Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 06, pp. 1–20.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.06

MULTIPLE POSITIVE NORMALIZED SOLUTIONS FOR KIRCHHOFF TYPE SYSTEM WITH VAN DER WAALS TYPE POTENTIALS

ZHEWEN CHEN, MUZI LI

Abstract. This article shows the existence of normalized solutions for Kirchhoff type system with van der Waals type potentials,

$$-(a+b\int_{\mathbb{R}^N}|\nabla u_1|^2dx)\Delta u_1=\lambda_1 u_1+\mu_1(I_\alpha*|u_1|^{p_1})|u_1|^{p_1-2}u_1+\Theta r_1(I_\beta*|u_2|^{r_2})|u_1|^{r_1-2}u_1,$$
$$-(a+b\int_{\mathbb{R}^N}|\nabla u_2|^2dx)\Delta u_2=\lambda_2 u_2+\mu_2(I_\alpha*|u_2|^{p_2})|u_2|^{p_2-2}u_2+\Theta r_2(I_\beta*|u_1|^{r_1})|u_2|^{r_2-2}u_2,$$
$$\int_{\mathbb{R}^N}|u_1|^2dx=d_1>0,\quad \int_{\mathbb{R}^N}|u_2|^2dx=d_2>0,$$

where $N=3,4$, $\mu_1,\mu_2,\Theta>0$, $\frac{N+\alpha}{N}<p_1,p_2<\frac{N+\alpha+2}{N}$, $2\cdot\frac{N+\beta}{N}<r_1+r_2<2\cdot 2^*_\beta=2\cdot\frac{N+\beta}{N-2}$, $0<\alpha,\beta<N$, I_α and I_β are the Riesz potentials. We show that the system has a positive least energy solution at negative energy level for Θ small. In addition, we also prove that the system admits a high energy positive solution at positive energy level in the special case.

1. Introduction and main results

In this article, we study the existence of the positive normalized solutions for Kirchhoff type system with van der Waals type potentials

$$\begin{aligned}
-(a+b\int_{\mathbb{R}^N}|\nabla u_1|^2dx)\Delta u_1&=\lambda_1 u_1+\mu_1(I_\alpha*|u_1|^{p_1})|u_1|^{p_1-2}u_1+\Theta r_1(I_\beta*|u_2|^{r_2})|u_1|^{r_1-2}u_1,\\
-(a+b\int_{\mathbb{R}^N}|\nabla u_2|^2dx)\Delta u_2&=\lambda_2 u_2+\mu_2(I_\alpha*|u_2|^{p_2})|u_2|^{p_2-2}u_2+\Theta r_2(I_\beta*|u_1|^{r_1})|u_2|^{r_2-2}u_2,
\end{aligned}\tag{1.1}$$

with the L^2-mass constraint

$$\int_{\mathbb{R}^N}|u_1|^2dx=d_1>0,\quad \int_{\mathbb{R}^N}|u_2|^2dx=d_2>0,\tag{1.2}$$

where $N=3,4$, $a,b>0$, $\alpha,\beta\in(0,N)$, I_α, I_β are the Riesz potentials defined for every $x\in\mathbb{R}^N\backslash\{0\}$ by

$$I_\alpha(x):=\frac{A_\alpha(N)}{|x|^{N-\alpha}},\quad A_\alpha(N):=\frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}\tag{1.3}$$

with Γ denoting the Gamma function. Throughout this paper, we always require that $a,b,\mu_1,\mu_2,\Theta>0$, and assume that $2<p_1,p_2<2+\frac{4}{N}$, $r_1,r_2>\frac{N+\beta}{N}$, $2\cdot\frac{N+\beta}{N}<r_1+r_2<2\cdot 2^*_\beta$.

Because of the appearance of the term $\int_{\mathbb{R}^N}|\nabla u|^2dx$, (1.1) is regard as a nonlocal problem, which implies that equation (1.1) is not a pointwise identity. Moreover, this phenomenon also leads to some mathematical difficulties that make the study of (1.1) more interesting. Problem (1.1) originates from the stationary analog of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{Y_0}{k}+\frac{R}{2J}\int_0^Q|\frac{\partial u}{\partial x}|^2dx\Big)\frac{\partial^2 u}{\partial x^2}=0,$$

2020 *Mathematics Subject Classification.* 35J20, 35J47, 35J50.
Key words and phrases. Ground state; normalized solution; Van der Waals type potential; Kirchhoff system.

Submitted July 2, 2025. Published January 19, 2026.

which was proposed by Kirchhoff [21] in 1883, and being an extension of the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In the last decades, Because of the strong background in physics, starting with the framework of Lions [23], many mathematicians have established many interesting conclusions about Kirchhoff-type problems [1, 16, 29, 21, 17, 18, 24, 14, 15].

In (1.1), if $\lambda \in \mathbb{R}$ is fixed, then we call (1.1) the fixed frequency problem. One can adopt the traditional variational method, looking for critical points of $F_\lambda(u_1, u_2)$, or fixed point theory, bifurcation, topological methods, Nehari manifold method and Lyapunov-Schmidt reduction, where

$$\begin{aligned} F_\lambda(u_1, u_2) := \frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\lambda_i\|u_i\|_2^2 - \sum_{i=1}^{2}\frac{\mu_i}{2p_i}\int_{\mathbb{R}^N}(I_\alpha * |u_i|^{p_i})|u_i|^{p_i}dx \\ - \Theta\int_{\mathbb{R}^N}(I_\beta * |u_1|^{p_1})|u_2|^{p_2}dx. \end{aligned}$$

In recent decades, because of the application to physics, mathematicians are interested in solutions that satisfy L^2-mass constraint (1.2). In this direction, the mass $d_1, d_2 > 0$ is prescribed, the frequency λ_i cannot be determined a priori, but is a part of unknown which appears as Lagrange multipliers. In this case, mathematicians often call (1.1)-(1.2) the fixed mass problem and the solution is called a normalized solution. One can get a normalized solution to problem (1.1) by looking for a critical point of the functional

$$\begin{aligned} &F(u_1, u_2) \\ &:= \frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{2p_i}\int_{\mathbb{R}^N}(I_\alpha * |u_i|^{p_i})|u_i|^{p_i}dx - \Theta\int_{\mathbb{R}^N}(I_\beta * |u_1|^{p_1})|u_2|^{p_2}dx \end{aligned}$$

constrained on $S(d_1) \times S(d_2)$, where $S(d) := \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N}|u|^2dx = d > 0\}$. It is standard to check that $F \in C^1$.

For the fixed mass problem, the L^2-mass constraint (1.2) presents some mathematical difficulties. As opposed to the fixed frequency problem, the fixed mass problem will have many technical difficulties when dealing with it in a variational framework: (I) the Nehari manifold method is inapplicable; (II) the Lagrange multipliers must be controlled; (III) for the fixed frequency problem, usually a nontrivial weak limit is also a solution. However, for the fixed mass problem, even though the weak limit is nontrivial, the L^2-mass constraint may be not satisfied; (IV) the L^2-mass critical exponent seriously affects the geometric structure of the functional.

The equation

$$i\partial_t\Psi + \Delta\Psi + (V * |\Psi|^p)|\Psi|^{p-2}\Psi = 0, \text{in}\mathbb{R}^+ \times \mathbb{R}^N, \tag{1.4}$$

has $\Psi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $V(x) = \delta_\zeta\frac{1}{|x|^\zeta} \pm \delta_\theta\frac{1}{|x|^\theta}(\alpha, \beta > 1)$ is the van der Waals type potential (see [30, 9]). The van der Waals coefficients δ_6, δ_8 and δ_{10} of alkaline-earth interactions calculated by Porsev and Derevianko using relativistic many-body perturbation theory are believed to be accurate to 1/100 (see[27]). As one of the van der Waals type potentials the Lennard-Jones potential

$$V_{LJ}(\nu) = \pm\xi(\frac{1}{\nu^\zeta} - \frac{1}{\nu^\theta})$$

with $\zeta = 6$ and $\theta = 12$, is often used as an approximate model for the isotropic part of a total van der Waals force as a function of distance (see [30, 9]).

When $a = 1$, $b = 0$ and the response function is a delta function, i.e. $I_\alpha(x) = I_\beta(x) = \delta(x)$, the nonlinear response is local and problem (1.1) with prescribed mass turns out to be

$$\begin{aligned} -\Delta u_1 &= \lambda_1 u_1 + \mu_1|u_1|^{p_1-2}u_1 + \Theta r_1|u_2|^{r_2}|u_1|^{r_1-2}u_1, \\ -\Delta u_2 &= \lambda_2 u_2 + \mu_2|u_2|^{p_2-2}u_2 + \Theta r_2|u_1|^{r_1}|u_2|^{r_2-2}u_2, \end{aligned} \tag{1.5}$$

with the L^2-mass constraint $\int_{\mathbb{R}^N}|u_1|^2dx = d_1 > 0$, $\int_{\mathbb{R}^N}|u_2|^2dx = d_2 > 0$. In recent years, mathematicians have drawn rich conclusions concerning the existence, multiplicity and qualitative properties of the normalized solutions of system (1.5) (see [3, 5, 6, 2, 4, 13]). Gou et al. [13] showed that the system admits two normalized solutions under the conditions that $2 < p_1, p_2 < 2 + \frac{4}{N} < r_1 + r_2 < 2^* = \frac{2N}{N-2}$ (resp. $2 + \frac{4}{N} < r_1 + r_2 < 2 + \frac{4}{N} < p_1, p_2 < 2^*$) by using minimizing methods, Pohožaev type manifold, Schwarz rearrangements and mountain pass theorem. Bartsch et al. [2] focused on the choice $p_1 = p_2 = 2^*$, and allowing $r_1 + r_2$ to be mass-subcritical, mass-critical or mass-supercritical. The authors proved the existence and non-existence of normalized ground state for different ranges of Θ.

When $a, b > 0$ and the response function is a delta function, i.e. $I_\alpha(x) = I_\beta(x) = \delta(x)$, system (1.1) becomes

$$\begin{aligned} -(a + b\int_{\mathbb{R}^N} |\nabla u_1|^2 dx)\Delta u_1 &= \lambda_1 u_1 + \mu_1 |u_1|^{p_1-2}u_1 + \Theta r_1 |u_2|^{r_2}|u_1|^{r_1-2}u_1 x \in \mathbb{R}^N, \\ -(a + b\int_{\mathbb{R}^N} |\nabla u_2|^2 dx)\Delta u_2 &= \lambda_2 u_2 + \mu_2 |u_2|^{p_2-2}u_2 + \Theta r_2 |u_1|^{r_1}|u_2|^{r_2-2}u_2 x \in \mathbb{R}^N. \end{aligned} \tag{1.6}$$

When $N \leq 3$ and $2 < p_1, p_2, r_1 + r_2 < 2 + \frac{8}{N}$, Cao et al. [8] proved that system (1.6) admits a positive normalized solution. When $N = 2, 3$, Yang [29] showed that system (1.6) admits a normalized ground state under the condition that $2 + \frac{8}{N} < p_1, p_2, r_1 + r_2 < 2^*$ and $2 < r_1 + r_2 < 2 + \frac{8}{N} < p_1, p_2 < 2^*$, respectively.

When $a = 1$, $b = 0$ and the response function is a delta function, i.e. $I_\alpha(x) = I_\beta(x) = \delta(x)$, system (1.1) becomes the Choquard system (Hartree system)

$$\begin{aligned} -\Delta u_1 &= \lambda_1 u_1 + \mu_1 (I_\alpha * |u_1|^{p_1})|u_1|^{p_1-2}u_1 + \Theta r_1 (I_\alpha * |u_2|^{r_2})|u_1|^{r_1-2}u_1 x \in \mathbb{R}^N, \\ -\Delta u_2 &= \lambda_2 u_2 + \mu_2 (I_\alpha * |u_2|^{p_2})|u_2|^{p_2-2}u_2 + \Theta r_2 (I_\alpha * |u_1|^{r_1})|u_2|^{r_2-2}u_2 x \in \mathbb{R}^N. \end{aligned} \tag{1.7}$$

When $\mu_1, \mu_2, \Theta > 0$ and $\frac{N+\alpha}{N} < r_1, r_2$, Geng et al. [10] proved that system (1.7) has a normalized ground state under the condition that $N \geq 3$ and $\frac{N+\alpha}{N} < p_1, p_2 < \frac{N+\alpha+2}{N} < r_1, r_2 < 2^*_\alpha$ by using Schwartz rearrangement. In addition, they proved that the system (1.7) has a second solution for $N = 3$ and $p_1 = p_2 = \alpha = 2$. They also proved that (1.7) has a second solution for $N = 5$, $\frac{N+\alpha}{N} < r_1, r_2 < \frac{N+\alpha+2}{N} < p_1, p_2 < 2^*_\alpha$ and $p_1 = p_2 = \alpha = 2$. For Hardy-Littlewood-Sobolev critical case, i.e. $p_1 = p_2 = 2^*_\alpha$, Zhang et al. [31] proved that system (1.7) has a normalized ground state for different ranges of Θ when $r_1 + r_2$ is set to be mass subcritical, mass critical and mass supercritical, respectively.

However, as far as we know, for the case of $a, b > 0$ and the response functions are different Riesz potential functions, the existence of the solution of the system is still unknown.

