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FINITE-TIME STABILITY FOR FUZZY FRACTIONAL DELAY DIFFERENTIAL EQUATIONS WITH GENERALIZED CAPUTO DERIVATIVES

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Abstract. This work is devoted to the analysis of fuzzy fractional delay differential equations (FDDEs) governed by the generalized Caputo fractional derivative (GCFD). By combining stepwise approximation methods with suitable Gronwall-type inequalities, we establish the existence and uniqueness of solutions. Furthermore, we derive explicit criteria that guarantee the finite-time stability. The theoretical contributions are illustrated with numerical simulations, confirm the analytical findings and demonstrate the effectiveness of the proposed framework in capturing the dynamical behavior of fuzzy fractional delay models.

1. Introduction

Over the past ten years, fuzzy fractional analysis and fuzzy fractional differential equations (FFDEs) have gained considerable attention for modeling dynamical systems with imprecise data. Pioneering studies by Allahviranloo et al. [5] explored explicit solutions and existence-uniqueness for FFDEs under Riemann-Liouville and Caputo fuzzy derivatives. Lupulescu [20] formulated a fractional calculus framework for interval-valued functions, later applied to FFDEs. Fard and Salehi [24] studied fuzzy variational problems using the Caputo derivative, while Hoa [14] extended Caputo fuzzy derivative theory to orders in $(1, 2)$, establishing solution uniqueness. Hoa and Vu [13] investigated qualitative properties of implicit FFDEs with Caputo, Riemann-Liouville, and Hadamard derivatives. Long et al. [18] introduced Caputo fuzzy partial derivatives for multivariable functions, addressing fuzzy fractional partial differential equations. Research on FFDEs has largely centered on operators such as the Riemann-Liouville, Caputo, Caputo-Hadamard, and Caputo-Katugampola derivatives within the generalized Hukuhara derivative framework. The choice of fractional derivative is guided by the alignment of experimental data with theoretical models, necessitating more flexible fractional operators that interpolate between classical forms. Samko [25] introduced the generalized fractional derivative, defined for a real-valued function w with respect to a function ψ. Almeida [6] advanced this by proposing the GCFD , which unifies various fractional derivative types. Recent studies have leveraged the GCFD Baleanu et al. [7] investigated the well-posedness of terminal value problems for nonlinear systems, Wu et al. [28] applied it to linear systems with non-homogeneous terms, and Luo et al. [19] used it to study the qualitative behavior of solutions to fractional uncertain differential equations driven by the Liu process. Furthermore, Fan et al. [12] laid the groundwork for general fractional calculus in the space of interval-valued functions, providing a theoretical basis for future advancements in FFDEs. Continued exploration of the references cited in these works offers valuable insights into the evolving landscape of generalized fractional operators and their applications. The reader may refer to the recent works [1, 3, 4, 21, 10, 11, 29] for more details.

Stability analysis is a cornerstone of control theory, especially in modeling dynamic systems. FTS, which ensures that system trajectories remain bounded within a specified time interval, has

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garnered significant interest due to its practical importance. Unlike asymptotic stability, which addresses long-term behavior, FTS focuses on state boundedness over a finite duration, making it ideal for time-critical applications. A system is deemed finite-time stable if its states stay within prescribed bounds during a given time window. Time delays, which account for memory effects in real-world systems, are often critical to include in dynamic models. In [17], Lazarević pioneered the study of FTS in fractional-order delay differential systems by establishing sufficient conditions. Since then, the field has advanced rapidly in both theoretical and applied domains, utilizing various fractional derivative definitions. The literature on FTS in fractional delay systems splits into two primary approaches Chen et al. in [9] use Hölder-type inequalities and analytical methods to derive straightforward sufficient conditions for FTS in nonlinear fractional systems with delays, while Zhang et al. [30] focus on systems with variable coefficients. Additionally, Naifar et al. [22] employed the fractional Gronwall inequality to study FTS in linear time-delay systems, an approach extended to nonlinear systems with time-varying delays by Phat et al. [23]. Extensive investigations into FTS across fractional differential systems have been conducted, yet the case of FDDEs with time-varying delays, approached through the fuzzy framework and the GCFD, remains largely uninvestigated. This important literature gap forms the central motivation for the current research.

In this article, we aim to fill this gap by investigating the finite-time stability (FTS) of fuzzy fractional delay differential equations (FDDEs) within the generalized Caputo fractional derivative (GCFD) framework. Specifically, we consider the system

$$\begin{gathered} {}^{C}\mathcal{D}_{0^+}^{\beta;\psi} w(\nu) = \mathcal{H}(\nu, w(\nu), w(\nu - r)), \quad \nu \in \mathcal{J} = [0, T], \\ w(\nu) = \varrho(\nu) \in \mathfrak{F}, \quad \nu \in [-r, 0], \end{gathered} \tag{1.1}$$

where

- $\beta \in (0, 1)$ and $r > 0$.
- The interval $\mathcal{J}$ is partitioned as $\mathcal{J} = \cup_{k=0}^{n} [kr, (k + 1)r] \cup [(n + 1)r, T]$.
- $\mathfrak{F}$ the space of fuzzy numbers.
- The function $\mathcal{H} : \mathcal{J} \times \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$, which is jointly continuous and satisfies the condition $\mathcal{H}(\nu, \hat{0}, \hat{0}) = \hat{0}$.
- The operator ${}^{C}\mathcal{D}_{0^+}^{\beta;\psi} w$ represents the GCFD of the fuzzy-valued function w with respect to the lower limit 0^+.
- The initial function $\varrho(\nu)$ is assumed to be continuous on the interval $[-r, 0]$.
- The function $\psi : \mathcal{J} \to \mathbb{R}$ belongs to the class $\mathcal{K}$, which consists of functions $\psi \in C^1$ that are strictly increasing, positive, and satisfy $\psi'(\nu) \neq 0$ for all $\nu \in (0, T)$.

Our work provides new insights into the interplay between fuzziness, memory effects, and delay phenomena in fractional dynamical systems. The structure of the paper is as follows: Section 2 covers the essential groundwork, including notations and fundamental concepts needed for the main results, and presents the explicit solution for (1.1). Section 3 investigates the existence and uniqueness of solutions, proposing two separate methods to establish the FTS of (1.1) with appropriate sufficient conditions. Section 4 offers a range of examples that substantiate the theoretical findings and illustrate the practical relevance of the suggested approaches.

2. Preliminaries

The following concepts are fundamental to understanding the structure and operations within $\mathfrak{F}$ we direct readers to [8].

For any fuzzy number $z \in \mathfrak{F}$, the ζ-cuts set for $\zeta \in (0, 1]$ is defined as

$$[z]^{\zeta} = \{x_0 \in \mathbb{R} \mid z(x_0) \geq \zeta\},$$

and the 0-cuts set is given by

$$[z]^{0} = \overline{\{x_0 \in \mathbb{R} \mid z(x_0) > 0\}}.$$

Each $[z]^\zeta$ forms a closed interval $[\underline{z}(\zeta), \overline{z}(\zeta)]$. The zero fuzzy number $\hat{0} \in \mathfrak{F}$ is defined by

$$\hat{0}(x_0) = \begin{cases} 1 & \text{if } x_0 = 0, \\ 0 & \text{otherwise.} \end{cases}$$

According to Zadeh's extension principle, addition and scalar multiplication in $\mathfrak{F}$ are defined for all $\zeta \in [0,1]$ as

$$[z_1 + z_2]^\zeta = [z_1]^\zeta + [z_2]^\zeta, \quad [\lambda z_1]^\zeta = \lambda [z_1]^\zeta,$$

where

$$[z_1]^\zeta + [z_2]^\zeta = [\underline{z_1}(\zeta) + \underline{z_2}(\zeta), \overline{z_1}(\zeta) + \overline{z_2}(\zeta)],$$

and

$$\lambda [z_1]^\zeta = \begin{cases} [\lambda \underline{z_1}(\zeta), \lambda \overline{z_1}(\zeta)] & \text{if } \lambda \geq 0, \\ [\lambda \overline{z_1}(\zeta), \lambda \underline{z_1}(\zeta)] & \text{if } \lambda < 0. \end{cases} \tag{2.1}$$

