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MULTIPLE SOLUTIONS FOR KIRCHHOFF TYPE SYSTEMS INVOLVING SINGULAR AND CRITICAL NONLINEARITIES

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Abstract. This article investigates the fractional singular Kirchhoff system

$$\begin{gathered} m(\mathcal{N}(\phi,\psi))\mathcal{L}^p_{\mathcal{K}}(\phi) = \lambda a(x)\phi^{-\gamma_1} + \frac{\theta_1}{p^\star_{N,s}}\phi^{\theta_1-1}\psi^{\theta_2} \quad \text{in } \mathcal{D} \\ m(\mathcal{N}(\phi,\psi))\mathcal{L}^p_{\mathcal{K}}(\psi) = \lambda b(x)\psi^{-\gamma_2} + \frac{\theta_2}{p^\star_{N,s}}\phi^{\theta_1}\psi^{\theta_2-1} \quad \text{in } \mathcal{D} \\ \phi > 0, \quad \psi > 0 \quad \text{in } \mathcal{D} \\ \phi = \psi = 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{D}, \end{gathered}$$

where

$$\mathcal{N}(\phi,\psi) = \int_{\mathbb{R}^{2N}} |\phi(x)-\phi(y)|^p \mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^{2N}} |\psi(x)-\psi(y)|^p \mathcal{K}(x,y)\,dx\,dy.$$

Here, $\mathcal{D}$ is a bounded domain in $\mathbb{R}^N$ with a Lipschitz boundary $\partial\mathcal{D}$. $\mathcal{L}^p_{\mathcal{K}}$ is a non-local operator with a singular kernel $\mathcal{K}$. $p > 1$, $\lambda > 0$, and m is a continuous function. $\gamma_1, \gamma_2 \in (0,1)$. and a, b are non-negative bounded functions. $\theta_1, \theta_2 > 1$ and $\theta_1 + \theta_2 = p^\star_{N_s}$, where $p^\star_{N_s}$ is the fractional critical Sobolev exponent $p^\star_{N_s} = \frac{Np}{N-sp}$. Our findings encompass the degenerate case in the fractional setting, allowing the Kirchhoff function m to take zero value at zero. We employ Kajikiya's version of the symmetric mountain pass lemma to prove the existence of a sequence of infinitely many small solutions with negative energy that converge to zero.

1. Introduction and main result

Recently, the study of non-local equations and systems has attracted a lot of attention. Various studies have been considered on this topic, see [5, 6, 9, 10, 12, 15, 20, 32, 31, 33, 35, 36, 37, 38]. Nonlocal equations have been widely used in many fields of sciences, including continuum mechanics, phase transition phenomena, population dynamics, and game theory, especially those involving fractional and nonlocal elliptic operators. As discussed in [1, 8], these operators naturally appear as stochastic stabilizers of Lévy processes..

On the other hand, a great deal of attention has been focused on nonlocal fractional equations with critical nonlinearities. This area of study has received significant attention in recent years leading to several advancements and investigations. Among the references we like to mention [2, 3, 5, 13, 16, 22, 24, 28, 30, 37]. These papers provide valuable insights and analysis of nonlocal fractional equations with critical nonlinearities, and they serve as important references for further exploration in this area.

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In this work, we study the existence of solutions for a fractional Kirchhoff system that involve singular and critical nonlinearities. The system studied is

$$\begin{gathered} m(\mathcal{N}(\phi,\psi))\mathcal{L}^p_{\mathcal{K}}(\phi) = \lambda a(x)\phi^{-\gamma_1} + \frac{\theta_1}{p^\star_{N,s}}\phi^{\theta_1-1}\psi^{\theta_2} \quad \text{in } \mathcal{D} \\ m(\mathcal{N}(\phi,\psi))\mathcal{L}^p_{\mathcal{K}}(\psi) = \lambda b(x)\psi^{-\gamma_2} + \frac{\theta_2}{p^\star_{N,s}}\phi^{\theta_1}\psi^{\theta_2-1} \quad \text{in } \mathcal{D} \\ \phi > 0, \quad \psi > 0 \quad \text{in } \mathcal{D} \\ \phi = \psi = 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{D}, \end{gathered} \tag{1.1}$$

where

$$\mathcal{N}(\phi,\psi) = \int_{\mathbb{R}^{2N}} |\phi(x)-\phi(y)|^p \mathcal{K}(x,y)\,dx\,dy + \int_{\mathbb{R}^{2N}} |\psi(x)-\psi(y)|^p \mathcal{K}(x,y)\,dx\,dy,$$

$\mathcal{D}$ is a smooth bounded domain in $\mathbb{R}^N$, $\lambda > 0$ is a real parameter, $p > 1$, $\gamma_1, \gamma_2 \in (0,1)$, and $\theta_1, \theta_2 > 1$ satisfy $\theta_1 + \theta_2 = p^\star_{N,s}$. The fractional critical Sobolev exponent $p^\star_{N,s}$ is defined as $p^\star_{N,s} = \frac{Np}{N-sp}$. $a, b \in L^\infty(\mathcal{D})$, $a > 0$, $b > 0$ a.e. in $\mathcal{D}$. $\mathcal{L}^p_{\mathcal{K}}$ is a non-local operator defined for any smooth functions $\phi : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{L}^p_{\mathcal{K}}\phi(x) = 2\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} |\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))\mathcal{K}(x,y)\,dy, \quad x \in \mathbb{R}^N.$$

Here, $\mathcal{K} : \mathbb{R}^{2N} \setminus \{0,0\} \to (0,+\infty)$ is a measurable function satisfying the following properties:

$$\begin{gathered} \mathcal{K}(x,y) \ge \mathcal{K}_0|x-y|^{-(N+sp)} \quad \text{for all } (x,y) \in \mathbb{R}^{2N},\ x \ne y \\ \sigma\mathcal{K} \in L^1(\mathbb{R}^{2N}) \quad \text{with } \sigma(x,y) = \min\{1, |x-y|^p\} \\ \mathcal{K}(x,y) = \mathcal{K}(y,x) \quad \text{for all } (x,y) \in \mathbb{R}^{2N}. \end{gathered}$$

A commonly used model for the kernel function $\mathcal{K}$ is given by the singular kernel $\mathcal{K}(x,y) = |x-y|^{-(sp+N)}$. In this case, the operator $\mathcal{L}^p_{\mathcal{K}}$ corresponds to the fractional p-Laplacian, which can be defined (up to normalization factors) for any $\phi \in C^\infty_0(\mathbb{R}^N)$ by:

$$(-\Delta_p)^s\phi(x) = 2\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} \frac{|\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))}{|x-y|^{sp+N}}\,dy, \quad x \in \mathbb{R}^N.$$

The Kirchhoff function m is assumed to satisfies the following assumptions:

- (A1) For each $\delta > 0$, there exists $\kappa_\delta > 0$ such that $m(z) \ge \kappa_\delta$ for all $z \ge \delta$.
- (A2) There exists $\sigma \in [1, \frac{p^\star_{N,s}}{p})$ such that

$$\sigma\mathscr{M}(z) = \sigma\int_0^z m(\varsigma)d\varsigma \ge zm(z), \quad \forall \xi \in \mathbb{R}_+.$$

- (A3) There exists $m_0 > 0$ such that $m(z) \ge m_0 z^{\sigma-1}$ for all $\xi \in [0,1]$.

A prototype for m, proposed by Kirchhoff, is

$$m(z) = \alpha + \beta z^{\sigma-1}, \quad \alpha, \beta \ge 0, \quad \alpha + \beta > 0, \quad \sigma > 1. \tag{1.2}$$

The Kirchhoff problem (1.1) is classified as non-degenerate if $m(z) \ge \overline{m} > 0$ for all $z \in \mathbb{R}_+$. A non-degenerate case can be achieved, for example, when $\alpha > 0$ and $\beta \ge 0$ in the model case (1.2). For recent results on non-degenerate Kirchhoff- type problems, we refer the reader to [22, 25, 26, 27, 30]. On the other hand, if $m(0) = 0$ but $m(z) > 0$ for all $z > 0$, the Kirchhoff problem is considered degenerate. This degenerate scenario occurs in the model case (1.2) when $\alpha = 0$ and $\beta > 0$. Relevant research papers on degenerate Kirchhoff-type problems include [4, 11, 13, 23, 31].

Fiscella and Valdinoci [18] presented a detailed examination of the physical understanding of fractional Kirchhoff problems and their applications. They considered a stationary Kirchhoff problem that capture the non-local component of tension caused by nonlocal measurements of a string's fractional length. In this case, the function m measures the change in tension on the string produced by differences in its length during vibration. The fact that $m(0) = 0$ implies that the string's base tension is zero, indicating a realistic model.

The aim of this article is to investigate the existence of solutions for system (1.1) by using variational methods and critical point theory. Before presenting our main result, we introduce the following notation and functional setting. Let us define the space

$$E = \Big\{\phi \in L^p(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^{2N}} |\phi(x) - \phi(y)|^p \mathcal{K}(x, y)\, dx\, dy < \infty\Big\},$$

equipped with the norm

$$\|\phi\|_E = \Big(\|\phi\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^{2N}} |\phi(x) - \phi(y)|^p \mathcal{K}(x, y)\, dx\, dy\Big)^{1/p}.$$

We also introduce the space

$$E_0(\mathcal{D}) = \Big\{\phi \in E : \phi = 0 \text{ a.e. } \mathbb{R}^N \setminus \mathcal{D}\Big\},$$

where $\mathcal{D}$ is a given domain. According to [17, Lemma 4], $E_0(\mathcal{D})$ is a separable and reflexive Banach space, which can be equipped with the norm

$$\|u\|_{E_0} = \Big(\int_{\mathbb{R}^{2N}} |\phi(x) - \phi(y)|^p \mathcal{K}(x, y)\, dx\, dy\Big)^{1/p}, \quad \forall \phi \in E_0(\mathcal{D}).$$

Given that $\mathcal{D}$ is a smooth bounded domain, it is well-known that the embedding $E_0(\mathcal{D}) \hookrightarrow L^\mu(\mathcal{D})$ holds continuously for $\mu \in [1, p^\star_{N,s}]$ (refer to [14], Lemma 2.3) and compactly for $\mu \in [1, p^\star_{N,s})$, where $p^\star_{N,s} = \frac{Np}{N-sp}$. Additionally, there exists a positive constant C_μ such that for any $\phi \in E_0(\mathcal{D})$, the following inequality holds:

$$\|\phi\|_{L^\mu(\Omega)} \le C_\mu \|\phi\|_{E_0}. \tag{1.3}$$

In what follows, $S_\star$ stands for the optimal constant for the Sobolev embedding $E_0(\mathcal{D}) \hookrightarrow L^{p^\star_{N,s}}(\mathcal{D})$, which is given by

$$S_\star = \inf_{\phi \in E_0(\mathcal{D})\setminus\{0\}} \frac{\big(\int_{\mathbb{R}^{2N}} |\phi(x) - \phi(y)|^p \mathcal{K}(x, y)\, dx\, dy\big)^{1/p}}{\big(\int_{\mathcal{D}} |\phi(x)|^{p^\star_{N,s}}\, dx\big)^{1/p^\star_{N,s}}}. \tag{1.4}$$