Definition 1.1. (u_1, u_2) is a *normalized ground state* to (1.1)-(1.2) if $F'|_{S(d_1)\times S(d_2)}(u_1, u_2) = 0$ and

$$F(u_1, u_2) = \inf\{F(v_1, v_2) : (v_1, v_2) \in S(d_1) \times S(d_2), F'|_{S(d_1)\times S(d_2)}(v_1, v_2) = 0\}.$$

Furthermore, (w_1, w_2) is a *high energy normalized solution* to (1.1)-(1.2) if $F'|_{S(d_1)\times S(d_2)}(w_1, w_2) = 0$ and

$$F(w_1, w_2) > \inf\{F(v_1, v_2) : (v_1, v_2) \in S(d_1) \times S(d_2), F'|_{S(d_1)\times S(d_2)}(v_1, v_2) = 0\}.$$

In this article, we demonstrate our main results in the following two cases:

(A1) $\frac{N+\alpha}{N} < p_1, p_2 < \frac{N+\alpha+2}{N}$, $r_1, r_2 > \frac{N+\beta}{N}$ and $2 \cdot \frac{N+\beta}{N} < r_1 + r_2 < 2 \cdot \frac{N+\beta+4}{N}$.

(A2) $\frac{N+\alpha}{N} < p_1, p_2 < \frac{N+\alpha+2}{N}$, $r_1, r_2 > \frac{N+\beta}{N}$ and $2 \cdot \frac{N+\beta+4}{N} < r_1 + r_2 < 2 \cdot 2^*_\beta$.

To restore some compactness, we search for critical points of F constrained on $S_r(d_1) \times S_r(d_2)$, where $S_r(d) := \{u \in H^1_r(\mathbb{R}^N) : \|u\|_2^2 = d\}$ and

$$H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : u \text{is a radially symmetric function}.$$

For case of (A_2), F is unbounded below in $S_r(d_1) \times S_r(d_2)$. Hence, we need the Pohožaev manifold $\mathcal{P}(d_1, d_2) := \{(u_1, u_2) \in S_r(d_1) \times S_r(d_2) : P(u_1, u_2) = 0\}$, where

$$P(u_1, u_2) = a\sum_{i=1}^{2} \|\nabla u_i\|_2^2 + b\sum_{i=1}^{2} \|\nabla u_i\|_2^4 - \sum_{i=1}^{2} \frac{\mu_i}{2p_i}(Np_i - N - \alpha)\int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i} dx$$

$$- \Theta \frac{Nr - 2(N+\beta)}{2} \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx$$

with $r = r_1 + r_2$. We define $W(k) := \{(u_1, u_2) \in S_r(d_1) \times S_r(d_2) : \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 < k\}$ and

$$\sigma(d_1, d_2) := \inf_{W(k)} F(u_1, u_2) < 0. \tag{1.8}$$

For each $u \in S_r(d)$, we define

$$\tau \star u_i := e^{\frac{N}{2}\tau} u_i(e^\tau x), \quad i = 1, 2.$$

Hence,

$$\begin{aligned} \Psi_{u_1,u_2}(\tau) &:= F(\tau \star u_1, \tau \star u_2) \\ &= \frac{a}{2} e^{2\tau} \sum_{i=1}^2 \|\nabla u_i\|_2^2 + \frac{b}{4} e^{4\tau} \sum_{i=1}^2 \|\nabla u_i\|_2^4 \\ &\quad - \sum_{i=1}^2 \frac{\mu_i}{2p_i} e^{(Np_i - N - \alpha)\tau} \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i}) |u_i|^{p_i} dx \\ &\quad - \Theta e^{\frac{Nr}{2} - N - \beta} \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx. \end{aligned} \tag{1.9}$$

By direct computations, we have $(\Psi_{u_1,u_2})'(0) = P(u_1, u_2)$. We divide $\mathcal{P}(d_1, d_2)$ into 3 parts,

$$\mathcal{P}(d_1, d_2) = \mathcal{P}_-(d_1, d_2) \cup \mathcal{P}_0(d_1, d_2) \cup \mathcal{P}_+(d_1, d_2),$$

where

$$\begin{aligned} \mathcal{P}_-(d_1, d_2) &= \{(u_1, u_2) \in \mathcal{P}(d_1, d_2) : (\Psi_{u_1,u_2})''(0) < 0\}, \\ \mathcal{P}_0(d_1, d_2) &= \{(u_1, u_2) \in \mathcal{P}(d_1, d_2) : (\Psi_{u_1,u_2})''(0) = 0\}, \\ \mathcal{P}_+(d_1, d_2) &= \{(u_1, u_2) \in \mathcal{P}(d_1, d_2) : (\Psi_{u_1,u_2})''(0) > 0\}, \end{aligned}$$

and

$$\begin{aligned} (\Psi_{u_1,u_2})''(0) &= 2a \sum_{i=1}^2 \|\nabla u_i\|_2^2 + 4b \sum_{i=1}^2 \|\nabla u_i\|_2^4 \\ &\quad - \sum_{i=1}^2 \frac{\mu_i}{2p_i} (Np_i - N - \alpha)^2 \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i}) |u_i|^{p_i} dx \\ &\quad - \Theta (\frac{Nr}{2} - N - \beta)^2 \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx. \end{aligned}$$

Theorem 1.2. *Assume that* (A1) *holds. Then, every minimizing sequence of* (1.8) *is compact, up to translation, in* $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. *Moreover, system* (1.1) *has a positive normalized ground state.*

Theorem 1.3. *Assume that* (A2) *holds and* $0 < \alpha - 2 \le \beta < \alpha < N$. *Then, there exist* $k_0 = k_0(d_1, d_2) > 0$, $\Theta_* = \Theta_*(d_1, d_2) > 0$, *such that for any* $0 < \Theta \le \Theta_*$, *system* (1.1)-(1.2) *has a positive radial solution* $(\upsilon_1, \upsilon_2) \in W(k_0)$ *at negative level* $F(\upsilon_1, \upsilon_2) < 0$ *for some* $\lambda_1, \lambda_2 < 0$.

Theorem 1.4. *Assume that* (A2) *holds,and* $a = 1$, $b = 0$, $N = 3$ *and* $p_1 = p_2 = \alpha = \beta = r_1 = r_2 = 2$. *Then, there exist* $k_0 = k_0(d_1, d_2) > 0$, $\Theta_* = \Theta_*(d_1, d_2) > 0$, *such that for any* $0 < \Theta \le \Theta_*$, *system* (1.1)-(1.2) *has a second positive radial solution* $(u_1, u_2) \in W(k_0)$ *at positive level* $F(u_1, u_2) > 0$ *for some* $\lambda_1, \lambda_2 < 0$.

Remark 1.5. Definition 1.1, Theorem 1.3 and **??** indicate that (1.1)-(1.2) admit a normalized ground state at negative level $F(\upsilon_1, \upsilon_2) < 0$ and a high energy normalized solution at positive level $F(u_1, u_2) > 0$, under the condition that (A2), $a = 1$, $b = 0$, $N = 3$ and $p_1 = p_2 = \alpha = \beta = r_1 = r_2 = 2$ hold.

Remark 1.6. For nonlinear classical Choquard system 1.7, the mass critical exponent is $\frac{N+\alpha+2}{N}$. In Kirchhoff-Choquard system, by the appearance of the $\int_{\mathbb{R}^N} |\nabla u|^4 dx$ term, the mass critical exponent becomes $\frac{N+\alpha+4}{N}$. In this paper, if $\frac{N+\alpha}{N} < p_1, p_2 < \frac{N+\alpha+4}{N}$, then Lemma 4.1 and Lemma 4.2 do not hold.

Remark 1.7. Since we do not know whether the inequality

$$\|\nabla\{u_1, u_2\}^*\|_2^4 \leq \|\nabla u_1\|_2^4 + \|\nabla u_2\|_2^4$$

holds, we can not use the Schwartz rearrangement method to prove the compactness of minimized sequence in this paper.

From a physical point of view, it is of great importance to study the solution of problem (1.1), as confirmed in [30, 9, 27]. We emphasize that this study seems to be the first contribution regarding existence of normalized ground states for a Kirchhoff system with van der Waals type potentials.

The rest of this article organized as follows: In Section 2, we present some preliminaries. In Section 3, we prove the existence of normalized ground states under the purely mass subcritical case. In Section 4, we prove the existence of the first solution, which is a local minimizer. In Section 5, the second solution is proved by using mountain pass theorem.

Notation: $L^s(\mathbb{R}^N)$ is the Lebesgue space with the norms $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s dx)^{1/s}$, $1 < s < \infty$. $H^1(\mathbb{R}^N)$ is the usual Sobolev space with norm $\|u\|_{H^1(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 dx)^{1/2}$.

2. Preliminaries

Lemma 2.1. *[Hardy-Littlewood-Sobolev inequality [22]] Let $N \geq 1$, $p, r > 1$, and $0 < \beta < N$ with $1/p + (N-\beta)/N + 1/r = 2$. Let $u \in L^p(\mathbb{R}^N)$ and $v \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C(N, p, \beta)$, independent of u and v, such that*

$$\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u(x)v(y)}{|x-y|^{N-\beta}} dxdy\Big| \leq C_{N,p,r,\beta}\|u\|_p\|v\|_r.$$

If $p = r = \frac{2N}{N+\beta}$, then

$$C_{N,p,r,\beta} = C_{N,\beta} = \pi^{\frac{N-\beta}{2}} \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{N+\beta}{2})} \Big\{\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\Big\}^{-\frac{\beta}{N}}.$$

Lemma 2.2 (Gagliardo-Nirenberg inequality of Power type [28])**.** *Let $N \geq 1$ and $2 \leq p < 2^*$, then the following sharp Gagliardo-Nirenberg inequality*

$$\|u\|_p \leq C_{N,p}\|u\|_2^{1-\delta_p}\|\nabla u\|_2^{\delta_p} \tag{2.1}$$

holds for any $u \in H^1(\mathbb{R}^N)$, where $\delta_p = \frac{N}{2} - \frac{N}{p}$, the sharp constant $C_{N,p}$ is

$$C_{N,p}^p = \frac{2p}{2N+(2-N)p}\Big(\frac{2N+(2-N)p}{N(p-2)}\Big)^{\frac{N(p-2)}{4}} \frac{1}{\|Q_p\|_2^{p-2}}$$

and Q_p is the unique positive radial solution of the equation

$$-\Delta Q + Q = |Q|^{p-2}Q.$$

Lemma 2.3 (Gagliardo-Nirenberg inequality of Choquard type[26])**.** *Let $N \geq 3$, $\alpha \in (0, N)$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, we have*

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p dx \leq C_{N,p}\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx\Big)^{\frac{Np-(N+\alpha)}{2}} \Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{\frac{N+\alpha-p(N-2)}{2}}, \tag{2.2}$$

where equality holds for $u = Q_p$, $C_{N,p} = \frac{p}{|Q_p|_2^{2p-2}}$ and Q_p is a nontrivial solution of

$$-\frac{N(p-2)+N-\alpha}{2}\Delta Q_p + \frac{N+\alpha-(N-2)p}{2}Q_p = (I_\alpha * |Q_p|^p)|Q_p|^{p-2}Q_p. \tag{2.3}$$

By [11, (3.3)], we have the following Lemma.

Lemma 2.4. *If $\frac{N+\alpha}{N} < r_1, r_2 < \frac{N+\alpha}{N-2}$, then*

$$\int_{\mathbb{R}^N} (I_\alpha * |u_1|^{r_1})|u_2|^{r_2} dx \leq \Big(\int_{\mathbb{R}^N} (I_\alpha * |u_1|^{r_1})|u_1|^{r_1} dx\Big)^{1/2} \Big(\int_{\mathbb{R}^N} (I_\alpha * |u_2|^{r_2})|u_2|^{r_2} dx\Big)^{1/2},$$

where I_α is given by (1.3).

Lemma 2.5 ([19, Lemma A.2])**.** *If $p \in (0, \frac{N}{N-2}]$ when $N \geq 3$ and $p \in (0, \infty)$ when $N = 1, 2$. Let $u \in L^p(\mathbb{R}^N)$ be a smooth nonnegative function satisfying $-\Delta u \geq 0$ in $\mathbb{R}^N$. Then $u \equiv 0$.*

We now consider the problem

$$\begin{gathered} -(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u = \lambda u + \mu(I_\alpha * |u|^p)|u|^{p-2}u, \\ \int_{\mathbb{R}^N} |u|^2 dx = d > 0, \end{gathered} \tag{2.4}$$

where $a, b > 0$, $\frac{N+\alpha}{N} < p < \frac{N+\alpha+2}{N}$ and $N = 3, 4$. The corresponding minimization problem of (2.4) is

$$\sigma_p^\mu(d) := \inf_{u \in S(d)} G_\mu(u), \tag{2.5}$$

where $G_\mu(u) = \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{\mu}{2p}\int_{\mathbb{R}^N}(I_\alpha * |u|^p)|u|^p dx$.

Lemma 2.6. (i) *For any $d > 0$, we have $\sigma_p^\mu(d) < 0$.*
(ii) *$\sigma_p^\mu(d)$ is continuous with respect to $d \geq 0$.*
(iii) *For any $d \geq m \geq 0$, then $\sigma_p^\mu(d) \leq \sigma_p^\mu(m) + \sigma_p^\mu(d - m)$.*

Proof. Item (i) follows from [24, Theorem 1.2]. For item (ii), we assume that $d^n = d + o_n(1)$. By the definition of $\sigma_p^\mu(d^n)$, for any $\varepsilon > 0$, there exists $u^n \in S_r(d^n)$ such that

$$G_\mu(u^n) \leq \sigma_p^\mu(d^n) + \varepsilon. \tag{2.6}$$

Let $w^n := \frac{u^n}{\|u^n\|_2 d^{1/2}}$. It is easy to check that $w^n \in S_r(d^n)$ and

$$\sigma_p^\mu(d) \leq G_\mu(w^n) = G_\mu(u^n) + o_n(1). \tag{2.7}$$

Combining (2.6) and (2.7), we have

$$\sigma_p^\mu(d) \leq \sigma_p^\mu(d^n) + \varepsilon + o_n(1).$$

Reversing the argument, we obtain similarly that

$$\sigma_p^\mu(d^n) \leq \sigma_p^\mu(d) + \varepsilon + o_n(1).$$

Thus, by the arbitrariness of ε, we have $\sigma_p^\mu(d^n) = \sigma_p^\mu(d) + o_n(1)$. This proves (ii).