Alternatively, these operations can be expressed as

$$[z_1]^\zeta + [z_2]^\zeta = \{x_0 + y_0 : x_0 \in [z_1]^\zeta, y_0 \in [z_2]^\zeta\}, \quad \lambda [z_1]^\zeta = \{\lambda x_0 : x_0 \in [z_1]^\zeta\}.$$

The diameter of the ζ-cuts set of z is

$$d([z]^\zeta) = \overline{z}(\zeta) - \underline{z}(\zeta), \quad \zeta \in [0,1].$$

The Hukuhara difference $z_1 \ominus z_2$ exists if there exists $z_3 \in \mathfrak{F}$ such that $z_1 = z_2 + z_3$. The generalized Hukuhara difference (gH-difference) extends this concept and is defined as

$$z_1 \ominus_{gH} z_2 = z_3 \Leftrightarrow \begin{cases} \text{(i) } z_1 = z_2 + z_3, & \text{or} \\ \text{(ii) } z_2 = z_1 + (-1) z_3. \end{cases} \tag{2.2}$$

Case (i) applies when $d([z_1]^\zeta) \geq d([z_2]^\zeta)$, and case (ii) when $d([z_1]^\zeta) \leq d([z_2]^\zeta)$.

The metric on $\mathfrak{F}$ is defined via the Hausdorff distance

$$D_0[z_1, z_2] = \sup_{\zeta \in [0,1]} H([z_1]^\zeta, [z_2]^\zeta),$$

where

$$H([z_1]^\zeta, [z_2]^\zeta) = \max\{|\underline{z_1}(\zeta) - \underline{z_2}(\zeta)|, |\overline{z_1}(\zeta) - \overline{z_2}(\zeta)|\}.$$

A function $w : [a,b] = \mathfrak{I} \to \mathfrak{F}$ with fuzzy values is defined as d-monotone if, for any fixed $\zeta \in [0,1]$, the function $\nu \mapsto d([w(\nu)]^\zeta)$ is monotonic across $\mathfrak{I}$.

Definition 2.1 ([8]). Consider a fuzzy-valued function $\mathfrak{g} : (a,b) \to \mathfrak{F}$. The gH-derivative of $\mathfrak{g}$ at a point $\nu \in (a,b)$ is expressed as

$$\mathfrak{g}'(\nu) = \lim_{h \to 0} \frac{\mathfrak{g}(\nu + h) \ominus_{gH} \mathfrak{g}(\nu)}{h}. \tag{2.2}$$

The function $\mathfrak{g}$ is deemed gH-differentiable at ν if the limit exists and belongs to $\mathfrak{F}$.

Let $C^1(\mathfrak{I}, \mathfrak{F})$ denote the space of all fuzzy functions $w : \mathfrak{I} \to \mathfrak{F}$ that are generalized Hukuhara differentiable (gH-differentiable) on $\mathfrak{I}$ and whose gH-derivative w' is continuous, i.e., $w' \in C(\mathfrak{I}, \mathfrak{F})$.

For $\mathfrak{g}, \mathfrak{f} \in C(\mathfrak{I}, \mathfrak{F})$, we define

$$D_*[\mathfrak{g}, \mathfrak{f}] = \sup_{\nu \in \mathfrak{I}} D_0[\mathfrak{g}(\nu), \mathfrak{f}(\nu)].$$

Under $(C(\mathfrak{I}, \mathfrak{F}), D_*)$ is a complete metric space.

A function $\mathcal{H} : \mathfrak{I} \times C(\mathfrak{I}, \mathfrak{F}) \to \mathfrak{F}$ is jointly continuous at (ν_0, w_0) if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } |\nu - \nu_0| + D_0[w, w_0] < \delta \implies D_0[\mathcal{H}(\nu, w), \mathcal{H}(\nu_0, w_0)] < \epsilon.$$

The notation $C_{\mathbb{J}}(\mathfrak{I}, \mathfrak{F})$ refers to the space of jointly continuous functions.

Definition 2.2 ([6])**.** Consider $\psi \in \mathcal{K}$, where $\mathcal{K}$ consists of strictly increasing, positive, C^1-functions on $\mathfrak{I}$ with non-vanishing derivatives on (a, b). The generalized Riemann-Liouville fuzzy fractional integral of order $\beta \in (0, 1)$ for a fuzzy-valued function $\mathfrak{g} : \mathfrak{I} \to \mathfrak{F}$ is given by

$$^c\mathcal{I}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu) = \frac{1}{\Gamma(\beta)} \int_a^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathfrak{g}(s)\, ds, \quad \text{for } \nu > a,$$

where $\mathfrak{D}_{\nu,s} = \psi(\nu) - \psi(s)$, and $\Gamma(\cdot)$ denotes the Gamma function.

Based on the GCFD framework introduced for real-valued functions in [6, 7] and extended to interval-valued functions in [12], we propose an extension of the generalized fractional derivative definition to the context of fuzzy-valued functions.

Definition 2.3. Suppose $\psi \in \mathcal{K}$, $\beta \in (0, 1)$, and let $\mathfrak{g} : \mathfrak{I} \to \mathfrak{F}$ be a fuzzy value function where $^c\mathcal{I}_{a^+}^{1-\beta;\psi}\mathfrak{g}(\nu)$ is gH-differentiable in $\mathfrak{I}$. The generalized Riemann–Liouville fractional derivative of order β is given by

$$^{RL}\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu) = \frac{1}{\psi'(\nu)}(^c\mathcal{I}_{a^+}^{1-\beta;\psi}\mathfrak{g}(\nu))',$$

equivalently written as

$$^{RL}\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu) = \frac{1}{\psi'(\nu)} \cdot \frac{1}{\Gamma(1-\beta)}(\int_a^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{-\beta}\mathfrak{g}(s)\, ds)', \quad \text{for } \nu > a. \tag{2.5}$$

Based on Definition 2.3, the GCFD of the fuzzy-valued function $\mathfrak{g}$ is defined in the weak sense as follows.

Definition 2.4. Let $\psi \in \mathcal{K}$, $\beta \in (0, 1)$, and let $\mathfrak{g} : \mathfrak{I} \to \mathfrak{F}$ be a fuzzy-valued function. If the generalized Riemann-Liouville fractional derivative $^{RL}\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu)$ exists on $\mathfrak{I}$, then the GCFD of $\mathfrak{g}$ is defined in the weak sense by

$$^C\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu) = {}^{RL}\mathcal{D}_{a^+}^{\beta;\psi}[\mathfrak{g}(\cdot) \ominus_{gH} \mathfrak{g}(a)],$$

which can be equivalently expressed as

$$^C\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu) = \frac{1}{\psi'(\nu)} \cdot \frac{1}{\Gamma(1-\beta)}\Big(\int_a^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{-\beta}[\mathfrak{g}(s) \ominus_{gH} \mathfrak{g}(a)]ds\Big)', \quad \nu > a. \tag{2.6}$$

Theorem 2.5 ([15])**.** *Let $\psi \in \mathcal{K}$ and $\beta \in (0, 1)$. If $\mathfrak{g} \in C^1(\mathfrak{I}, \mathfrak{F})$ is a d-monotone fuzzy function, then the GCFD of $\mathfrak{g}$ is given by*

$$^C\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu) = {}^c\mathcal{I}_{a^+}^{1-\beta;\psi}(\frac{1}{\psi'(\nu)}\mathfrak{g}'(\nu)) = \frac{1}{\Gamma(1-\beta)} \int_a^\nu \mathfrak{D}_{\nu,s}^{-\beta}\mathfrak{g}'(s)\, ds, \quad \nu > a, \tag{2.3}$$

where the term $\frac{1}{\psi'(\nu)}\mathfrak{g}'(\nu)$ denotes the scalar multiplication of a real-valued function with a fuzzy-valued function.

