $\overline{\xi}_{B^c} : B^c \to \mathbb{C}^\Omega$ *is univalent.*

Proof. Let $z = x+iy \in B^c$.

(1) Since the principal logarithm is analytic, e_z is analytic for $z \in \mathbb{C}_{\mu(\mathbb{T})}$ [8], and the nonregressive points lie in B, it follows that $\overline{\xi}_{B^c}$ is analytic.

(2) To show univalence, from [8], the function $m_z : \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}$,

$$m_z(s,t) = \int_s^t \frac{\Delta\tau}{1+z\mu(\tau)},$$

is analytic in z, so

$$\begin{aligned}\overline{\xi}'_{B^c}(z) &= \frac{1}{L}\int_0^L \frac{\Delta\tau}{1+z\mu(\tau)}\\ &= \frac{1}{L}\int_0^L \frac{(1+x\mu(\tau))\Delta\tau}{|1+(x+iy)\mu(\tau)|^2} - \frac{i}{L}\int_0^L \frac{y\mu(\tau))\Delta\tau}{|1+(x+iy)\mu(\tau)|^2}.\end{aligned} \tag{3.2}$$

If $\mathbb{T} = \mathbb{R}$, univalence follows immediately since $\overline{\xi}_{B^c}(z) = z$. If $\mathbb{T} \ne \mathbb{R}$, we consider three cases: $y < 0, y = 0$, and $y > 0$. If $y > 0$, let $U := \{x+iy \in \mathbb{C} \mid y > 0\}$. Then, for $x+iy \in U$,

$$\operatorname{Re}(i\,\overline{\xi}'_{B^c}(x+iy)) = \frac{1}{L}\int_0^L \frac{y\mu(\tau)\Delta\tau}{|1+(x+iy)\mu(\tau)|^2} > 0,$$

since the integrand is nonnegative and not identically zero over $[0,L]_\mathbb{T}$ due to the fact that $\mu(t) \not\equiv 0$, $y > 0$, and (crucially) that

$$\inf_{\tau\in[0,L]_\mathbb{T}} \frac{1}{|1+(x+iy)\mu(\tau)|^2} > 0,$$

by [8, Lemma 2.1]. Since U is convex, by the Noshiro-Warschawski criterion (with $\alpha = \pi/2$), $\overline{\xi}_{B^c}$ is univalent on U.

Similarly, if $y < 0$, $\overline{\xi}_{B^c}$ is univalent on the lower half-plane. Note that if $z_1, z_2 \in \mathbb{C}$ with $\operatorname{Im}(z_1) > 0$ while $\operatorname{Im}(z_2) < 0$, then $\overline{\xi}_{B^c}(z_1) \neq \overline{\xi}_{B^c}(z_2)$ since $\omega_{B^c}(x,y) > 0$ for $y > 0$ and $\omega_{B^c}(x,y) < 0$ for $y < 0$.

Finally, if $y = 0$ and $x > -1/\mu_{\max}$, $\omega_{B^c}(x,0) = 0$ while $\overline{\xi}'_{B^c}(x) > 0$, which implies $\overline{\xi}_{B^c}$ is one-to-one for $x > -1/\mu_{\max}$. Since ω_{B^c} is positive in the upper half-plane, negative in the lower-half plane, and zero on the ray $(-1/\mu_{\max}, \infty)$, and since $\overline{\xi}_{B^c}$ is univalent on each of these spaces, it follows that $\overline{\xi}_{B^c}$ is globally univalent. □

We now show that $\overline{\xi}_{B^c}$ can be extended to a map $\overline{\xi} : \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}$ which is globally univalent. We will call $\overline{\xi}$ the *ergodic cylinder transformation*. To do so, we treat the behavior along the branch cut B carefully. Since the branch cut consists of intervals between nonregressive points, and since the properties of these intervals depend on properties of the time scale, we establish some notation and technical lemmas to aid in the proof.

Let $M = \{\mu_n\}$ be the collection of distinct, positive graininesses of $\mathbb{T} \neq \mathbb{R}$, ordered as

$$\mu_{\max} = \mu_1 > \mu_2 > \cdots .$$

Let $\mu_* = \inf_n \mu_n$ so that

$$-\frac{1}{\mu_*} < \cdots < -\frac{1}{\mu_2} < -\frac{1}{\mu_1} < 0.$$

Let

$$I_n := \Big(-\frac{1}{\mu_{n+1}}, -\frac{1}{\mu_n} \Big), \ n = 1, 2, \dots,$$
$$I_0 := \begin{cases} (-\infty, -1/\mu_*), & \mu_* > 0, \\ \emptyset, & \mu_* = 0. \end{cases}$$

Lemma 3.4. *Assume* (P). *Then M is either finite or countably infinite. Moreover,*

(1) *If M is finite,*

$$B \setminus \Big\{ -\frac{1}{\mu_n} \Big\} = I_0 \cup \left(\bigcup_{n=1}^{N-1} I_n \right).$$

(2) *If M is countably infinite,*

$$B \setminus \Big\{ -\frac{1}{\mu_n} \Big\} = \bigcup_{n}^{\infty} I_n.$$

Proof. An arbitrary time scale has at most countably many graininesses.

(1) If M is finite, then $\mu_* > 0$ so $I_0 \neq \emptyset$ and the result follows.
(2) If M is countably infinite, then we still have $L = \sum_n \mu_n < \infty$, so it must be that $\mu_* = 0$. Hence, $I_0 = \emptyset$ and the result follows.

□

Theorem 3.5. *$\overline{\xi}_{B^c} : B^c \to \mathbb{C}^\Omega$ can be extended to a globally univalent function $\overline{\xi} : \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}^\Omega$.*

Proof. We present the proof in two cases.

Case 1: M finite. Since M is finite, we can list the N distinct graininesses in order as

$$\mu_{\max} = \mu_1 > \mu_2 > \mu_3 > \cdots > \mu_N,$$

so that the corresponding non-regressive points satisfy

$$-\frac{1}{\mu_N} < \cdots < -\frac{1}{\mu_2} < -\frac{1}{\mu_1} < 0.$$

Then, by Lemma 3.4, we have

$$B \setminus \Big\{-\frac{1}{\mu_n}\Big\} = I_0 \cup \Big(\bigcup_{n=1}^{N-1} I_n\Big).$$

To extend $\overline{\xi}_{B^c}$, we consider

$$\overline{\xi}^+_{B^c}(x) := \lim_{y\to 0^+} \overline{\xi}_{B^c}(x+iy), \quad \text{for } x \in B \setminus \{-1/\mu_n\}. \tag{3.3}$$

Taking real and imaginary parts of $\overline{\xi}^+_{B^c}(x)$ gives

$$\begin{aligned}
\gamma_B(x) := \operatorname{Re}\overline{\xi}^+_{B^c}(x) &= \frac{1}{L}\int_{\{t\in[0,L]:\mu(t)=0\}} x\,\Delta t + \frac{1}{L}\int_{\{t\in[0,L]:\mu(t)>0\}} \frac{\ln|1+\mu(t)x|}{\mu(t)}\Delta t,\\
\omega_B(x) := \operatorname{Im}\overline{\xi}^+_{B^c}(x) &= \frac{\pi}{L}\int_{\{t\in[0,L]:\mu(t)>0\}} \frac{\mathbf{1}_{\{x<-1/\mu(t)\}}}{\mu(t)}\,\Delta t\\
&= \frac{\pi}{L}\sum_{\{t\in[0,L]:\mu(t)>0\}} \mathbf{1}_{\{x<-1/\mu(t)\}}\\
&= \frac{\pi}{L}\begin{cases} \sum_{\{t\in[0,L]:\mu(t)\ge\mu_n\}} 1, & x \in I_n,\\ \sum_{\{t\in[0,L]:\mu(t)>0\}} 1, & x\in I_0,\end{cases}
\end{aligned}$$

where $\mathbf{1}_A$ denotes the characteristic function on the set A. Thus, ω_B is constant on each I_n; write $\omega_n := \omega_B(x)$ for any $x \in I_n$. Also, ω_B is constant on I_0 as well, so we let $\omega_0 := \omega_B(x)$ for $x \in I_0$. Then

$$\omega_1 < \omega_2 < \cdots < \omega_{N-1} < \omega_0.$$

For $x \in B \setminus \{-1/\mu_n\}$, differentiating under the integral yields

$$\begin{aligned}
\gamma_B'(x) &= \frac{1}{L}\int_{\{[0,L]:\mu(t)=0\}} 1\,\Delta t + \frac{1}{L}\int_{\{[0,L]:\mu(t)>0\}} \frac{1}{1+\mu(t)x}\,\Delta t\\
&= p_0 + \frac{1}{L}\int_{\{[0,L]:\mu(t)>0\}} \frac{1}{1+\mu(t)x}\,\Delta t,\\
\gamma_B''(x) &= -\frac{1}{L}\int_{\{[0,L]:\mu(t)>0\}} \frac{\mu(t)}{(1+\mu(t)x)^2}\,\Delta t \;<\; 0,
\end{aligned}$$

where

$$p_0 := \frac{1}{L}\int_{\{[0,L]:\mu(t)=0\}} 1\,\Delta t$$

is a constant. Thus, γ_B' is strictly decreasing on each I_n, $n = 1,2,\dots$. Moreover, on I_n, we have the end behavior

$$\lim_{x\downarrow -1/\mu_{n+1}} \gamma_B'(x) = +\infty, \quad \lim_{x\uparrow -1/\mu_n} \gamma_B'(x) = -\infty.$$