By the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, for any $\varepsilon > 0$, there exist $\tilde{\psi}$, $\bar{\psi} \in C_0^\infty(\mathbb{R}^N) \times C_0^\infty(\mathbb{R}^N)$ with $\|\tilde{\psi}\|_2^2 = m$, $\|\bar{\psi}\|_2^2 = d - m$, $i = 1, 2$ such that

$$\begin{gathered} G_\mu(\tilde{\psi}) \leq \sigma_p^\mu(m) + \varepsilon, \\ G_\mu(\bar{\psi}) \leq \sigma_p^\mu(d - m) + \varepsilon. \end{gathered}$$

Since G_μ is invariant by translation, without loss of generality, we may assume that $\operatorname{supp}\tilde{\psi} \cap \operatorname{supp}\bar{\psi} = \emptyset$, and then $\|\tilde{\psi} + \bar{\psi}\|_2^2 = \|\tilde{\psi}\|_2^2 + \|\bar{\psi}\|_2^2 = d$,

$$\sigma_p^\mu(d) \leq G_\mu(\tilde{\psi} + \bar{\psi}) \leq \sigma_p^\mu(m) + \sigma_p^\mu(d - m) + 2\varepsilon.$$

Therefore,

$$\sigma_p^\mu(d) \leq \sigma_p^\mu(m) + \sigma_p^\mu(d - m).$$

This proof is complete. □

Lemma 2.7 ([?, Lemma 2.5]linjie). *Let $\frac{N+\alpha}{N} < r_1, r_2 < \frac{N+\alpha}{N-2}$, if*

$$(u_1^n, u_2^n) \rightharpoonup (u_1, u_2) in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N),$$

then, up to a subsequence,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_1^n|^{r_1})|u_2^n|^{r_2} dx = \int_{\mathbb{R}^N} (I_\alpha * |u_1|^{r_1})|u_2|^{r_2} dx + o_n(1). \tag{2.8}$$

Lemma 2.8. *If* (A1) *holds, then* $\inf_{W(k)} F(u_1, u_2) < 0$ *for all* $k > 0$.

Proof. Let $u^\tau(x) = \tau^{\frac{N}{2}} u(\tau x)$. Then it is easy to check that

$$(u_1, u_2) \in S_r(d_1) \times S_r(d_2), \quad (u_1^\tau, u_2^\tau) \in W(k)$$

when τ is sufficiently small. From

$$\begin{aligned} F(u_1^\tau, u_2^\tau) = &\frac{a}{2}\tau^2 \sum_{i=1}^2 \|\nabla u_i\|_2^2 + \frac{b}{4}\tau^4 \sum_{i=1}^2 \|\nabla u_i\|_2^4 - \sum_{i=1}^2 \frac{\mu_i}{2p_i}\tau^{Np_i - N - \alpha} \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i} dx \\ &- \Theta \tau^{\frac{Nr}{2} - N - \beta} \int_{\mathbb{R}^N} (I_\beta * |u_1|^{p_1})|u_2|^{p_2} dx, \end{aligned}$$

$\frac{N+\alpha}{N} < p_1, p_2 < \frac{N+\alpha+2}{N}$ and $\frac{N+\beta}{N} < r_1, r_2 < 2_\beta^*$, we have $\Phi_{(u_1,u_2)}(\tau) < 0$ for τ small enough. □

Lemma 2.9. *Assume that* (A2) *holds. Then there exist* $k_0 = k_0(d_1, d_2) > 0$, $\Theta_* = \Theta_*(d_1, d_2) > 0$ *such that for any* $0 < \Theta < \Theta_\star$,

$$\inf_{W(2k_0)\setminus W(k_0)} F(u_1, u_2) > 0.$$

And there exists $\varepsilon_0 > 0$ *small enough, such that*

$$\sigma(d_1, d_2) < \inf_{\overline{W(k_0)}\setminus A(k_0 - \varepsilon_0)} F(u_1, u_2).$$

Proof. For $(u_1, u_2) \in S_r(d_1) \times S_r(d_2)$, let $k = \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2$. Then by Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} &F(u_1, u_2) \\ &= \frac{a}{2}k + \frac{b}{4}\sum_{i=1}^2 \|\nabla u_i\|_2^4 - \sum_{i=1}^2 \frac{\mu_i}{2p_i}\int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i} dx - \Theta \int_{\mathbb{R}^N} (I_\beta * |u_1|^{p_1})|u_2|^{p_2} dx \\ &\geq \frac{b}{8}k^2 - \sum_{i=1}^2 B_i \|\nabla u_i\|_2^{Np_i - N - \alpha} - \Theta B_3 k^{\frac{Nr - 2(N+\beta)}{4}} \\ &\geq \frac{b}{8}k^2 - \sum_{i=1}^2 B_i k^{\frac{Np_i - N - \alpha}{2}} - \Theta B_3 k^{\frac{Nr - 2(N+\beta)}{4}} := g(k), \end{aligned} \tag{2.9}$$

where $B_i = \frac{\mu_i}{2p_i} C_{N,p_i} d_i^{\frac{N+\alpha-p_i(N-2)}{2}}$ $(i = 1, 2)$, and $B_3 = C_{N,r_1} C_{N,r_2} d_1^{\frac{N+\beta-r_1(N-2)}{4}} d_2^{\frac{N+\beta-r_2(N-2)}{4}}$. By (A2), we have

$$0 < \frac{Np_i - N - \alpha}{2} < 1 \quad \text{and} \quad \frac{Nr - 2(N+\beta)}{4} > 2, \quad i = 1, 2.$$

We fix $k = k_0$ sufficiently large such that

$$\sum_{i=1}^2 B_i k_0^{\frac{Np_i - N - \alpha - 4}{2}} \leq \frac{b}{24},$$

and fix $\Theta = \Theta_*$ sufficiently small such that

$$\Theta_* B_3 (2k_0)^{\frac{Nr - 2(N+\beta) - 8}{4}} \leq \frac{b}{24}.$$

Therefore, for each $0 < \Theta \le \Theta_*$ and $(u_1, u_2) \in W(2k_0) \setminus W(k_0)$, we obtain

$$\begin{aligned} F(u_1,u_2) &\ge \frac{b}{8}k^2 - \sum_{i=1}^{2} B_i k^{\frac{Np_i-N-\alpha}{2}} - \Theta B_3 k^{\frac{Nr-2(N+\beta)}{4}} \\ &= k^2\Big(\frac{b}{8} - \sum_{i=1}^{2} B_i k^{\frac{Np_i-N-\alpha-4}{2}} - \Theta B_3 k^{\frac{Nr-2(N+\beta)-8}{4}}\Big) \\ &\ge bk_0^2\Big(\frac{1}{8} - \frac{1}{24} - \frac{1}{24}\Big) = \frac{b}{24}k_0^2. \end{aligned}$$

Next, by continuity of $g(k)$ and $g(k_0) > 0$, there exists $\varepsilon_0 > 0$ sufficiently small such that $g(k) \ge 0$ when $k \in [k_0 - \varepsilon, k_0]$. Hence,

$$F(u_1,u_2) \ge g(k) \ge 0 > \sigma(d_1,d_2)$$

for any $(u_1,u_2) \in \overline{W(k_0)} \setminus W(k_0 - \varepsilon_0)$. This proof is complete. ☐

Lemma 2.10. *Let $\frac{N+\alpha}{N} < p_1, p_2 < 2^*_\alpha$ and $\frac{N+\beta}{N} < r_1, r_2 < 2^*_\beta$. If $(u_1,u_2) \ne (0,0)$ is a solution of* (1.1) *for some $(\lambda_1,\lambda_2) \in \mathbb{R}^2$, then $\lambda_1 < 0$ or $\lambda_2 < 0$. Furthermore, if $u_1 > 0$ and $u_2 \ge 0$, then $\lambda_1 < 0$; if $u_1 \ge 0$ and $u_2 > 0$, then $\lambda_2 < 0$.*

Proof. Testing (1.1) by (u_1,u_2) and integrating in $\mathbb{R}^N$, one has

$$\begin{aligned} &\lambda_1\|u_1\|_2^2 + \lambda_2\|u_2\|_2^2 \\ &= a\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + b\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\mu_i\int_{\mathbb{R}^N}(I_\alpha * |u_i|^{p_i})|u_i|^{p_i}dx - \Theta r\int_{\mathbb{R}^N}(I_\beta * |u_1|^{r_1})|u_2|^{r_2}dx. \end{aligned}$$

Combining $P(u_1,u_2) = 0$, $2 < p_1, p_2 < 2^*$ and $\frac{N+\beta}{N} < r_1, r_2 < 2^*_\beta$, we have

$$\begin{aligned} \lambda_1\|u_1\|_2^2 + \lambda_2\|u_2\|_2^2 &= \sum_{i=1}^{2}\mu_i(Np_i - N - \alpha - 1)\int_{\mathbb{R}^N}(I_\alpha * |u_i|^{p_i})|u_i|^{p_i}dx \\ &\quad + \Theta\left(\frac{Nr - 2(N+\beta)}{4} - r\right)\int_{\mathbb{R}^N}(I_\beta * |u_1|^{p_1})|u_2|^{p_2}dx < 0. \end{aligned}$$

Thus, at least one of λ_1 and λ_2 is negative.

Then, we argue by contradiction, suppose $\lambda_1 \ge 0$. In view $u_1 > 0$, we have

$$-\Big(a + b\int_{\mathbb{R}^N}|\nabla u_1|^2dx\Big)\Delta u_1 = \lambda_1 u_1 + \mu_1(I_\alpha * |u_1|^{p_1})|u_1|^{p_1-2}u_1 + \Theta r_1(I_\beta * |u_2|^{p_2})|u_1|^{p_1-2}u_1 \ge 0.$$

By Lemma 2.5, we obtain $u_1 = 0$, which contradicts $u_1 \ne 0$. Therefore, $\lambda_1 < 0$. The other case can be proved in the same way. ☐

Lemma 2.11. *Assume that $2 < p_1, p_2 < \frac{2N}{N-2}$ and $\frac{N+\beta}{N} < r_1, r_2 < \frac{N+\beta}{N-2}$. For any bounded Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for F on $S_r(d_1) \times S_r(d_2)$, then there exist $(u_1,u_2) \in H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ and a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$, such that up to a subsequence*

(a) *$(u_1^n, u_2^n) \rightharpoonup (u_1,u_2)$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$, $(u_1^n, u_2^n) \to (u_1,u_2)$ in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ for $p \in (2, \frac{2N}{N-2})$.*

(b) *$(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$ in $(\mathbb{R}^2)$.*

(c) *$F'(u_1^n, u_2^n) - \lambda_1^n(u_1^n, 0) - \lambda_2^n(0, u_2^n) \to 0$ in $H_r^{-1}(\mathbb{R}^N) \times H_r^{-1}(\mathbb{R}^N)$.*

(d) *(u_1^n, u_2^n) is a solution to the system* (1.1) *for some $\lambda_1, \lambda_2 \le 0$ if (u_1,u_2) satisfies the additional property $P(u_1^n, u_2^n) \to 0$, where (λ_1,λ_2) is given by (b).*

Proof. Obviously, item (a) is true Since $\{(u_1^n, u_2^n)\} \subset H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ is bounded, by [7], we have $(F|_{S_r(d_1)\times S_r(d_2)})'(u_1^n, u_2^n) \to 0$ in $H_r^{-1}(\mathbb{R}^N) \times H_r^{-1}(\mathbb{R}^N)$ is equivalent to

$$F'(u_1^n,u_2^n) - \frac{1}{\|u_1^n\|_2^2}\langle F'(u_1^n,u_2^n),(u_1^n,0)\rangle(u_1^n,0) - \frac{1}{\|u_2^n\|_2^2}\langle F'(u_1^n,u_2^n),(0,u_2^n)\rangle(0,u_2^n) \to 0$$

in $H_r^{-1}(\mathbb{R}^N) \times H_r^{-1}(\mathbb{R}^N)$. Thus, we have

$$F'(u_1^n, u_2^n) - \lambda_1^n(u_1^n, 0) - \lambda_2^n(0, u_2^n) \to 0 \quad \text{in } H_r^{-1}(\mathbb{R}^N) \times H_r^{-1}(\mathbb{R}^N)$$

with

$$\begin{aligned} \lambda_1^n = \frac{1}{\|u_1^n\|_2^2}\Big(& a\|\nabla u_1^n\|_2^2 + b\|\nabla u_1^n\|_2^4 - \mu_1 \int_{\mathbb{R}^N} (I_\alpha * |u_1^n|^{p_1})|u_1^n|^{p_1} dx \\ & - \Theta r_1 \int_{\mathbb{R}^N} (I_\beta * |u_1^n|^{p_1})|u_2^n|^{p_2} dx\Big) - o_n(1), \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \lambda_2^n = \frac{1}{\|u_2^n\|_2^2}\Big(& a\|\nabla u_2^n\|_2^2 + b\|\nabla u_2^n\|_2^4 - \mu_2 \int_{\mathbb{R}^N} (I_\alpha * |u_2^n|^{p_2})|u_2^n|^{p_2} dx \\ & - \Theta r_2 \int_{\mathbb{R}^N} (I_\beta * |u_1^n|^{p_1})|u_2^n|^{p_2} dx\Big) - o_n(1). \end{aligned} \tag{2.11}$$