Remark 2.6. Equation (2.3) defines $^C\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu)$ as the strong-sense generalized Caputo fractional derivative. Moreover, strong GCFD differentiability entails weak GCFD differentiability for all $\mathfrak{g} \in C^1(\mathfrak{I}, \mathfrak{F})$.

Using a proof of technique analogous to that employed in [15, Prop. 1 and Corollary 1], we can establish the following results.

Theorem 2.7. *Let $\psi \in \mathcal{K}$, $\beta \in (0, 1)$, and let $\mathfrak{g} \in C(\mathfrak{I}, \mathfrak{F})$ be a d-monotone fuzzy function on the interval $\mathfrak{I}$. Then, the following properties hold*

(1) $^c\mathcal{I}_{a^+}^{\beta;\psi}(^C\mathcal{D}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu)) = \mathfrak{g}(\nu) \ominus_{gH} \mathfrak{g}(a)$;
(2) $^{RL}\mathcal{D}_{a^+}^{\beta;\psi}(^c\mathcal{I}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu)) = \mathfrak{g}(\nu)$;
(3) $^C\mathcal{D}_{a^+}^{\beta;\psi}(^c\mathcal{I}_{a^+}^{\beta;\psi}\mathfrak{g}(\nu)) = \mathfrak{g}(\nu)$.

We set $\mathbb{G}(\nu) = \mathcal{H}(\nu, w(\nu), w(\nu - r))$. Similar to the [15, Theorem 3], we also obtain the lemma below.

Lemma 2.8. *Assume that $\mathbb{G} : \mathcal{J} \to \mathfrak{F}$ is jointly continuous. Then, a d-monotone function $w \in C([-r,T],\mathfrak{F})$ solves problem (1.1) if and only if it satisfies*

$$\begin{aligned} w(\nu) &= \varrho(\nu), \quad \nu \in [-r,0] \\ w(\nu) \ominus_{gH} \varrho(0) &= \frac{1}{\Gamma(\beta)} \int_0^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathbb{G}(s)\,ds, \quad \nu \in \mathcal{J} \end{aligned} \tag{2.4}$$

assuming the right-hand side integral exhibits d-monotonicity on $(0,T]$.

Proof. Consider $w \in C([-r,T],\mathfrak{F})$ as a d-monotone solution to problem (1.1) on $\mathcal{J}$. For $\nu \in [-r,0]$, it holds that $w(\nu) = \varrho(\nu)$. Then, for $\nu \in \mathcal{J}$, applying the operator ${}^c\mathcal{I}_{0^+}^{\beta;\psi}$ to both sides of (1.1) and according to property (i) of Theorem 2.7, we derive

$$w(\nu) \ominus_{gH} \varrho(0) = \frac{1}{\Gamma(\beta)} \int_0^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathbb{G}(s)\,ds, \quad \nu \in \mathcal{J} \tag{2.5}$$

Consequently, w is a solution to (2.4). It is also known that the d- increasing property of (2.5)'s left-hand side over $\mathcal{J}$ ensures the same d-increasing behavior for its right-hand side over $\mathcal{J}$.

Conversely, suppose that $w \in C([-r,T],\mathfrak{F})$ is a d-monotone fuzzy function that satisfies equation (2.4), and that the integral on the right-hand side is d-increasing on $\mathcal{J}$. It is straightforward to verify that $w(\nu) = \varrho(\nu)$ for all $\nu \in [-r,0]$. Since the fuzzy function $\mathbb{G}$ is jointly continuous, the fractional integral ${}^c\mathcal{I}_{0^+}^{\beta;\psi}\mathbb{G}(\nu)$ is also continuous, and satisfies ${}^c\mathcal{I}_{0^+}^{\beta;\psi}\mathbb{G}(0) = \hat{0}$. Consequently, we have $w(0) = \varrho(0)$.

In addition, since the mapping $\nu \mapsto {}^c\mathcal{I}_{0^+}^{\beta;\psi}\mathbb{G}(\nu)$ is d-increasing on $\mathcal{J}$, applying the generalized Riemann-Liouville derivative ${}^{RL}\mathcal{D}_{0^+}^{\beta;\psi}$ to equation (2.4) and invoking part (ii) of Theorem 2.7 along with Definition 2.4, we obtain

$${}^C\mathcal{D}_{0^+}^{\beta;\psi}w(\nu) = \mathbb{G}(\nu),$$

which confirms that equation (1.1) is satisfied. □

Definition 2.9. Let $w : [-r,T] \to \mathfrak{F}$ be a function that solves problem (1.1). It is a solution if $w \in C([-r,T],\mathfrak{F})$, adheres to $w(\nu) = \varrho(\nu)$ for $\nu \in [-r,0]$, and satisfies ${}^C\mathcal{D}_{0^+}^{\beta;\psi}w(\nu) = \mathbb{G}(\nu)$ on $\mathcal{J}$. We classify w as d-monotone if it is either d-increasing or d-decreasing throughout $\mathcal{J}$.

Remark 2.10. According to equation (2.2), which defines the generalized Hukuhara difference, the integral equation (2.4) can be recast as follows

If $d([w(\nu)]^\zeta) \geq d([\varrho(0)]^\zeta)$ for all $\nu \in \mathcal{J}$ and every $\zeta \in [0,1]$, so that $\nu \mapsto d([w(\nu)]^\zeta)$ is nondecreasing on $\mathcal{J}$, then (2.4) is expressed as

$$\begin{aligned} w(\nu) &= \varrho(\nu), \quad \nu \in [-r,0] \\ w(\nu) &= \varrho(0) + \frac{1}{\Gamma(\beta)} \int_0^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathbb{G}(s)\,ds, \quad \nu \in \mathcal{J} \end{aligned} \tag{2.10}$$

If $d([w(\nu)]^\zeta) \leq d([\varrho(0)]^\zeta)$ for all $\nu \in \mathcal{J}$ and every $\zeta \in [0,1]$, indicating that $\nu \mapsto d([w(\nu)]^\zeta)$ is nonincreasing on $\mathcal{J}$, then (2.4) becomes

$$\begin{aligned} w(\nu) &= \varrho(\nu), \quad \nu \in [-r,0] \\ w(\nu) &= \varrho(0) \ominus \frac{(-1)}{\Gamma(\beta)} \int_0^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathbb{G}(s)\,ds, \quad \nu \in \mathcal{J} \end{aligned} \tag{2.11}$$

Assuming the Hukuhara difference in (2.11) exists on $\mathcal{J}$, the two cases of the generalized Hukuhara difference imply that (2.4) acts as a general form encompassing one of these integral equations.

Let $\mathcal{X}^{(i)}$ be the collection of fuzzy functions $w \in C([-r,T],\mathfrak{F})$ such that ${}^C\mathcal{D}_{0^+}^{\beta;\psi}w \in C(\mathcal{J},\mathfrak{F})$. Further, let $\mathcal{X}^{(ii)}$ denote the subset of $\mathcal{X}^{(i)}$ where the Hukuhara difference

$$w(\nu) = \varrho(0) \ominus \frac{(-1)}{\Gamma(\beta)} \int_0^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathbb{G}(s)\,ds,$$

is well-defined for $\nu \in \mathcal{J}$ and $\psi \in \mathcal{K}$.