Hence, there exists a unique $\alpha_n \in I_n$ with $\gamma_B'(\alpha_n) = 0$. Consequently,

$$\gamma_B'(x) > 0 \text{ on } \big(-\tfrac{1}{\mu_{n+1}},\, \alpha_n\big), \quad \gamma_B'(x) < 0 \text{ on } \big(\alpha_n,\, -\tfrac{1}{\mu_n}\big).$$

We can similarly analyze the behavior of γ_B on I_0. If $p_0 = 0$, then $\gamma_B'(x) < 0$ for $x \in I_0$. If $p_0 > 0$, then there exists a unique $\alpha_0 \in (-\infty, -1/\mu_N)$ such that $\gamma_B'(\alpha_0) = 0$ and $\gamma_B'(x) > 0$ for $x \in (-\infty, \alpha_0)$ while $\gamma_B'(x) < 0$ for $x \in (\alpha_0, -1/\mu_N)$.

We define $\overline{\xi}_B : B \setminus \{-1/\mu_n\} \to \mathbb{C}$ by

$$\overline{\xi}_B(x) = \begin{cases}
\gamma_B(x) + i\,\omega_0, & p_0 > 0,\ x \in (-\infty, \alpha_0],\\
\gamma_B(x) - i\,\omega_0, & p_0 > 0,\ x \in (\alpha_0, -1/\mu_N),\\
\gamma_B(x) + i\,\omega_0, & p_0 = 0,\ x \in (-\infty, -1/\mu_N),\\
\gamma_0(x) - i\,\omega_n, & x \in \big(-\tfrac{1}{\mu_{n+1}},\, \alpha_n\big],\ n = 1,\dots,N-1,\\
\gamma_B(x) + i\,\omega_n, & x \in \big(\alpha_n,\, -\tfrac{1}{\mu_n}\big),\ n = 1,\dots,N-1,
\end{cases} \tag{3.4}$$

Note that on each I_n, we use $-\overline{\xi}^{+}_{B^c}(x)$ for $x \in (-1/\mu_{n+1}, \alpha_n]$, but use $\overline{\xi}^{+}_{B^c}(x)$ for $x \in (\alpha_n, -1/\mu_n)$. As a preference, on I_0, we use $\overline{\xi}^{+}_{B^c}(x)$ for $x \in (-\infty, \alpha_0]$, but use $-\overline{\xi}^{+}_{B^c}(x)$ for $x \in (\alpha_0, -1/\mu_N)$.

Finally, define $\overline{\xi} : \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}^{\Omega}$ by

$$\overline{\xi}(z) = \begin{cases} \overline{\xi}_{B^c}(z), & z \in B^c, \\ \overline{\xi}_B(x), & z = x \in B \setminus \{-1/\mu_n\}. \end{cases}$$

We now show that $\overline{\xi}$ is globally univalent. We consider three subcases.

Subcase 1: Two interior points. If $z_1, z_2 \in B^c$ and $\overline{\xi}(z_1) = \overline{\xi}(z_2)$, then $\overline{\xi}_{B^c}(z_1) = \overline{\xi}_{B^c}(z_2)$. By the univalence of $\overline{\xi}_{B^c}$ on B^c, we get $z_1 = z_2$.

Subcase 2: One interior point, one branch cut point. Suppose $z \in B^c$ and $x \in B \setminus \{-1/\mu_n\}$. The image $\overline{\xi}(B^c) = \overline{\xi}_{B^c}(B^c)$ is open. The value $\overline{\xi}_B(x)$ is a one-sided (non-tangential) limit of $\overline{\xi}_{B^c}$ at $x \in \partial B^c$, hence $\overline{\xi}_B(x) \in \partial\, \overline{\xi}_{B^c}(B^c)$ and is not an interior point. Therefore $\overline{\xi}(z) \neq \overline{\xi}(x)$.

Subcase 3: Two branch cut points. Let $x_1 \neq x_2$ in $B \setminus \{-1/\mu_n\}$. Since $\overline{\xi}(x) = \overline{\xi}_B(x)$ on $B \setminus \{-1/\mu_n\}$, it suffices to show $\overline{\xi}_B(x_1) \neq \overline{\xi}_B(x_2)$.

Suppose x_1, x_2 lie in one of the same half-intervals in (3.4). Then, $\operatorname{Im} \overline{\xi}_B$ is constant while $\gamma'_B(x)$ has a fixed nonzero sign; hence $\overline{\xi}_B$ is strictly monotone along a horizontal line and $\overline{\xi}_B(x_1) \neq \overline{\xi}_B(x_2)$.

On the other hand, if x_1, x_2 are in the two different halves of the same I_n, then $\operatorname{Im} \overline{\xi}_B(x_1) = -\omega_n$ and $\operatorname{Im} \overline{\xi}_B(x_2) = +\omega_n$, so $\overline{\xi}_B(x_1) \neq \overline{\xi}_B(x_2)$.

If x_1, x_2 lie in distinct intervals $I_j \neq I_n$, then $\operatorname{Im} \overline{\xi}_B(x_\ell) \in \{\pm\omega_j, \pm\omega_n\}$ with $\omega_j \neq \omega_n$ since the sequence $\{\omega_n\}$ is strictly increasing in n. Again, the imaginary parts differ, so the values are distinct.

Combining the three cases shows $\overline{\xi}$ is injective on $\mathbb{C}_{\mu(\mathbb{T})} = \mathbb{C} \setminus \{-1/\mu_n\}$ when M is finite.

Case 2: M countably infinite. If M is countably infinite, by Lemma 3.4,

$$B \setminus \left\{-\frac{1}{\mu_n}\right\} = \bigcup_{n=1}^{\infty} I_n.$$

Since $I_0 = \emptyset$, we can define this simpler version of $\overline{\xi}_B : B \setminus \{-1/\mu_n\} \to \mathbb{C}$ by

$$\overline{\xi}_B(x) = \begin{cases} \gamma_B(x) - i\,\omega_n, & x \in \big(-\frac{1}{\mu_{n+1}}, \alpha_n\big], \\ \gamma_B(x) + i\,\omega_n, & x \in \big(\alpha_n, -\frac{1}{\mu_n}\big), \end{cases} \tag{3.5}$$

for all n for which μ_{n+1} exists. Then

$$\overline{\xi}(z) = \begin{cases} \overline{\xi}_{B^c}(z), & z \in B^c, \\ \overline{\xi}_B(x), & z = x \in B \setminus \{-1/\mu_n\}. \end{cases}$$

The argument that $\overline{\xi}$ is injective in this situation is already contained in the previous argument. $\square$

The construction of the ergodic cylinder transformation in Theorem 3.5 gives us a way to extend the definition of γ_{B^c} and ω_{B^c} to the branch cut via the functions γ_B and ω_B. We can tie them into global functions via

$$\gamma(x, y) := \operatorname{Re}(\overline{\xi}(x + iy), \quad \omega(x, y) := \operatorname{Im}(\overline{\xi}(x + iy)), \qquad \text{for all } x + iy \in \mathbb{C}_{\mu(\mathbb{T})}.$$

Let $x + iy \in \mathbb{C}_{\mu(\mathbb{T})}$ be fixed. By Theorem 3.5, $\overline{\xi}^{-1} : \overline{\xi}(\mathbb{C}_{\mu(\mathbb{T})}) \to \mathbb{C}_{\mu(\mathbb{T})}$ exists and is defined by

$$\overline{\xi}^{-1}(\gamma + i\omega) = x + iy.$$

where $\gamma(x, y) = \gamma$ and $\omega(x, y) = \omega$. The level curves of $\gamma(x, y)$ and $\omega(x, y)$ therefore form a coordinate system on $\mathbb{C}_{\mu(\mathbb{T})}$ which is an orthogonal curvilinear coordinate system on B^c by Theorem 3.3. See Figures 2 and 3.

