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ON NODAL GROUND STATES FOR SCHRÖDINGER SYSTEMS

ANNA LISA AMADORI

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Abstract. In this article we characterize the least energy nodal and semi-nodal solutions to some Schrödinger system as the minimum on constrained Nehari sets of codimension 4 and 3, respectively; thus allowing to compute their Morse index and the exact number of nodal domains. Next the focus is on the symmetry properties of the sign-changing solutions. We show that, even though the domain is a ball, ground states are not radial, and produce other non-radial solutions with the given symmetry.

1. Introduction

We study the Schrödinger system

$$\begin{gathered} -\Delta u + \lambda_1 u = \mu_1 |u|^{2q-2} u + \beta |u|^{q-2} |v|^q u \quad \text{in } \Omega, \\ -\Delta v + \lambda_2 v = \mu_2 |v|^{2q-2} v + \beta |u|^q |v|^{q-2} v \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\lambda_1, \lambda_2 \geq 0$, $\mu_1, \mu_2 > 0$, Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N = 2, 3$), and $2 \leq q < q_N$, ($q_2 = \infty$, $q_3 = 3$). The parameter β measures the coupling of the system: we are interested in $\beta < 0$ (competing system) or $\beta > 0$ but "small" (weakly cooperative system).

A Schrödinger system of type (1.1) arises as a model for various physical phenomena, in particular in the study of standing waves for a mixture of Bose-Einstein condensates of two states, see for example [9]. From a mathematical viewpoint, it has been so extensively studied that narrowing down the most significant works is necessarily reductive. The most extensively studied case is $q = 2$, also known as Gross-Pitaewskij equation, in particular concerning existence and multiplicity of positive solutions; we mention [14, 4, 15], among others. More recently, sign-changing and semi-nodal solutions have attracted the attention of many mathematicians. When $\beta > 0$ is sufficiently small, radially symmetric sign-changing solutions with any given number of nodal domains were constructed in [16] on $\mathbb{R}^N$ for $q = 2$, and a similar result for $N = 3$ and $2 \leq q < 3$ was proved in [23]. As for bounded (non spherically symmetric) domains, the existence of infinitely many sign-changing and seminodal solutions has been proved in [5] in the weakly cooperative regime, and in [6] and [13] in the competitive regime. These contributions address the case $q = 2$ and investigate appropriate constrained problems by means of the notion of vector genus introduced by [22]. We also mention [19, 7, 20] for some related systems of $d \geq 2$ equations.

Here we focus on least energy nodal and semi-nodal solutions, and we characterize them as constrained minima on a Nehari type set of codimension 4 and 3, respectively. The solutions display the usual inf-sup characterization of mountain-pass type solutions, with the paths interpreted here as 4- (respectively 3-) dimensional surfaces rather than curves. To manage the complexity of the four (or three) dimensions, we impose the additional condition $q \geq 2$. In that way, the problem is subcritical only for $N \leq 3$. A similar assumption was also made in [15, 23]. Even though some

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technical details differ if $\beta < 0$ or $\beta > 0$, the competitive and weakly cooperative regimes can be handled simultaneously.

This approach allows to easily compute the number of nodes and the Morse index of least energy solutions, in particular we see that nodal least energy solutions have Morse index 4, and each component has exactly two nodal zones (Proposition 3.7). Likewise, semi-nodal solutions have Morse index 3 and their components have respectively two and one nodal zones (Proposition 3.10). Furthermore, the Nehari construction can be adapted to subspaces of functions with pre-assigned symmetries: we give here two examples.

When Ω is a ball, it is easy to show the existence of a pure vector (resp., a nodal or a semi-nodal) solution with radial symmetry, which has the least energy among all radial and nontrivial (resp., nodal or seminodal) solutions and has *radial* Morse index 2 (resp., 4 or 3). Observe that the radial Morse index can be less than the Morse index. The question is whether this provides new solutions or, rather, a symmetry property of the least energy solutions.