Let $X = E_0(\mathcal{D}) \times E_0(\mathcal{D})$, equipped with the usual norm

$$\|(\phi, \psi)\| = \Big(\|\phi\|^p_{E_0} + \|\psi\|^p_{E_0}\Big)^{1/p}, \quad \text{for } (\phi, \psi) \in X.$$

Definition 1.1. We say that $(\phi, \psi) \in X$ is a weak solution of system (1.1) if

$$\begin{aligned}
& m(\mathcal{N}(\phi, \psi)) \int_{\mathbb{R}^{2N}} |\phi(x) - \phi(y)|^{p-2}(\phi(x) - \phi(y))(\varphi_1(x) - \varphi_1(y))\mathcal{K}(x, y)\, dx\, dy \\
& + m(\mathcal{N}(\phi, \psi)) \int_{\mathbb{R}^{2N}} |\psi(x) - \psi(y)|^{p-2}(\psi(x) - \psi(y))(\varphi_2(x) - \varphi_2(y))\mathcal{K}(x, y)\, dx\, dy \\
& = \lambda \int_{\mathcal{D}} a(x)\phi^{-\gamma_1}\varphi_1\, dx + \int_{\mathcal{D}} b(x)\psi^{-\gamma_2}\varphi_2\, dx \\
& + \frac{\theta_1}{p^\star_{N,s}} \int_{\mathcal{D}} \phi^{\theta_1 - 1}\psi^{\theta_2}\varphi_1\, dx + \frac{\theta_2}{p^\star_{N,s}} \int_{\mathcal{D}} \phi^{\theta_1}\psi^{\theta_2 - 1}\varphi_2\, dx, \quad \forall (\varphi_1, \varphi_2) \in X.
\end{aligned} \tag{1.5}$$

To obtain the existence of weak solutions for system (1.1), we seek critical points of the associated energy functional $\mathcal{J}_\lambda : X \to \mathbb{R}$ given by

$$\mathcal{J}_\lambda(\phi, \psi) = \frac{1}{p}\mathscr{M}(\mathcal{N}(\phi, \psi)) - \lambda\Phi(\phi, \psi) - \Psi(\phi, \psi),$$

where

$$\Phi(\phi, \psi) = \frac{1}{1-\gamma_1}\int_{\mathcal{D}} a(x)(\phi^+)^{1-\gamma_1}\, dx + \frac{1}{1-\gamma_2}\int_{\mathcal{D}} b(x)(\psi^+)^{1-\gamma_2}\, dx,$$

and

$$\Psi(\phi,\psi)=\frac{1}{p^{\star}_{N,s}}\int_{\mathcal{D}}(\phi^{+})^{\theta_1}(\psi^{+})^{\theta_2}\,dx.$$

Our main result can be summarized as follows.

Theorem 1.2. *Under assumptions* (A1)–(A3) *there is a positive constant* $\lambda_0>0$ *such that for any* $\lambda\in(0,\lambda_0)$*, system* (1.1) *possesses a sequence of weak solutions* $(\phi_k,\psi_k)_{k\in\mathbb{N}}\subset X$ *such that* $\mathcal{J}_\lambda(\phi_k,\psi_k)<0$ *and* $\|(\phi_k,\psi_k)\|\to 0$ *as* $k\to+\infty$.

The remainder of this paper is organized as follows: In section 2, we present some preliminary results. In section 3, we show the existence of infinitely many small weak solutions for an associated approximating problem. In section 4, we prove our main result.

2. Preliminaries

In this section, we give some important results that will be useful in the proof of our main result.

Lemma 2.1. *For each* $(\phi,\psi)\in X$*, there exists* $S>0$ *such that*

$$\|(\phi,\psi)\|\geq S\int_{\mathcal{D}}|\phi(x)|^{\theta_1}|\psi(x)|^{\theta_2}\,dx. \tag{2.1}$$

Proof. Let $(\phi,\psi)\in X$. By using Hölder's inequality, one has

$$\begin{aligned}\int_{\mathcal{D}}|\phi(x)|^{\theta_1}|\psi(x)|^{\theta_2}\,dx&\leq\Big(\int_{\mathcal{D}}|\phi(x)|^{p^{\star}_{N,s}}\,dx\Big)^{\frac{\theta_1}{p^{\star}_{N,s}}}\Big(\int_{\mathcal{D}}|\psi(x)|^{p^{\star}_{N,s}}\,dx\Big)^{\frac{\theta_2}{p^{\star}_{N,s}}}\\&\leq S_\star^{-(\theta_1+\theta_2)}\|\phi\|_{E_0}^{\theta_1}\|\psi\|_{E_0}^{\theta_2}\\&\leq\frac{1}{S}\|(\phi,\psi)\|^{p^{\star}_{N,s}},\end{aligned}$$

where $S=S^{\theta_1+\theta_2}$. □

In the following Lemma, we establishes a key result related to the convergence of a sequence in $E_0(\mathcal{D})$ to a limit within the same space.

Lemma 2.2. *Let* $(\phi_n)_{n\in\mathbb{N}}\subset E_0(\mathcal{D})$ *be such* $\phi_n\to\phi$ *weakly in* $E_0(\mathcal{D})$ *for some* $\phi\in E_0(\mathcal{D})$*. Then, we have*

$$\|\phi_n-\phi\|^p_{E_0}+\|\phi\|^p_{E_0}=\|\phi_n\|^p_{E_0}+o_n(1).$$

Proof. We use a result of Brézis-Lieb's lemma [7]. It states that if we have a bounded sequence $(\Phi_n)_{n\in\mathbb{N}}$ in the space $L^p(\mathbb{R}^d)$, where $p>1$, and Φ_n converges to Φ almost everywhere in $\mathbb{R}^d$, then we have

$$|\Phi_n-\Phi|^p_{L^p(\mathbb{R}^d)}+|\Phi|^p_{L^p(\mathbb{R}^d)}=|\Phi_n|^p_{L^p(\mathbb{R}^d)}+o_n(1).$$

We define the function

$$\Phi_n=(\phi_n(x)-\phi_n(y))\sqrt[p]{\mathcal{K}(x,y)},\quad\text{where } d=2N.$$

Using this definition, we can write

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}|(\phi_n-\phi)(x)-(\phi_n-\phi)(y)|^p\mathcal{K}(x,y)\,dx\,dy+\int_{\mathbb{R}^{2N}}|\phi(x)-\phi(y)|^p\mathcal{K}(x,y)\,dx\,dy\\&=\int_{\mathbb{R}^{2N}}|\phi_n(x)-\phi_n(y)|^p\mathcal{K}(x,y)\,dx\,dy+o_n(1).\end{aligned}$$

This completes the proof. □

Lemma 2.3. *Let* $\alpha,\beta\in(1,+\infty)$*. Given* $\epsilon>0$*, there exists* $C>0$ *such that for each* ϕ_1,ϕ_2,v_1,v_2*, we have*

$$\Big||\phi_1+\psi_1|^\alpha|\phi_2+\psi_2|^\beta-|\phi_1|^\alpha|\phi_2|^\beta\Big|\leq\epsilon|\phi_1|^\alpha|\phi_2|^\beta+C(|\phi_1|^\alpha|\psi_2|^\beta+|\psi_1|^\alpha|\phi_2|^\beta+|\psi_1|^\alpha|\psi_2|^\beta).$$

Proof. Fix $0 < \eta < \min\{1, \frac{\epsilon}{1+2^\alpha}\}$. Then there exists $\overline{C} > 0$ such that

$$\begin{aligned}
&\Big||\phi_1 + \psi_1|^\alpha |\phi_2 + \psi_2|^\beta - |\phi_1|^\alpha |\phi_2|^\beta\Big| \\
&\leq |\phi_1 + \psi_1|^\alpha \Big||\phi_2 + \psi_2|^\beta - |\phi_2|^\beta\Big| + \Big||\phi_1 + \psi_1|^\alpha - |\phi_1|^\alpha |\phi_2|^\beta\Big| \\
&\leq 2^\alpha (|\phi_1|^\alpha + |\psi_1|^\alpha)(\eta|\phi_2|^\beta + \overline{C}|\psi_2|^\beta) + (\eta|\phi_1|^\alpha + \overline{C}|\psi_1|^\alpha)|\phi_2|^\beta, \\
&\leq \epsilon|\phi_1|^\alpha|\phi_2|^\beta + 2^\alpha\overline{C}|\phi_1|^\alpha|\psi_2|^\beta + (2^\alpha + \overline{C})\psi_1|^\alpha|\phi_2|^\beta + 2^\alpha\overline{C}\psi_1|^\alpha|\psi_2|^\beta,
\end{aligned}$$

as claimed. □

The following lemma presents a key result regarding the convergence properties of a sequence (ϕ_n, ψ_n) in the space X, under weak convergence and almost everywhere convergence in $\mathbb{R}^N$.

Lemma 2.4. *Let $(\phi_n, v_n) \rightharpoonup (\phi, v)$ weakly in X and $(\phi_n, v_n) \to (\phi, v)$ a.e. in $\mathbb{R}^N$. Then, for each $\alpha, \beta > 1$ with $\alpha + \beta \leq p^\star_{N,s}$, up to a subsequence, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\phi_n(x) - \phi(x)|^\alpha |\psi_n(x) - \psi(x)|^\beta \,dx + \int_{\mathbb{R}^N} |\phi(x)|^\alpha |\psi(x)|^\beta \,dx \\
&= \int_{\mathbb{R}^N} |\phi_n(x)|^\alpha |\psi_n(x)|^\beta \,dx + o_n(1).
\end{aligned}$$

Proof. Passing to a subsequence, we have that

$$\begin{gathered}
\phi_n \to \phi,\ \psi_n \to \psi, \quad \text{a.e. in } \Omega, \\
\phi_n \to \phi \quad \text{in} L^\alpha_{\rm loc}(\mathbb{R}^N), \quad \text{and} \quad \psi_n \to \psi \quad \text{in } L^\alpha_{\rm loc}(\mathbb{R}^N).
\end{gathered}$$

Let $\epsilon > 0$, and let $C > 0$ be a fixed real number as in Lemma 2.3. We define

$$\begin{aligned}
W_n = &\Big||\phi_n|^\alpha|\psi_n|^\beta - |\phi_n - \phi|^\alpha|\psi_n - \psi|^\beta - |\phi|^\alpha|\psi|^\beta\Big| - \epsilon|\phi_n - \phi|^\alpha|\psi_n - \psi|^\beta \\
&- C(|\phi|^\alpha|\psi_n - \psi|^\beta + |\phi_n - \phi|^\alpha|\psi|^\beta).
\end{aligned} \tag{2.2}$$