This proves (c). By the boundedness of u_1^n, u_2^n in $H_r^1(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$, Lemma 2.3 and Lemma 2.4, we obtain $\{\lambda_1^n\}$, $\{\lambda_2^n\}$ are bounded. This proves (b). Combining (b) and (c), it is now standard to deduce (d). □

Lemma 2.12. *Under the conditions of Lemma 2.11, $u_1^n \to u_1$ in $H_r^1(\mathbb{R}^N)$ if $\lambda_1 < 0$. Similarly, $u_2^n \to u_2$ in $H_r^1(\mathbb{R}^N)$ if $\lambda_2 < 0$.*

Proof. In view of Lemmas 2.7 and 2.11, we obtain

$$\int_{\mathbb{R}^N} (I_\alpha * |u_i^n|^{p_i})|u_i^n|^{p_i} dx \to \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i} dx (\frac{N+\alpha}{N} < p_i < 2_\alpha^*, \quad i = 1, 2), \tag{2.12}$$

$$\int_{\mathbb{R}^N} (I_\beta * |u_1^n|^{r_1})|u_2^n|^{r_2} dx \to \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1})|u_2|^{r_2} dx (\frac{N+\beta}{N} < r_1, r_2 < 2_\beta^*), \tag{2.13}$$

$$\langle F'(u_1^n, u_2^n) - \lambda_1^n(u_1^n, 0), (u_1^n, 0)\rangle \to 0 = \langle F'(u_1, u_2) - \lambda_1^n(u_1, 0), (u_1, 0)\rangle. \tag{2.14}$$

Therefore,

$$a\|\nabla u_1^n\|_2^2 + b\|\nabla u_1^n\|_2^4 - \lambda_1^n\|u_1^n\|_2^2 \to a\|\nabla u_1\|_2^2 + b\|\nabla u_1\|_2^4 - \lambda_1\|u_1\|_2^2.$$

Since

$$\|u_1\|_2^2 \le \varliminf_{n\to+\infty} \|u_1^n\|_2^2, \quad \|\nabla u_1\|_2^2 \le \varliminf_{n\to+\infty} \|\nabla u_1^n\|_2^2, \tag{2.15}$$

it follows that

$$\|\nabla u_1^n\|_2^2 \to \|\nabla u_1\|_2^2, \quad \|u_1^n\|_2^2 \to \|u_1\|_2^2.$$

This proof is complete. □

3. Proof of Theorem 1.2

Lemma 3.1. *Assume that* (A1) *holds. Then F is bounded form below and coercive on $S_r(d_1) \times S_r(d_2)$. Furthermore, there exists a bounded Palais-Smale sequence $\{(u_1^n, u_2^n)\} \subset S_r(d_1) \times S_r(d_2)$, which satisfies $(u_1^n)^- \to 0$ and $(u_2^n)^- \to 0$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$.*

Proof. Since $\frac{N+\alpha}{N} < p_1, p_2 < \frac{N+\alpha+2}{N}$, $2 \cdot \frac{N+\beta}{N} < r_1 + r_2 < 2 \cdot \frac{N+\beta+4}{N}$, we have $0 < Np_i - N - \alpha < 2$ and $\frac{Nr-2(N+\beta)}{4} < 2$. From (2.9), we know that F is bounded form below and coercive on $S_r(d_1) \times S_r(d_2)$.

Next, let $\{(u_1^n, u_2^n)\} \subset S_r(d_1) \times S_r(d_2)$ be a minimizing sequence for F. By the coerciveness of F, the sequence is bounded. Since the functional F is even, we may assume that $w_1^n \ge 0$ and $w_2^n \ge 0$. It is easy to check that F is a C^1-manifold in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$, then by Ekeland's variational principle, there exists a minimizing sequence $\{(u_1^n, u_2^n)\} \subset S_r(d_1) \times S_r(d_2)$, which is the Palais-Smale sequence for F restricted to $S_r(d_1) \times S_r(d_2)$ and satisfies $\|(u_1^n, u_2^n) - (w_1^n, w_2^n)\|_{H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)} \to 0$ as $n \to +\infty$. From $w_1^n \ge 0$ and $w_2^n \ge 0$, we have $(u_1^n)^- \to 0$ and $(u_2^n)^- \to 0$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. □

Proof of Theorem 1.2. By Lemma 3.1, there exists a bounded Palais-Smale sequence $\{(u_1^n, u_2^n)\} \subset S_r(d_1)\times S_r(d_2)$ and $(u_1, u_2) \in H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N)$ with $u_1 \geq 0$ and $u_2 \geq 0$, such that $\lim_{n\to+\infty}(u_1^n, u_2^n) = \sigma(d_1, d_2)$, $(u_1^n, u_2^n) \rightharpoonup (u_1, u_2)$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. Hence, it suffices to prove that $(u_1^n, u_2^n) \to (u_1, u_2)$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. Indeed, if this holds, then we have $\{(u_1^n, u_2^n)\} \subset S_r(d_1) \times S_r(d_2)$ and $F(u_1, u_2) = \sigma(d_1, d_2)$. Furthermore, by the strong maximum principle, we have $u_1 > 0$ and $u_2 > 0$. From $\Theta > 0$, obviously, we have $\sigma(d_1, d_2) \leq \sigma_{p_1}^{\mu_1}(d_1) + \sigma_{p_2}^{\mu_2}(d_2)$. By Lemma 2.8, we know $\sigma(d_1, d_2) < 0$. We divide four cases to have that $u_1 > 0$ and $u_2 > 0$.

Case I: $u_1 = 0$ and $u_2 = 0$. Obviously, we can obtain that $\lim_{n\to+\infty} F(u_1^n, u_2^n) = 0$, which is contradicts $\sigma(d_1, d_2) < 0$.

Case II: $u_1 = 0$ and $u_2 \neq 0$. Combining (2.12) and (2.15), we have

$$\varliminf_{n\to+\infty} F(u_1^n, u_2^n) \geq \frac{a}{2}\|\nabla u_2\|_2^2 + \frac{b}{4}\|\nabla u_2\|_2^4 - \frac{\mu_2}{2p_2}\int_{\mathbb{R}^N}(I_\alpha * |u_2|^{p_2})|u_2|^{p_2}dx \geq \sigma_{p_2}^{\mu_2}(d_3), \tag{3.1}$$

where $d_3 := \|u_2\|_2^2 \leq d_2$. By (i) and (iii) of Lemma 2.6, we have $\sigma_{p_2}^{\mu_2}(d_3) \geq \sigma_{p_2}^{\mu_2}(d_2)$. Thus, by (3.1), we have $\sigma(d_1, d_2) \geq \sigma_{p_2}^{\mu_2}(d_3)$. On the other hand, using Lemma 2.6 again, we have $\sigma_{p_1}^{\mu_1}(d_1) < 0$. In view of $\sigma(d_1, d_2) \leq \sigma_{p_1}^{\mu_1}(d_1) + \sigma_{p_2}^{\mu_2}(d_2)$, we have $\sigma(d_1, d_2) < \sigma_{p_2}^{\mu_2}(d_2)$. This is a contradiction.

Case III: $u_1 \neq 0$ and $u_2 = 0$. By the same proof as in Case II, we can obtain a contradiction.

Case IV: $u_1 \neq 0$ and $u_2 \neq 0$. From Lemma 2.10, we have $\lambda_1 < 0$ and $\lambda_2 < 0$. Thus, by Lemma 2.12, we obtain $u_1^n \to u_1$ and $u_2^n \to u_2$ in $H_r^1(\mathbb{R}^N)$. This proof is complete. □

4. Proof of Theorem 1.3

Let $k = \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2$, then for each $(u_1, u_2) \in S_r(d_1) \times S_r(d_2)$, by the same proof as in Lemma 2.9, we obtain

$$\begin{aligned}
&F(u_1, u_2)\\
&= \frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{2p_i}\int_{\mathbb{R}^N}(I_\alpha * |u_i|^{p_i})|u_i|^{p_i}dx - \Theta\int_{\mathbb{R}^N}(I_\beta * |u_1|^{p_1})|u_2|^{p_2}dx\\
&\geq \frac{b}{8}k^2 - B_1 k^{\frac{Np_1-N-\alpha}{2}} - B_2 k^{\frac{Np_2-N-\alpha}{2}} - \Theta B_3 k^{\frac{Nr-2(N+\beta)}{4}} := g(k),
\end{aligned}$$

where $B_i = \frac{\mu_i}{2p_i}C_{N,p_i}d_i^{\frac{N+\alpha-p_i(N-2)}{2}}$ $(i = 1, 2)$, and $B_3 = C_{N,r_1}C_{N,r_2}d_1^{\frac{N+\beta-r_1(N-2)}{4}}d_2^{\frac{N+\beta-r_2(N-2)}{4}}$.

By Lemmas 2.8 and 2.9, we consider the minimization problem

$$\sigma(d_1, d_2) := \inf_{W(k_0)} F(u_1, u_2) < 0 \text{for any} 0 < \Theta \leq \Theta_*.$$

Lemma 4.1. *Assume that* (A1) *holds and* $0 < \alpha - 2 \leq \beta < \alpha < N$. *Then there exists* $\Theta_1 > 0$, *such that if* $0 < \Theta < \Theta_1$, *then the function* $g(k)$ *has a unique local minimum point at the negative level and a unique global maximum point at the positive level. Moreover, there exists* $0 < k_0 < k_1$, *such that* $g(k_0) = g(k_1) = 0$ *and* $g(k) > 0$ *if and only if* $k \in (k_0, k_1)$.

Proof. Without loss of generality, we assume that $p_1 \geq p_2$.

Case I: $p_1 = p_2 = p$. For $k > 0$, we have

$$\begin{aligned}
g(k) &= \frac{b}{8}k^2 - (B_1 + B_2)k^{\frac{Np-N-\alpha}{2}} - \Theta B_3 k^{\frac{Nr-2(N+\beta)}{4}}\\
&= k^{\frac{Np-N-\alpha}{2}}\Big(\frac{b}{8}k^{\frac{4+N+\alpha-Np}{2}} - (B_1 + B_2) - \Theta B_3 k^{\frac{Nr-2Np+2(\alpha-\beta)}{4}}\Big).
\end{aligned}$$

Define $h(k) = \frac{b}{8}k^{\frac{4+N+\alpha-Np}{2}} - \Theta B_3 k^{\frac{Nr-2Np+2(\alpha-\beta)}{4}}$, then $g(k) > 0 \Leftrightarrow h(k) > B_1 + B_2$. Setting

$$\tilde{k} = \Big(\frac{b(4 + N + \alpha - Np)}{4\Theta B_3(Nr - 2Np + 2(\alpha - \beta))}\Big)^{\frac{4}{Nr-2(N+\beta+4)}},$$

it is easy to check that $h(k)$ is increasing in $(0, \tilde{k})$, decreasing in $(\tilde{k}, +\infty)$ and

$$h(\tilde{k}) = M\Theta^{-\frac{2(4+N+\alpha-Np)}{Nr-2(N+\beta+4)}},$$

where M is a positive constant. Hence, we can take Θ small enough, such that $h(\tilde{k}) > B_1 + B_2$. Thus, there exists $\Theta_1 > 0$, such that $0 < \Theta < \Theta_1$, which implies that $g(k) > 0$ on $k \in (k_0, k_1)$. Since $g(k) \to 0^-$ as $k \to 0^+$, $g(k)$ has a local minimum point on $(0, k_0)$. Therefore, $g(k)$ has at least two critical points.

For $k > 0$, we define

$$\varphi(k) = \frac{b}{4} k^{\frac{4+N+\alpha-Np}{2}} - \frac{Nr - 2(N+\beta)}{4} \Theta B_3 k^{\frac{Nr-2Np+2(\alpha-\beta)}{4}}.$$

By direct computation, $g'(k) = k^{\frac{Np-N-\alpha-2}{2}}(\varphi(k) - (B_1 + B_2)(Np - N - \alpha))$ and $g'(k) = 0 \Leftrightarrow \varphi(k) = (B_1 + B_2)(Np - N - \alpha)$. Since $\varphi(k)$ has a unique global maximum point, $\varphi(k) = (B_1 + B_2)(Np - N - \alpha)$ has at most two solutions, i.e., $g(k)$ has at most two critical points. Thus, $g(k)$ has a unique local minimum point at the negative level and a unique global maximum point at the positive level.

Case II: $p_1 > p_2$. In this case, we have

$$\begin{aligned} g(k) &= \frac{b}{8} k^2 - B_1 k^{\frac{Np_1-N-\alpha}{2}} - B_2 k^{\frac{Np_2-N-\alpha}{2}} - \Theta B_3 k^{\frac{Nr-2(N+\beta)}{4}} \\ &= k^{\frac{Np_2-N-\alpha}{2}} \Big(\frac{b}{8} k^{\frac{4+N+\alpha-Np_2}{2}} - B_1 k^{\frac{Np_1-Np_2}{2}} - B_2 - \Theta B_3 k^{\frac{Nr-2Np_2+2(\alpha-\beta)}{4}} \Big) \\ &=: k^{\frac{Np_2-N-\alpha}{2}} (Q(k) - B_2). \end{aligned}$$

By calculation and analysis, $Q(k)$ has a unique global maximum point $\bar{k}$ with

$$\bar{k} > \Big(\frac{b(4+N+\alpha-Np_1)(4+N+\alpha-Np_2)}{2B_3\Theta(Nr-2Np_1+2(\alpha-\beta))(Nr-2Np_2+2(\alpha-\beta))} \Big)^{\frac{4}{Nr-2(N+\beta+4)}} := k_*,$$

and $Q(\bar{k}) > Q(k_*) > \bar{k}^{\frac{N(p_2-p_1)}{2}}$ for Θ small enough. Therefore, we can take Θ small enough, such that $Q(\bar{k}) > B_2$. The rest of the proof is similar to the case $p_1 = p_2$. □

Lemma 4.2. *Assume that* (A1) *holds and* $0 < \alpha - 2 \leq \beta < \alpha < N$. *Then there exists* $\Theta_2 > 0$, *such that if* $0 < \Theta < \Theta_2$, *then* $\mathcal{P}_0(d_1, d_2) = \emptyset$, *and* $\mathcal{P}$ *is a submanifold of* $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$.