For the scalar Schrodinger equation corresponding to $\beta = 0$, this fact is well understood. The least energy solution on the ball (which has fixed sign) is radial by the celebrated result by Gidas, Ni and Niremberg [11]. Conversely, comparing the information on the Morse index one sees that any sign changing radial solution is not the least energy one in the whole set of sign changing solutions, see [1]. Though, the least energy nodal solution inherits some symmetry property of the domain, precisely it is foliated Schwarz symmetric, because its Morse index is not greater than N, see [18].

For cooperative Schrödinger systems, a similar sufficient condition for Schwarz symmetry has been established in [10]: it applies to pure vector ground states, but not to sign changing solutions in dimension 2 and 3. The estimation of the Morse index of radial solutions is, in turn, a delicate issue, and warrants further study. We limit ourselves here to a perturbative approach, which provides a result concerning small interaction parameter.

Theorem 1.1. *Let Ω be a ball in dimension $N = 2$ or 3, and $2 \le q < q_N$. There exist $b_1, b_2 > 0$ such that sign-changing (resp., semi-nodal) ground states are not radial when $|\beta| < b_1$ (resp., $|\beta| < b_2$).*

If g is any subgroup of the orthogonal group $O(N)$ one can address to the subspace of g-invariant functions and easily prove the existence of pure vector, nodal and semi-nodal g-invariant solutions which minimize the energy in the respective families. Though, as any radial function is g-invariant, all these g-invariant least energy solutions could coincide with the radial ones. Comparing Morse indices can yield insights into the multiplicity of solutions, but this relies on knowing not only the precise Morse index of the radial solution, but also the symmetry features of the associated eigenfunctions. Studies in this direction have been carried out in the papers [2] and [12], concerning the Schrödinger equation. In particular [2], relying on a singular eigenvalue problem, provides a decomposition of the spectrum according to spherical coordinates that naturally extends to systems without substantial modifications, see also [3]. The paper [12], by means of a fine asymptotic study of radial solutions to the Lane-Emden problem in the disc, computes the exact Morse index and deduce a multiplicity result. Their result carries over to systems of type (1.1) with small interaction parameter, thanks to a stability argument.

To state the obtained result, we assume that Ω is a disc in $\mathbb{R}^2$, write r and θ for the usual polar coordinates,

$$H_k := \left\{(u,v) \in \left(H_0^1(\Omega)\right)^2 : u \text{ and } v \text{ are even and } \frac{2\pi}{k}\text{-periodic w.r.t. } \theta\right\}$$

for every $k \in \mathbb{Z}_+$, and call *k-symmetric* any function in H_k.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a disc and $\lambda_1 = \lambda_2 = 0$. There exists $q^* \ge 2$ such that for every $q \ge q^*$ and $|\beta| < \bar\beta(q)$ the sign-changing k-symmetric least energy solution is not radial for $k = 1, \dots 5$.*

We also mention that in higher dimension, and for $\beta < 0$, nonradial solutions with a different kind of symmetry have been produced in [7].

The paper is organized as follows. In Section 2 we describe the variational structure and recollect some facts about ground state solutions (with fixed sign). Next in Section 3 we introduce the constrained nodal Nehari set (3.1), describe its geometric properties and prove the existence (Theorem 3.5) and some further properties (Subsection 3.3) of sign-changing solutions, including the computation of their Morse index and of the number of nodes, in full details; moreover in Subsection 3.4 we apply similar reasoning to produce a semi-nodal least energy solution. Section 4 is devoted to symmetric solutions and proves Theorems 1.1 and 1.2.

2. Preliminary remarks on the variational setting and fixed-sign solutions

Henceforth we let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$ and

$$q_N = \begin{cases} +\infty & \text{for } N = 2, \\ \frac{N}{N-2} & \text{for } N \geq 3. \end{cases}$$

For $i = 1, 2$ let $E_1 = H_0^1(\Omega)$ equipped with the inner product

$$\langle w, \phi \rangle_{E_i} = \int_\Omega \nabla w \nabla \phi + \lambda_i \int_\Omega w\phi$$

and the induced norm $\|w\|_{E_i}^2 = \langle w, w \rangle_{E_i}$, which is equivalent to the standard one. Both E_1 and E_2 compactly embed into L^{2q} for $1 < q < q_N$. We denote by E the product space $E_1 \times E_2$ with the natural inner product and norm.