Theorem 2.11 (Schaefer's fixed point theorem, [26]). *Let the operator $\mathcal{Q}:\mathcal{X}\to\mathcal{X}$ be completely continuous, where $\mathcal{X}$ is a Banach space. We denote*

$$\mathfrak{P}(\mathcal{Q})=\{v\in\mathcal{X}\mid v=\lambda\mathcal{Q}(v)\ \text{for some}\ 0<\lambda<1\}.$$

Then, if $\mathfrak{P}(\mathcal{Q})$ is bounded, $\mathcal{Q}$ has at least one fixed point.

To analyze the existence, uniqueness, and FTS of solutions to problem (1.1), we use the following assumptions:

- (A1) The fuzzy mapping $\mathcal{H}:\mathcal{J}\times\mathfrak{F}\times\mathfrak{F}\to\mathfrak{F}$ is jointly continuous.
- (A2) A constant $L>0$ exists such that
$$D_0[\mathcal{H}(\nu,z_1,z_2),\mathcal{H}(\nu,w_1,w_2)]\le L(D_0[z_1(\nu),w_1(\nu)]+D_0[z_2(\nu-r),w_2(\nu-r)]).$$
- (A3) There exist positive constants $\hat{M}_1$ and $\hat{M}_2\in(0,1)$ such that
$$D_0[\mathcal{H}(\nu,z_1,z_2),\hat{0}]\le\hat{M}_1D_0[z_1(\nu),\hat{0}]+\hat{M}_2D_0[z_2(\nu-r),\hat{0}],\forall\nu\in\mathcal{J}.$$

3. Existence and uniqueness results

We establish the existence of solutions to problem (1.1). To achieve this, we define two operators as follows.

To secure a d-increasing solution, we define the operator $\mathcal{Q}:\mathcal{X}^{(i)}\to\mathcal{X}^{(i)}$ as

$$(\mathcal{Q}w)(\nu)=\begin{cases}\varrho(\nu), & \nu\in[-r,0]\\ \varrho(0)+\frac{1}{\Gamma(\beta)}\int_0^\nu\psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathbb{G}(s)\,ds, & \nu\in\mathcal{J}\end{cases}\tag{3.1}$$

For the existence of the d-decreasing solution, we consider the operator $\mathcal{Q}:\mathcal{X}^{(ii)}\to\mathcal{X}^{(ii)}$ given by

$$(\mathcal{Q}w)(\nu)=\begin{cases}\varrho(\nu), & \nu\in[-r,0]\\ \varrho(0)\ominus\frac{(-1)}{\Gamma(\beta)}\int_0^\nu\psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1}\mathbb{G}(s)\,ds, & \nu\in\mathcal{J}\end{cases}\tag{3.1}$$

where $\mathbb{G}(\nu)=\mathcal{H}(\nu,w(\nu),w(\nu-r))$.

Let $T^*\in(0,T]$ such that $\lambda(\hat{M}_1+\hat{M}_2)\mathfrak{D}_{T^*,0}^{\beta}<\Gamma(\beta+1)$, where $\beta\in(0,1)$ and $\lambda\in(0,1)$.

Theorem 3.1. *If* (A1)–(A3) *are satisfied, then a solution to problem* (1.1) *exists on* $[0,T^*]$ *for every case under study.*

Proof. Since the proof strategy is the same for both cases in Remark 2.10, we focus on a representative one. The existence of a solution to (1.1) in (A1)–(A3) follows from Schaefer's fixed point theorem.

We reexamine the operator $\mathcal{Q}$ from (3.1) and verify that it is correctly defined by confirming $\mathcal{Q}(\mathcal{X}^{(ii)})\subseteq\mathcal{X}^{(ii)}$. This conclusion follows immediately from the continuity of $\mathbb{G}$, which ensures the continuity of $(\mathcal{Q}w)(\nu)$ for $\nu\in[-r,T]$. Applying ${}^C\mathcal{D}_{0^+}^{\beta;\psi}$ to both sides of (3.1) gives

$${}^C\mathcal{D}_{0^+}^{\beta;\psi}(\mathcal{Q}w)(\nu)=\mathbb{G}(\nu),\quad\forall\nu\in\mathcal{J},$$

which implies that ${}^C\mathcal{D}_{0^+}^{\beta;\psi}(\mathcal{Q}w)$ is continuous on $\mathcal{J}$. The verification of the hypotheses of Theorem 2.11 will now be carried out in four steps.

Step 1. The operator $\mathcal{Q}:\mathcal{X}^{(ii)}\to\mathcal{X}^{(ii)}$ is continuous. To demonstrate this, take a sequence $\{w_n\}_{n\in\mathbb{N}}\subset\mathcal{X}^{(ii)}$ such that $w_n\to w\in\mathcal{X}^{(ii)}$ as $n\to\infty$. For $\nu\in[-r,0]$, it holds that

$$\mathcal{Q}(w_n)(\nu)=\varrho(\nu)\to\mathcal{Q}(w)(\nu)=\varrho(\nu),$$

since $\mathcal{Q}$ is defined to yield $\varrho(\nu)$ for all functions in $\mathcal{X}^{(ii)}$ on $[-r,0]$.

$$D_0[(\mathcal{Q}w_n)(\nu),(\mathcal{Q}w)(\nu)]=0,$$

and for any $\nu\in\mathcal{J}$, one also obtains

$$D_0[(\mathcal{Q}w_n)(\nu),(\mathcal{Q}w)(\nu)]\le\frac{2L}{\Gamma(\beta+1)}\mathfrak{D}_{\nu,0}^{\beta}\sup_{\nu\in[-r,T]}D_0[w_n(\nu),w(\nu)],$$

where we use assumption (A2). Hence, it yields

$$D_*[\mathcal{Q}w_n, \mathcal{Q}w] \le \frac{2L\mathfrak{D}^{\beta}_{\nu,0}}{\Gamma(\beta+1)} D_*[w_n, w], \quad \forall \nu \in [-r, T].$$

Because $w_n \to w$ as $n \to \infty$, one can imply that $D_*[\mathcal{Q}w_n, \mathcal{Q}w] \to 0$ on $\mathcal{X}^{(ii)}$ as $n \to \infty$. Therefore, the operator $\mathcal{Q}$ is continuous on $\mathcal{X}^{(ii)}$.

Step 2. Consider the ball $\mathbb{B}(\hat{0}, \rho) \subset \mathcal{X}^{(ii)}$, defined by $D_0[\mathfrak{g}(\nu), \hat{0}] \le \rho$. We aim to prove that $\mathcal{Q}$ is a bounded operator on bounded subsets, i.e., there exists $\hat{\rho} > 0$ such that $D_*[\mathcal{Q}v, \hat{0}] \le \hat{\rho}$. This follows from the behavior of $\mathcal{Q}$ on $[-r, 0]$ and from assumptions (A1) and (A2) on $\mathcal{J}$.

$$\begin{aligned} D_0[(\mathcal{Q}w)(\nu), \hat{0}] &\le D_0[\varrho(0), \hat{0}] + \frac{L}{\Gamma(\beta)} \Big(\sup_{\nu \in [-r,T]} D_0[w(\nu), \hat{0}] + \sup_{\nu \in [-r,T]} D_0[w(\nu - r), \hat{0}] \Big) \\ &\quad \times \int_0^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s}\, ds + \frac{1}{\Gamma(\beta)} \int_0^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[\mathcal{H}(s, \hat{0}, \hat{0}), \hat{0}]\, ds \\ &\le \rho + \frac{2L\mathfrak{D}^{\beta}_{\nu,0}}{\Gamma(\beta+1)} D_*[w, \hat{0}]. \end{aligned}$$

Therefore, by choosing

$$\hat{\rho} = \rho + \frac{2L\mathfrak{D}^{\beta}_{\nu,0} D_*[w, \hat{0}]}{\Gamma(\beta+1)},$$

we ensure that the operator $\mathcal{Q}$ maps any function $w \in \mathbb{B}(\hat{0}, \rho)$ into a function whose distance from $\hat{0}$ remains bounded by $\hat{\rho}$.

$$D_*[\mathcal{Q}w, \hat{0}] \le \hat{\rho} < \infty, \quad \forall \nu \in \mathcal{J}.$$

This shows that the operator $\mathcal{Q}$ maps bounded sets into bounded sets of $\mathcal{X}^{(ii)}$.