Proof. We argue by contradiction. Suppose that there exists $(u_1, u_2) \in \mathcal{P}_0(d_1, d_2)$. Then

$$\begin{aligned} & a\sum_{i=1}^2 \|\nabla u_i\|_2^2 + b\sum_{i=1}^2 \|\nabla u_i\|_2^4 - \sum_{i=1}^2 \frac{\mu_i}{2p_i}(Np_i - N - \alpha) \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i} dx \\ & - \Theta\Big(\frac{N}{2} - N - \beta\Big) \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1})|u_2|^{r_2} dx = 0 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} & 2a\sum_{i=1}^2 \|u_i\|_2^2 + 4b\sum_{i=1}^2 \|u_i\|_2^4 - \sum_{i=1}^2 \frac{\mu_i}{2p_i}(Np_i - N - \alpha)^2 \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i} dx \\ & - \Theta\Big(\frac{Nr}{2} - N - \beta\Big)^2 \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1})|u_2|^{r_2} dx = 0. \end{aligned} \tag{4.2}$$

By (4.1) and (4.2), we have

$$\begin{aligned} & \Big(\frac{Nr}{2} - N - \beta - 2\Big) a \sum_{i=1}^2 \|u_i\|_2^2 + (\frac{Nr}{2} - N - \beta - 4) b \sum_{i=1}^2 \|u_i\|_2^4 \\ & = \sum_{i=1}^2 \frac{\mu_i}{2p_i} \Big(\frac{Nr}{2} - Np_i + \alpha - \beta\Big) \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i} dx. \end{aligned} \tag{4.3}$$

On the one hand, by (4.3), we have

$$\frac{b}{2}(\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2)^2 \leq b(\|\nabla u_1\|_2^4 + \|\nabla u_2\|_2^4) + \frac{\frac{Nr}{2} - N - \beta - 2}{\frac{Nr}{2} - N - \beta - 4} a \sum_{i=1}^2 \|\nabla u_i\|_2^2$$

$$
\begin{aligned}
&= \frac{1}{\frac{Nr}{2} - N - \beta - 4} \sum_{i=1}^{2} \frac{\mu_i}{2p_i} \Big(\frac{Nr}{2} - Np_i + \alpha - \beta \Big) \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i}) |u_i|^{p_i} dx \\
&\le \sum_{i=1}^{2} \mathbb{K}_i (\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2)^{\frac{Np_i - N - \alpha}{2}},
\end{aligned}
$$

where

$$
\mathbb{K}_i := \frac{1}{\frac{Nr}{2} - N - \beta - 4} \frac{\mu_i}{2p_i} \Big(\frac{Nr}{2} - Np_i + \alpha - \beta \Big) C_{N,p_i,\alpha}^{p_i} d_i^{\frac{N+\alpha - p_i(N-2)}{2}}.
$$

Thus, there exists $\mathbb{K}_3 > 0$, such that

$$
\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 \le \mathbb{K}_3. \tag{4.4}
$$

On the other hand, combining (4.1) and (4.2), we have

$$
\begin{aligned}
&2b \sum_{i=1}^{2} \|u_i\|_2^4 + \sum_{i=1}^{2} \frac{\mu_i}{2p_i} (Np_i - N - \alpha)(N + \alpha + 2 - Np_i) \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i}) |u_i|^{p_i} dx \\
&= \Theta \Big(\frac{Nr}{2} - N - \beta \Big) \Big(\frac{Nr}{2} - N - \beta - 2 \Big) \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\sum_{i=1}^{2} \frac{\mu_i}{2p_i} (Np_i - N - \alpha)(N + \alpha + 2 - Np_i) \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i}) |u_i|^{p_i} dx \\
&\le 2b \sum_{i=1}^{2} \|u_i\|_2^4 + \sum_{i=1}^{2} \frac{\mu_i}{2p_i} (Np_i - N - \alpha)(N + \alpha + 2 - Np_i) \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i}) |u_i|^{p_i} dx \\
&= \Theta \Big(\frac{Nr}{2} - N - \beta \Big) \Big(\frac{Nr}{2} - N - \beta - 2 \Big) \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx.
\end{aligned} \tag{4.5}
$$

If $\int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx = 0$, by (4.5), we have $u_1 = u_2 = 0$, which is a contradiction. By (4.5), we obtain

$$
\begin{aligned}
&\frac{b}{2} (\|u_1\|_2^2 + \|u_2\|_2^2)^2 \\
&\le b(\|u_1\|_2^4 + \|u_2\|_2^4) + a(\|u_1\|_2^2 + \|u_2\|_2^2) \\
&= \sum_{i=1}^{2} \frac{\mu_i}{2p_i} (Np_i - N - \alpha) \int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i}) |u_i|^{p_i} dx + \Theta \Big(\frac{Nr}{2} - N - \beta \Big) \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx \\
&\le \Theta \Big(\frac{Nr}{2} - N - \beta \Big) \Big(\frac{\frac{Nr}{2} - N - \beta - 2}{N + \alpha + 2 - N \cdot \min\{p_1, p_2\}} + 1 \Big) \int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1}) |u_2|^{r_2} dx \\
&\le \Theta \mathbb{K}_4 (\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2)^{\frac{Nr - 2(N+\beta)}{4}}.
\end{aligned}
$$

Since $\frac{rN - 2(N+\beta)}{4} > 2$, then we have

$$
\|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 \ge \Big(\frac{b}{2\mathbb{K}_4 \Theta} \Big)^{\frac{4}{Nr - 2(N+\beta+4)}}. \tag{4.6}
$$

Combining (4.4) and (4.6), we obtain

$$
\Big(\frac{b}{2\mathbb{K}_4 \Theta} \Big)^{\frac{4}{Nr - 2(N+\beta+4)}} \le \|\nabla u_1\|_2^2 + \|\nabla u_2\|_2^2 \le \mathbb{K}_3;
$$

it does not hold for Θ small enough.

Next, we show that $\mathcal{P}(d_1, d_2)$ is a submanifold of $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. Note that

$$
\mathcal{P}(d_1, d_2) := \{(u_1, u_2) \in H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N) : P(u_1, u_2) = 0, \mathbb{S}_1(u_1) = 0, \mathbb{S}_2(u_2) = 0\},
$$

where $\mathbb{S}_1(u_1) = \|u_1\|_2^2 - d_1$, $\mathbb{S}_2(u_2) = \|u_2\|_2^2 - d_2$. It suffices to show that $d(P, \mathbb{S}_1, \mathbb{S}_2) : H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N) \to \mathbb{R}^3$ is surjective. Otherwise, by the independence of $d\mathbb{S}_1(u_1)$ and $d\mathbb{S}_2(u_2)$, $dP(u_1, u_2)$

must be a linear combination of $d\mathbb{S}_1(u_1)$ and $d\mathbb{S}_2(u_2)$, i.e., there exists $\kappa_1, \kappa_2 \in \mathbb{R}$, such that (u_1, u_2) is a weak solution of

$$
\begin{aligned}
&-(a+2b\int_{\mathbb{R}^N}|\nabla u_1|^2dx)\Delta u_1\\
&=\kappa_1 u_1+\mu_1\frac{Np_1-N-\alpha}{2}(I_\alpha*|u_1|^{p_1})|u_1|^{p_1-2}u_1+\Theta r_1\frac{Nr-2(N+\beta)}{4}(I_\beta*|u_2|^{p_2})|u_1|^{p_1-2}u_1,\\
&-(a+2b\int_{\mathbb{R}^N}|\nabla u_2|^2dx)\Delta u_2\\
&=\kappa_2 u_2+\mu_2\frac{Np_2-N-\alpha}{2}(I_\alpha*|u_2|^{p_2})|u_2|^{p_2-2}u_2+\Theta r_2\frac{Nr-2(N+\beta)}{4}(I_\beta*|u_1|^{p_1})|u_2|^{p_2-2}u_2.
\end{aligned}
$$

Testing the above system with (u_1, u_2) and combining with Pohožaev identity, we obtain $(\Phi_{(u_1,u_2)})''(0) = 0$. Then $(u_1, u_2) \in \mathcal{P}_0(d_1, d_2) = 0$, which is contradicted to $\mathcal{P}_0(d_1, d_2) = \emptyset$. The proof is complete. □

Let $\Theta_* = \min\{\Theta_1, \Theta_2\}$, $\Pi_1 := \{(u_1, u_2) \in H^1_r(\mathbb{R}^N) \times H^1_r(\mathbb{R}^N) : \int_{\mathbb{R}^N}(I_\beta * |u_1|^{r_1})|u_2|^{r_2}dx > 0\}$ and $\Pi_2 := \{(u_1, u_2) \in H^1_r(\mathbb{R}^N) \times H^1_r(\mathbb{R}^N) : \int_{\mathbb{R}^N}(I_\beta * |u_1|^{r_1})|u_2|^{r_2}dx = 0\}$. Next, we analyze the geometric structure of $\Phi_{(u_1,u_2)}(\tau)$.

Lemma 4.3. *Assume that* (A1) *holds and let $0 < \Theta < \Theta_*$. Then for each $(u_1, u_2) \in S_r(d_1) \times S_r(d_2) \cap \Pi_1$, $\Phi_{(u_1,u_2)}(\tau)$ has exactly two critical points $\tau_{(u_1,u_2)} < s_{(u_1,u_2)}$ and two zeros $e_{(u_1,u_2)} < f_{(u_1,u_2)}$ with $\tau_{(u_1,u_2)} < e_{(u_1,u_2)} < s_{(u_1,u_2)} < f_{(u_1,u_2)}$. Moreover,*

- (i) *$\tau \star (u_1, u_2) \in \mathcal{P}_+(d_1, d_2)$ if and only if $\tau = \tau_{(u_1,u_2)}$, $\tau \star (u_1, u_2) \in \mathcal{P}_-(d_1, d_2)$ if and only if $\tau = s_{(u_1,u_2)}$.*
- (ii) *If $\tau < e_{(u_1,u_2)}$, then $\|\nabla(\tau\star u_1)\|_2^2+\|\nabla(\tau\star u_2)\|_2^2 \le k_0$, and $F(\tau_{(u_1,u_2)}\star(u_1,u_2)) = \min\{F(\tau\star(u_1,u_2)) : \tau\in\mathbb{R}, \|\nabla(\tau\star u_1)\|_2^2+\|\nabla(\tau\star u_2)\|_2^2 < k_0\} < 0$. Moreover, $F(s_{(u_1,u_2)}\star(u_1,u_2)) = \max_{\tau\in\mathbb{R}} F(\tau\star(u_1,u_2))$.*
- (iii) *The maps $(u_1,u_2)\mapsto \tau_{(u_1,u_2)}$ and $(u_1,u_2)\mapsto s_{(u_1,u_2)}$ are of class C^1.*

Proof. For $(u_1, u_2) \in S_r(d_1, d_2)$, we obtain

$$
\begin{aligned}
(\Phi_{(u_1,u_2)})'(\tau) &= ae^{2\tau}\sum_{i=1}^2\|\nabla u_i\|_2^2+be^{4\tau}\sum_{i=1}^2\|\nabla u_i\|_2^4-\sum_{i=1}^2\frac{\mu_i}{2p_i}e^{(Np_i-N-\alpha)\tau}\int_{\mathbb{R}^N}(I_\alpha*|u_i|^{p_i})|u_i|^{p_i}dx\\
&\quad-\Theta(\frac{Nr}{2}-N-\beta)e^{(\frac{Nr}{2}-N-\beta)\tau}\int_{\mathbb{R}^N}(I_\beta*|u_1|^{r_1})|u_2|^{r_2}dx\\
&=a\sum_{i=1}^2\|\nabla(\tau\star u_i)\|_2^2+b\sum_{i=1}^2\|\nabla(\tau\star u_i)\|_2^4\\
&\quad-\sum_{i=1}^2\frac{\mu_i}{2p_i}(Np_i-N-\alpha)\int_{\mathbb{R}^N}(I_\alpha*|\tau\star u_i|^{p_i})|\tau\star u_i|^{p_i}dx\\
&\quad-\Theta(\frac{Nr}{2}-N-\beta)\int_{\mathbb{R}^N}(I_\beta*|\tau\star u_1|^{r_1})|\tau\star u_2|^{r_2}dx\\
&=P(\tau\star u_1,\tau\star u_2).
\end{aligned}
$$

Therefore, $\tau \star (u_1, u_2) \in \mathcal{P}(d_1, d_2)$ if and only if $(\Phi_{(u_1,u_2)})'(\tau) = 0$. Obviously,

$$
\Phi_{(u_1,u_2)}(\tau) = F(\tau\star u_1, \tau\star u_2) \ge g(e^{2\tau}k).
$$