It is easy to see that, $w_n \to 0$ a.e. in $\mathbb{R}^N$ and $w_n \in L^1(\mathbb{R}^N)$. Furthermore, utilizing Lemma 2.3 with $\phi_1 = \phi_n - \phi$, $\phi_2 = \psi_n - \psi$, $\psi_1 = \phi$, and $\psi_2 = \psi$, we obtain

$$W_n \leq (C+1)|\phi|^\alpha|\psi|^\beta.$$

Then, we apply the dominated convergence theorem to get

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} W_n^+ \,dx = 0, \tag{2.3}$$

where $W_n^+ = \max\{W_n, 0\}$. Fix $T > 0$ large enough so that

$$C\int_{|x|\geq T} |\phi|^\alpha|\psi_n - \psi|^\beta dx \leq C|\psi_n - \psi|^\beta_{L^{p^\star_s}(\mathbb{R}^N)} \Big(\int_{|x|\geq T} |\phi|^{p^\star_s}\Big)^{\frac{\alpha}{p^\star_s}} < \epsilon, \tag{2.4}$$

and

$$C\int_{|x|\geq T} |\phi_n - \phi|^\alpha|\psi|^\beta dx \leq C|\phi_n - \phi|^\alpha_{L^{p^\star_s}(\mathbb{R}^N)} \Big(\int_{|x|\geq T} |\psi|^{p^\star_s}\Big)^{\frac{\beta}{p^\star_s}} < \epsilon. \tag{2.5}$$

Then, for n large enough, we have

$$C\int_{|x|\leq T} |\phi|^\alpha|\psi_n - \psi|^\beta dx \leq C\max_{|x|\leq T} |\phi(x)|^\alpha \int_{|x|\leq T} |\psi_n - \psi|^\beta dx < \epsilon, \tag{2.6}$$

and

$$C\int_{|x|\leq T} |\phi_n - \phi|^\alpha|v|^\beta dx \leq C\max_{|x|\leq T} |v(x)|^\beta \int_{|x|\leq T} |\phi_n - \phi|^\alpha dx < \epsilon. \tag{2.7}$$

Now, combining equations (2.2)-(2.7), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big||\phi_n|^\alpha|v_n|^\beta - |\phi_n - \phi|^\alpha|v_n - v|^\beta - |\phi|^\alpha|v|^\beta\Big| \,dx \\
&\leq \epsilon\int_{\mathbb{R}^N} |\phi_n - \phi|^\alpha|v_n - v|^\beta \,dx + C\int_{\mathbb{R}^N} |\phi|^\alpha|v_n - v|^\beta + |\phi_n - \phi|^\alpha|v|^\beta \,dx + \int_{\mathbb{R}^N} W_n^+ \,dx,
\end{aligned}$$

$$< \tilde{C}\epsilon,$$

for n large enough, as claimed. □

Now, we recall some of topological techniques introduced by Krasnoselskii in [34]. These methods have been widely recognized as powerful tools to obtain solutions to a variety of mathematical problems. Specifically, in our case, we aim to prove the existence of a sequence of small solutions. Let $(X, |\cdot|_X)$ be a real Banach space. We introduce the class Σ, which consists of all closed subsets $A \subset X \setminus 0$ that exhibit symmetry with respect to the origin. In other words, if $u \in A$, then $-u \in A$. This symmetry property is a key characteristic of the sets in Σ.

Now, to characterize the properties and complexity of sets in Σ, we introduce the notion of Krasnoselskii's genus.

Definition 2.5. Let $A \in \Sigma$. The Krasnoselskii's genus $\gamma(A)$ of A is defined as the smallest positive integer n such that there exists an odd mapping $h \in C(A, \mathbb{R}^n)$ satisfying $h(x) \neq 0$ for all $x \in A$. If such an integer n does not exist, we set $\gamma(A) = +\infty$. Furthermore, we define $\gamma(\emptyset) = 0$.

Next, we will outline the essential properties of the genus that will be employed throughout this work. For more comprehensive information on this topic, readers are encouraged to consult the reference [34].

Proposition 2.6. *Let A and B be symmetric closed subsets of E that do not include the origin. The following properties hold*

1. *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leq \gamma(B)$.*
2. *If there is an odd homeomorphism from A onto B, then $\gamma(A) = \gamma(B)$.*
3. *If $\gamma(A) < \infty$, then $\gamma(A \setminus B) = \gamma(A) - \gamma(B)$.*
4. *The n-dimensional sphere $\mathbb{S}^n$ has a genus of $(n+1)$ as a consequence of the Borsuk-Ulam Theorem.*

Now we present a variant of the symmetric mountain pass lemma, originally formulated by Kajikiya [21]. This lemma offers a powerful tool for analysing critical points of functionals with symmetric features.

Lemma 2.7 (Kajikiya's Variant of the Symmetric Mountain Pass Lemma)**.** *Let X be an infinite-dimensional Banach space, and let $\mathcal{J} \in C^1(X, \mathbb{R})$ be a functional satisfying the following conditions*

1. *$\mathcal{J}$ is even, bounded from below, and $\mathcal{J}(0) = 0$.*
2. *$\mathcal{J}$ satisfies the local Palais-Smale condition, which means that for some $\overline{c} \in \mathbb{R}$, any sequence $(\phi_n) \subset X$ satisfying $\mathcal{J}(\phi_n) \to c < \overline{c}$ and $\mathcal{J}'(\phi_n) \to 0$ in X' has a convergent subsequence.*
3. *For each $n \in \mathbb{N}$, there exists a set $A_n \subset \Gamma_n$ such that $\sup_{\phi \in A_n} \mathcal{J}(\phi) < 0$.*

Then, there exists a sequence (ϕ_n) such that $\mathcal{J}'(\phi_n) = 0$, $\mathcal{J}(\phi_n) < 0$, and $\phi_n \to 0$ in X.

3. Auxiliary problem

The classic variational theory cannot be used to the energy functional $\mathcal{J}_\lambda$ since it is not Fréchet differentiable due to the singular term. So, to prove our main result, we introduce the perturbed problem

$$\begin{gathered}
m(\mathcal{N}(\phi,\psi))\mathcal{L}^p_{\mathcal{K}}(\phi) = \lambda a(x)(\phi^+ + \varepsilon)^{-\gamma_1} + \frac{\theta_1}{p^\star_{N,s}}(\phi^+)^{\theta_1 - 1}(\psi^+)^{\theta_2} \quad \text{in } \mathcal{D} \\
m(\mathcal{N}(\phi,\psi))\mathcal{L}^p_{\mathcal{K}}(\psi) = \lambda b(x)(\psi^+ + \varepsilon)^{-\gamma_2} + \frac{\theta_2}{p^\star_{N,s}}(\phi^+)^{\theta_1}(\psi^+)^{\theta_2 - 1} \quad \text{in } \mathcal{D} \\
\phi = \psi = 0 \quad \text{in } \mathbb{R}^N \setminus \mathcal{D},
\end{gathered} \tag{3.1}$$

where $\varepsilon \in (0,1)$ is a real number. The energy functional associated with problem (3.1) is

$$\mathcal{J}_{\lambda,\varepsilon}(\phi,\psi) = \frac{1}{p}\mathscr{M}(\mathcal{N}(\phi,\psi)) - \lambda\Phi_\varepsilon(\phi,\psi) - \Psi(\phi,\psi),$$

where

$$\Phi_\varepsilon(\phi,\psi) = \frac{1}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\phi^+ + \varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx + \frac{1}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\psi^+ + \varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx,$$

It is easy to see that the functional $\mathcal{J}_{\varepsilon,\lambda}$ is of C^1 in X, and for any $(\phi,\psi)\in X$ and $(\varphi_1,\varphi_2)\in X$, we have

$$\begin{aligned}
&\mathcal{J}'_{\lambda,\varepsilon}(\phi,\psi)\cdot(\varphi_1,\varphi_2)\\
&= m(\mathcal{N}(\phi,\psi))\int_{\mathbb{R}^{2N}} |\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))(\varphi_1(x)-\varphi_1(y))\mathcal{K}(x,y)\,dx\,dy\\
&\quad + m(\mathcal{N}(\phi,\psi))\int_{\mathbb{R}^{2N}} |\psi(x)-\psi(y)|^{p-2}(\psi(x)-\psi(y))(\varphi_2(x)-\varphi_2(y))\mathcal{K}(x,y)\,dx\,dy\\
&\quad - \lambda\int_{\mathcal{D}} a(x)(\phi^+ + \varepsilon)^{-\gamma_1}\varphi_1\,dx - \lambda\int_{\mathcal{D}} b(x)(\psi^+ + \varepsilon)^{-\gamma_2}\varphi_2\,dx\\
&\quad - \frac{\theta_1}{p^\star_{N,s}}\int_{\mathcal{D}} (\phi^+)^{\theta_1-1}(\psi^+)^{\theta_2}\varphi_1\,dx - \frac{\theta_2}{p^\star_{N,s}}\int_{\mathcal{D}} (\phi^+)^{\theta_1}(\psi^+)^{\theta_2-1}\varphi_2\,dx.
\end{aligned} \tag{3.2}$$

Now, we present the following lemma concerning the Palais Smale condition, which will play a crucial role in the proof of our main result.

Definition 3.1. A functional $\mathcal{J}$ is said to be satisfying the Palais-Smale condition at level $L_\lambda\in\mathbb{R}$ if any sequence $(\phi_n,\psi_n)_{n\in\mathbb{N}}$ in X with

$$\mathcal{J}(\phi_n,\psi_n)\to L_\lambda \quad\text{and}\quad \mathcal{J}'(\phi_n,\psi_n)\to 0 \quad\text{in } X' \quad\text{as } n\to+\infty,$$

possesses a convergent subsequence in X.

Lemma 3.2. *Under assumptions* (A1)–(A3)*, there exists $\lambda_0>0$ such that for any $\lambda\in(0,\lambda_0)$, the functional $\mathcal{J}_{\varepsilon,\lambda}$ satisfies the Palais-Smale condition at any level $L_\lambda<0$.*

Proof. Let (ϕ_n,ψ_n) be a sequence of X such that

$$\mathcal{J}_{\lambda,\varepsilon}(\phi_n,\psi_n)\to L_\lambda \quad\text{and}\quad \mathcal{J}'_{\lambda,\varepsilon}(\phi_n,\psi_n)\to 0 \quad\text{in } X' \quad\text{as } n\to+\infty, \tag{3.3}$$

Due to the degenerate type of system (3.1), two situations have to be considered: either

$$\inf_{n\in\mathbb{N}}\mathcal{N}(\phi_n,\psi_n)=\delta>0, \quad\text{and}\quad \inf_{n\in\mathbb{N}}\mathcal{N}(\phi_n,\psi_n)=0.$$

For this, we divide the proof into two cases.