Problem (1.1) has a variational structure and the related energy is $\mathcal{I} : E \to \mathbb{R}$,

$$\mathcal{I}(u, v) = \int_\Omega |\nabla u|^2 + |\nabla v|^2 + \lambda_1 u^2 + \lambda_2 v^2 - \frac{1}{2q} \int_\Omega \mu_1 |u|^{2q} + \mu_2 |v|^{2q} + 2\beta |uv|^q. \tag{2.1}$$

Note that weak solutions are critical points for $\mathcal{I}$. A ground state solution attains the minimum of $\mathcal{I}$ among all solutions (except the trivial one $u = v = 0$), i.e. realizes

$$c_{\text{full}} = \inf\{\mathcal{I}(u, v) : (u, v) \in E \setminus \{0\}, \mathcal{I}'(u, v) = 0\}.$$

As $1 < q < q_N$, it is completely standard producing ground state solutions by minimizing $\mathcal{I}$ on the Nehari set

$$\mathcal{N}_{\text{full}} = \big\{(u, v) \in E \setminus \{0\} : I'(u, v)\,(u, v) = 0\big\}. \tag{2.2}$$

But this procedure can lead to a so called semi-trivial solution of type $(u, 0)$ or $(0, v)$, where u and v are the least energy solutions of the single equations obtained letting $\beta = 0$. It happens, for instance, if the parameter β is negative or positive but small, see for instance [4].

In this range, though, nontrivial or "pure vector" solutions, i.e. with $u \neq 0, v \neq 0$, do exist. In particular, there is a nontrivial ground state, which attains

$$c = \inf\{\mathcal{I}(u, v) : (u, v) \in E,\ u, v \neq 0,\ \mathcal{I}'(u, v) = 0\}, \tag{2.3}$$

and can be characterized as the minimum in a constrained Nehari set

$$\gamma = \inf\left\{\mathcal{I}(u, v) : (u, v) \in \mathcal{N}\right\}, \tag{2.4}$$

$$\mathcal{N} = \{(u, v) \in E : u, v \neq 0,\ \mathcal{I}'(u, v)(u, 0) = \mathcal{I}'(u, v)(0, v) = 0\}. \tag{2.5}$$

Proposition 2.1. *Let $N = 2, 3$ and $2 \leq q < q_N$. There exists $\bar\beta > 0$ such that $c = \gamma$ is attained by a function $(u, v) \in \mathcal{N}$ which solves (1.1), for every $\beta < \bar\beta$. Moreover*

$$c = \gamma = \inf_{(u,v)\in\mathcal{A}} \sup_{s,t\geq 0} \mathcal{I}(s^{\frac{1}{q}} u, t^{\frac{1}{q}} v),$$

with

$$\mathcal{A} = \begin{cases} \big\{(u, v) \in E : \mu_1 \|u\|_4^4 \|v\|_{E_2}^2 > \beta \|uv\|_2^2 \|u\|_{E_1}^2, \\ \quad \mu_2 \|v\|_4^4 \|u\|_{E_1}^2 > \beta \|uv\|_2^2 \|v\|_{E_2}^2\big\} & \text{if } q = 2 \text{ and } 0 < \beta < \bar\beta, \\ E \setminus \{0\} & \text{otherwise.} \end{cases}$$

In the case $q = 2$, the proof of Proposition 2.1 consists in a mere rearrangement of the arguments employed in [14, Section 2]. Modifications are necessary to handle the case $q > 2$, that we do not report here because they are explained in detail in the following section about sign-changing solutions.

Starting from the characterization of the ground state as a minimum on the Nehari set, one can compute the Morse index and the number of nodal zones. We recall that, as the energy functional $\mathcal{I}$ is of class C^2 on E, the Morse index of any solution (u, v) is the maximal dimension of a subspace of E where the quadratic form related to $\mathcal{I}''(u, v)$ is negative defined.