Definition 3.6. Let $\mathbb{T}$ be a time scale. The regressive complex plane $\mathbb{C}_{\mu(\mathbb{T})}$ equipped with the (γ, ω) coordinate system induced by $\overline{\xi}^{-1}$ is called the *ergodic complex plane* for $\mathbb{T}$.

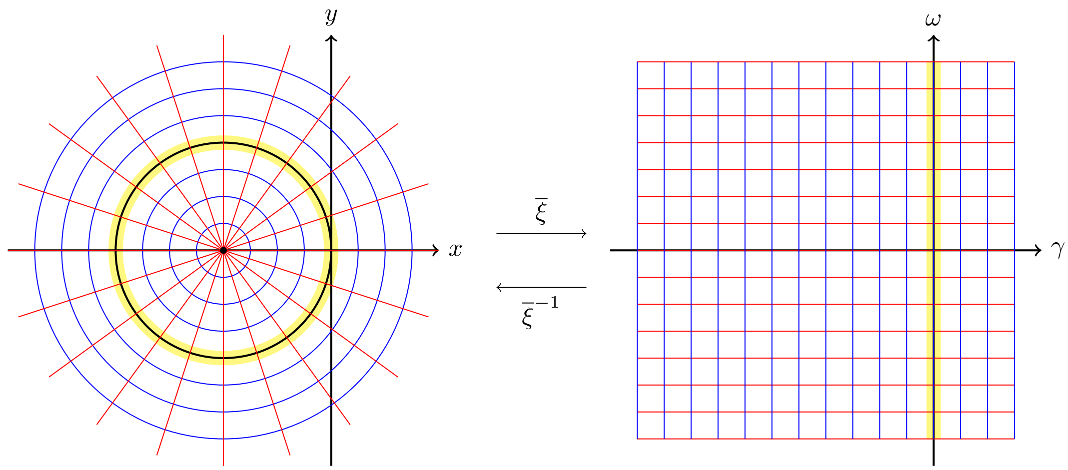

FIGURE 2. The ergodic cylinder transformation on $\mathbb{T} = \mathbb{Z}$. Here, $\bar{\xi}^{-1}$ maps lines of constant average exponential growth rate (blue) and constant average exponential frequency (red) in the γ-ω plane to the level curves of $\gamma(x, y)$ (blue) and $\omega(x, y)$ (red) in the x-y plane to form a new a curvilinear coordinate system on $\mathbb{C}$. We call this the ergodic complex plane for $\mathbb{T}$. Note that $\bar{\xi}^{-1}$ maps the imaginary axis in γ-ω coordinates to $\partial\mathcal{S}(\mathbb{T})$ in the ergodic complex plane.

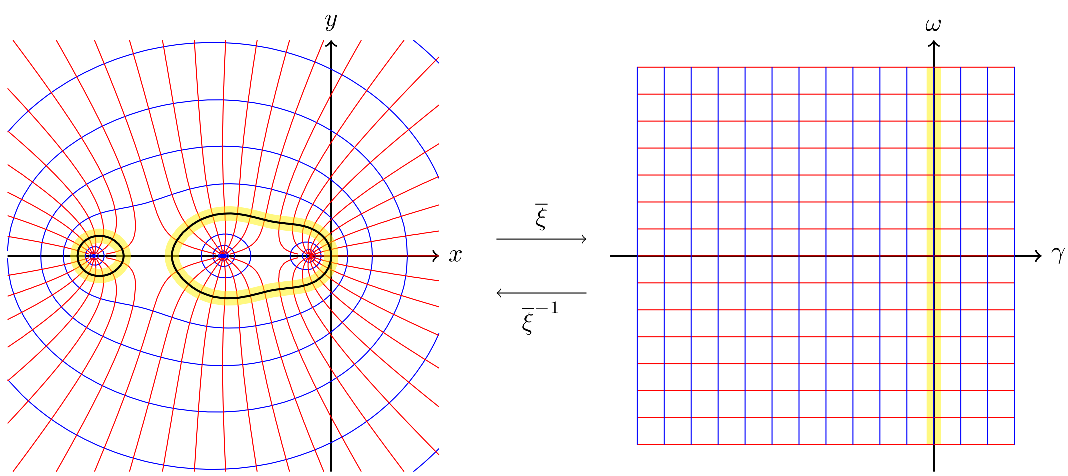

FIGURE 3. The ergodic cylinder transformation for $\mathbb{T}_{1,5,0.45}$. The solid black curves are where $\gamma(x, y) = 0$, i.e. $\partial\mathcal{S}(\mathbb{T})$.

Theorem 3.7. *Suppose* (P) *holds. Let* $\mathbb{T}$ *be a time scale which is unbounded above but with bounded graininess. Let*

$$\begin{aligned}\Gamma_0 &:= \{(x, y) : \gamma(x, y) = 0\},\\ \Gamma_- &:= \{(x, y) : \gamma(x, y) < 0\},\\ \Gamma_+ &:= \{(x, y) : \gamma(x, y) > 0\}.\end{aligned}$$

Then:

(1) $\Gamma_0 = \partial\mathcal{S}(\mathbb{T})$, Γ_- *is the interior of* $\mathcal{S}(\mathbb{T})$, *and* Γ_+ *is the exterior of* $\mathcal{S}(\mathbb{T})$.
(2) $\overline{\xi}(\Gamma_0) = \{(0,\omega) : -\Omega < \omega \le \Omega\}$, $\overline{\xi}(\Gamma_-) = \{(\gamma,\omega) : \gamma < 0\}$, *and* $\overline{\xi}(\Gamma_+) = \{(\gamma,\omega) : \gamma > 0\}$.
(3) *Let* $\Gamma_c := \{(x,y) : -1/\mu_{\max} < x < 0, y = 0\}$. *Then* $\overline{\xi}(\Gamma_c) = \mathbb{R}^-$.

Proof. (1) Assuming (P), on Γ_0, (2.7) vanishes. The second claim is a restatement of [9, Lemma 3.4b]. The third follows from the fact that (C1) is necessary for exponential stability, and (C1) becomes (2.9) due to (P).
(2) and (3) follow from the definitions of the sets involved and $\overline{\xi}$. □

Just as Hilger's pure imaginary number parameterizes the Hilger circle via

$$\mathbb{I}_{\mu(t)} = \{\overset{o}{\iota}\,\omega \mid -\pi/\mu(t) < \omega \le \pi/\mu(t)\},$$

the ergodic cylinder transformation allows us to parameterize $\partial\mathcal{S} = \Gamma_0$ as the preimage of the imaginary axis in $\mathbb{C}^\Omega$.

Definition 3.8. Suppose $-\Omega < \omega \le \Omega$ such that $\overline{\xi}^{-1}(i\omega)$ exists. The *ergodic pure imaginary number* $\overset{\odot}{\imath}\omega \in \partial\mathcal{S} = \Gamma_0$ is defined as

$$\overset{\odot}{\imath}\omega := \overline{\xi}^{-1}(i\omega).$$

On $\mathbb{R}$, $\overset{\odot}{\imath}\omega$ is $i\omega$ on the imaginary axis, and on $h\mathbb{Z}$, $\overset{\odot}{\imath}\omega$ is $\overset{o}{\iota}\,\omega$, the Hilger pure imaginary number, on $\mathbb{I}_h$. This characterization allows us to parameterize Γ_0 as

$$\Gamma_0 = \{\overset{\odot}{\imath}\omega \mid -\Omega < \omega \le \Omega \text{ and } \overset{\odot}{\imath}\omega \text{ exists}\}.$$

This parameterization allows us to prove properties of Γ_0, as the next two propositions show.