Case 1: $\inf_{n\in\mathbb{N}}\mathcal{N}(\phi_n,\psi_n)=\delta>0$. At the beginning, we show that (ϕ_n,ψ_n) is bounded in X. By assumption (A1), there exists $\kappa_\delta>0$ such that

$$m(\mathcal{N}(\phi_n,\psi_n))\ge\kappa_\delta \quad\text{for all } n\in\mathbb{N}. \tag{3.4}$$

By Equation (3.3), and (3.4) with (A2), and the elementary inequality

$$(\alpha+\beta)^{1-\gamma}-\beta^{1-\gamma}\le\alpha^{1-\gamma}, \quad \alpha,\beta\ge0,\ \gamma\in(0,1), \tag{3.5}$$

we have

$$\begin{aligned}
L_\lambda + o_n(1) &= \mathcal{J}_{\lambda,\varepsilon}(\phi_n,\psi_n) - \frac{1}{p^\star_{N,s}}\mathcal{J}'_{\lambda,\varepsilon}(\phi_n,\psi_n)\cdot(\phi_n,\psi_n) \\
&= \frac{1}{p}\mathscr{M}(\mathcal{N}(\phi_n,\psi_n)) - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\phi_n^+ + \varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx \\
&\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\psi_n^+ + \varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx - \frac{1}{p^\star_{N,s}}\int_{\mathcal{D}}(\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2}\,dx \\
&\quad - \frac{1}{p^\star_{N,s}} m(\mathcal{N}(\phi_n,\psi_n))\mathcal{N}(\phi_n,\psi_n) + \frac{\lambda}{p^\star_{N,s}}\int_{\mathcal{D}} a(x)(\phi_n^+ + \varepsilon)^{-\gamma_1}\phi_n\,dx \qquad (3.6)\\
&\quad + \frac{\lambda}{p^\star_{N,s}}\int_{\mathcal{D}} b(x)(\psi_n^+ + \varepsilon)^{-\gamma_2}\psi_n\,dx + \frac{1}{p^\star_{N,s}}\int_{\mathcal{D}}(\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2}\,dx \\
&\ge \Big(\frac{\kappa_\delta}{p\sigma} - \frac{1}{p^\star_{N,s}}\Big)\|\phi_n,\psi_n\|^p - \lambda\Big(\frac{1}{p^\star_{N,s}} + \frac{1}{1-\gamma_1}\Big)\int_{\mathcal{D}} a(x)(\phi_n^+)^{1-\gamma_1}\,dx \\
&\quad - \lambda\Big(\frac{1}{p^\star_{N,s}} + \frac{1}{1-\gamma_2}\Big)\int_{\mathcal{D}} b(x)(\psi_n^+)^{1-\gamma_2}\,dx
\end{aligned}$$

On the other hand, by using Hölder's inequality, and (1.3) we obtain

$$\begin{aligned}
\int_{\mathcal{D}} a(x)(\phi_n^+)^{1-\gamma_1}dx &\le |a|_{L^\infty(\mathcal{D})}|\mathcal{D}|^{\frac{p+\gamma_1-1}{p}}|\phi_n|^{1-\gamma_1}_{L^p(\mathcal{D})} \\
&\le |a|_{L^\infty(\mathcal{D})}|\mathcal{D}|^{\frac{p+\gamma_1-1}{p}}C_p^{1-\gamma_1}|\phi_n|^{1-\gamma_1}_{E_0} \qquad (3.7)\\
&\le C_a\|\phi_n,\psi_n\|^{1-\gamma_1},
\end{aligned}$$

and

$$\begin{aligned}
\int_{\mathcal{D}} b(x)(\psi_n^+)^{1-\gamma_2}dx &\le |b|_{L^\infty(\mathcal{D})}|\mathcal{D}|^{\frac{p+\gamma_2-1}{p}}|\psi_n|^{1-\gamma_2}_{L^p(\mathcal{D})} \\
&\le |b|_{L^\infty(\mathcal{D})}|\mathcal{D}|^{\frac{p+\gamma_2-1}{p}}C_p^{1-\gamma_2}|\psi_n|^{1-\gamma_2}_{E_0} \qquad (3.8)\\
&\le C_b\|\phi_n,\psi_n\|^{1-\gamma_2},
\end{aligned}$$

where

$$C_a = |a|_{L^\infty(\mathcal{D})}|\mathcal{D}|^{\frac{p+\gamma_1-1}{p}}C_p^{1-\gamma_1} \quad\text{and}\quad C_b = |b|_{L^\infty(\mathcal{D})}|\mathcal{D}|^{\frac{p+\gamma_2-1}{p}}C_p^{1-\gamma_2}.$$

Now, combining equations (3.7) and (3.8) with (3.6), we obtain the boundedness of (ϕ_n,ψ_n). Now, since X is a reflexive space, there exist $(\phi,\psi)\in X$ such that, up to a subsequence (still denoted by (ϕ_n,ψ_n)), $(\phi_n,\psi_n)\rightharpoonup(\phi,\psi)$ weakly in X, strongly in $L^\alpha(\mathcal{D})\times L^\beta(\mathcal{D})$, for any $(\alpha,\beta)\in[1,p^\star_{N,s})^2$, almost every where in $\mathcal{D}\times\mathcal{D}$ as $n\to+\infty$. Furthermore, there exist two positive real numbers η and μ such that

$$\mathcal{N}(\phi_n,\psi_n)\to\mu, \quad\text{and}\quad \int_{\mathcal{D}}|\phi_n^+ - \phi^+|^{\theta_1}|\psi_n^+ - \psi^+|^{\theta_2}\,dx \to \eta, \qquad (3.9)$$

as $n\to+\infty$. Clearly, $\mu>0$ since we are in the case where $\delta>0$. Therefore, the weak convergence of $(\phi_n,\psi_n))$ give that the sequence (Φ_n,Ψ_n) defined in $\mathbb{R}^{2N}\setminus\operatorname{diag}\{\mathbb{R}^{2N}\}$ by

$$\begin{aligned}
A_n(x,y) &= |\phi_n(x)-\phi_n(y)|^{p-2}(\phi_n(x)-\phi_n(y))\mathcal{K}(x,y)^{\frac{1}{p'}}, \\
B_n(x,y) &= |\psi_n(x)-\psi_n(y)|^{p-2}(\psi_n(x)-\psi_n(y))\mathcal{K}(x,y)^{\frac{1}{p'}},
\end{aligned}$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$, as well as

$$\begin{aligned}
A_n(x,y) &\longrightarrow |\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))\mathcal{K}(x,y)^{\frac{1}{p'}}\text{a.e. in } \mathbb{R}^{2N}, \\
B_n(x,y) &\longrightarrow |\psi(x)-\psi(y)|^{p-2}(\psi(x)-\psi(y))\mathcal{K}(x,y)^{\frac{1}{p'}}\text{a.e. in } \mathbb{R}^{2N}.
\end{aligned}$$

Here $p'=\frac{p}{p-1}$ which is defined the conjugate of p. Then, going if necessary to a further subsequence, we have

$$A_n(x,y)\longrightarrow |\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))\mathcal{K}(x,y)^{\frac{1}{p'}}$$

and

$$B_n(x,y) \longrightarrow |\psi(x)-\psi(y)|^{p-2}(\psi(x)-\psi(y))\mathcal{K}(x,y)^{\frac{1}{p'}}$$

in $L^{p'}(\mathbb{R}^{2N})$. Consequently, as $n \to +\infty$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} |\phi_n(x)-\phi_n(y)|^{p-2}(\phi_n(x)-\phi_n(y))(\phi_1(x)-\phi(y))\mathcal{K}(x,y)\,dx\,dy\\ &= \int_{\mathbb{R}^{2N}} |\phi(x)-\phi(y)|^{p-2}(\phi(x)-\phi(y))(\phi(x)-\phi(y))\mathcal{K}(x,y)\,dx\,dy + o_n(1)\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} |\psi_n(x)-\psi_n(y)|^{p-2}(\psi_n(x)-\psi_n(y))(\psi(x)-\psi(y))\mathcal{K}(x,y)\,dx\,dy\\ &= \int_{\mathbb{R}^{2N}} |\psi(x)-\psi(y)|^{p-2}(\psi(x)-\psi(y))(\psi(x)-\psi(y))\mathcal{K}(x,y)\,dx\,dy + o_n(1).\end{aligned} \tag{3.11}$$

On the other hand, by Hölder's inequality and Equation (2.1), we have

$$\begin{aligned}\int_{\mathcal{D}} \Big|(\phi_n^+)^{\theta_1-1}(\psi_n^+)^{\theta_2}\Big|^{\frac{p^\star_{N,s}}{p^\star_{N,s}-1}}\,dx &\le \Big(\int_{\mathcal{D}} |\phi_n|^{p^\star_{N,s}}\,dx\Big)^{\frac{(\theta_1-1)}{p^\star_{N,s}-1}}\Big(\int_{\mathcal{D}} |\psi_n|^{p^\star_{N,s}}\,dx\Big)^{\frac{\theta_2}{p^\star_{N,s}-1}}\\ &\le \Big(S_\star^{-\frac{(\theta_1-1)}{p^\star_{N,s}-1}}\|\phi_n\|_{E_0}^{\frac{(\theta_1-1)}{p^\star_{N,s}}}\Big)\Big(S_\star^{-\frac{\theta_2 p^\star_{N,s}}{p^\star_{N,s}-1}}\|\psi_n\|_{E_0)}^{\frac{\theta_2 p^\star_{N,s}}{p^\star_{N,s}}}\Big) \le C,\end{aligned}$$

for some $C>0$. Similarly, one has

$$\int_{\mathcal{D}} \Big|(\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2-1}\Big|^{\frac{p^\star_{N,s}}{p^\star_{N,s}-1}}\,dx \le C.$$

Therefore, as $n \to +\infty$, we have

$$\int_{\mathcal{D}} (\phi_n^+)^{\theta_1-1}(\psi_n^+)^{\theta_2}\phi(x)\,dx = \int_{\mathcal{D}} (\phi^+)^{\theta_1}(\psi^+)^{\theta_2}\,dx + o_n(1), \tag{3.12}$$

$$\int_{\mathcal{D}} (\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2-1}\psi(x)\,dx = \int_{\mathcal{D}} (\phi^+)^{\theta_1}(\psi^+)^{\theta_2}\,dx + o_n(1). \tag{3.13}$$