By Lemma 4.1, we have

$$
\Phi_{(u_1,u_2)}(\tau) > 0, \quad \text{for all } \tau \in \Big(\frac12\ln\frac{k_0}{k}, \frac12\ln\frac{k_1}{k}\Big).
$$

In view of $\Phi_{(u_1,u_2)}(-\infty) = 0^-$ and $\Phi_{(u_1,u_2)}(+\infty) = -\infty$, it follows that $\Phi_{(u_1,u_2)}(\tau)$ has at least two critical points $\tau_{u_1,u_2} < s_{(u_1,u_2)}$, where $s_{(u_1,u_2)}$ is global maximum point at positive level, $\tau_{(u_1,u_2)}$ is a local minimum point on $(-\infty, \frac{1}{2}\ln\frac{k_0}{k})$ at negative level. Similar to Lemma 4.1, $\Phi_{(u_1,u_2)}(\tau)$ has at most two critical points. Therefore, $\Phi_{(u_1,u_2)}(\tau)$ has exactly the two critical points $\tau_{(u_1,u_2)}$ and $s_{(u_1,u_2)}$. From $\tau\star(u_1,u_2)\in\mathcal{P}(d_1,d_2)$ if and only $\Phi'_{(u_1,u_2)}(\tau)=0$, then $\tau\star(u_1,u_2)\in\mathcal{P}(d_1,d_2)$ if and only $\tau=\tau_{(u_1,u_2)}$ or $\tau=s_{(u_1,u_2)}$. Since $\tau_{(u_1,u_2)}$ is a local minimum point, we have that

$$(\Phi_{\tau_{(u_1,u_2)}\star(u_1,u_2)})''(0)=(\Phi''_{(u_1,u_2)}(\tau_{(u_1,u_2)})\ge 0.$$

Since $\mathcal{P}_0(d_1,d_2)=\emptyset$, it follows that $(\Phi_{\tau_{(u_1,u_2)}\star(u_1,u_2)})''(0)>0$. Then $\tau_{(u_1,u_2)}\star(u_1,u_2)\in\mathcal{P}_+(d_1,d_2)$ and $s_{(u_1,u_2)}\star(u_1,u_2)\in\mathcal{P}_-(d_1,d_2)$. By the monotonicity, $\Phi_{(u_1,u_2)}(\tau)$ has exactly two zeros $e_{(u_1,u_2)}<f_{(u_1,u_2)}$ with $\tau_{(u_1,u_2)}<e_{(u_1,u_2)}<s_{(u_1,u_2)}<f_{(u_1,u_2)}$. Finally, let $\Psi(\tau,u_1,u_2):=(\Phi_{(u_1,u_2)})'(\tau)$. Then

$$\begin{gathered}\Psi(\tau_{(u_1,u_2)},u_1,u_2)=\Psi(s_{(u_1,u_2)},u_1,u_2)=0\\ \partial_\tau\Psi(\tau_{(u_1,u_2)},u_1,u_2)=(\Phi_{(u_1,u_2)})''(\tau_{(u_1,u_2)})>0,\\ \partial_s\Psi(s_{(u_1,u_2)},u_1,u_2)=(\Phi_{(u_1,u_2)})''(s_{(u_1,u_2)})<0.\end{gathered}$$

Applying the implicit function theorem, the maps $(u_1,u_2)\mapsto\tau_{(u_1,u_2)}$ and $(u_1,u_2)\mapsto s_{(u_1,u_2)}$ are of class C^1. □

Lemma 4.4. *Assume that* (A1) *holds and let* $0<\Theta<\Theta_*$. *Then for each* $(u_1,u_2)\in S_r(d_1)\times S_r(d_2)\cap\Pi_2$, $\Phi_{(u_1,u_2)}(\tau)$ *has a unique critical point* $\tau_{(u_1,u_2)}$ *and a zero* $e_{(u_1,u_2)}$ *with* $\tau_{(u_1,u_2)}<e_{(u_1,u_2)}$. *Moreover,*

- (i) $\mathcal{P}(d_1,d_2)=\mathcal{P}_+(d_1,d_2)$ *and* $\tau\star(u_1,u_2)\in\mathcal{P}_+(d_1,d_2)$ *if and only if* $\tau=\tau_{(u_1,u_2)}$.
- (ii) $F(\tau_{(u_1,u_2)}\star(u_1,u_2))=\min_{\tau\in\mathbb{R}}F(\tau\star(u_1,u_2))$.
- (iii) $\|\nabla(\tau\star u_1)\|_2^2+\|\nabla(\tau\star u_2)\|_2^2<k_0$ *for all* $\tau<e_{(u_1,u_2)}$.

Proof. Without loss of generality, we assume that $p_1\ge p_2$. It is easy to see that $\Phi_{(u_1,u_2)}(\tau)\to 0^-$ as $\tau\to-\infty$, and $\Phi_{(u_1,u_2)}(\tau)\to+\infty$ as $\tau\to+\infty$ for any $(u_1,u_2)\in S_r(d_1,d_2)\cap\Pi_2$. Hence, $\Phi_{(u_1,u_2)}(\tau)$ has a global minimum point $\tau_{(u_1,u_2)}$ at negative level. Note that $(\Phi_{(u_1,u_2)})'(\tau)=0$ if and only if

$$\begin{aligned}&ae^{(N+\alpha+2-Np_2)\tau}\sum_{i=1}^2\|\nabla u_i\|_2^2+be^{(N+\alpha+4-Np_2)\tau}\sum_{i=1}^2\|\nabla u_i\|_2^4\\&-\frac{\mu_1}{2p_1}(Np_1-N-\alpha)e^{N(p_1-p_2)\tau}\int_{\mathbb{R}^N}(I_\alpha*|u_1|^{p_1})|u_1|^{p_1}dx\\&=\frac{\mu_2}{2p_2}(Np_2-N-\alpha)\int_{\mathbb{R}^N}(I_\alpha*|u_2|^{p_2})|u_2|^{p_2}dx.\end{aligned}$$

By direct calculation, it is not difficult to check that equation has exactly one solution. Hence, $\tau\star(u_1,u_2)\in\mathcal{P}(d_1,d_2)$ if and only if $\tau=\tau_{(u_1,u_2)}$. Since $\tau_{(u_1,u_2)}$ is global minimum point of $\Phi_{(u_1,u_2)}(\tau)$, then $(\Phi_{(u_1,u_2)})''(\tau_{(u_1,u_2)})\ge 0$. Since $\mathcal{P}_0(d_1,d_2)=\emptyset$, we obtain that $(\Phi_{\tau_{(u_1,u_2)}\star(u_1,u_2)})''(0)>0$, and thus $\tau_{(u_1,u_2)}\star(u_1,u_2)\in\mathcal{P}_+(d_1,d_2)$. Furthermore, using the monotonicity and the behavior at infinity, $\Phi_{(u_1,u_2)}(\tau)$ has a unique zero $e_{(u_1,u_2)}$ with $\tau_{(u_1,u_2)}<e_{(u_1,u_2)}$. By $\Phi_{(u_1,u_2)}(\tau)\ge g(e^{2\tau}(\|\nabla u_1\|_2^2+\|\nabla u_2\|_2^2))$, then $\Phi_{(u_1,u_2)}(\tau)\ge g(k_0)=0$ at $\tau=\frac{1}{2}\ln\frac{k_0}{\|\nabla u_1\|_2^2+\|\nabla u_2\|_2^2}$. Therefore, $\|\nabla(\tau\star u_1)\|_2^2+\|\nabla(\tau\star u_2)\|_2^2<k_0$ for all $\tau<e_{(u_1,u_2)}$. □

Lemma 4.5. *Assume that* (A1) *holds. If* $0<\Theta<\Theta_*$, *then*

$$\sigma(d_1,d_2)=\inf_{\mathcal{P}(d_1,d_2)}F(u_1,u_2)=\inf_{\mathcal{P}_+(d_1,d_2)}F(u_1,u_2),$$

and there exists $\varepsilon_0>0$ *small enough, such that*

$$\sigma(d_1,d_2)<\inf_{\overline{W(k_0)}\setminus W(k_0-\varepsilon_0)}F(u_1,u_2).$$

Proof. We first prove that $\sigma(d_1,d_2)=\inf_{\mathcal{P}_+(d_1,d_2)}F(u_1,u_2)$. For any $(u_1,u_2)\in\mathcal{P}_+(d_1,d_2)$, $\tau_{(u_1,u_2)}=0$. By Lemma 4.3, $0<\frac{1}{2}\ln\frac{k_0}{\|\nabla u_1\|_2^2+\|\nabla u_2\|_2^2}$, so $k_0>\|\nabla u_1\|_2^2+\|\nabla u_2\|_2^2$. Hence, $\mathcal{P}_+(d_1,d_2)\subset W(k_0)$ and

$$\sigma(d_1,d_2)\le\inf_{\mathcal{P}_+(d_1,d_2)}F(u_1,u_2).$$

On the other hand, for any $(u_1,u_2)\in W(k_0)$, there exists a unique $\tau_{(u_1,u_2)}\in\mathbb{R}$, such that

$$\tau_{(u_1,u_2)}\star(u_1,u_2)\in\mathcal{P}_+\subset W(k_0).$$

Using (ii) in Lemma 4.3 and Lemma 4.4, we have

$$F(\tau_{(u_1,u_2)}\star(u_1,u_2))=\min\{F(\tau\star(u_1,u_2)):\tau\in\mathbb{R},\|\nabla(\tau\star u_1)\|_2^2+\|\nabla(\tau\star u_2)\|_2^2<k_0\}\le F(u_1,u_2).$$

Hence, $\sigma(d_1,d_2)\ge\inf_{\mathcal{P}_+(d_1,d_2)}F(u_1,u_2)$. Therefore, $\sigma(d_1,d_2)=\inf_{\mathcal{P}_+(d_1,d_2)}F(u_1,u_2)$. By (ii) in Lemma 4.3, we obtain

$$\inf_{\mathcal{P}(d_1,d_2)}F(u_1,u_2)=\inf_{\mathcal{P}_+(d_1,d_2)}F(u_1,u_2).$$

This proof is complete. By $g(k_0)=0$ and continuity of g, there is $\varepsilon_0>0$ such that $g(k)\ge\frac{\sigma(d_1,d_2)}{2}$ if $k\in[k_0-\varepsilon,k_0]$. Therefore,

$$F(u_1,u_2)\ge g(k)\ge\frac{\sigma(d_1,d_2)}{2}>\sigma(d_1,d_2)\quad\text{for all }(u_1,u_2)\in\overline{W(k_0)}\setminus W(k_0-\varepsilon_0).$$

□

Lemma 4.6. *Assume that* (A1) *holds. Let* $0<\Theta<\Theta_*$, *then* $\sigma(d_1,d_2)<\min\{\sigma_0(d_1,0),\sigma_0(d_2,0)\}$.

Proof. For each $(u_1,u_2)\in W(k_0)$, we have

$$\begin{aligned}F(u_1,u_2)&=G_{\mu_1}(u_1)+G_{\mu_2}(u_2)-\Theta\int_{\mathbb{R}^N}(I_\beta*|u_1|^{r_1})|u_2|^{r_2}dx\\&\le G_{\mu_1}(u_1)+G_{\mu_2}(u_2).\end{aligned}$$

Hence, $\sigma(d_1,d_2)\le\inf_{W(k_0)}(G_{\mu_1}(u_1)+G_{\mu_2}(u_2))$. For any $u_1\in S_r(d_1)$ with $\|\nabla u_1\|_2^2=k_0$, then

$$\begin{aligned}G_{\mu_1}(u_1)&=\frac{a}{2}\|\nabla u_1\|_2^2+\frac{b}{4}\|\nabla u_1\|_2^4-\frac{\mu_1}{p_1}\int_{\mathbb{R}^N}(I_\alpha*|u_1|^{2p_1})|u_1|^{p_1}dx\\&\ge\frac{b}{4}k_0^2-B_i\|\nabla u_1\|_2^{\delta_{p_i}p_i}\ge g(k_0)=0,\end{aligned}$$

where B_i is defined by Lemma 2.9. Therefore,

$$\inf_{S_r(d_1)}G_{\mu_1}=\inf_{E(d_1,k_0)}G_{\mu_1}(u_1)<0,$$

where $E(d,k):=\{u\in S_r(d):\|\nabla u\|_2^2<k\}$. Since the map $k\mapsto\frac{b}{4}k_0^2-B_i\|\nabla u_1\|_2^{\delta_{p_i}p_i}$ is continuity, by similar proof in Lemma 4.5, there is $\varepsilon_0>0$ such that

$$\sigma(d_1,\mu_1)<\inf_{\overline{E(d_1,k_0)}\setminus E(d_1,k_0-\varepsilon_0)}G_{\mu_1}(u_1),$$

and $\inf_{E(d_2,\varepsilon_0)}G_{\mu_2}(u_2)<0$. We define $\Lambda:=\{(u_1,u_2):u_1\in E(d_1,k_0-\varepsilon_0),u_2\in E(d_2,\varepsilon_0)\}$; then $\Lambda\subset W(k_0)$. Therefore,

$$\begin{aligned}\inf_{W(k_0)}(G_{\mu_1}(u_1)+G_{\mu_2}(u_2))&\le\inf_{\Lambda}(G_{\mu_1}(u_1)+G_{\mu_2}(u_2))\\&=\inf_{E(d_1,k_0-\varepsilon_0)}G_{\mu_1}(u_1)+\inf_{E(d_2,\varepsilon_0)}G_{\mu_2}(u_2)\\&<\inf_{E(d_1,k_0-\varepsilon_0)}G_{\mu_1}(u_1)\\&=\sigma_0(d_1,0).\end{aligned}$$

Thus, $\sigma(d_1,d_2)<\sigma_0(d_1,0)$. It can be shown that $\sigma(d_1,d_2)<\sigma_0(d_2,0)$ in the same way as above. □

Lemma 4.7. *Let* $\{(u_1^n,u_2^n)\}\subset S_r(d_1)\times S_r(d_2)$ *be a Palais-Smale sequence for* $F|_{S_r(d_1)\times S_r(d_2)}$ *at level* $\sigma_\beta(d_1,d_2)$, *and* $P(u_1^n,u_2^n)\to0$ *as* $n\to+\infty$. *Then,* $(u_1^n,u_2^n)\to(u_1,u_2)$ *in* $H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N)$.