Proposition 2.2. *For $q \in [2, q_N)$ and $\beta < \bar{\beta}$, a nontrivial ground state has Morse index 2.*

Proof. The set $\mathcal{N}$ is a regular manifold of codimension 2 in E, hence the Morse index of any solution is at most 2. On the other hand, one sees that the Morse index is at least 2 by showing that the quadratic form associated to $\mathcal{I}''(u, v)$ is negative on the 2-dimensional space spanned by $(u, 0)$ and $(0, v)$. For every $(t, s) \in \mathbb{R}^2$ $(t, s) \neq (0, 0)$ we have

$$\begin{aligned}\langle \mathcal{I}''(u,v)(tu,sv),(tu,sv)\rangle &= t^2\left(\|u\|_{E_1}^2 - (2q-1)\mu_1\|u\|_{2q}^{2q} - \beta(q-1)\|uv\|_q^q\right)\\ &\quad + s^2\left(\|v\|_{E_2}^2 - (2q-1)\mu_2\|v\|_{2q}^{2q} - \beta(q-1)\|uv\|_q^q\right) - 2\beta qts\|uv\|_q^q\\ &= -2(q-1)\left(t^2\|u\|_{E_1}^2 + s^2\|v\|_{E_2}^2\right) + \beta q(t-s)^2\|uv\|_q^q,\end{aligned}$$

since (u, v) solves (1.1).

When $\beta \le 0$ the above quantity certainly is negative. When $\beta > 0$, instead, using again that (u, v) solves (1.1) we write

$$\begin{aligned}&\langle \mathcal{I}''(u,v)(tu,sv),(tu,sv)\rangle\\ &= -2(q-1)\int_\Omega \left(\mu_1 t^2|u|^{2q} + \mu_2 s^2|v|^{2q}\right) - \left(\beta(q-2)(t^2+s^2) + 2\beta qts\right)\int_\Omega |uv|^q\\ &= -2(q-1)\int_\Omega \left(\sqrt{\mu_1}|t||u|^q - \sqrt{\mu_2}|s||v|^q\right)^2\\ &\quad - \beta(q-2)(t-s)^2\int_\Omega |uv|^q - 4(q-1)\left(\sqrt{\mu_1\mu_2}|ts| + \beta ts\right)\int_\Omega |uv|^q < 0,\end{aligned}$$

because $\beta < \bar{\beta} \le \sqrt{\mu_1\mu_2}$. ☐

We remark that for large positive values of β the ground state solution which realizes the minimum on $\mathcal{N}_{\text{full}}$ is nontrivial and has Morse index 1, see [15].

Knowing the Morse index, some qualitative properties follows. First, the components of a least energy nontrivial solution have fixed sign (note that if (u, v) is any solution to (1.1) then also $(\pm u, \pm v)$ solve (1.1)).

Corollary 2.3. *For $q \in [2, q_N)$ and $\beta < \bar{\beta}$, a nontrivial ground state (u, v) is (component-wise) sign definite, i.e. $\pm u > 0$ and $\pm v > 0$ on Ω.*

Proof. Assume by contradiction that neither u^+ nor u^- is identically zero. Then the quadratic form associated to $\mathcal{I}''(u, v)$ is negative defined on the three-dimensional space spanned by $(u^+, 0)$, $(u^-, 0)$, $(0, v)$. The computations are very similar to the ones in the proof of Proposition 2.2 and we do not repeat them (see also Proposition 3.7 later on). But this contradicts the fact that (u, v) has Morse index 2. Hence $\pm u, \pm v \ge 0$, and using Hopf boundary Lemmas it is easy to show that $\pm u, \pm v > 0$ on Ω. ☐

When the set Ω is radially symmetric, a ball or an annulus, then the nontrivial ground state inherits some symmetry property of the domain, at least in the cooperative regime. Recall that (u, v) is said *foliated Schwarz symmetric* if there exist a unitary vector p in $\mathbb{R}^N$ such that $u(x)$ and $v(x)$ depend only on $r = |x|$ and $\varphi = \arccos(x \cdot p/|x|)$ and are nonincreasing in θ. Applying [10, Theorem 1.1] gives the following result.