In addition, we have

$$\Big|\int_{\mathcal{D}} a(x)(\phi_n^+ + \varepsilon)^{-\gamma_1}(\phi_n-\phi)\,dx\Big| \le |a|_{L^\infty(\mathcal{D})}\varepsilon^{-\gamma_1}\int_{\mathcal{D}} |\phi_n-\phi|\,dx = o_n(1), \tag{3.14}$$

$$\Big|\int_{\mathcal{D}} b(x)(\psi_n^+ + \varepsilon)^{-\gamma_2}(\psi_n-\psi)\,dx\Big| \le |b|_{L^\infty(\mathcal{D})}\varepsilon^{-\gamma_2}\int_{\mathcal{D}} |\psi_n-\psi|\,dx = o_n(1), \tag{3.15}$$

as $n \to +\infty$. Combining equations (3.3) with (3.9)-(3.15), we obtain

$$\begin{aligned}o_n(1) &= \mathcal{J}'_{\lambda,\varepsilon}(\phi_n,\psi_n)\cdot(\phi_n-\phi,\psi_n-\psi)\\ &= m(\mathcal{N}(\phi_n,\psi_n))\Big(\mathcal{N}(\phi_n,\psi_n)-\mathcal{N}(\phi,\psi)\Big)\\ &\quad - \int_{\mathcal{D}} a(x)(\phi_n^+ + \varepsilon)^{-\gamma_1}(\phi_n-\phi)\,dx\\ &\quad - \int_{\mathcal{D}} b(x)(\psi_n^+ + \varepsilon)^{-\gamma_2}(\psi_n-\psi)\,dx\\ &\quad - \int_{\mathcal{D}} (\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2}\,dx + \int_{\mathcal{D}} (\phi^+)^{\theta_1}(\psi^+)^{\theta_2}\,dx + o_n(1)\\ &= m(\mu)\|\phi_n-\phi,\psi_n-\psi\|^p - \int_{\Omega} |\phi_n^+(x)-\phi^+(x)|^{\theta_1}|\psi_n^+(x)-\psi^+(x)|^{\theta_2}\,dx + o_n(1),\end{aligned}$$

thanks to Lemmas 2.2 and 2.4. Consequently,

$$m(\mu)\|\phi_n-\phi,\psi_n-\psi\|^p = \int_{\Omega} |\phi_n^+(x)-\phi^+(x)|^{\theta_1}|\psi_n^+(x)-\psi^+(x)|^{\theta_2}\,dx + o_n(1). \tag{3.16}$$

When $\eta = 0$, since m possesses a unique zero at 0 and $\mu > 0$, Equation (3.16) gives

$$\|\phi_n - \phi, \psi_n - \psi\|^p = o_n(1),$$

concluding the proof. Assuming, by contradiction that $\eta > 0$. From equation (3.16), it follows that

$$\eta = m(\mu)\left(\mu - \|\phi, \psi\|^p\right). \tag{3.17}$$

In addition, by equation (2.1) and Equation (3.16), we have:

$$\eta^{\frac{p^\star_{N,s}-p}{p^\star_{N,s}}} \geq m(\mu) S^{\frac{p}{p^\star_{N,s}}}. \tag{3.18}$$

Now, utilizing (3.17) and (3.18), we have

$$m(\mu) S^{\frac{p}{p^\star_{N,s}}} \leq \eta^{\frac{p^\star_s - p}{p^\star_s}} = \left[m(\mu)\left(\mu - \|u, v\|^p\right)\right]^{\frac{p^\star_{N,s}-p}{p^\star_{N,s}}}. \tag{3.19}$$

The latter gives

$$\left(\mu - \|\phi, \psi\|^p\right)^{\frac{p^\star_{N,s}-p}{p}} \geq m(\mu) S. \tag{3.20}$$

Since the exact behavior of m is unknown, we need to consider two additional cases, either $\mu \in (0,1)$ or $\mu \geq 1$. To proceed, we divide the proof of the first case into two subcases.

Subcase $\mu \in (0,1)$: By condition (A3) and inequality (3.20), we have

$$\mu^{\frac{p^\star_{N,s}-p}{p}} \geq \left(\mu - \|u, v\|^p\right)^{\frac{p^\star_{N,s}-p}{p}} \geq m(\mu) S \geq m_0 \mu^{\sigma - 1} S.$$

Since $\theta\sigma < p^\star_{N,s}$, it follows that:

$$\mu \geq \left(m_0 S\right)^{\frac{p}{p^\star_{N,s} - \sigma p}}. \tag{3.21}$$

By utilizing condition (A3), inequalities (3.18), and (3.21), one has

$$\begin{aligned}
\eta &\geq \Big(m(\mu) S^{\frac{p}{p^\star_{N,s}}}\Big)^{\frac{p^\star_s}{p^\star_{N,s}-p}} \geq \Big(m_0 \mu^{\sigma-1} S^{\frac{p}{p^\star_{N,s}}}\Big)^{\frac{p^\star_{N,s}}{p^\star_{N,s}-p}}, \\
&\geq \Big[m_0 \big(S m_0\big)^{\frac{(\sigma-1)p}{p^\star_{N,s}-\sigma p}} S^{\frac{p}{p^\star_{N,s}}}\Big]^{\frac{p^\star_{N,s}}{p^\star_{N,s}-p}}, \\
&= \left(m_0^{p^\star_s} S^{\sigma p}\right)^{\frac{1}{p^\star_{N,s}-\sigma p}}.
\end{aligned} \tag{3.22}$$

By considering assumptions (A1) and (A3), we obtain the inequality

$$\mathscr{M}(\mu) \geq \frac{1}{\sigma} m(\mu)\mu \geq \frac{1}{\sigma} m_0 \mu^\sigma. \tag{3.23}$$

Let $\sigma_0 > \sigma$ be such that $p\sigma_0 < p^\star_{N,s}$. From equations (3.3), (3.5), (3.7), (3.8), (3.23), we deduce that

$$
\begin{aligned}
L_\lambda + o_n(1) &= \mathcal{J}_{\lambda,\varepsilon}(\phi_n,\psi_n) - \frac{1}{p\sigma_0}\mathcal{J}'_{\lambda,\varepsilon}(\phi_n,\psi_n)\cdot(\phi_n,\psi_n) \\
&= \frac{1}{p}\mathscr{M}(\mathcal{N}(\phi_n,\psi_n)) - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\phi_n^+ + \varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx \\
&\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\psi_n^+ + \varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx - \frac{1}{p^\star_{N,s}}\int_{\mathcal{D}}(\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2}\,dx \\
&\quad - \frac{1}{p\sigma_0} m(\mathcal{N}(\phi_n,\psi_n))\mathcal{N}(\phi_n,\psi_n) \\
&\quad + \frac{\lambda}{p\sigma_0}\int_{\mathcal{D}} a(x)(\phi_n^+ + \varepsilon)^{-\gamma_1}\phi_n\,dx + \frac{\lambda}{p\sigma_0}\int_{\mathcal{D}} b(x)(\psi_n^+ + \varepsilon)^{-\gamma_2}\psi_n\,dx \\
&\quad + \frac{1}{p\sigma_0}\int_{\mathcal{D}}(\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2}\,dx \qquad (3.24)\\
&\geq \frac{1}{p}\mathscr{M}(\mu) - \frac{1}{p\sigma_0}m(\mu)\mu + \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\int_{\mathcal{D}}(\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2}\,dx \\
&\quad - 2\lambda|a|_{L^\infty(\mathcal{D})}\int_{\mathcal{D}}(\phi_n^+)^{1-\gamma_1}\,dx - 2\lambda|b|_{L^\infty(\mathcal{D})}\int_{\mathcal{D}}(\psi_n^+)^{1-\gamma_2}\,dx + o_n(1), \\
&\geq \frac{1}{p}\mathscr{M}(\mu) - \frac{1}{p\sigma_0}m(\mu)\mu + \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\int_{\mathcal{D}}(\phi_n^+)^{\theta_1}(\psi_n^+)^{\theta_2}\,dx \\
&\quad - 2\lambda C_a\|\phi_n,\psi_n\|^{1-\gamma_1} - 2\lambda C_b\|\phi_n,\psi_n\|^{1-\gamma_2} + o_n(1) \\
&\geq \Big(\frac{1}{\sigma} - \frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\theta + \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\eta - \lambda C\max\{\mu^{\frac{1-\gamma_1}{p}},\mu^{\frac{1-\gamma_2}{p}}\} + o_n(1).
\end{aligned}
$$

Here $C = 2(C_a + C_b)$. Without loos of generality, we assume may that $\gamma_1 \leq \gamma_2$. Thus, since $\mu \in (0,1)$, one has

$$
L_\lambda \geq \Big(\frac{1}{\sigma} - \frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\theta + \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\eta - \lambda C\mu^{\frac{1-\gamma_2}{p}}.
$$

By employing the Young inequality, we obtain

$$
\begin{aligned}
L_\lambda &\geq \Big(\frac{1}{\theta} - \frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\sigma + \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\eta - \lambda C\mu^{\frac{1-\gamma_2}{p}}, \\
&\geq \Big(\frac{1}{\sigma} - \frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\sigma + \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\eta - \Big(\frac{1}{\sigma} - \frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\sigma \\
&\quad - (\lambda C)^{\frac{\sigma p}{\theta p-(1-\gamma_2)}}\Big((\frac{1}{\sigma} - \frac{1}{\sigma_0})\frac{m_0}{p}\Big)^{-\frac{1-\gamma_2}{\theta p-(2-\gamma_1-\gamma_2)}}, \\
&\geq \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\eta - (\lambda C)^{\frac{\sigma p}{\sigma p-(1-\gamma_2)}}\Big((\frac{1}{\sigma} - \frac{1}{\sigma_0})\frac{m_0}{p}\Big)^{-\frac{2-\gamma_1-\gamma_2}{\sigma p-(1-\gamma_2)}},
\end{aligned}
$$

Let

$$
\lambda' = \frac{1}{C}\Big((\frac{1}{\sigma} - \frac{1}{\sigma_0})\frac{m_0}{p}\Big)^{\frac{1-\gamma_2}{\sigma p}}\Big[(\frac{1}{\sigma p} - \frac{1}{p^\star_{N,s}})(m_0^{p_s^\star}S^{p^\star_{N,s}\sigma})^{\frac{1}{p^\star_{N,s}-\sigma p}}\Big]^{\frac{\sigma p-(1-\gamma_2)}{\sigma p}}. \qquad (3.25)
$$

Then, we deduce that for any $\lambda < \lambda'$,

$$
0 > L_\lambda \geq \Big(\frac{1}{\sigma_0 p} - \frac{1}{p^\star_{N,s}}\Big)\eta - (\lambda C)^{\frac{\sigma p}{\sigma p-(1-\gamma_2)}}((\frac{1}{\sigma} - \frac{1}{\sigma_0})\frac{m_0}{p})^{-\frac{1-\gamma_2}{\sigma p-(1-\gamma_2)}} > 0.
$$

Thanks to equation (3.22), we obtain our contradiction concluding the proof of the first sub-case.