Proposition 3.9. *Assume* (P). *Let* $\mathbb{T} = \mathbb{T}_{\mu_1,\mu_2,\dots,\mu_n}$. *Then* Γ_0 *is bounded.*

Proof. Fix $\omega \in (-\Omega,\Omega]$ such that $\overline{\xi}^{-1}(i\omega)$ exists and consider the point $z = 0 + i\omega \in \mathbb{C}^\Omega$. Let $\mu_{\min} := \min \mu_k$. Then $\overline{\xi}^{-1}(z) = \overset{\odot}{\imath}\omega \in \Gamma_0$, so

$$\begin{aligned}
0 &= \int_0^L \frac{\ln|1 + \overset{\odot}{\imath}\omega\mu(t)|}{\mu(t)}\Delta t \\
&\ge \int_0^L \frac{2\operatorname{Re}(\overset{\odot}{\imath}\omega) + |\overset{\odot}{\imath}\omega|^2\mu(t)}{|1 + \overset{\odot}{\imath}\omega\mu(t)|^2}\Delta t \\
&\ge \int_0^L \frac{2\operatorname{Re}(\overset{\odot}{\imath}\omega) + |\overset{\odot}{\imath}\omega|^2\mu_{\min}}{|1 + \overset{\odot}{\imath}\omega\mu(t)|^2}\Delta t.
\end{aligned}$$

Since the denominator is positive, this implies $2\operatorname{Re}(\overset{\odot}{\imath}\omega) + |\overset{\odot}{\imath}\omega|^2\mu_{\min} < 0$, which means $\overset{\odot}{\imath}\omega$ is in a disk of radius $1/\mu_{\min}$ centered at $-1/\mu_{\min}$ (which is a Hilger disk of curvature $\mu_{\min}$). □

Proposition 3.10. Γ_0 *is symmetric about the real axis. That is, for* $\omega \in (-\Omega,\Omega]$ *and* $\overset{\odot}{\imath}\omega \notin \mathbb{R}$, $\overline{\overset{\odot}{\imath}\omega} = \overset{\odot}{\imath}(-\omega)$.

Proof. Write $\overset{\odot}{\imath}\omega = x + iy$. Note $\overline{\xi}(x+iy) = \gamma(x,y) + i\omega(x,y) = i\omega$. Then, calculations for $\overline{\overset{\odot}{\imath}\omega} = x - iy$ yield

$$\begin{aligned}
\gamma(x,-y) &= \frac{1}{L}\int_0^L \lim_{s\searrow\mu(t)} \frac{\ln|\overline{1 + (x+iy)s}|}{s}\Delta s \\
&= \frac{1}{L}\int_0^L \lim_{s\searrow\mu(t)} \frac{\ln|1 + (x+iy)s|}{s}\Delta s \\
&= \gamma(x,y) = 0,
\end{aligned}$$

and

$$\begin{aligned}\omega(x,-y) &= \frac{1}{L}\int_0^L \lim_{s\searrow\mu(t)} \frac{\operatorname{Arg}(\overline{1+(x+iy)s})}{s}\Delta s \\ &= -\frac{1}{L}\int_0^L \lim_{s\searrow\mu(t)} \frac{\operatorname{Arg}(1+(x+iy)s)}{s}\Delta s \\ &= -\omega(x,y) \\ &= -\omega\end{aligned}$$

Therefore, $\overset{\odot}{\imath}(-\omega) = x - iy = \overline{\overset{\odot}{\imath}\omega}$. $\square$

Finally, we now have a way of characterizing the time scale exponential function in terms of its ergodic growth rate and ergodic frequency via the ergodic cylinder transformation and its inverse.

Remark 3.11. Assuming (P), an exponential function on $\mathbb{T}$ with ergodic growth rate γ and ergodic frequency ω and with constant subscript is given by

$$e_{\bar{\xi}^{-1}(\gamma+i\omega)}(t,t_0), \tag{3.6}$$

provided that $\bar{\xi}^{-1}(\gamma+i\omega)$ exists. In fact, among the family of functions of the form $e_\alpha(t,t_0)$ where $\alpha\in\mathbb{C}$ is constant, the formulation in (3.6) allows us to fully understand the qualitative behavior of $e_\alpha(t,t_0)$ in terms of the real and imaginary parts of α.

Example 3.12.

(1) The ergodic complex plane for $\mathbb{T}=\mathbb{T}_{1,2}$, the discrete periodic time scale with graininesses $\{1,2\}$, is shown in Figure 4. On this time scale, $\Omega = 2\pi/3$.

(2) The ergodic complex plane for $\mathbb{T}=\mathbb{T}_{1,4,4}$, the discrete periodic time scale with graininesses $\{1,4,4\}$, is shown in Figure 4. On this time scale, $\Omega=\pi/3$. In particular, the boundary of the region of exponential stability on $\mathbb{T}_{1,4,4}$ contains a saddle point. The value of ω at the saddle point is $2\pi/9$. Therefore, $\overset{\odot}{\imath}2\pi/9$ exists but $\overset{\odot}{\imath}(-2\pi/9)$ does not exist.

(3) The ergodic complex plane for $\mathbb{T}=\mathbb{P}_{1/4,1}$ is shown in Figure 5. Note that the boundary of the region of exponential stability on $\mathbb{P}_{1/4,1}$ is disconnected. On the real axis, $\overset{\odot}{\imath}4\pi/5$ is the first intersection of the boundary with $\mathbb{R}^-$, while $\overset{\odot}{\imath}(-4\pi/5)$ is the second intersection of the boundary with $\mathbb{R}^-$, illustrating how we define ω on $\mathbb{R}^-$. On this time scale, $\Omega=\infty$.

(4) Consider $\mathbb{T}=C_{1/3}$, the time scale consisting of repeated middle-third Cantor sets. The ergodic complex plane for $\mathbb{T}=C_{1/3}$ is shown in Figure 5. On this time scale, $\Omega=\infty$.

Example 3.13. The map $\bar{\xi}$ is in general not bijective. To see this, consider $\mathbb{T}=\mathbb{T}_{1,2}$. The only saddle point of $\bar{\xi}_B$ is given by $\alpha_1=-3/4$. Since we make the choice to map saddle points of $\bar{\xi}_B$ to positive frequencies, we have $\bar{\xi}(\alpha_1) = -\ln(2)+i\pi/3$. But, this means that the point $z^* = -\ln 2 - i\pi/3 \in \mathbb{C}^\Omega$ has no preimage under $\bar{\xi}$, that is, $\bar{\xi}^{-1}(z^*)$ does not exist, and hence $\bar{\xi}$ is not a bijection between $\mathbb{C}_{\mu(\mathbb{T})}$ and $\mathbb{C}^\Omega$.

In general, there are as many points in $\mathbb{C}_{\mu(\mathbb{T})}$ that do not have an image under $\bar{\xi}$ as there are saddle points of $\bar{\xi}$, so for the time scales considered in this paper, the number of such points is finite, and each of these points is isolated.

We conclude this section by considering discrete time scales, which are prominent in time series data. We show that our results must adhere to major results in sampling theory.

Proposition 3.14. *Assume* (P). *Let* $\mathbb{T}=\mathbb{T}_{\mu_1,\mu_2,\dots,\mu_N}$. *Then* $L=\sum_{n=1}^N \mu_n$. *Set* $a := \frac{L}{N}$. *Then for* $x+iy\in\mathbb{C}_{\mu(t)}$, $-\pi/a < \omega(x,y) \le \pi/a$. *That is,* $\Omega = \pi/a$.

Proof. Since $\mathbb{T}$ is discrete, for $x+iy\in B^c$,

$$\omega_{B^c}(x,y) = \frac{1}{L}\int_0^L \operatorname{Im}_{\mu(t)}(x+iy)\Delta t = \frac{1}{L}\sum_{k=1}^N \operatorname{Arg}(1+(x+iy)\mu_k) \le \frac{N\pi}{L} = \frac{\pi}{a}.$$

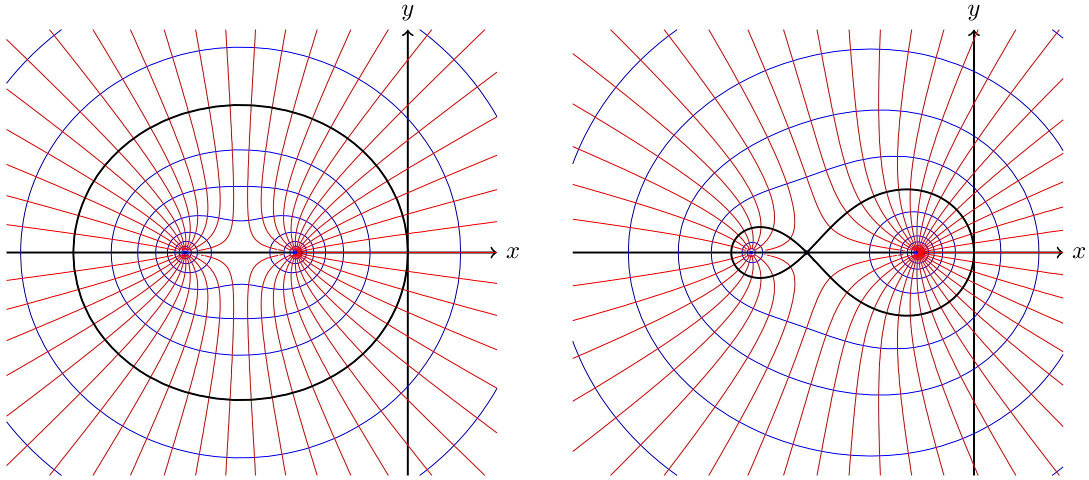

FIGURE 4. The ergodic complex plane for $\mathbb{T}_{1,2}$ (left) and $\mathbb{T}_{1,4,4}$ (right). The solid black curve is where $\gamma(x, y) = 0$, i.e. $\partial\mathcal{S}(\mathbb{T})$.