Step 3. We now show that the operator $\mathcal{Q}$ maps bounded sets into equi-continuous subsets of $\mathcal{X}^{(ii)}$. To this end, consider two time points $\nu_1, \nu_2 \in \mathcal{J}$ such that $\nu_1 < \nu_2$, and let $w \in \mathbb{B}(\hat{0}, \rho)$. Then, based on assumptions (A1) and (A2), one can derive that

$$\begin{aligned} &D_0[(\mathcal{Q}w)(\nu_2), (\mathcal{Q}w)(\nu_1)] \\ &\le \frac{1}{\Gamma(\beta)} \int_0^{\nu_1} \psi'(s) (\mathfrak{D}^{\beta-1}_{\nu_2,s} - \mathfrak{D}^{\beta-1}_{\nu_1,s}) D_0[\mathbb{G}(s), \hat{0}]\, ds + \frac{1}{\Gamma(\beta)} \int_{\nu_1}^{\nu_2} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu_2,s} D_0[\mathbb{G}(s), \hat{0}]\, ds, \\ &\le \frac{2\rho L \mathfrak{D}^{\beta}_{\nu_2,\nu_1}}{\Gamma(\beta+1)}. \end{aligned}$$

where we use the estimate below via using the assumption (A2)

$$D_0[\mathbb{G}(\nu), \hat{0}] \le 2L D_*[w, \hat{0}] = 2\rho L, \quad \forall \nu \in \mathcal{J} \tag{3.2}$$

Because ψ belongs to $\mathcal{K}$, the distance between $(\mathcal{Q}w)(\nu_2)$ and $(\mathcal{Q}w)(\nu_1)$ vanishes as $\nu_2 \to \nu_1$, which implies the equi-continuity of $\mathcal{Q}$. From the Ascoli-Arzelà theorem, it follows that the set $\mathcal{Q}(\mathbb{B}(\hat{0}, \rho))$ is relatively compact in $\mathcal{X}^{(ii)}$. This analysis shows that $\mathcal{Q}$ is a continuous and completely continuous operator.

Step 4. The final step in invoking Theorem 2.11 consists of establishing the boundedness of the set defined by

$$\mathfrak{P}(\mathcal{Q}) = \{ w \in \mathbb{B}(\hat{0}, \rho) : u = \lambda \mathcal{Q}(u), \lambda \in (0,1) \},$$

is bounded in $\mathcal{X}^{(ii)}$. Let $w \in \mathfrak{P}(\mathcal{Q})$ and $\lambda \in (0,1)$, then one notices $w = \lambda \mathcal{Q}(u)$. Hence, we have that

$$w(\nu) = \lambda(\varrho(0) \ominus \frac{(-1)}{\Gamma(\beta)} \int_0^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} \mathbb{G}(s)\, ds), \quad \nu \in \mathcal{J} \tag{3.3}$$

By (A3), we obtain

$$D_0[\mathbb{G}(\nu), \hat{0}] \le (\hat{M}_1 + \hat{M}_2) D_*[w, \hat{0}], \quad \forall \nu \in \mathcal{J} \tag{3.4}$$

A straightforward combination of (3.3) and (3.4) leads to

$$D_0[w(\nu),\hat{0}] \le \lambda(D_0[\varrho(0),\hat{0}] + (\hat{M}_1+\hat{M}_2)\frac{\mathfrak{D}^{\beta}_{\nu,0}}{\Gamma(\beta+1)})D_*[w,\hat{0}],$$

which implies

$$D_*[w,\hat{0}] \le \lambda(D_0[\varrho(0),\hat{0}] + (\hat{M}_1+\hat{M}_2)\frac{\mathfrak{D}^{\beta}_{\nu,0}}{\Gamma(\beta+1)})D_*[w,\hat{0}],$$

where ψ is nondecreasing and $\exists T^* \in \mathcal{J}$ such that $\lambda(\hat{M}_1+\hat{M}_2)\mathfrak{D}^{\beta}_{T^*,0} < \Gamma(\beta+1)$, where $\beta \in (0,1)$ and $\lambda \in (0,1)$. Then, we obtain

$$D_*[w,\hat{0}] \le \lambda D_0[\varrho(0),\hat{0}](1-\lambda(\hat{M}_1+\hat{M}_2)\frac{\mathfrak{D}^{\beta}_{T^*,0}}{\Gamma(\beta+1)})^{-1} < \infty.$$

The boundedness of $\mathfrak{P}(\mathcal{Q})$ implies, via Theorem 2.11 (Schaefer), that (1.1) possesses at least one solution in $\mathcal{J}^* = [0,T^*]$. □

Theorem 3.2. *Under assumptions* (A1)–(A3)*, system* (1.1) *admits a unique solution on* $\mathcal{J}^*$*,* $T^* \in (0,T]$*, for all cases.*

Proof. Assume $T^* > r$ with the partition

$$\mathcal{J}^* = \cup_{k=0}^{n}[kr,(k+1)r] \cup [(n+1)r,T^*],$$

where $(n+1)r < T^* \le (n+2)r$. For any two solutions w and $\mathfrak{g}$ of (1.1) satisfying $w(\nu) = \varrho_1(\nu)$ and $\mathfrak{g}(\nu) = \varrho_2(\nu)$ for all $\nu \in [-r,0]$, Lemma 2.8 yields the following:

For $\nu \in [0,r]$,

$$\begin{aligned} D_0[w(\nu),\mathfrak{g}(\nu)] \le D_0[w(0),\mathfrak{g}(0)] &+ \frac{L}{\Gamma(\beta)}\int_0^{\nu}\psi'(s)\mathfrak{D}^{\beta-1}_{\nu,s}D_0[w(s-r),\mathfrak{g}(s-r)]\,ds \\ &+ \frac{L}{\Gamma(\beta)}\int_0^{\nu}\psi'(s)\mathfrak{D}^{\beta-1}_{\nu,s}D_0[w(s),\mathfrak{g}(s)]\,ds. \end{aligned}$$

Considering $r \in (0,r_{\max}]$, we note that

$$D_0[w(\nu),\mathfrak{g}(\nu)] \le D_0[w(0),\mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{\nu,0}}{\Gamma(\beta+1)}D_r[\varrho_1,\varrho_2] + \frac{L}{\Gamma(\beta)}\int_0^{\nu}\psi'(s)\mathfrak{D}^{\beta-1}_{\nu,s}D_0[w(s),\mathfrak{g}(s)]\,ds.$$

Using the generalized Gronwall inequality in [27], we obtain

$$D_0[w(\nu),\mathfrak{g}(\nu)] \le \mathbb{A}_1(r)E_{\beta,1}(L\mathfrak{D}^{\beta}_{r,0}), \quad \forall \nu \in [0,r], \tag{3.5}$$

where

$$\mathbb{A}_1(r) = D_0[w(0),\mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)}D_r[\varrho_1,\varrho_2]. \tag{3.6}$$