Subcase $\mu \geq 1$: From condition (A1) for $\delta = 1$, there exists $\kappa_\delta > 0$ such that

$$
m(\mu) \geq \kappa_\delta \quad \text{for all } \mu \geq \delta. \qquad (3.26)
$$

Combining equation (3.26) with equation (3.18), we obtain

$$\eta \geq \Big(m(\mu) S^{\frac{p}{p^\star_{N,s}}}\Big)^{\frac{p^\star_{N,s}}{p^\star_{N,s}-p}} \geq \Big(\kappa_\delta S^{\frac{p}{p^\star_{N,s}}}\Big)^{\frac{p^\star_s}{p^\star_{N,s}-p}}. \tag{3.27}$$

Proceeding as earlier, with equations (3.26) and (3.27) we obtain

$$L_\lambda \geq \Big(\frac{1}{\sigma}-\frac{1}{\sigma_0}\Big)\frac{\kappa_\delta}{p}\mu + \Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\eta - \lambda C \mu^{\frac{1-\gamma_2}{p}}, \geq \Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\eta - \Big[\frac{\lambda C}{\big(\frac{1}{\sigma}-\frac{1}{\sigma_0}\big)\frac{\kappa_\delta}{p}}\Big]^{\frac{p}{p-(1-\gamma_2)}}.$$

Let

$$\lambda'' = \Big(\frac{1}{\sigma}-\frac{1}{\sigma_0}\Big)\frac{\kappa_\delta}{pC}\Big[\Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\Big(\kappa_1 S_\star^{\frac{p}{p^\star_s}}\Big)^{\frac{p^\star_s}{p^\star_{N,s}-p}}\Big]^{\frac{p-(1-\gamma_2)}{p}}. \tag{3.28}$$

It follows from equation (3.27) that for any $\lambda < \lambda''$, one has

$$0 > L_\lambda \geq \Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\Big(\kappa_\delta S^{\frac{p}{p^\star_{N,s}}}\Big)^{\frac{p^\star_{N,s}}{p^\star_{N,s}-p}} - \Big[\frac{\lambda C}{\big(\frac{1}{\sigma}-\frac{1}{\sigma_0}\big)\frac{\kappa_\delta}{p}}\Big]^{\frac{p}{p-(1-\gamma)}} > 0.$$

Hence, we still have a contradiction, which concludes the proof of the first case.

Case 2: $\inf_{n\in\mathbb{N}} \mathcal{N}(\phi_n,\psi_n) = 0$. In this case, we have two possibilities. Either $(0,0)$ is an accumulation point for the sequence (ϕ_n,ψ_n), and so that, there exists a subsequence of (ϕ_n,ψ_n) (still denoted by (ϕ_n,ψ_n)), that strongly converges to $(\phi,\psi)=(0,0)$, or $(0,0)$ is an isolated point of (ϕ_n,ψ_n).

The first case cannot occur because it would give that the trivial solution $(0,0)$ is a critical point at the level L_λ. However, this is impossible since we have $\mathcal{J}_{\lambda,\varepsilon}(0,0) = L_\lambda < 0$. Thus, only the latter case can occur, which means that there exists a subsequence, (still denoted by (ϕ_n,ψ_n)), such that $\inf_{n\in\mathbb{N}} \mathcal{N}(\phi_n,\psi_n) = \delta > 0$. We can proceed as before by considering this subsequence.

This completes the proof of the second case. Hence, $\mathcal{J}_{\lambda,\varepsilon}$ satisfies the Palais-Smale condition at any level $L_\lambda < 0$ for any $\lambda \in (0,\lambda_0)$, where $\lambda_0 = \min\{\lambda',\lambda''\}$. □

4. Infinitely many small solutions with negative energy

Let us note that the functional $\mathcal{J}_{\lambda,\varepsilon}$ is not bounded from below in X. Indeed, let $(\phi,\psi)\in X$ with $\|(\phi,\psi)\| \leq 1$. From (A1), we have $m(t) > 0$ for any $t > 0$. On the other hand, condition (A2) gives that $\frac{M(t)}{\mathscr{M}(t)} \leq \frac{\sigma}{t}$. Integrating over $[1,t]$ with $t > 1$, we obtain

$$\mathscr{M}(1)t^\sigma \geq \mathscr{M}(t), \quad \text{for all } t \geq 1.$$

The latter gives

$$\begin{aligned}
\mathcal{J}_{\lambda,\varepsilon}(t\phi,t\psi) &= \frac{1}{p}\mathscr{M}(\mathcal{N}(t\phi,t\psi)) - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(t\phi^+ + \varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx \\
&\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(t\psi^+ + \varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx - \frac{1}{p^\star_{N,s}}\int_{\mathcal{D}} (t\phi^+)^{\theta_1}(t\psi^+)^{\theta_2}\,dx \\
&\leq \frac{\mathscr{M}(1)}{p}t^{p\sigma} + \frac{\lambda\varepsilon^{1-\gamma_1}}{1-\gamma_1}\int_{\mathcal{D}} a(x)\,dx + \frac{\lambda\varepsilon^{1-\gamma_2}}{1-\gamma_2}\int_{\mathcal{D}} b(x)\,dx \\
&\quad - \frac{1}{p^\star_{N,s}} t^{\theta_1+\theta_2}\int_{\mathcal{D}} (\phi^+)^{\theta_1}(\psi^+)^{\theta_2}\,dx,
\end{aligned}$$

so that $\mathcal{J}_{\lambda,\varepsilon}(t\phi,t\psi) \to -\infty$ as $t\to+\infty$, thanks to the fact that $p\sigma < \theta_1+\theta_2$. Hence, to obtain the existence of weak solutions to problem (3.1), we introduce a suitable truncated functional related to $\mathcal{J}_{\lambda,\varepsilon}$ that satisfies the conditions of Lemma 2.7.

By using equations (2.1), (3.7) and (3.8), for any $(\phi,\psi)\in X$, we have

$$\begin{aligned}
\mathcal{J}_{\lambda,\varepsilon}(\phi,\psi) &= \frac{1}{p}\mathscr{M}(\mathcal{N}(\phi,\psi)) - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\phi^+ + \varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx \\
&\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\psi^+ + \varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{p^\star_{N,s}} \int_{\mathcal{D}} (\phi^+)^{\theta_1} (\psi^+)^{\theta_2}\, dx \\
&\geq \frac{1}{p} \mathscr{M}(\|\phi,\psi\|^p) - \lambda c_a \|\phi,\psi\|^{1-\gamma_1} - \lambda c_b \|\phi,\psi\|^{1-\gamma_2} - \frac{1}{S p^\star_{N,s}} \|\phi,\psi\|^{p^\star_{N,s}}.
\end{aligned}
$$

On the other hand, by (A1) and (A2), we have:

If $\|\phi,\psi\| \leq 1$, then

$$
\mathcal{J}_{\lambda,\varepsilon}(\phi,\psi) \geq \frac{\mathscr{M}(1)}{p} \|\phi,\psi\|^{\sigma p} - \lambda c_a \|\phi,\psi\|^{1-\gamma_1} - \lambda c_b \|\phi,\psi\|^{1-\gamma_2} - \frac{S^{-1}}{p^\star_{N,s}} \|\phi,\psi\|^{p^\star_{N,s}}.
$$

If $\|\phi,\psi\| > 1$, then

$$
\mathcal{J}_{\lambda,\varepsilon}(\phi,\psi) \geq \frac{1}{p} \kappa \|\phi,\psi\|^{p} - \lambda c_a \|\phi,\psi\|^{1-\gamma_1} - \lambda c_b \|\phi,\psi\|^{1-\gamma_2} - \frac{S^{-1}}{p^\star_{N,s}} \|\phi,\psi\|^{p^\star_{N,s}}.
$$

We define

$$
H(\xi) = \begin{cases} \frac{\mathscr{M}(1)}{p} \xi^{p\theta} - \lambda c_a \xi^{1-\gamma_1} - \lambda c_b \xi^{1-\gamma_2} - \frac{S^{-1}}{p^\star_{N,s}} \xi^{p^\star_{N,s}} & \text{if } \xi \leq 1 \\ \frac{\kappa}{p} \xi^{p} - \lambda c_a \xi^{1-\gamma_1} - \lambda c_b \xi^{1-\gamma_2} - \frac{S^{-1}}{p^\star_{N,s}} \xi^{p^\star_{N,s}} & \text{if } \xi > 1. \end{cases}
$$

It is easy to see that for any $\lambda > 0$ sufficiently small enough, there exist $\xi_1, \xi_2 \in (0,1)$ with $\xi_1 < \xi_2$ such that $H(\xi_1) = H(\xi_2) = 0$ and

$$
H(\xi) \begin{cases} < 0 & \text{if } 0 < \xi < \xi_1 \\ > 0 & \text{if } \xi_1 < \xi < \xi_2 \,. \end{cases}
$$

Now, following the same approach as in [37], we introduce the following truncated functional $\mathcal{T}_\lambda : X \to \mathbb{R}$ defined as

$$
\begin{aligned}
\mathcal{T}_\lambda(\phi,\psi) = {}& \frac{1}{p} \mathscr{M}(\mathcal{N}(\phi,\psi)) - \frac{\lambda}{1-\gamma_1} \int_{\mathcal{D}} a(x) \Big[(\phi^+ + \varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1} \Big]\, dx \\
& - \frac{\lambda}{1-\gamma_2} \int_{\mathcal{D}} b(x) \Big[(\psi^+ + \varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2} \Big]\, dx - \frac{\chi(\|\phi,\psi\|)}{p^\star_{N,s}} \int_{\mathcal{D}} (\phi^+)^{\theta_1} (\psi^+)^{\theta_2}\, dx
\end{aligned}
$$

Here, $\chi : \mathbb{R}+ \to [0,1]$ is a non-increasing smooth function such that $\chi(\xi) = 0$ if $\xi \geq \xi_2$ and $\chi(\xi) = 1$ if $\xi \leq \xi_1$.

By the construction of $\mathcal{T}_\lambda$ with Lemma 3.3, it can be easily verified that $\mathcal{T}_\lambda$ possesses the following properties:

Lemma 4.1. (1) *The functional $\mathcal{T}_\lambda$ is of C^1, even, and bounded from below on X.*
(2) *If $\mathcal{T}_\lambda(\phi,\psi) < 0$, then $\|\phi,\psi\| < \xi_1$, and $\mathcal{T}_\lambda(\phi,\psi) = \mathcal{J}_{\lambda,\varepsilon}(\phi,\psi)$.*
(3) *For any $\lambda \in (0,\lambda_0)$, $\mathcal{T}_\lambda$ satisfies a local Palais-Smale condition for $L_\lambda < 0$.*

Let us define the set

$$
\Sigma_k = \{ A \in X \setminus \{0\} : A \text{ is closed}, A = -A, \gamma(A) \geq k \},
$$

where $\gamma(A)$ denotes the genus of A, (see Definition 2.5). For $k \in \mathbb{N}$, we define the number

$$
C_k = \inf_{A \in \Sigma_k} \sup_{(\phi,\psi) \in A} \mathcal{T}_\lambda(\phi,\psi).
$$

The significance of the real number C_k is that it provides a lower bound for the critical values of $\mathcal{T}_\lambda$ restricted to certain subsets of X with higher Lusternik-Schnirelmann category.