Corollary 2.4. *Under the assumptions of Proposition 2.1, if $\beta > 0$, then a nontrivial ground state is foliated Schwarz symmetric. Moreover if it is not radial, it is (component-wise) strictly decreasing in the angular variable.*

See also the example in [10, Section 4].

3. Nodal solutions

Given a function w we write w^+ and w^- for its positive and negative part, i.e. $w^\pm = \max\{\pm w, 0\}$. We will say that (u, v) is (component-wise) sign-changing, or nodal, if $u^\pm$ and $v^\pm$ are all non-trivial, and write $E_{\rm nod}$ for the subset of E made up by sign-changing functions, i.e.

$$E_{\rm nod} = \{(u, v) \in E : u^\pm, v^\pm \neq 0\}.$$

A function $(u, v) \in E_{\rm nod}$ which solves (1.1) is said a nodal least energy solution if $\mathcal{I}(u, v)$ is equal to

$$c_{\rm nod} = \inf \{\mathcal{I}(u, v) : (u, v) \in E_{\rm nod}, \mathcal{I}'(u, v) = 0\}.$$

One can not expect to produce a nodal neither a semi-nodal solution by minimization in the standard nodal Nehari set

$$\mathcal{M} = \{(u, v) \in E : (u^+, v^+) \neq 0,\ (u^-, v^-) \neq 0,\ \mathcal{I}'(u, v)(u^+, v^+) = \mathcal{I}'(u, v)(u^-, v^-) = 0\}.$$

Indeed, if $(\bar u, \bar v)$ is a nontrivial least energy solution, then $(|\bar u|, -|\bar v|)$ is itself a least energy solution and belongs to $\mathcal{M}$, but none of its components are sign-changing.

Here we produce a nodal least energy solution by minimizing the energy functional on a constrained set of codimension 4. This section is organized as follows. First we introduce the nodal Nehari set and describe its geometrical properties, next in Subsection 3.2 we adapt the standard Nehari technique to produce a least energy nodal solution, and in Subsection 3.3 we deduce some further properties. At last we illustrate how to obtain similar results for semi-nodal solutions in Subsection 3.4.

3.1. Nodal Nehari set.

We define the nodal Nehari set as

$$\begin{aligned}\mathcal{N}_{\rm nod} &:= \big\{(u, v) \in E_{\rm nod} : \mathcal{I}'(u, v)(u^\pm, 0) = \mathcal{I}'(u, v)(0, v^\pm) = 0\big\}\\ &= \big\{(u, v) \in E_{\rm nod} : \|u^\pm\|_{E_1}^2 = \mu_1\|u^\pm\|_{2q}^{2q} + \beta\|u^\pm v\|_q^q, \|v^\pm\|_{E_2}^2 = \mu_2\|v^\pm\|_{2q}^{2q} + \beta\|uv^\pm\|_q^q\big\}.\end{aligned} \tag{3.1}$$

To describe the geometric properties of $\mathcal{N}_{\rm nod}$, we fix $(u, v) \in E_{\rm nod}$ and define the function $\theta : [0, \infty)^4 \to \mathbb{R}$

$$\begin{aligned}\theta(a, b, c, d) &= \mathcal{I}(a^{\frac1q} u^+ - b^{\frac1q} u^-, c^{\frac1q} v^+ - d^{\frac1q} v^-)\\ &= \frac12 W \cdot (a^{\frac2q}, b^{\frac2q}, c^{\frac2q}, d^{\frac2q}) - \frac{1}{2q} M(a, b, c, d) \cdot (a, b, c, d)\end{aligned} \tag{3.2}$$

where

$$W = W(u, v) = \left(\|u^+\|_{E_1}^2, \|u^-\|_{E_1}^2, \|v^+\|_{E_2}^2, \|v^-\|_{E_2}^2\right), \tag{3.3}$$

$$M = M(u, v) = \begin{pmatrix} \mu_1\|u^+\|_{2q}^{2q} & 0 & \beta\|u^+v^+\|_q^q & \beta\|u^+v^-\|_q^q\\ 0 & \mu_1\|u^-\|_{2q}^{2q} & \beta\|u^-v^+\|_q^q & \beta\|u^-v^-\|_q^q\\ \beta\|u^+v^+\|_q^q & \beta\|u^-v^+\|_q^q & \mu_2\|v^+\|_{2q}^{2q} & 0\\ \beta\|u^+v^-\|_q^q & \beta\|u^-v^-\|_q^q & 0 & \mu_2\|v^-\|_{2q}^{2q}\end{pmatrix} \tag{3.4}$$