For $\nu \in (r,2r]$, combining (3.5) with the inequality $\lambda_1^{\beta} - \lambda_2^{\beta} \le (\lambda_1-\lambda_2)^{\beta}$ (valid for $0 \le \lambda_2 \le \lambda_1$, $\beta \in (0,1)$) yields

$$\begin{aligned} D_0[w(\nu),\mathfrak{g}(\nu)] \le D_0[w(0),\mathfrak{g}(0)] &+ \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)}D_r[\varrho_1,\varrho_2] + \mathbb{A}_1(r)\frac{L\mathfrak{D}^{\beta}_{\nu,r}}{\Gamma(\beta+1)}E_{\beta,1}(L\mathfrak{D}^{\beta}_{r,0}) \\ &+ \frac{L}{\Gamma(\beta)}\int_0^{\nu}\psi'(s)\mathfrak{D}^{\beta-1}_{\nu,s}D_0[w(s),\mathfrak{g}(s)]\,ds, \end{aligned}$$

since $(s-r) \in (-r,0]$ and $(s-r) \in (0,r]$ for $s \in (0,r]$ and $s \in (r,2r]$, respectively. Through the generalized Gronwall inequality, we establish the following crucial estimate

$$D_0[w(\nu),\mathfrak{g}(\nu)] \le \mathbb{A}_2(r)E_{\beta,1}(L\mathfrak{D}^{\beta}_{2r,0}), \quad \forall \nu \in (r,2r], \tag{3.7}$$

where

$$\mathbb{A}_2(r) = D_0[w(0),\mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)}D_r[\varrho_1,\varrho_2] + \mathbb{A}_1(r)\frac{L\mathfrak{D}^{\beta}_{2r,r}}{\Gamma(\beta+1)}E_{\beta,1}(L\mathfrak{D}^{\beta}_{r,0}). \tag{3.8}$$

By induction on $k \geq 3$, assume $\forall \nu \in ((k-1)r, kr]$ the inequality holds

$$D_0[w(\nu), \mathfrak{g}(\nu)] \leq \mathbb{A}_k(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{kr,0}), \quad \forall \nu \in ((k-1)r, kr], \tag{3.9}$$

where

$$\mathbb{A}_k(r) = D_0[w(0), \mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)} D_r[\varrho_1, \varrho_2] + \frac{L}{\Gamma(\beta+1)} \sum_{i=1}^{k-1} \mathfrak{D}^{\beta}_{(i+1)r,ir} \mathbb{A}_i(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{ir,0}). \tag{3.10}$$

For $\nu \in (kr, (k+1)r]$ $(k \geq 3)$, using the inequality $\mathrm{a}^{\beta} - \mathrm{b}^{\beta} \leq (\mathrm{a}-\mathrm{b})^{\beta}$ with $r \in [0, r_0]$ and (3.9), we obtain

$$\begin{aligned}
&D_0[w(\nu), \mathfrak{g}(\nu)] \\
&\leq D_0[w(0), \mathfrak{g}(0)] + \frac{L}{\Gamma(\beta)} \sum_{i=0}^{k-1} \int_{ir}^{(i+1)r} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s-r), \mathfrak{g}(s-r)]\, ds \\
&\quad + \frac{L}{\Gamma(\beta)} \int_{kr}^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s-r), \mathfrak{g}(s-r)]\, ds + \frac{L}{\Gamma(\beta)} \int_0^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s), \mathfrak{g}(s)]\, ds, \\
&\leq D_0[w(0), \mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)} D_r[\varrho_1, \varrho_2] + \frac{L}{\Gamma(\beta+1)} \sum_{i=1}^{k-1} \mathfrak{D}^{\beta}_{(i+1)r,ir} \mathbb{A}_i(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{ir,0}) \\
&\quad + \frac{L}{\Gamma(\beta+1)} (\psi(\nu) - \psi(kr))^{\beta} \mathbb{A}_k(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{kr,0}) + \frac{L}{\Gamma(\beta)} \int_0^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s), \mathfrak{g}(s)]\, ds.
\end{aligned}$$

It follows from the generalized Gronwall inequality that

$$D_0[w(\nu), \mathfrak{g}(\nu)] \leq \mathbb{A}_{k+1}(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{(k+1)r,0}), \quad \forall \nu \in (kr, (k+1)r] \tag{3.11}$$

where

$$\mathbb{A}_{k+1}(r) = D_0[w(0), \mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)} D_r[\varrho_1, \varrho_2] + \frac{L}{\Gamma(\beta+1)} \sum_{i=1}^{k} \mathfrak{D}^{\beta}_{(i+1)r,ir} \mathbb{A}_i(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{ir,0}). \tag{3.12}$$

Similarly, for $\nu \in ((n+1)r, T^*]$, we also have

$$\begin{aligned}
&D_0[w(\nu), \mathfrak{g}(\nu)] \\
&\leq D_0[w(0), \mathfrak{g}(0)] + \frac{L}{\Gamma(\beta)} \sum_{i=0}^{n} \int_{ir}^{(i+1)r} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s-r), \mathfrak{g}(s-r)]\, ds \\
&\quad + \frac{L}{\Gamma(\beta)} \int_{(n+1)r}^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s-r), \mathfrak{g}(s-r)]\, ds + \frac{L}{\Gamma(\beta)} \int_0^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s), \mathfrak{g}(s)]\, ds, \\
&\leq D_0[w(0), \mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)} D_r[\varrho_1, \varrho_2] + \frac{L}{\Gamma(\beta+1)} \sum_{i=1}^{n} (\psi((i+1)r - \psi(ir))^{\beta} \mathbb{A}_i(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{ir,0}) \\
&\quad + \frac{L}{\Gamma(\beta+1)} \mathfrak{D}^{\beta}_{T^*,(n+1)r} \mathbb{A}_{n+1}(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{(n+1)r,0}) + \frac{L}{\Gamma(\beta)} \int_0^{\nu} \psi'(s) \mathfrak{D}^{\beta-1}_{\nu,s} D_0[w(s), \mathfrak{g}(s)]\, ds,
\end{aligned}$$

with $\mathbb{A}_{n+1}(r)$ given by (3.12). An application of the generalized Gronwall inequality

$$D_0[w(\nu), \mathfrak{g}(\nu)] \leq \mathbb{A}_{T^*}(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{T^*,0}), \quad \forall \nu \in ((n+1)r, T^*], \tag{3.13}$$

where

$$\begin{aligned}
\mathbb{A}_{T^*}(r) &= D_0[w(0), \mathfrak{g}(0)] + \frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)} D_r[\varrho_1, \varrho_2] \\
&\quad + \frac{L}{\Gamma(\beta+1)} \sum_{i=1}^{n} \mathfrak{D}^{\beta}_{(i+1)r,ir} \mathbb{A}_i(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{ir,0}) \\
&\quad + \frac{L}{\Gamma(\beta+1)} \mathfrak{D}^{\beta}_{T^*,(n+1)r} \mathbb{A}_{n+1}(r) E_{\beta,1}(L\mathfrak{D}^{\beta}_{(n+1)r,0}).
\end{aligned}$$

Assuming $\varrho_1(\nu) = \varrho_2(\nu)$ for all $\nu \in [-r, 0]$, we observe that

$$\mathbb{A}_1(r) = \mathbb{A}_2(r) = \cdots = \mathbb{A}_{n+1}(r) = \mathbb{A}_{T^*}(r) = 0.$$

Consequently, we obtain the identity $w(\nu) = \mathfrak{g}(\nu)$ for all $\nu \in [-r, T^*]$. □

4. Finite time stability result

Definition 4.1. Problem (1.1) is said to exhibit FTS with respect to the set $\{0, \mathcal{J}, \delta, \varepsilon, r\}$ if the following conditions are satisfied

1. For any initial function ϱ such that $\sup_{\nu\in[-r,0]} D_0[\varrho(\nu), \hat{0}] \leq \delta$,
2. The corresponding solution w satisfies $D_0[w(\nu), \hat{0}] \leq \varepsilon$ for all $\nu \in \mathcal{J}$.