Proof. By $F(|u_1^n|,|u_2^n|) = F(u_1^n,u_2^n)$, then we can assume that $u_1^n, u_2^n \geq 0$. Since $\{(u_1^n,u_2^n)\} \subset S_r(d_1)\times S_r(d_2)$ be a Palais-Smale sequence for $F|_{S_r(d_1)\times S_r(d_2)}$ at level $\sigma(d_1,d_2)$, we have

$$\begin{aligned}
\sigma(d_1,d_2)+o_n(1) &= \frac{a}{2}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 + \frac{b}{4}\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4 - \sum_{i=1}^{2}\frac{\mu_i}{2p_i}\int_{\mathbb{R}^N}(I_\alpha * |u_i^n|^{p_i})|u_i^n|^{p_i}dx \\
&\quad - \Theta\int_{\mathbb{R}^N}(I_\beta * |u_1^n|^{p_1})|u_2^n|^{p_2}dx \\
&= a\Big(\frac{1}{2}-\frac{2}{Nr-2(N+\beta)}\Big)\sum_{i=1}^{2}\|\nabla u_i^n\|_2^2 + b\Big(\frac{1}{4}-\frac{2}{Nr-2(N+\beta)}\Big)\sum_{i=1}^{2}\|\nabla u_i^n\|_2^4 \\
&\quad - \sum_{i=1}^{2}\frac{\mu_i}{2p_i}\Big(1-\frac{2(Np_i-N-\alpha)}{Nr-2(N+\beta)}\Big)\int_{\mathbb{R}^N}(I_\alpha * |u_i^n|^{p_i})|u_i^n|^{p_i}dx \\
&\geq a\Big(\frac{1}{2}-\frac{2}{Nr-2(N+\beta)}\Big)k + \frac{b}{2}\Big(\frac{1}{2}-\frac{2}{Nr-2(N+\beta)}\Big)k^2 \\
&\quad - \sum_{i=1}^{2}\frac{\mu_i}{2p_i}M_i\Big(1-\frac{2(Np_i-N-\alpha)}{Nr-2(N+\beta)}\Big)\|\nabla u_i^n\|_2^{Np_i-N-\alpha},
\end{aligned}$$

where $k = \|\nabla u_1^n\|_2^2 + \|\nabla u_2^n\|_2$, $M_i := C_{N,p_i}^{p_i} d_i^{\frac{N+\alpha-(N-2)p_i}{2}}$ $(i=1,2)$, $8 < Nr-2(N+\beta) < 4\cdot 2_\beta^*$ and $0 < Np_i - N - \alpha < 2$. We obtain the sequence $\{(u_1^n,u_2^n)\}$ is bounded in $H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N)$. Hence, we have

$$\begin{gathered}
(u_1^n,u_2^n) \rightharpoonup (u_1,u_2) \quad \text{in } H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N) \\
(u_1^n,u_2^n) \to (u_1,u_2) \quad \text{in } L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)\text{for} p\in(2,2^*).
\end{gathered}$$

By (c) in Lemma 2.11, there exist two sequences of real numbers $\{\lambda_1^n\}$, $\{\lambda_2^n\}$, such that

$$\begin{aligned}
& a\sum_{i=1}^{2}\int_{\mathbb{R}^N}\nabla u_i^n\nabla\phi_i dx + b\sum_{i=1}^{2}\int_{\mathbb{R}^N}|\nabla u_i^n|^2dx\int_{\mathbb{R}^N}\nabla u_i^n\nabla\phi_i dx \\
& - \sum_{i=1}^{2}\mu_i\int_{\mathbb{R}^N}(I_\alpha * |u_i^n|^{p_i})|u_i^n|^{p_i-2}u_i^n\phi_i dx - \Theta r_1\int_{\mathbb{R}^N}(I_\beta * |u_2^n|^{r_2})|u_1^n|^{r_1-2}u_1^n\phi_1 dx \\
& - \Theta r_2\int_{\mathbb{R}^N}(I_\beta * |u_1^n|^{r_1})|u_2^n|^{r_2-2}u_2^n\phi_1 dx - \sum_{i=1}^{2}\int_{\mathbb{R}^N}\lambda_i^n u_i^n\phi_i dx \\
& = o_n(1)\|(\phi,\phi_2)\|_{H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N)},
\end{aligned}$$

where $o_n(1)\to 0$ as $n\to+\infty$. By Lemma 2.11, $\lambda_i^n \to \lambda_i$ $(i=1,2)$, and (u_1,u_2) is a solution to (1.1). Since

$$F(u_1,u_2) \leq \varliminf_{n\to+\infty} F(u_1^n,u_2^n) = \sigma(d_1,d_2) < 0, \tag{4.7}$$

we have $(u_1,u_2)\neq(0,0)$. From Lemma 2.10, we obtain that $\lambda_1<0$ or $\lambda_2<0$. Without loss of generality, we assume that $\lambda_1<0$. Using Lemma 2.12, $u_1^n\to u_1$ in $H_r^1(\mathbb{R}^N)$. Suppose by contradiction that $\lambda_2\geq 0$. Then

$$-(a+b\int_{\mathbb{R}^N}|\nabla u_2|^2dx)\Delta u_2 = \lambda_2u_2 + \mu_2(I_\alpha * |u_2|^{p_2})|u_2|^{p_2-2}u_2 + \Theta r_2(I_\beta * |u_1|^{r_1})|u_2|^{r_2-2}u_2 \geq 0.$$

By Lemma 2.5, we have $u_2=0$. And u_1 satisfies

$$\begin{gathered}
-(a+b\int_{\mathbb{R}^N}|\nabla u_1|^2dx)\Delta u = \lambda_1u_1 + \mu_1(I_\alpha * |u_1|^{p_1})|u_1|^{p_1-2}u_1, \\
\int_{\mathbb{R}^N}|u|^2dx = d_1 > 0.
\end{gathered}$$

Hence,

$$\begin{aligned}\sigma_\beta(d_1,d_2) &= \lim_{n\to+\infty} F(u_1^n,u_2^n)\\ &= \lim_{n\to+\infty} \frac{a}{2}\sum_{i=1}^2 \|\nabla u_i\|_2^2 + \frac{b}{4}\sum_{i=1}^2 \|\nabla u_i\|_2^4 - \sum_{i=1}^2 \frac{\mu_i}{2p_i}\int_{\mathbb{R}^N} (I_\alpha * |u_i|^{p_i})|u_i|^{p_i}dx\\ &\quad - \Theta\int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1})|u_2|^{r_2}dx\\ &= \lim_{n\to+\infty} \frac{a}{2}\sum_{i=1}^2 \|\nabla u_i\|_2^2 + \frac{b}{4}\sum_{i=1}^2 \|\nabla u_i\|_2^4 - \frac{\mu_1}{2p_1}\int_{\mathbb{R}^N} (I_\alpha * |u_1|^{p_1})|u_1|^{p_1}dx\\ &\ge \frac{a}{2}\|\nabla u_1\|_2^2 + \frac{b}{4}\|\nabla u_1\|_2^4 - \frac{\mu_1}{2p_1}\int_{\mathbb{R}^N} (I_\alpha * |u_1|^{p_1})|u_1|^{p_1}dx\\ &= G_{\mu_1}(d_1)\\ &\ge \sigma(d_1,0),\end{aligned}$$

which contradicts Lemma 4.6. Hence, $\lambda_2 < 0$, and then, $u_2^n \to u_2$ in $H_r^1(\mathbb{R}^N)$. □

Proof of Theorem 1.3. Let $\{(u_1^n,u_2^n)\} \subset W(k_0)$ be a minimizing sequence for $\sigma(d_1,d_2)$, i.e. $F(u_1^n,u_2^n) \to \sigma(d_1,d_2)$. By Lemma 4.3, $\tau_{(u_1^n,u_2^n)} \star (u_1^n,u_2^n) \in \mathcal{P}_+(d_1,d_2)$ for every n, $\|\nabla(\tau_{(u_1^n,u_2^n)} \star u_1^n)\|_2^2 + \|\nabla(\tau_{(u_1^n,u_2^n)} \star u_2^n)\|_2^2 < k_0$ and

$$F(\tau_{(u_1^n,u_2^n)} \star u_1^n, \tau_{(u_1^n,u_2^n)} \star u_2^n) \le F(u_1^n,u_1^n).$$

Let $(\varpi_1^n,\varpi_2^n) := (\tau_{(u_1^n,u_2^n)} \star u_1^n, \tau_{(u_1^n,u_2^n)} \star u_2^n)$, then $\{\varpi_1^n,\varpi_2^n\} \subset W(k_0)$ is a minimizing sequence for $\sigma_\beta(d_1,d_2)$ and $(\varpi_1^n,\varpi_2^n) \in \mathcal{P}_+(d_1,d_2)$. By Lemma 4.5, $\{(\varpi_1^n,\varpi_2^n)\} \subset W(k_0-\varepsilon_0)$. Therefore, by Ekeland's variational principle, there is a radial symmetric Palais-Smale sequence $(\upsilon_1^n,\upsilon_2^n)$ for $F|_{S_r(d_1)\times S_r(d_1)}$ satisfying $\|(\varpi_1^n,\varpi_2^n) - (\upsilon_1^n,\upsilon_2^n)\|_{H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N)} \to 0$ as $n \to +\infty$. Thus, $\{(\upsilon_1^n,\upsilon_2^n)\} \subset W(k_0)$ and $P(\upsilon_1^n,\upsilon_2^n) \to 0$ as $n \to +\infty$. Now Lemma 4.7 implies that there exists $\upsilon_1,\upsilon_2 > 0$ such that $(\upsilon_1^n,\upsilon_2^n) \to (\upsilon_1,\upsilon_2)$ in $H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N)$, and then (υ_1,υ_2) is a local minimizer for $F|_{W(k_0)}$. Therefore, (υ_1,υ_2) is a positive radial solution to (1.1) for some $\lambda_1,\lambda_2 < 0$. □

5. Proof of Theorem 1.4

In this section, we prove the existence of the second normalized solution. By [25], the Choquard equation

$$-\Delta u + u = \int_{\mathbb{R}^N} \frac{|u(y)|^2}{|x-y|^{N-2}}dyu \quad \text{in } \mathbb{R}^3 \tag{5.1}$$

has a unique positive solution, which is often a strong help to obtain the second solution of (1.1).

By Lemma 2.8 and Lemma 2.9, we introduce a minimax structure of mountain pass type. There exists $k^* \in (0,k_0)$, such that for any $0 < \Theta \le \Theta_*$

$$\delta(s_1,d_2) := \inf_{h\in\Gamma}\max_{t\in[0,1]} F(h(t)) > \max\{F(h(0)),F(h(1))\},$$

where $\Gamma := \{h \in C([0,1],S_r(d_1)\times S_r(d_2)) : h(0) \in W(\bar{k}), h(1) \notin \overline{W(k_0)}, F(h(1)) < 0\}$.