Moreover, on B, for $x \in I_0$ (that is, for x to the left of all nonregressive points), $\omega_B(x) = \pi/a$. For $x \in I_k, k \geq 1$, by construction $\omega_B(x) < \pi/a$. Thus $\omega(x, y) \leq \pi/a$. The proof to show $\omega(x, y) > -\pi/a$ is similar. □

Remark 3.15. If $\mathbb{T} = \mathbb{T}_{\mu_1,\mu_2,\dots,\mu_N}$ and $t \in \mathbb{T}$ is measured in seconds, then we have the following:

(1) The average sampling rate is $1/a$ samples per second.

(2) The Nyquist-Shannon Sampling Theorem [11] says that we can perfectly reconstruct a continuous time signal sampled on the time scale that contains no frequencies higher than $\frac{1}{2a}$ hertz. That is, we can perfectly reconstruct a continuous time signal sampled on the time scale that contains no frequencies higher than $\pi/a = \Omega$, so Ω is the Nyquist frequency for sampling of a real signal done on time scale points.

(3) Because of (2), the definition of the ergodic frequency captures a key feature of frequency from the perspective of signal processing.

4. The box plus operation

In this section, we propose a generalization of the time scale circle plus binary operator for the ergodic complex plane.

The cylinder strip $\mathbb{C}^{\Omega(h)}$ is a manifestation of $\mathbb{C}^{\Omega}$ when $\mathbb{T} = h\mathbb{Z}$, where h is constant, and therefore its coordinates can be thought of as a *local* growth rate on the real axis and *local* frequency on the imaginary axis. Therefore, by (1.3), one way to understand the action of $\oplus_h$ is that it maps $z, w \in \mathbb{C}_h$ to $\mathbb{C}^{\Omega}$ for $\mathbb{T} = h\mathbb{Z}$, adds the local growth rate of z to the local growth rate of w and adds the local frequency of z to the local frequency of w (mod $2\pi i/h$), and finally pulls back to the unique point in $\mathbb{C}_h$ that would map to that new growth rate and frequency pair under ξ_h.

We can generalize $\oplus_h$ by employing $\overline{\xi}$ rather than ξ_h in (1.3). Since $\overline{\xi}$ is not a bijection (see Example 3.13), addition (mod $2i\Omega$) in $\mathbb{C}_{\Omega}$ is not a group since the sum of two points may not correspond to the image of a point under $\overline{\xi}$. Thus $(\mathbb{C}^{\Omega}, +\ (\text{mod } 2i\Omega))$ is a groupoid, and $+$ (mod $2i\Omega$) is a partial function.

Example 4.1. To demonstrate the lack of closure under addition (mod $2i\Omega$) in $\mathbb{C}^{\Omega}$, consider again the time scale $\mathbb{T}_{1,2}$ from Example 3.13.

Fixing $z \in \mathbb{C}^{\Omega}$, there is just one point v such that $z + v$ does not exist. This point v satisfies

$$v = -\ln 2 - i\pi/3 - z.$$

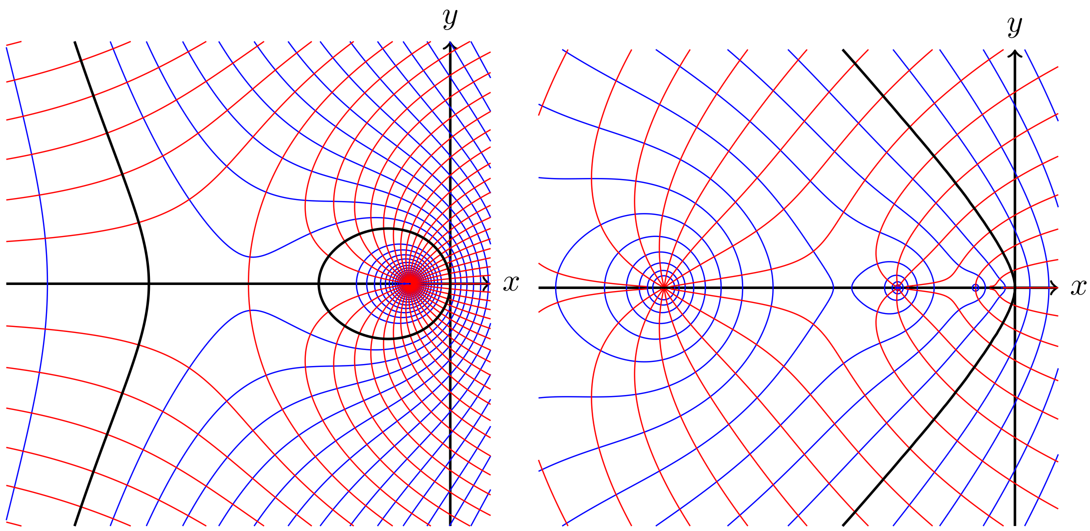

FIGURE 5. The ergodic complex plane for $\mathbb{T} = \mathbb{P}_{1/4,1}$ (left) and $\mathbb{T} = C_{1/3}$ (right).

There are a few properties we want to point out in this example that are illustrative for later proofs. Notice that $\overline{\xi}^{-1}(\overline{\xi}(-3/4)) = \overline{\xi}^{-1}(-\ln 2 + \pi i/3) = -3/4$. Conversely, $\overline{\xi}(\overline{\xi}^{-1}(-\ln 2 - \pi i/3))$ does not exist. Finally, $\overline{\xi}(\overline{\xi}^{-1}(\overline{\xi}(-3/4))) = \overline{\xi}(-3/4) = -\ln 2 + \pi i/3$. This example gives us intuition for which compositions of $\overline{\xi}$ and $\overline{\xi}^{-1}$ exist in general.

Crucially, $\overline{\xi}^{-1}(\overline{\xi}(z))$ always exists, but $\overline{\xi}(\overline{\xi}^{-1}(z))$ may not exist. Moreover,

$$\overline{\xi}(\overline{\xi}^{-1}(\overline{\xi}(z))) = \overline{\xi}(z).$$

Since $\overline{\xi}$ is defined as an integral over the time scale, $\overline{\xi}$ can act on functions as well so long as the resulting integral converges. In what follows, for fixed z, w, we will evaluate $\overline{\xi}$ at $z \oplus_{\mu(t)} w$, which is a function of t.