Lemma 4.2. *Assuming that* (A1)–(A3) *hold. Then for each $\lambda > 0$ sufficiently small enough and $k \in \mathbb{N}$, $C_k < 0$, each C_k is a critical value of $\mathcal{T}_\lambda$.*

Proof. Let $\xi \in (0,\xi_1)$ and $(\phi,\psi) \in X$ with $\|\phi,\psi\| = 1$, we have $\chi(\|\xi\phi,\xi\psi\|) = 1$. Now, let's fix $k \in \mathbb{N}$ and define $X^{(k)}$ as a k-dimensional subspace of $\mathcal{C}_0^\infty(\mathcal{D}) \times \mathcal{C}_0^\infty(\mathcal{D})$. Since all the norms in $X^{(k)}$ are equivalent, there exists $r_k \in (0,1)$ such that for any $(\phi,\psi) \in X^{(k)}$, we have

$$
\|\phi,\psi\| \leq r_k \implies |\phi,\psi|_{L^\infty(\mathcal{D}\times\mathcal{D})} \leq 1.
$$

Let us consider the set

$$\mathscr{S}_{r_k}^{(k)} = \{(\phi,\psi) \in X^{(k)} : \|\phi,\psi\| = r_k\}.$$

We choose $(\phi^{(k)},\psi^{(k)}) \in \mathscr{S}_{r_k}^{(k)}$ such that $\phi^{(k)} > 0$, $\psi^{(k)} > 0$, and $\xi \in (0,\xi_1)$. Then,

$$\begin{aligned}
&\mathcal{T}_\lambda(\xi\phi^{(k)},\xi\psi^{(k)})\\
&= \frac{1}{p}\mathscr{M}(\mathcal{N}(\xi\phi^{(k)},\xi\psi^{(k)})) - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\xi\phi^{(k)}+\varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx\\
&\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\xi\psi^{(k)}+\varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx - \frac{\xi^{p^\star_{N,s}}}{p^\star_{N,s}}\int_{\mathcal{D}} (\phi^{(k)})^{\theta_1}(\psi^{(k)})^{\theta_2}\,dx\\
&\le \frac{\xi^p}{p}\Big(\max_{z\in(0,\xi_1)} m(z)\Big)\|\phi^{(k)},\psi^{(k)}\|^p\\
&\quad - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\xi\phi^{(k)}+\varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx\\
&\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\xi\psi^{(k)}+\varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx\\
&\le \frac{\xi^p}{p}\Big(\max_{z\in(0,\xi_1)} m(z)\Big)\|\phi^{(k)},\psi^{(k)}\|^p\\
&\quad - \xi^{\frac{1-\gamma_1}{p}}\varepsilon^{\frac{(1-\gamma_1)(p-1)}{p}}\int_{\mathcal{D}} a(x)\big(\phi^{(k)}\big)^{\frac{1-\gamma_1}{p}}\,dx - \xi^{\frac{1-\gamma_2}{p}}\varepsilon^{\frac{(1-\gamma_2)(p-1)}{p}}\int_{\mathcal{D}} b(x)\big(\psi^{(k)}\big)^{\frac{1-\gamma_2}{p}}\,dx,
\end{aligned}$$

thanks to the elementary inequality

$$(z+t)^{1-\gamma} - t^{1-\gamma} \ge (1-\gamma)z^{\frac{1-\gamma}{p}}t^{\frac{(1-\gamma)(p-1)}{p}}, \quad p>1,\ t>0,\ z>0 \text{ large enough}.$$

Since $p > \frac{1-\gamma_1}{p}$ and $p > \frac{1-\gamma_2}{p}$, then we can find $\xi_k \in (0,t_1)$ and τ_k such that

$$\mathcal{T}_\lambda(\xi_k\phi^{(k)},\xi_k\psi^{(k)}) \le -\tau_k < 0, \quad \forall(\phi^{(k)},\psi^{(k)}) \in \mathscr{S}_{r_k}^{(k)},$$

which gives

$$\mathcal{T}_\lambda(\phi^{(k)},\psi^{(k)}) \le -\tau_k < 0, \quad \forall(\phi^{(k)},\psi^{(k)}) \in \mathscr{S}_{\xi_k\rho_k}^{(k)}.$$

Thus, $C_k < 0$ for all $k \in \mathbb{N}$. By applying Lemma 4.2, we can deduce that $\mathcal{I}_\lambda$ is bounded from below and satisfies the Palais-Smale condition at the level C_k for any $\lambda > 0$ small enough. Furthermore, Proposition 2.6 states that $\gamma(\mathscr{S}_{r_k}^{(k)}) = k$. Consequently, according to Lemma 2.7, we can conclude that each C_k, $k \in \mathbb{N}$, is a critical value of $\mathcal{T}_\lambda$. Consequently, according to (2) of Lemma 4.1, $\mathcal{J}_{\lambda,\varepsilon}$ admits a sequence of critical points $(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) \subset X$ that converges to zero. Now, to show the positivity of the solutions $(\phi_{k,\varepsilon},\psi_{k,\varepsilon})$, we replace the test function (φ_1,φ_2) in the equation (3.2) by $\phi_{k,\varepsilon}^- = \max\{-\phi_{k,\varepsilon},0\}$, $\psi_{k,\varepsilon}^- = \max\{-\psi_{k,\varepsilon},0\}$ and using the elementary inequality

$$(a-b)(a^- - b^-) \le -(a^- - b^-)^2,$$

we obtain $\|\phi_{k,\varepsilon}^-,\psi_{k,\varepsilon}^-\| = 0$ implying that $(\phi_{k,\varepsilon})$ and $(\psi_{k,\varepsilon})$ are two nonnegative functions. By applying the maximum principle (Proposition 2.17, [29]), we conclude that $(\phi_{k,\varepsilon},\psi_{k,\varepsilon})$ is a sequence of positive solutions for system (3.1). This completes the proof. □

5. Proof of our main result

In this section, we will show that system (1.1) possesses a sequence of nontrivial weak solutions (ϕ_k,ψ_k) in the space X as a limit of the solutions of problem (3.1) obtained in the previous section. Let $\lambda \in (0,\lambda_0)$ be small enough and for $k \in \mathbb{N}$, let $(\phi_{k,\varepsilon},\psi_{k,\varepsilon})_{\varepsilon>0}$ be a family of positive weak solutions of problem (3.1).

Case 1: $\inf_{\varepsilon>0}\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) = \delta_k > 0$. From (A1), there exists $\kappa_k > 0$ such that

$$m(\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon})) \ge \kappa_k, \quad \text{for all } \varepsilon > 0.$$

From this, (A2), and equations (3.7) and (3.8), we have

$$\begin{aligned}
C_k + o_\varepsilon(1) &= \mathcal{J}_{\lambda,\varepsilon}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) - \frac{1}{p^\star_{N,s}}\mathcal{J}'_{\lambda,\varepsilon}(\phi_{k,\varepsilon},\psi_{k,\varepsilon})\cdot(\phi_{k,\varepsilon},\psi_{k,\varepsilon})\\
&= \frac{1}{p}\mathscr{M}(\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon})) - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\phi_{k,\varepsilon}+\varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx\\
&\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\psi_{k,\varepsilon}+\varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx - \frac{1}{p^\star_{N,s}}\int_{\mathcal{D}}\phi_{k,\varepsilon}^{\theta_1}\psi_{k,\varepsilon}^{\theta_2}\,dx\\
&\quad - \frac{1}{p^\star_{N,s}} m(\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}))\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) + \frac{\lambda}{p^\star_{N,s}}\int_{\mathcal{D}} a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}\phi_{k,\varepsilon}\,dx\\
&\quad + \frac{\lambda}{p^\star_{N,s}}\int_{\mathcal{D}} b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}\psi_{k,\varepsilon}\,dx + \frac{1}{p^\star_{N,s}}\int_{\mathcal{D}}\phi_{k,\varepsilon}^{\theta_1}\psi_{k,\varepsilon}^{\theta_2}\,dx\\
&\ge (\frac{\kappa_\delta}{p\sigma} - \frac{1}{p^\star_{N,s}})\|\phi_{k,\varepsilon},\psi_{k,\varepsilon}\|^p - \lambda(\frac{1}{p^\star_{N,s}} + \frac{1}{1-\gamma_1})\int_{\mathcal{D}} a(x)(\phi_n^+)^{1-\gamma_1}\,dx\\
&\quad - \lambda(\frac{1}{p^\star_{N,s}} + \frac{1}{1-\gamma_2})\int_{\mathcal{D}} b(x)(\psi_n^+)^{1-\gamma_2}\,dx\\
&\ge (\frac{\kappa_\delta}{p\sigma} - \frac{1}{p^\star_{N,s}})\|\phi_{k,\varepsilon},\psi_{k,\varepsilon}\|^p - \lambda(\frac{1}{p^\star_{N,s}} + \frac{1}{1-\gamma_1})C_a\|\phi_{k,\varepsilon},\psi_{k,\varepsilon}\|^{1-\gamma_1}\\
&\quad - \lambda(\frac{1}{p^\star_{N,s}} + \frac{1}{1-\gamma_2})C_b\|\phi_{k,\varepsilon},\psi_{k,\varepsilon}\|^{1-\gamma_2}
\end{aligned}$$

This gives the boundedness of $(\phi_{k,\varepsilon},\psi_{k,\varepsilon})_{\varepsilon>0}$. Since the space X is reflexive, up to a subsequence, still denoted by $(\phi_{k,\varepsilon},\psi_{k,\varepsilon})_{\varepsilon>0}$, there exists $(\phi_k,\psi_k)\in X$ such that, $(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) \rightharpoonup (\phi_k,\psi_k)$ weakly in X, $(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) \to (\phi_k,\psi_k)$ strongly in $L^\alpha(\mathcal{D})\times L^\beta(\mathcal{D})$ for all $(\alpha,\beta)\in[1,p^\star_{N,s})^2$, $(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) \to (\phi_k,\psi_k)$ a.e. in $\mathcal{D}\times\mathcal{D}$, as $\varepsilon\to 0^+$. In addition, there exist $\mu,\eta>0$ and $f_1,f_2\in L^1(\mathcal{D})$ such that $\phi_{k,\varepsilon}\le f_1$, $\psi_{k,\varepsilon}\le f_2$, and

$$\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) \to \mu, \quad \text{and} \quad \int_{\mathcal{D}} |\phi_{k,\varepsilon}-\phi_k|^{\theta_1}|\psi_{k,\varepsilon}-\psi_k|^{\theta_2}\,dx \to \eta. \tag{5.1}$$