Theorem 4.2. *Under Assumptions*(A1)–(A3) *there exists a solution to* (1.1) *on* $\mathcal{J}^*$. *This solution is finite-time stable with respect to* $\{0, \mathcal{J}^*, \delta, \varepsilon, r\}$ *if*

$$\delta E_{\beta,1}(2L\mathfrak{D}^{\beta}_{T^*,0}) \leq \varepsilon. \tag{4.1}$$

Proof. Suppose that w solves system (1.1) over the interval $\mathcal{J}^*$. We define the function $\mathcal{Z}(\nu)$ to measure the distance between $w(\nu)$ and the zero element. Then, by applying the explicit solution formula from (2.4) and using assumption (A2), we can establish the following estimate for all $\nu \in \mathcal{J}^*$,

$$\mathcal{Z}(\nu) \leq \mathcal{Z}(0) + \frac{1}{\Gamma(\beta)} \int_0^{\nu} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu,s}[L\mathcal{Z}(s) + L\mathcal{Z}(s-r)]\, ds. \tag{4.2}$$

Let $\mathcal{Z}_*(\nu) = \sup_{\xi\in[-r,t]} \mathcal{Z}(\xi)$, for all $\nu \in \mathcal{J}^*$. Clearly, $\mathcal{Z}_*(\nu)$ increases with t, and we note that

$$\mathcal{Z}(\nu - r) \leq \mathcal{Z}_*(\nu) \quad \text{and} \quad \mathcal{Z}(\nu) \leq \mathcal{Z}_*(\nu). \tag{4.3}$$

Moreover, we observe that $\mathcal{Z}(s - r) \leq \mathcal{Z}_*(s)$ for all $s \in \mathcal{J}$. Consequently, inequality (4.2) implies that

$$\mathcal{Z}(\nu) \leq \mathcal{Z}(0) + \frac{1}{\Gamma(\beta)} \int_0^{\nu} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu,s} 2L\mathcal{Z}_*(s)\, ds. \tag{4.4}$$

Next, we set $C = 2L$ and $\mathcal{U}(\nu) = \frac{C}{\Gamma(\beta)} \int_0^{\nu} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu,s}\mathcal{Z}_*(s)\, ds$. We want to show that $\mathcal{U}(\nu)$ gets larger as time t increases within $\mathcal{J}^*$. Take two times ν_1 and ν_2 such that $\nu_1 \leq \nu_2$. Since

$$\frac{\mathfrak{D}^{\beta}_{\nu_2,0}}{\Gamma(\beta+1)} \geq \frac{\mathfrak{D}^{\beta}_{\nu_1,0}}{\Gamma(\beta+1)},$$

it follows that

$$\frac{C}{\Gamma(\beta)} \int_{\nu_1}^{\nu_2} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_2,0}\, ds \geq \frac{C}{\Gamma(\beta)} \int_0^{\nu_1} [\psi'(s)\mathfrak{D}^{\beta-1}_{\nu_1,0} - \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_2,0}]\, ds.$$

Since $\mathcal{Z}_*$ is increasing on $\mathcal{J}^*$, one has

$$\frac{C}{\Gamma(\beta)} \int_{\nu_1}^{\nu_2} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_2,0}\mathcal{Z}_*(s)\, ds \geq \frac{C}{\Gamma(\beta)} \int_0^{\nu_1} [\psi'(s)\mathfrak{D}^{\beta-1}_{\nu_1,0} - \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_2,0}]\mathcal{Z}_*(s)\, ds.$$

Therefore,

$$\begin{aligned}
&\mathcal{U}(\nu_2) - \mathcal{U}(\nu_1) \\
&= \frac{C}{\Gamma(\beta)} \int_0^{\nu_2} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_2,0}\mathcal{Z}_*(s)\, ds - \frac{C}{\Gamma(\beta)} \int_0^{\nu_1} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_1,0}\mathcal{Z}_*(s)\, ds, \\
&= \frac{C}{\Gamma(\beta)} \int_{\nu_1}^{\nu_2} \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_2,0}\mathcal{Z}_*(s)\, ds - \frac{C}{\Gamma(\beta)} \int_0^{\nu_1} [\psi'(s)\mathfrak{D}^{\beta-1}_{\nu_1,0} - \psi'(s)\mathfrak{D}^{\beta-1}_{\nu_2,0}]\mathcal{Z}_*(s)\, ds \geq 0.
\end{aligned}$$

The above inequality shows that

$$\mathcal{U}(\nu_2) - \mathcal{U}(\nu_1) \geq 0 \quad \text{for all } \nu_2 \geq \nu_1.$$

This monotonicity follows from $\mathcal{Z}_*(\nu)$ being increasing, and the positivity of the generalized fractional kernel $(\psi(\nu) - \psi(s))^{\beta-1}$, since $\psi \in \mathcal{K}$ is strictly increasing and $\psi'(s) > 0$. Hence, the generalized fractional integral operator preserves monotonicity, and $\mathcal{U}(\nu)$ increasing on $\mathcal{J}^*$. Applying inequality (4.4), we obtain

$$\mathcal{Z}(\xi) \le \mathcal{Z}(0) + \frac{C}{\Gamma(\beta)} \int_0^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1} \mathcal{Z}_*(s)\, ds. \tag{4.5}$$

for all $\xi \in \mathcal{J}$. Thus, we have

$$\mathcal{Z}_*(\nu) = \sup_{\xi\in[-r,t]} \mathcal{Z}(\xi) \le \mathcal{Z}(0) + \frac{C}{\Gamma(\beta)} \int_0^\nu \psi'(s)\mathfrak{D}_{\nu,s}^{\beta-1} \mathcal{Z}_*(s)\, ds. \tag{4.6}$$

By applying the generalized Gronwall inequality, we obtain that

$$\mathcal{Z}_*(\nu) \le \mathcal{Z}(0)E_{\beta,1}(C\mathfrak{D}_{\nu,0}^{\beta}). \tag{4.7}$$

So, this means that

$$\mathcal{Z}(\nu) \le \mathcal{Z}_*(\nu) = \sup_{\xi\in[-r,t]} \mathcal{Z}(\xi) \le \mathcal{Z}(0)E_{\beta,1}(C\mathfrak{D}_{T^*,0}^{\beta}),$$

where $\mathcal{Z}(0) = D_0[\varrho(0), \hat{0}]$. Then, if $D_0[\varrho(0), \hat{0}] \le \delta$, it follows from (4.1) that

$$D_0[w(\nu), \hat{0}] \le \varepsilon, \quad \forall \nu \in \mathcal{J}^*.$$

This yields the FTS of the solution of problem (1.1). □

Theorem 4.3. *Under assumptions* (A1)–(A3)*there exists a solution to* (1.1) *on* $\mathcal{J}^*$. *This solution is finite-time stable with respect to* $\{0, \mathcal{J}^*, \delta, \varepsilon, r\}$ *if*

$$\mathbb{E}_{T^*}(r)E_{\beta,1}(L\mathfrak{D}_{T^*,0}^{\beta}) \le \varepsilon, \tag{4.8}$$

where $\mathbb{E}_{T^*}(r)$ *is defined in the proof below.*

Proof. Assume that $T > r$. We partition the interval $\mathcal{J}$ as

$$\mathcal{J} = \cup_{k=0}^{n}[kr, (k+1)r] \cup [(n+1)r, T],$$

where $(n+1)r < T \le (n+2)r$. By applying the method of steps in conjunction with the generalized Gronwall inequality, we establish the FTS of system (1.1), as outlined below. Additionally, we assume that the initial condition satisfies $D_0[\varrho(\nu), \hat{0}] \le \delta$ for all $\nu \in [-r, 0]$.