Proposition 4.2. *Let $z, w \in B^c$. Then*

$$\overline{\xi}_{B^c}(z \oplus_{\mu(t)} w) = \overline{\xi}_{B^c}(z) + \overline{\xi}_{B^c}(w) \pmod{2i\Omega},$$
$$\overline{\xi}_{B^c}(z \ominus_{\mu(t)} w) = \overline{\xi}_{B^c}(z) - \overline{\xi}_{B^c}(w) \pmod{2i\Omega}.$$

Proof. Let $z, w \in \mathbb{C}_{\mu(\mathbb{T})}$. Then z, w are regressive for each $t \in \mathbb{T}$, so $z \oplus_{\mu(t)} w$ is defined for each $t \in \mathbb{T}$. Hence,

$$\begin{aligned} \overline{\xi}_{B^c}(z \oplus_{\mu(t)} w) &:= \frac{1}{L}\int_0^L \xi_{\mu(t)}(z \oplus_{\mu(t)} w)\,\Delta t \pmod{2i\Omega} \\ &= \frac{1}{L}\int_0^L \xi_{\mu(t)}(z) + \xi_{\mu(t)}(w)\,\Delta t \pmod{2i\Omega} \\ &= \frac{1}{L}\int_0^L \xi_{\mu(t)}(z)\,\Delta t + \frac{1}{L}\int_0^L \xi_{\mu(t)}(w)\,\Delta t \pmod{2i\Omega} \\ &= \overline{\xi}_{B^c}(z) + \overline{\xi}_{B^c}(w) \pmod{2i\Omega}. \end{aligned}$$

The proof for $\ominus_{\mu(t)}$ follows similarly. □

Although according to Proposition 4.2, the function $\overline{\xi}_{B^c}$ appears to be operation preserving, it is important to realize that $z \oplus_{\mu(t)} w$ is not a point in $\mathbb{C}_{\mu(\mathbb{T})}$, but instead is a function of t. Nevertheless, if $\overline{\xi}^{-1}(\overline{\xi}(z) + \overline{\xi}(w) \pmod{2i\Omega})$ exists, there is a unique point in $\mathbb{C}_{\mu(\mathbb{T})}$ that maps to

$\overline{\xi}(z)+\overline{\xi}(w) \pmod{2i\Omega}$. This fact, along with Proposition 4.2, motivates us to call this point the *box plus sum of z and w*, denoted $z \boxplus_{\mathbb{T}} w$. We can similarly define the *box minus difference of z and w*, denoted $z \boxminus_{\mathbb{T}} w$.

Definition 4.3. Let $z, w \in \mathbb{C}_{\mu(\mathbb{T})}$ and suppose $\overline{\xi}^{-1}(\overline{\xi}(z)+\overline{\xi}(w) \pmod{2i\Omega})$ exists. We define the partial function $\boxplus_{\mathbb{T}} : \mathbb{C}_{\mu(\mathbb{T})} \times \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}_{\mu(\mathbb{T})}$ by

$$z \boxplus_{\mathbb{T}} w := \overline{\xi}^{-1}(\overline{\xi}(z)+\overline{\xi}(w) \pmod{2i\Omega}).$$

Similarly, if $z, w \in \mathbb{C}_{\mu(\mathbb{T})}$ and $\overline{\xi}^{-1}(\overline{\xi}(z)-\overline{\xi}(w) \pmod{2i\Omega})$ exists, then define the partial function $\boxminus_{\mathbb{T}} : \mathbb{C}_{\mu(\mathbb{T})} \times \mathbb{C}_{\mu(\mathbb{T})} \to \mathbb{C}_{\mu(\mathbb{T})}$ by

$$z \boxminus_{\mathbb{T}} w := \overline{\xi}^{-1}(\overline{\xi}(z)-\overline{\xi}(w) \pmod{2i\Omega}).$$

Finally, if $z \in \mathbb{C}_{\mu(\mathbb{T})}$ and $\overline{\xi}^{-1}(-\overline{\xi}(w))$ exists, define $\boxminus_{\mathbb{T}} z := 0 \boxminus_{\mathbb{T}} z$. Note the mod condition is not necessary in the last definition since $\overline{\xi}(w) \in \mathbb{C}^{\Omega}$ implies $-\overline{\xi}(w) \in \mathbb{C}^{\Omega}$.

Now, we have the following useful identities.

Proposition 4.4. *Let $z, w, u \in \mathbb{C}_{\mu(\mathbb{T})}$. Then*

1. $0 \boxplus_{\mathbb{T}} z = z$
2. *If $z \boxplus_{\mathbb{T}} w$, $w \boxplus_{\mathbb{T}} u$, $(z \boxplus_{\mathbb{T}} w) \boxplus_{\mathbb{T}} u$, and $z \boxplus_{\mathbb{T}} (w \boxplus_{\mathbb{T}} u)$ exist, then*
$$(z \boxplus_{\mathbb{T}} w) \boxplus_{\mathbb{T}} u = z \boxplus_{\mathbb{T}} (w \boxplus_{\mathbb{T}} u).$$
3. *If $z \boxplus_{\mathbb{T}} w$ exists, then $z \boxplus_{\mathbb{T}} w = w \boxplus_{\mathbb{T}} z$.*
4. *If $(\boxminus_{\mathbb{T}} z)$ exists, then $z \boxminus_{\mathbb{T}} z = z \boxplus_{\mathbb{T}} (\boxminus_{\mathbb{T}} z) = 0$.*
5. *If $(\boxminus_{\mathbb{T}} z)$ exists, then $\boxminus_{\mathbb{T}}(\boxminus_{\mathbb{T}} z) = z$.*
6. *If $z \boxminus_{\mathbb{T}} w$ and $w \boxminus_{\mathbb{T}} z$ exist, then $\boxminus_{\mathbb{T}}(z \boxminus_{\mathbb{T}} w) = w \boxminus_{\mathbb{T}} z$.*
7. *If $(\boxminus_{\mathbb{T}} z)$ and $(\boxminus_{\mathbb{T}} w)$ exist, then $\boxminus_{\mathbb{T}}(z \boxplus_{\mathbb{T}} w) = (\boxminus_{\mathbb{T}} z) \boxplus_{\mathbb{T}} (\boxminus_{\mathbb{T}} w)$.*
8. *If $z \in \Gamma_0/\mathbb{R}$, then $\boxminus_{\mathbb{T}} z$ exists and $\overline{z} = \boxminus_{\mathbb{T}} z$. Thus, if $z = \overset{\odot}{i}\omega$, then $\overset{\odot}{i}(-\omega) = \boxminus_{\mathbb{T}} \overset{\odot}{i}\omega$.*

We will prove (1), (2), (5), and (8). The proofs of (3), (4), (6) and (7) are similar to the proof of (2), in the sense the proof involves compositions and cancelations of $\overline{\xi}$ and $\overline{\xi}^{-1}$.

Proof. (1) $0 \boxplus_{\mathbb{T}} z = \overline{\xi}^{-1}(\overline{\xi}(0)+\overline{\xi}(z) \pmod{2i\Omega}) = \overline{\xi}^{-1}(0+\overline{\xi}(z)) = \overline{\xi}^{-1}(\overline{\xi}(z)) = z$.

(2) Since $z \boxplus_{\mathbb{T}} w$ exists, $\overline{\xi}(\overline{\xi}^{-1}(\overline{\xi}(z)+\overline{\xi}(w) \pmod{2i\Omega})) = \overline{\xi}(z)+\overline{\xi}(w) \pmod{2i\Omega}$. A similar expression holds for $w \boxplus_{\mathbb{T}} u$. Then

$$\begin{aligned}
(z \boxplus_{\mathbb{T}} w) \boxplus_{\mathbb{T}} u &= \overline{\xi}^{-1}(\overline{\xi}(\overline{\xi}^{-1}(\overline{\xi}(z)+\overline{\xi}(w) \pmod{2i\Omega})) + \overline{\xi}(u) \pmod{2i\Omega}) \\
&= \overline{\xi}^{-1}((\overline{\xi}(z)+\overline{\xi}(w) \pmod{2i\Omega}) + \overline{\xi}(u) \pmod{2i\Omega}) \\
&= \overline{\xi}^{-1}(\overline{\xi}(z) + (\overline{\xi}(w)+\overline{\xi}(u) \pmod{2i\Omega}) \pmod{2i\Omega}) \\
&= \overline{\xi}^{-1}(\overline{\xi}(z) + \overline{\xi}(\overline{\xi}^{-1}(\overline{\xi}(w)+\overline{\xi}(u) \pmod{2i\Omega})) \pmod{2i\Omega}) \\
&= z \boxplus_{\mathbb{T}} (w \boxplus_{\mathbb{T}} u).
\end{aligned}$$

(5) Consider

$$\boxminus_{\mathbb{T}}(\boxminus_{\mathbb{T}} z) = \boxminus_{\mathbb{T}}(\overline{\xi}^{-1}(-\overline{\xi}(z)) = \overline{\xi}^{-1}(-\overline{\xi}(\overline{\xi}^{-1}(-\overline{\xi}(z)))) = \overline{\xi}^{-1}(\overline{\xi}(z)) = z.$$

(8) Since $z \in \Gamma_0/\mathbb{R}$, there is a unique $\omega \in (-\Omega, \Omega]$ such that $z = \overline{\xi}^{-1}(i\omega) = \overset{\odot}{i}\omega$. Then

$$\boxminus_{\mathbb{T}} z = \boxminus_{\mathbb{T}} \overset{\odot}{i}\omega = \overline{\xi}^{-1}(-\overline{\xi}(\overline{\xi}^{-1}(i\omega))) = \overline{\xi}^{-1}(-i\omega) = \overset{\odot}{i}(-\omega) = \overline{z},$$

where we use Proposition 3.10 for the last equality. □

Thus, $\boxplus_{\mathbb{T}}$ provides a global analogue of the local action of $\oplus_{\mu(t)}$. The $\boxplus_{\mathbb{T}}$ operator maps $z, w \in \mathbb{C}_{\mu}$ to $\mathbb{C}_{\Omega}$, adds the ergodic growth rate of z to the ergodic growth rate of w and adds the ergodic frequency of z to the ergodic frequency of w (mod $2i\Omega$), and finally pulls back to the unique point (if it exists) in $\mathbb{C}_{\mu(\mathbb{T})}$ that would map to that new growth rate and frequency pair under $\overline{\xi}$.