Let

$$E_0 = \{(u, v) \in E_{\rm nod} : M(u, v) \text{ is positive defined}\}. \tag{3.5}$$

This set is nonempty because it contains any couple $(u, v) \in E_{\rm nod}$ such that u and v have disjoint support. Furthermore the following holds.

Lemma 3.1. *If $|\beta| < \sqrt{\mu_1\mu_2}/2$, then $E_0 = E_{\rm nod}$. If $\beta \le 0$, then E_0 contains any $(u, v) \in E_{\rm nod}$ such that*

$$\mu_1\|u^\pm\|_{2q}^{2q} + \beta\|u^\pm v\|_q^q > 0, \quad \mu_2\|v^\pm\|_{2q}^{2q} + \beta\|uv^\pm\|_q^q > 0. \tag{3.6}$$

In particular $\mathcal{N}_{\rm nod} \subset E_0$.

Proof. For every $(u,v)\in E_{\rm nod}$ and $(a,b,c,d)\in\mathbb{R}^4\setminus\{0\}$ we have

$$\begin{aligned}&M(u,v)(a,b,c,d)\cdot(a,b,c,d)\\&=\mu_1\|u^+\|_{2q}^{2q}a^2+\mu_1\|u^-\|_{2q}^{2q}b^2+\mu_2\|v^+\|_{2q}^{2q}c^2\mu_2\|v^-\|_{2q}^{2q}d^2\\&\quad+2\beta\|u^+v^+\|_q^qac+2\beta\|u^+v^-\|_q^qad+2\beta\|u^-v^+\|_q^qbc+2\beta\|u^-v^-\|_q^qbc\end{aligned}$$

and Holder's inequality implies

$$\begin{aligned}&\ge\mu_1\|u^+\|_{2q}^{2q}a^2+\mu_1\|u^-\|_{2q}^{2q}b^2+\mu_2\|v^+\|_{2q}^{2q}c^2\mu_2\|v^-\|_{2q}^{2q}d^2\\&\quad-2|\beta|\left(\|u^+\|_{2q}^q\|v^+\|_{2q}^q|ac|+\|u^+\|_{2q}^q\|v^-\|_{2q}^q|ad|+\|u^-\|_{2q}^q\|v^+\|_{2q}^q|bc|+\|u^-\|_{2q}^q\|v^-\|_{2q}^q|bd|\right)\\&>\frac12\left(\sqrt{\mu_1}\|u^+\|_{2q}^q|a|-\sqrt{\mu_2}\|v^+\|_{2q}^{2q}|c|\right)^2+\frac12\left(\sqrt{\mu_1}\|u^+\|_{2q}^q|a|-\sqrt{\mu_2}\|v^-\|_{2q}^{2q}|d|\right)^2\\&\quad+\frac12\left(\sqrt{\mu_1}\|u^-\|_{2q}^q|b|-\sqrt{\mu_2}\|v^+\|_{2q}^{2q}|c|\right)^2+\frac12\left(\sqrt{\mu_1}\|u^-\|_{2q}^q|b|-\sqrt{\mu_2}\|v^-\|_{2q}^{2q}|d|\right)^2\ge0,\end{aligned}$$

if $2|\beta|<\sqrt{\mu_1\mu_2}$. Otherwise, if $\beta<0$ and (u,v) satisfies (3.6), then

$$\begin{aligned}M(u,v)(a,b,c,d)\cdot(a,b,c,d)>-\beta\Big(&\|u^+v^+\|_q^q(a-c)^2+\|u^+v^-\|_q^q(a-d)^2+\|u^-v^+\|_q^q(b-c)^2\\&+\|u^-v^-\|_q^q(b-d)^2\Big)\ge0.\end{aligned}$$