Consider $\nu \in [0, r]$. Applying the same estimation method developed for inequality (3.5) in the proof of Theorem 3.2, and noting that $r \le r_{\max}$, we establish

$$D_0[w(\nu), \hat{0}] \le \mathbb{E}_1(r)E_{\beta,1}(L\mathfrak{D}_{r,0}^{\beta}), \quad \forall \nu \in [0, r], \tag{3.24}$$

where

$$\mathbb{E}_1(r) = \delta(1 + \frac{L\mathfrak{D}_{r,0}^{\beta}}{\Gamma(\beta+1)}). \tag{4.9}$$

For $\nu \in (r, 2r]$, as in the estimate (3.7), we obtain

$$D_0[w(\nu), \hat{0}] \le \mathbb{E}_2(r)E_{\beta,1}(L\mathfrak{D}_{2r,0}^{\beta}), \quad \forall \nu \in (r, 2r], \tag{4.10}$$

where

$$\mathbb{E}_2(r) = \delta(1 + \frac{L\mathfrak{D}_{r,0}^{\beta}}{\Gamma(\beta+1)}) + \mathbb{E}_1(r)\frac{L\mathfrak{D}_{2r,r}^{\beta}}{\Gamma(\beta+1)}E_{\beta,1}(L\mathfrak{D}_{r,0}^{\beta}).$$

For $\nu \in (kr, (k+1)r]$, where $k = 3, 4, 5, \dots$. Similar to the estimate (3.11), the following estimate holds,

$$D_0[w(\nu), \hat{0}] \le \mathbb{E}_{k+1}(r)E_{\beta,1}(L\mathfrak{D}_{(k+1)r,0}^{\beta}), \quad \forall \nu \in (kr, (k+1)r] \tag{4.11}$$

where

$$\mathbb{E}_{k+1}(r) = \delta(1 + \frac{L\mathfrak{D}_{r,0}^{\beta}}{\Gamma(\beta+1)}) + \frac{L}{\Gamma(\beta+1)} \sum_{i=1}^{k} \mathfrak{D}_{(i+1)r,ir}^{\beta} \mathbb{E}_i(r)E_{\beta,1}(L\mathfrak{D}_{ir,0}^{\beta}).$$

For $\nu \in ((k+1)r, T^*]$, following the same bounding technique as in (3.13), we establish

$$D_0[w(\nu), \hat{0}] \le \varepsilon,$$

which verifies the FTS of solution to problem (5.1).

$$D_0[w(\nu),\hat{0}] \le \mathbb{E}_{T^*}(r)E_{\beta,1}(L\mathfrak{D}^{\beta}_{T^*,0}), \quad \forall \nu \in ((n+1)r, T^*], \tag{4.12}$$

where

$$\begin{aligned}\mathbb{E}_{T^*}(r) &= \delta(1+\frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)}) + \frac{L}{\Gamma(\beta+1)}\sum_{i=1}^{n}\mathfrak{D}^{\beta}_{(i+1)r,ir}\mathbb{E}_i(r)E_{\beta,1}(L\mathfrak{D}^{\beta}_{ir,0})\\ &\quad + \frac{L}{\Gamma(\beta+1)}\mathfrak{D}^{\beta}_{T^*,(n+1)r}\mathbb{E}_{n+1}(r)E_{\beta,1}(L\mathfrak{D}^{\beta}_{(n+1)r,0}).\end{aligned}$$

For $\nu \in ((k+1)r, T^*]$, akin to the bound in (3.13), we derive $D_0[w(\nu),\hat{0}] \le \varepsilon$, which ensures the FTS of the solution to system (5.1). □

Remark 4.4. Using the analytical technique from Theorem 4.3 applied to the interval $T^* \in (0, r]$, we arrive at the estimate

$$D_0[w(\nu),\hat{0}] \le \delta(1+\frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)})E_{\beta,1}(L\mathfrak{D}^{\beta}_{r,0}).$$

Consequently, system (1.1) exhibits FTS under the following sufficient condition.

$$\delta(1+\frac{L\mathfrak{D}^{\beta}_{r,0}}{\Gamma(\beta+1)})E_{\beta,1}(L\mathfrak{D}^{\beta}_{r,0}) \le \varepsilon. \tag{4.13}$$

5. Examples

In this section, we illustrate the validate of our theoretical results. While the computational framework developed in Theorem 4.3 appears more involved than that of Theorem 4.2, we emphasize that its FTS conditions are actually less restrictive, offering broader applicability in practice. We examine the concrete problem

$$\begin{gathered} {}^{C}\mathcal{D}^{\beta;\psi}_{0^+}w(\nu) = \mathcal{H}(\nu, w(\nu), w(\nu-r)), \quad \nu \in [0, 0.3],\\ w(\nu) = \varrho(\nu) \in \mathfrak{F}, \quad \nu \in [-r, 0],\end{gathered} \tag{5.1}$$

For the function $\mathcal{H}(\nu, w(\nu), w(\nu-r)) = \sin(\nu)w(\nu) + \cos(\nu)w(\nu-r)$, where the delay is constant at $r = 0.1$, we suppose $\sup_{\nu\in[-r,0]} D_0[\varrho(\nu),\hat{0}] \le \delta$, with δ being a positive real number. Due to the difficulty in deriving an analytical solution for (5.1), this example provides sufficient conditions for the FTS of the solution, setting $L = 1$. The solution is presumed to exist over the interval $[0, 0.3]$.

From Theorem 4.2, the solution to (5.1) is FTS with respect to $\{0, [0, 0.3], \delta, \varepsilon, 0.1\}$ provided the specified conditions hold.

$$E_{\beta,1}(2\mathfrak{D}^{\beta}_{0.3,0}) \le \varepsilon/\delta.$$

According to Theorem 4.3, the solution of system (5.1) demonstrates FTS with respect to $\{0, [0, 0.3], \delta, \varepsilon, 0.1\}$ when the given conditions are satisfied.

$$\mathbb{E}_{0.3}(0.1)E_{\beta,1}(\mathfrak{D}^{\beta}_{0.3,0}) \le \varepsilon,$$

where

$$\begin{aligned}\mathbb{E}_1(0.1) &= \delta(1+\frac{\mathfrak{D}^{\beta}_{0.1,0}}{\Gamma(\beta+1)}),\\ \mathbb{E}_2(0.1) &= \mathbb{E}_1(0.1) + \mathbb{E}_1(0.1)\frac{\mathfrak{D}^{\beta}_{0.2,0.1}}{\Gamma(\beta+1)}E_{\beta,1}(\mathfrak{D}^{\beta}_{0.1,0}),\\ \mathbb{E}_{0.3}(0.1) &= \mathbb{E}_2(0.1) + \mathbb{E}_2(0.1)\frac{\mathfrak{D}^{\beta}_{0.3,0.2}}{\Gamma(\beta+1)}E_{\beta,1}(\mathfrak{D}^{\beta}_{0.2,0}).\end{aligned}$$

To illustrate the applicability of Theorems 4.2 and 4.3 in analyzing the FTS of system (5.1), Table 1 presents numerical results for different functions ψ, with $\beta = 0.5$.

11.2

TABLE 1. Evaluation of Theorems 4.2 and 4.3 for fuzzy time-scale stability with $\delta = 20$.

Sufficient conditions on ψ	t	$\log(t+1)$	$\sqrt{t^2+1}$
Theorem 4.2	1.247477×10^2	1.058240×10^2	34.51709
Theorem 4.3	2.6871×10^3	2.1038×10^3	161.9795

Conclusion

In this article, we studied fuzzy fractional delay differential equations within the framework of the generalized Caputo fractional derivative. By employing stepwise approximation techniques together with Gronwall-type inequalities, we established the existence and uniqueness of solutions. Furthermore, sufficient conditions ensuring finite-time stability were derived. Theoretical findings were supported with computational simulations, which validated the analytical results and demonstrated the effectiveness of the proposed framework. The obtained results highlight the interplay between fuzziness, memory effects, and delays in fractional dynamical systems, providing new insights into their qualitative behavior.

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