Lemma 5.1. *Assume that* (A2) *holds. Then for each $0 < \Theta < \Theta_*$, there exists a Palais-Smale sequence $\{(u_1^n,u_2^n)\}$ for $F|_{S_r(d_1)\times S_r(d_2)}$ at the level $\delta(d_1,d_2)$, which satisfies $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$ and $P(u_1^n,u_2^n) \to 0$.*

Proof. We recall the stretched functional first introduced in [20],

$$\tilde{F} : \mathbb{R}\times\Big(H_r^1(\mathbb{R}^N)\times H_r^1(\mathbb{R}^N)\Big) \to \mathbb{R}, \quad (s,(u_1,u_2)) \mapsto F(s\star u_1, s\star u_2).$$

We define

$$\begin{aligned}\tilde{\Gamma} := \big\{&\tilde{h} \in C([0,1],S_r(d_1)\times S_r(d_2)) : \tilde{h}(0) \in (0,h(0)), \tilde{h}(1) = (0,h(1)),\\ &h(0) \in W(k^*), h(1) \notin \overline{W(k_0)}, F(h(1)) < 0\big\}\end{aligned}$$

and

$$\tilde{\delta}(d_1, d_2) := \inf_{\tilde{h}\in\tilde{\Gamma}} \max_{t\in[0,1]} \tilde{F}(\tilde{h}(t)).$$

It is not difficult to check that $\tilde{\delta}(d_1, d_2) = \delta(d_1, d_2)$. Indeed, by the definitions of $\tilde{\delta}(d_1, d_2)$ and $\delta(d_1, d_2)$, this identity follows immediately from the fact that the maps

$$\phi : \Gamma \to \tilde{\Gamma}, \quad h \mapsto \phi(h) : (0, h),$$

and

$$\varphi : \tilde{\Gamma} \to \Gamma, \quad \tilde{h} = (\varpi, h) \mapsto \varphi(\tilde{h}) := \varpi \star h \quad \text{with } (\varpi \star h)(t) = \varpi(t) \star h(t)$$

satisfying

$$\tilde{F}(\phi(h)) = F(h), \quad F(\varphi(\tilde{h})) = \tilde{F}(\tilde{h}).$$

Then, we obtain a sequence $\{(v_1^n, v_2^n)\} \subset \Gamma$ such that

$$\max_{t\in[0,1]} \tilde{F}(0, (v_1^n(t), v_2^n(t))) \to \tilde{\delta}(d_1, d_2).$$

In view of $F(u_1, u_2) = F(|u_1|, |u_2|)$ for $(u_1, u_2) \in S_r(d_1) \times S_r(d_2)$, then we can assume that $v_1^n(t) \ge 0$ and $v_2^n(t) \ge 0$ for $t \in [0, 1]$. By [12, Theorem 4.1], there exists a Palais-Smale sequence $\{(s_n, (u_1^n, u_2^n))\}$ for $\tilde{F}|_{\mathbb{R}\times(S_r(d_1)\times S_r(d_2))}$ at the level $\delta(d_1, d_2)$, such that $s_n \to 0$ and $\|(u_1^n, u_2^n) - (v_1^n, v_2^n)\| \to 0$. It follows that $(u_1^n)^- \to 0$, $(u_1^n)^- \to 0$. Since $\tilde{F}(s, (u_1, u_2)) = \tilde{F}(0, s \star (u_1, u_2))$, we have

$$(\partial_s \tilde{F})(s, (u_1, u_2)) = (\partial_s \tilde{F})(0, s \star (u_1, u_2))$$

and $(\partial_u \tilde{F})(s, u)[\psi] = (\partial_u \tilde{F})(0, s \star u)[s \star \psi]$ for $u = (u_1, u_2)$, $\psi = (\psi_1, \psi_2)$. Therefore, $\{(0, s_n \star (u_1^n, u_2^n))\}$ is also a Palais-Smale sequence for $\tilde{F}|_{\mathbb{R}\times(S_r(d_1)\times S_r(d_2))}$ at the level $\delta(d_1, d_2)$. Then we may assume that $s_n = 0$, which implies that $\{(u_1^n, u_2^n)\} \subset S_r(d_1) \times S_r(d_2)$ is a Palais-Smale sequence for $F|_{S_r(d_1)\times S_r(d2)}$ at the level $\delta(d_1, d_2)$ and $(\partial_s \tilde{F})(0, (u_1^n, u_2^n)) \to 0$, that $P(u_1^n, u_2^n) \to 0$ holds. □

Lemma 5.2. *Assume that* (A1) *holds and* $0 < \Theta \le \Theta_*$, *then there exists a positive and radial solution* (u_1, u_2) *to the system* (1.1) *for some* (λ_1, λ_2), *and* $F(u_1, u_2) = \delta(d_1, \delta_2)$.

Proof. By proof similar to that of Lemma 4.7, the Palais-Smale sequence $\{(u_1^n, u_2^n)\}$ for F restricted to $S_r(d_1) \times S_r(d_2)$ at the level $\delta(d_1, d_2)$ is bounded in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$. Now we can assume that

$$\begin{gathered}(u_1^n, u_2^n) \rightharpoonup (u_1, u_2) \quad \text{in } H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N), \\ (u_1^n, u_2^n) \to (u_1, u_2) \quad \text{in } L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \text{ for } p \in (2, 2^*).\end{gathered}$$

From Lemma 2.11, there exists a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R}^2$, such that $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$ in $\mathbb{R}^2$, (u_1, u_2) is a solution to the system (1.1) and $P(u_1, u_2) = 0$. By $(u_1^n)^- \to 0$, $(u_2^n)^- \to 0$, then $u_1, u_2 \ge 0$.

Next, we prove that $F(u_1, u_2) = \delta(d_1, d_2)$. By $P(u_1^n, u_2^n) \to 0$, i.e.,

$$\begin{aligned} a\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 &= \sum_{i=1}^{2} \frac{\mu_i}{2p_i}(Np_i - N - \alpha)\int_{\mathbb{R}^N} (I_\alpha * |u_i^n|^{p_i})|u_i^n|^{p_i} dx \\ &\quad + \Theta\frac{Nr - 2(N+\beta)}{2}\int_{\mathbb{R}^N} (I_\beta * |u_1^n|^{r_1})|u_2^n|^{p_2} dx. \end{aligned} \tag{5.2}$$

By Lemmas 2.7 and 2.11, we have

$$\begin{aligned} &\sum_{i=1}^{2} \frac{\mu_i}{2p_i}(Np_i - N - \alpha)\int_{\mathbb{R}^N} (I_\alpha * |u_i^n|^{p_i})|u_i^n|^{p_i} dx + \Theta\frac{Nr - 2(N+\beta)}{2}\int_{\mathbb{R}^N} (I_\beta * |u_1^n|^{r_1})|u_2^n|^{r_2} dx \\ &\to \sum_{i=1}^{2} \frac{\mu_i}{2p_i}(Np_i - N - \alpha)\int_{\mathbb{R}^N} (I_\alpha * |u_i^n|^{p_i})|u_i^n|^{p_i} dx + \Theta\frac{Nr - 2(N+\beta)}{2}\int_{\mathbb{R}^N} (I_\beta * |u_1|^{r_1})|u_2|^{r_2} dx. \end{aligned}$$

Combining $P(u_1, u_2) = 0$, we have

$$\lim_{n\to+\infty} a\sum_{i=1}^{2} \|\nabla u_i^n\|_2^2 + b\sum_{i=1}^{2} \|\nabla u_i^n\|_2^4 = a\sum_{i=1}^{2} \|\nabla u_i\|_2^2 + b\sum_{i=1}^{2} \|\nabla u_i\|_2^4.$$

Hence, $F(u_1^n, u_2^n) = F(u_1, u_2)$, and then $F(u_1, u_2) = \delta(d_1, d_2)$. □

Proof of Theorem 1.4. By Lemma 5.2, we only need to prove $(u_1, u_2) \in S_r(d_1) \times S_r(d_2)$. Since (u_1, u_2) is solution of (1.1), by Lemma 2.10, we have $\lambda_1 < 0$ or $\lambda_2 < 0$. Without loss of generality, we can assume $\lambda_1 < 0$. By Lemma 2.12, we obtain that $u_1^n \to u_1$ in $H_r^1(\mathbb{R}^N)$ and then $u_1 \in S_r(d_1)$. Suppose by contradiction that $\lambda_2 \geq 0$, then

$$-\Delta u_2 = \lambda_2 u_2 + \mu_2(I_\alpha * |u_2|^{p_2})|u_2|^{p_2-2}u_2 + \Theta r_2(I_\beta * |u_1|^{p_1})|u_2|^{p_2-2}u_2 \geq 0.$$

By Lemma 2.5, we have $u_2 = 0$. Therefore, $F(u_1, u_2) = F(u_1, 0)$, and $u_1 \in S_r(d_1)$ satisfies (2.4). By Lemma 2.6, $F(u_1, 0) = \sigma_{p_1}^{\mu_1}(d_1) < 0$ if $a = 1$, $b = 0$, $N = 3$ and $p_1 = p_2 = r_1 = r_2 = \alpha = \beta = 2$, which is in contradiction to the fact $F(u_1, 0) = \delta(d_1, d_2) > 0$. Thus, $\lambda_2 < 0$, and then $u_2 \in S_r(d_2)$. By the maximum principle, we obtain the $u_1, u_2 > 0$ in $\mathbb{R}^3$. □

Acknowledgements. This study was funded by the Fujian Provincial Natural Science of China (No. 2025J08201). Zhewen Chen thanks Professor Jianqing Chen, at the Fujian normal University, for his valuable comments in preparing this article.

References

[1] A. Arosio, S. Panizzi; *On the well-posedness of the Kirchhoff string.* Trans. Am. Math. Soc., 348 (1996), 305-330.

[2] T. Bartsch, H. W. Li, W. M. Zou; *Existence and asymptotic behavior of normalized ground states for Sobolev critical Schrödinger systems*, Calc. Var. Partial Differential Equations, 62 (2023) 9.

[3] T. Bartsch, L. Jeanjean, N. Soave; *Normalized solutions for a system of coupled cubic Schrödinger equations on* $\mathbb{R}^3$, J. Math. Pures Appl., 106 (2016) 583-614.

[4] T. Bartsch, N. Soave; *A nature constraint approach to normalized solutions of nonlinear Schrödinger equations and systems*, J. Funct. Anal., 272 (2017) 4998-5037.

[5] T. Bartsch, N. Soave; *Multiple normalized solutions for a competing system of Schrödinger equations*, Calc. Var. Partial Differential Equations 58 (2019) 22.

[6] T. Bartsch, X. X. Zhong, W. M. Zou; *Normalized solutions for a coupled Schrödinger system*, Math. Ann., 380 (2021) 1713-1740.

[7] H. Berestycki, P. L. Lions; *Nonlinear scalar field equations II*, Arch. Ration. Mech. Anal., 82(1983) 347-376.

[8] X. Cao, J. Xu, J. Wang; *The existence of solutions with prescribed* L^2*-norm for Kirchhoff type system.* J. Math. Phys., 58 (2017) 041502.

[9] I. E. Dzyaloshinskii, E. M. Lifshitz, Lev P. Pitaevskii; *General theory of van der Waals' forces.* Sov. Phys. Usp., 4 (1961) 153.

[10] Q. P. Geng, Y. Y. Tu, J. Wang; *Existence and multiplicity of the positive normalized solutions to the coupled Hartree-Fock type nonlocal elliptic system*, J. Fixed Point Theory Appl., 24 (2022) 83.

[11] M. Ghimenti, J. Van Schaftingen; *Nodal solutions for the Choquard equation*, J. Funct. Anal. 271 (2016) 107-135.

[12] N. Ghoussoub; *Duality and perturbation methods in critical point theory*, Cambridge Tracts in Mathematics, vol. 107 (Cambridge University Press, 1993).

[13] T. X. Gou, L. Jenajean; *Multiple positive normalized solutions for nonlinear Schrödinger systems*, Nonlinearity, 31 (2018) 2319-2345.

[14] Z. Y. Guo, L. J. Zhao; *Existence of ground states for fractional Choquard-Kirchhoff equations with magnetic fields and critical exponents*, Period. Math. Hungar. 87 (2023) 468-483.

[15] Z. Y. Guo, T. Q. Zhang; *Normalized solutions of fractional Kirchhoff equations in the defocusing case*, Electronic J. Differ. Equ., 2025 (2025) No. 75, 1-16.

[16] J. Q. Hu, A. M. Mao; *Normalized solutions to nonlocal Schrödinger systems with* L^2*-subcritical and supercritical nonlinearities*, J. Fixed Point Theory Appl., 25 (2023), 77.

[17] Q. H. He, Z. Y. Lv, Z. W. Tang; *The existence of normalized solutions to the Kirchhoff equation with potential and Sobolev critical nonlinearities*, J. Geom. Anal., 33 (2023), 236.

[18] Q. H. He, Z. Y. Lv, Y. M. Zhang, X. X. Zhong; *Existence and blow up behavior of positive normalized solution to the Kirchhoff equation with general nonlinearities: Mass super-critical case*, J. Differ. Equ., 356 (2023), 375-406.

[19] N. Ikoma; *Compactness of minimizing sequences in nonlinear Schrödinger systems under multiconstraint conditions*, Adv. Nonlinear Stud., 14 (2014), 115-136.

[20] L. Jeanjean; *Existence of solutions with prescribed norm for semilinear elliptic equations.* Nonlin. Analysis, 28 (1997), 1633-1659.
[21] G. Kirchhoff; *Mechanik.* Teubner, Leipzig, 1883.
[22] E. H. Lieb, M. Loss; *Analysis*, Gradute Studies in Mathematics, AMS, Providence, Rhode island, 2001.
[23] J. Lions; *On some questions in boundary value problems of mathematical physics.* North-Holland Math. Stud., 30 (1978), 284-346.
[24] Z. Liu; *Multiple normalized solutions for Choquard equations involving Kirchhoff type perturbation*, Topol. Methods Nonlinear Anal., 54 (2019) 297-319.
[25] L. Ma, L. Zhao; *Classiflcation of positive solitary solutions of the nonlinear Choquard equation*, Arch. Ration. Mech. Anal., 195 (2010), 455-467.
[26] V. Moroz, J. Van Schaftingen; *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Maths. Soc., 367 (2015), 6557-6579.
[27] S. G. Porsev, A. Derevianko; *High-accuracy calculations of dipole, quadrupole, and octupole electric dynamic polarizabilities and van der Waals coefficients C_6, C_8, and C_{10} for alkaline-earth dimers*, J. Exp. Theor. Phys. 102 (2006), 195-205.
[28] M. I. Weinstein; *Nonlinear Schrödinger equations and sharp interpolation estimates.* Comm. Math. Phys. 87 (1983), 567-576.
[29] Z. Yang; *Normalized ground state solutions for Kirchhoff type systems.* J. Math. Phys., 62 (2021), 031504.
[30] Y. Zheng, A. Narayanaswamy; *Lifshitz theory of van der Waals pressure in dissipative media.* Phys. Rev. A, 83 (2011), 042504.
[31] H. Zhang, J. J. Zhang, X. X. Zhong; *Normalized solutions for critical Choquard systems*, arXiv:2307.01483v2

Zhewen Chen (corresponding author)
College of Science, Jimei University, Xiamen 361021, China
Email address: zhewenchen@jmu.edu.cn

Muzi Li
College of Mathematics and Statistics, Fujian Normal University, Qishan Campus, Fuzhou 350117, China
Email address: mutoudededemuzi@163.com