We aim to show that $(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) \rightharpoonup (\phi_k,\psi_k)$ strongly in X. We observe that

$$\begin{aligned}
|a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}\phi_{k,\varepsilon}| &\le a(x)\phi_{k,\varepsilon}^{1-\gamma_1},\\
|b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}\psi_{k,\varepsilon}| &\le b(x)\psi_{k,\varepsilon}^{1-\gamma_2},
\end{aligned}$$

a.e. in $\mathcal{D}$, so by the Vitali convergence theorem, we obtain

$$\int_{\mathcal{D}} a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}\phi_{k,\varepsilon}\,dx = \int_{\mathcal{D}} a(x)\phi_k^{1-\gamma_1}\,dx, \tag{5.2}$$

$$\int_{\mathcal{D}} b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}\psi_{k,\varepsilon}\,dx = \int_{\mathcal{D}} b(x)\psi_k^{1-\gamma_2}\,dx. \tag{5.3}$$

On the other hand, a simple calculation in (3.1), we have

$$\begin{aligned}
m(\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}))\mathcal{L}_p(\phi_{k,\varepsilon}) &= \lambda(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1} + \frac{\theta_1}{p^\star_{N,s}}\phi_{k,\varepsilon}^{\theta_1-1}\psi_{k,\varepsilon}^{\theta_2} \ge \min\{\frac{\lambda}{2^{\gamma_1}},\frac{\theta_1}{p^\star_{N,s}}\}\\
m(\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}))\mathcal{L}_p(\psi_{k,\varepsilon}) &= \lambda(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2} + \frac{\theta_2}{p^\star_{N,s}}\phi_{k,\varepsilon}^{\theta_1}\psi_{k,\varepsilon}^{\theta_2-1} \ge \min\{\frac{\lambda}{2^{\gamma_2}},\frac{\theta_1}{p^\star_{N,s}}\}.
\end{aligned}$$

Therefore, since $\inf_{\varepsilon>0}\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon}) = \delta_k > 0$ and using the strong maximum principle (see [29]), there exist $\mathcal{D}_0\subset\mathcal{D}$, $\mathcal{D}_1\subset\mathcal{D}$ and a constants $d_0>0$, d_1 that are independent of ε such that for any $\varepsilon>0$, we have

$$\phi_{k,\varepsilon}\ge d_0 \text{ a.e. in } \mathcal{D}_0, \quad \text{and} \quad \psi_{k,\varepsilon}\ge d_1 \text{ a.e.in } \mathcal{D}_1. \tag{5.4}$$

Now, let us consider a function $\varphi_1 \in \mathcal{C}_0^\infty(\mathcal{D}$ with $\operatorname{supp}(\varphi_1) \subset \mathcal{D}_0$ and a function $\varphi_2 \in \mathcal{C}_0^\infty(\mathcal{D}$ with $\operatorname{supp}(\varphi_2) \subset \mathcal{D}_0$. We observe that from (5.4) we have

$$\begin{aligned} |a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}\varphi_1| &\le a(x)d_0^{-\gamma_1}|\varphi_1|, \quad \text{a.e. in } \mathcal{D}_0, \\ |b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}\varphi_2| &\le b(x)d_1^{-\gamma_2}|\varphi_2|, \quad \text{a.e. in } \mathcal{D}_1. \end{aligned}$$

Consequently, the dominated convergence theorem implies that

$$\int_{\mathcal{D}} a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}\varphi_1\,dx = \int_{\mathcal{D}} a(x)\phi_k^{-\gamma_1}\varphi_1\,dx + o_\varepsilon(1), \tag{5.5}$$

$$\int_{\mathcal{D}} b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}\varphi_2\,dx = \int_{\mathcal{D}} b(x)\psi_k^{-\gamma_2}\varphi_2\,dx + o_\varepsilon(1), \tag{5.6}$$

as $\varepsilon \to 0^+$. Since $\partial\mathcal{D}$ is continuous, the space $\mathcal{C}_0^\infty(\mathcal{D})$ is dense in $E_0(\mathcal{D})$ (see [19, Theorem 6]). Therefore, by a standard density argument, equations (5.5) and (5.6) hold true for any $(\varphi_1, \varphi_2) \in X$. Thus, by combining (5.2)-(5.3) with (5.5)-(5.6) with $\varphi_1 = \phi_k$ and $\varphi_2 = \psi_k$, as $\varepsilon \to 0^+$, we obtain

$$\begin{aligned} \int_{\mathcal{D}} a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}(\phi_{k,\varepsilon}-\phi_k)\,dx &= o_\varepsilon(1), \\ \int_{\mathcal{D}} b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}(\psi_{k,\varepsilon}-\psi_k)\,dx &= o_\varepsilon(1). \end{aligned}$$

Consequently, from (5.1), as $\varepsilon \to 0^+$ we have

$$\begin{aligned} o_\varepsilon(1) &= \mathcal{J}'_{\varepsilon,\lambda}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon}) \cdot (\phi_{k,\varepsilon}-\phi_k, \psi_{k,\varepsilon}-\psi_k) \\ &= m(\mathcal{N}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon}))\Big(\mathcal{N}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon}) - \mathcal{N}(\phi_k, \psi_k)\Big) \\ &\quad - \int_{\mathcal{D}} a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}(\phi_{k,\varepsilon}-\phi_k)\,dx - \int_{\mathcal{D}} b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}(\psi_{k,\varepsilon}-\psi_k)\,dx \\ &\quad - \int_{\mathcal{D}} (\phi_{k,\varepsilon})^{\theta_1}(\psi_{k,\varepsilon})^{\theta_2}\,dx + \int_{\mathcal{D}} (\phi_k)^{\theta_1}(\psi_k)^{\theta_2}\,dx + o_\varepsilon(1) \\ &= m(\mu)\|\phi_{k,\varepsilon}-\phi_k, \phi_{k,\varepsilon}-\psi_k\|^p - \int_\Omega |\phi_{k,\varepsilon}(x)-\phi_k(x)|^{\theta_1}|\psi_{k,\varepsilon}(x)-\psi_k(x)|^{\theta_2}\,dx + o_\varepsilon(1), \end{aligned}$$

so that

$$m(\mu)\|\phi_{k,\varepsilon}-\phi_k, \phi_{k,\varepsilon}-\psi_k\|^p = \eta + o_\varepsilon(1).$$

Now, let us show that $\eta = 0$. By contradiction, i.e., we assume that $\eta > 0$. Arguing as in Lemma 3.2, we can show that

$$\eta^{\frac{p^\star_{N,s}-p}{p^\star_{N,s}}} \ge m(\mu)S^{\frac{p}{p^\star_{N,s}}}. \tag{5.7}$$

Now, consider $\sigma_0 > \sigma$ be such that $p\sigma_0 < p^\star_{N,s}$. Then, similarly to (3.24), as $\varepsilon \to 0^+$ we have

$$\begin{aligned} C_k + o_\varepsilon(1) &= \mathcal{J}_{\lambda,\varepsilon}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon}) - \frac{1}{\sigma_0}\mathcal{J}'_{\lambda,\varepsilon}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon}) \cdot (\phi_{k,\varepsilon}, \psi_{k,\varepsilon}) \\ &= \frac{1}{p}\mathscr{M}(\mathcal{N}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon})) - \frac{\lambda}{1-\gamma_1}\int_{\mathcal{D}} a(x)\Big[(\phi_{k,\varepsilon}+\varepsilon)^{1-\gamma_1} - \varepsilon^{1-\gamma_1}\Big]\,dx \\ &\quad - \frac{\lambda}{1-\gamma_2}\int_{\mathcal{D}} b(x)\Big[(\psi_{k,\varepsilon}+\varepsilon)^{1-\gamma_2} - \varepsilon^{1-\gamma_2}\Big]\,dx - \frac{1}{p^\star_{N,s}}\int_{\mathcal{D}} \phi_{k,\varepsilon}^{\theta_1}\psi_{k,\varepsilon}^{\theta_2}\,dx \\ &\quad - \frac{1}{\sigma_0}m(\mathcal{N}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon}))\mathcal{N}(\phi_{k,\varepsilon}, \psi_{k,\varepsilon}) + \frac{\lambda}{\sigma_0}\int_{\mathcal{D}} a(x)(\phi_{k,\varepsilon}+\varepsilon)^{-\gamma_1}\phi_{k,\varepsilon}\,dx \\ &\quad + \frac{\lambda}{p^\star_{N,s}}\int_{\mathcal{D}} b(x)(\psi_{k,\varepsilon}+\varepsilon)^{-\gamma_2}\psi_{k,\varepsilon}\,dx + \frac{1}{\sigma_0}\int_{\mathcal{D}} \phi_{k,\varepsilon}^{\theta_1}\psi_{k,\varepsilon}^{\theta_2}\,dx \\ &\ge \frac{1}{p}\mathscr{M}(\mu) - \frac{1}{p\sigma_0}m(\mu)\mu + \Big(\frac{1}{p\sigma_0} - \frac{1}{p^\star_{N,s}}\Big)\int_{\mathcal{D}} |\phi_{k,\varepsilon}-\phi_k|^{\theta_1}|\psi_{k,\varepsilon}-\psi_k|^{\theta_2}\,dx \end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\int_{\mathcal{D}}|\phi_k|^{\theta_1}|\psi_k|^{\theta_2}\,dx-\lambda C\Big(\|\phi_n,\psi_n\|^{1-\gamma_1}+\|\phi_n,\psi_n\|^{1-\gamma_2}\Big)+o_\varepsilon(1)\\
&\geq\Big(\frac{1}{\sigma}-\frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\theta+\Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\eta-\lambda C\Big(\|\phi_n,\psi_n\|^{1-\gamma_1}+\|\phi_n,\psi_n\|^{1-\gamma_2}\Big)+o_\varepsilon(1).
\end{aligned}
$$

That is,

$$
C_k\geq\Big(\frac{1}{\sigma}-\frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\theta+\Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\eta-\lambda C\max\{\mu^{\frac{1-\gamma_1}{p}},\mu^{\frac{1-\gamma_2}{p}}\}.
$$

Without loss of generality, we may assume that $\sigma_1\leq\sigma_2$. In particular, for any $\lambda\in(0,\lambda_0)$, we obtain

$$
0>C_k\geq\Big(\frac{1}{\sigma}-\frac{1}{\sigma_0}\Big)\frac{m_0}{p}\mu^\theta+\Big(\frac{1}{p\sigma_0}-\frac{1}{p^\star_{N,s}}\Big)\eta-\lambda C\mu^{\frac{1-\gamma_2}{p}}>0,
$$

thanks to (3.25) and (3.28). Which leads to a contradiction concluding $\eta=0$. Hence $(\phi_{k,\varepsilon},\psi_{k,\varepsilon})\rightharpoonup(\phi_k,\psi_k)$ strongly in X.

Case 2: $\inf_{\varepsilon>0}\mathcal{N}(\phi_{k,\varepsilon},\psi_{k,\varepsilon})=0$. In this case, we can apply a similar approach to that used in the proof of Lemma 4.2, allowing us to deduce that $(\phi_{k,\varepsilon},\psi_{k,\varepsilon})\rightharpoonup(\phi_k,\psi_k)$ strongly in X. This complete the proof of Theorem 1.2.

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