The operations $\boxminus_{\mathbb{T}}$ and $\ominus_{\mu}$ also interact with time scale exponential functions in an interesting way. To show that, we need the following result.

Proposition 4.5. *Let $\mathbb{T}$ be a time scale and let $-\Omega < \omega \le \Omega$ such that $\overset{\odot}{\imath}\omega$ exists. When $\overset{\odot}{\imath}(-\omega) = \boxminus_{\mathbb{T}}(\overset{\odot}{\imath}\omega)$ exists, we have*

$$\overline{e_{\overset{\odot}{\imath}\omega}(t,t_0)} = e_{\boxminus\overset{\odot}{\imath}\omega}(t,t_0).$$

Proof. (1) This follows since $\overline{e_z(t,t_0)} = e_{\overline{z}}(t,t_0)$ for regressive z and since $\overset{\odot}{\imath}(-\omega) = \overline{\overset{\odot}{\imath}\omega} = \boxminus_{\mathbb{T}}\overset{\odot}{\imath}\omega$ by Propositions 3.10 and 4.4. □

We now can show the relationship between $\ominus_{\mu(t)}$ and $\boxminus_{\mathbb{T}}$ in the time scale exponential. Since $\ominus_{\mu(t)}$ is a *local* operation while $\boxplus_{\mathbb{T}}$ is a *global* operation, the following proposition provides a bridge between the local and the global behavior of the time scale exponential function.

Proposition 4.6. *Let $\mathbb{T}$ be a time scale and let $-\Omega < \omega \le \Omega$ such that $\overset{\odot}{\imath}\omega$ exists. When $\boxminus_{\mathbb{T}}(\overset{\odot}{\imath}\omega)$ exists,*

$$e_{\ominus_{\mu(t)}\overset{\odot}{\imath}\omega}(t,t_0) = e_{\boxminus_{\mathbb{T}}\overset{\odot}{\imath}\omega}(t,t_0)Q_\omega(t,t_0),$$

where $Q_\omega(t,t_0) := 1/|e_{\overset{\odot}{\imath}\omega}(t,t_0)|^2$.

Proof. By direct calculation,

$$e_{\ominus_{\mu(t)}\overset{\odot}{\imath}\omega}(t,t_0) = \frac{1}{e_{\overset{\odot}{\imath}\omega}(t,t_0)} = \frac{\overline{e_{\overset{\odot}{\imath}\omega}(t,t_0)}}{|e_{\overset{\odot}{\imath}\omega}(t,t_0)|^2} = e_{\boxminus_{\mathbb{T}}\overset{\odot}{\imath}\omega}(t,t_0)Q_\omega(t,t_0).$$

□

Notice in Proposition 4.6 that $\boxminus_{\mathbb{T}}\overset{\odot}{\imath}\omega$ is a number, while $\ominus_{\mu(t)}\overset{\odot}{\imath}\omega$ is a function of t. The effect that the time variation in $\ominus_{\mu(t)}\overset{\odot}{\imath}\omega$ has on the time scale exponential is embedded in $Q_\omega(t,t_0)$.

As explained in Remark 3.11, each point $z \in \mathbb{C}_{\mu(\mathbb{T})}$ has an ergodic growth rate and ergodic frequency. Using $\boxplus_{\mathbb{T}}$, we can explicitly write z in an analogue of the rectangular form, much like the Hilger decomposition.

Generalizing the Hilger real and imaginary parts in the spirit of Lemma 1.2, we have the following definition.

Definition 4.7. Let $z \in \mathbb{C}_{\mu(\mathbb{T})}$. We define the *ergodic real part* of z, denoted $\mathrm{Re}_{\mathbb{T}}(z)$, by

$$\mathrm{Re}_{\mathbb{T}}(z) := \overline{\xi}^{-1}(\mathrm{Re}(\overline{\xi}(z))),$$

and the *ergodic imaginary part* of z, denoted $\mathrm{Im}_{\mathbb{T}}(z)$, by

$$\mathrm{Im}_{\mathbb{T}}(z) := \mathrm{Im}(\overline{\xi}(z)).$$

Note that $\mathrm{Re}_{\mathbb{T}}(z)$ is the right-most point on the real axis along the contour $\gamma = \overline{\xi}(z)$ by virtue of having zero ergodic imaginary part. In particular, this point is to the right of $-1/\mu_{\max}$, which means $\overline{\xi}^{-1}(\mathrm{Re}(\overline{\xi}(z)))$ always exists. Also, the ergodic growth rate $\gamma(x,y)$ and ergodic real part $\mathrm{Re}_{\mathbb{T}}(x+iy)$ are related by

$$\overline{\xi}(\mathrm{Re}_{\mathbb{T}}(x+iy)) = \mathrm{Re}(\overline{\xi}(x+iy)) = \gamma(x,y).$$

We are now able to give a rectangular ergodic decomposition for $z \in \mathbb{C}_{\mu(\mathbb{T})}$ in terms of $\boxplus_{\mathbb{T}}$. See Figure 6.

Proposition 4.8. *Let $z \in \mathbb{C}_{\mu(\mathbb{T})}$. Then $z = \mathrm{Re}_{\mathbb{T}}(z) \boxplus_{\mathbb{T}} \overset{\odot}{\imath}\,\mathrm{Im}_{\mathbb{T}}(z)$.*

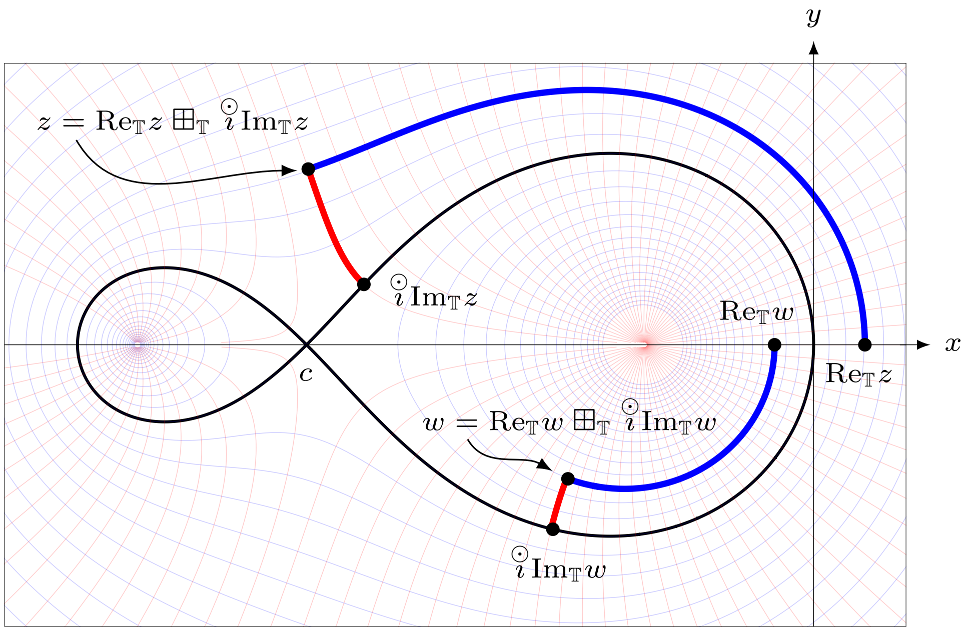

FIGURE 6. The ergodic decomposition $z = \operatorname{Re}_{\mathbb{T}} z \boxplus_{\mathbb{T}} \operatorname{Im}_{\mathbb{T}} z$. $\operatorname{Re}_{\mathbb{T}} z$ is found by projecting along the appropriate level curve of $\gamma(x,y)$ until it reaches the x-axis while $\operatorname{Im}_{\mathbb{T}} z$ is found by projecting along the appropriate level curve of $\omega(x,y)$ until it reaches Γ_0. If $z \in \Gamma_-$, then $\operatorname{Re}_{\mathbb{T}} z < 0$, while if $z \in \Gamma_+$, then $\operatorname{Re}_{\mathbb{T}} z > 0$. If $\operatorname{Im} z > 0$, then $\operatorname{Im}_{\mathbb{T}} z > 0$, while if $\operatorname{Im} z < 0$, then $\operatorname{Im}_{\mathbb{T}} z < 0$. Care must be taken at self-crossings such as c, where our definition chooses $\operatorname{Im}_{\mathbb{T}} c > 0$.

Proof.

$$
\begin{aligned}
\operatorname{Re}_{\mathbb{T}}(z) \boxplus_{\mathbb{T}} \overset{\odot}{i} \operatorname{Im}_{\mathbb{T}}(z) &= \overline{\xi}^{-1}(\overline{\xi}(\operatorname{Re}_{\mathbb{T}}(z)) + \overline{\xi}(\overset{\odot}{i} \operatorname{Im}_{\mathbb{T}}(z)) \pmod{2i\Omega}) \\
&= \overline{\xi}^{-1}(\operatorname{Re}(\overline{\xi}(z)) + i \operatorname{Im}_{\mathbb{T}}(z) \pmod{2i\Omega}) \\
&= \overline{\xi}^{-1}(\operatorname{Re}(\overline{\xi}(z)) + i \operatorname{Im}(\overline{\xi}(z)) \pmod{2i\Omega}) \\
&= \overline{\xi}^{-1}(\overline{\xi}(z)) = z.
\end{aligned}
$$

□

Example 4.9. *Again, consider $\mathbb{T}_{1,2}$. The rectangular ergodic decomposition exists even for $\alpha_1 = -3/4$. Indeed, by the calculations in Example 4.1, $\operatorname{Re}_{\mathbb{T}_{1,2}}(z) = \overline{\xi}^{-1}(-\ln 2) \approx -0.396447$ and $\operatorname{Im}_{\mathbb{T}_{1,2}}(z) = \frac{\pi}{3}$. Thus,*

$$
\alpha_1 = \overline{\xi}^{-1}(-\ln(2)) \boxplus_{\mathbb{T}_{1,2}} \left(\overset{\odot}{i} \frac{\pi}{3}\right) \approx (-0.396447) \boxplus_{\mathbb{T}_{1,2}} (-0.75 + 0.661438i).
$$

Remark 4.10. We can recover a group structure for the box plus operator by extending $\overline{\xi}$ so its codomain is all of $\mathbb{C}_{\mu(\mathbb{T})}$ by changing the domain so that a point is also associated with its frequency. Consider the set $\mathbb{C}^g_{\mu(\mathbb{T})} \subset \mathbb{C}_{\mu(\mathbb{T})} \times (-\Omega, \Omega]$ with the condition that $(z,\omega) \in \mathbb{C}^g_{\mu(\mathbb{T})}$ if and only if $\operatorname{Im}(\overline{\xi}(z)) = \omega$ when z is not a saddle point of $\overline{\xi}$ and $\operatorname{Im}(\overline{\xi}(z)) = \pm\omega$ when z is a saddle point of $\overline{\xi}$. Then define $\overline{\xi}^g : \mathbb{C}^g_{\mu(\mathbb{T})} \to \mathbb{C}^{\Omega}$ by

$$
\overline{\xi}^g(z,\omega) = \begin{cases} \gamma + i\omega, & \overline{\xi}(z) = \gamma + i\omega, \\ \gamma + i\omega, & \overline{\xi}(z) = \gamma - i\omega. \end{cases}
$$

Then, for a saddle point α_k of $\overline{\xi}_B$ satisfying $\overline{\xi}_B(\alpha_k) = \gamma + i\omega$,

$$(\overline{\xi}^g)^{-1}(\gamma - i\omega) = (\alpha_k, -\omega),$$

and for all other points $(\overline{\xi}^g)^{-1}$ agrees with $\overline{\xi}^{-1}$.

With this, define $\boxplus^g_{\mathbb{T}} : \mathbb{C}^g_{\mu(\mathbb{T})} \times \mathbb{C}^g_{\mu(\mathbb{T})} \to \mathbb{C}^\Omega$ by

$$(z_1, \omega_1) \boxplus^g_{\mathbb{T}} (z_2, \omega_2) := (\overline{\xi}^g)^{-1}(\overline{\xi}^g(z_1, \omega_1) + \overline{\xi}^g(z_2, \omega_2)),$$

where the addition is mod $2i\Omega$ in the first argument of $\overline{\xi}^g$ and is mod 2Ω in the second argument of $\overline{\xi}^g$. Then $(\mathbb{C}^g_{\mu(\mathbb{T})}, \boxplus^g_{\mathbb{T}})$ is a group with identity $(0, 0)$ and with inverse of (z, ω) being $((\overline{\xi}^g)^{-1}(-\overline{\xi}^g(z)), -\omega)$.

Remark 4.11. For an easier group structure involving box plus, we can generalize the observation that the Hilger circle with the circle plus, $(\mathcal{H}_{\mu(t)}, \oplus_{\mu(t)})$, is a subgroup of $(\mathbb{C}_{\mu(t)}, \oplus_{\mu(t)})$ and is isomorphic to $((-\pi/\mu(t), \pi/\mu(t)], + \pmod{2\pi/\mu(t)})$. Notice that as long as Γ_0 does not have any self-intersections, $\overline{\xi} : \Gamma_0 \to \overline{\xi}(\Gamma_0)$ is a bijection. In this case, $(\Gamma_0, \boxplus_{\mathbb{T}})$ is a group which is isomorphic to $((-\Omega, \Omega], + \pmod{2\Omega})$.

Remark 4.12. We can extend the range of ω from the Nyquist range $(-\Omega, \Omega]$ to $\mathbb{R}$. Fix the positive principal branch of $\overline{\xi}^{-1}$ on the Nyquist segment $i(-\Omega, \Omega] \setminus \{i\omega_-\}$, where $i\omega_-$ is the parameter corresponding to a self-intersection point of Γ_0. Analytic continuation along the imaginary axis lifts this branch to the Riemann surface $\mathcal{R}_{\overline{\xi}^{-1}}$, giving a single-valued map

$$\overline{\xi}^{-1}_{\text{lift}} : i\big(\mathbb{R} \setminus \Lambda\big) \to \mathcal{C}, \quad \Lambda := \{\omega_- + 2k\Omega : k \in \mathbb{Z}\},$$

which satisfies $\overline{\xi}^{-1}_{\text{lift}}\big(i(\omega + 2\Omega)\big) = \overline{\xi}^{-1}_{\text{lift}}(i\omega)$ for every $\omega \in \mathbb{R} \setminus \Lambda$. Projecting to the plane extends $\overline{\xi}^{-1}$ to the punctured imaginary axis while identifying all inputs that differ by 2Ω. Let $Z = \operatorname{im} \overline{\xi}^{-1}_{\text{lift}}$. Because the branch on $i(-\Omega, \Omega] \setminus \{i\omega_-\}$ is injective, every $z \in Z$ has a unique representative $\omega \in (-\Omega, \Omega] \setminus \{\omega_-\}$. We define

$$z_1 \boxplus z_2 := \overline{\xi}^{-1}_{\text{lift}}\big(i(\omega_1 + \omega_2)\big), \quad \text{whenever } \omega_1 + \omega_2 \notin \Lambda.$$

Hence $\overline{\xi}^{-1}_{\text{lift}}$ is an isomorphism onto $(Z, \boxplus)$. On the punctured circle this provides a global operation; on all of $\mathbb{R} \setminus 2\Omega\mathbb{Z}$, it is only partial because inputs whose sum is in Λ are excluded. However, a construction similar to that in Remark 4.10 will create a global group isomorphism.

5. Conclusion

In this paper we introduced the ergodic complex plane as a global analogue of Hilger's complex plane on time scales. By averaging the cylinder transformation, we defined the ergodic growth rate and ergodic frequency, and showed that together they generate the ergodic cylinder transformation, an analytic, globally univalent map that yields an orthogonal curvilinear coordinate system on the regressive complex plane. This framework provides an ergodic decomposition analogous to Hilger's as well as the global box plus operation as an extension of the local circle plus operation. These results unify local and global perspectives on growth and frequency, and open the way for further developments in harmonic and spectral analysis on general time scales.

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