□

Lemma 3.2. *Let $q\in(2,q_N)$ and $\beta<\sqrt{\mu_1\mu_2}/2$, or $q=2$ and $\beta\le0$. Then for every $(u,v)\in E_0$ the function θ has a unique maximum point in $(0,\infty)^4$ characterized by the condition $(a^{\frac1q}u^+-b^{\frac1q}u^-,c^{\frac1q}v^+-d^{\frac1q}v^-)\in\mathcal{N}_{\rm nod}$. In particular $(u,v)\in\mathcal{N}_{\rm nod}$ if and only if θ achieves its maximum at $(1,1,1,1)$.*

Proof. From the representation in (3.2), and the fact that M is positive defined, it follows that $\theta(a,b,c,d)\to-\infty$ if $a+b+c+d\to+\infty$. Hence θ has a maximum point $(a,b,c,d)\in[0,\infty)^4$. Furthermore $\theta(a,a,a,a)>0=\theta(0,0,0,0)$ for sufficiently small a, hence the maximum point does not fall in the origin.

Let us check that it is an interior point. If, for instance, $a=0$, then

$$0\ge\lim_{h\to0^+}\frac1h\left[\theta(h,b,c,d)-\theta(0,b,c,d)\right]=\lim_{h\to0^+}\frac12\|u^+\|_{E_1}^2h^{\frac{2-q}{q}}-\frac{\beta}{q}\Big[c\int_\Omega|u^+v^+|^q+d\int_\Omega|u^+v^-|^q\Big],$$

which implies that either $q<2$ or $q=2$ and $\beta>0$.

So under the present assumptions θ attains its maximum at some $(a,b,c,d)\in(0,\infty)^4$, and $\nabla\theta(a,b,c,d)=0$, i.e.

$$\Big(\|u^+\|_{E_1}^2a^{\frac{2-q}{q}},\|u^-\|_{E_1}^2b^{\frac{2-q}{q}},\|v^+\|_{E_2}^2c^{\frac{2-q}{q}},\|v^-\|_{E_2}^2d^{\frac{2-q}{q}}\Big)=M(a,b,c,d),$$

which is equivalent to $(a^{\frac1q}u^+-b^{\frac1q}u^-,c^{\frac1q}v^+-d^{\frac1q}v^-)\in\mathcal{N}_{\rm nod}$ by a trivial computation.

There are no other critical points because θ is a strictly concave function. Indeed for every $(a,b,c,d)\in(0,\infty)^4$ we have

$$-D^2\theta(a,b,c,d)=\frac{q-2}{q^2}\operatorname{diag}\Big(\|u^+\|_{E_1}^2a^{\frac{2(1-q)}{q}},\|u^-\|_{E_1}^2b^{\frac{2(1-q)}{q}},\|v^+\|_{E_2}^2c^{\frac{2(1-q)}{q}},\|v^-\|_{E_2}^2d^{\frac{2(1-q)}{q}}\Big)+\frac1qM.$$

□

Note that Proposition 3.2 ensures that $\mathcal{N}_{\rm nod}$ is not empty. Proposition 3.2 does not hold for $q=2$ and $\beta>0$ because there are functions $(u,v)\in E_0$ for which θ attains its maximum at the boundary of $[0,\infty)^4$. We therefore argue in the subset

$$\mathcal{O}=\{(u,v)\in E_0: M^{-1}W\in(0,\infty)^4\}. \tag{3.7}$$

where M and W are defined by (3.3), (3.4).

Lemma 3.3. *The set $\mathcal{O}$ is nonempty and $\mathcal{N}_{\rm nod}\subset\mathcal{O}$.*

Proof. Take w_1, w_2, z_1, z_2 any smooth nonnegative functions with disjoint supports and define $w = w_1 - w_2$, $z = z_1 - z_2$. Now $M(w,z)$ is diagonal and it is clear that $(w,z) \in \mathcal{O}$.

Next, for every $(u,v) \in E_{\rm nod}$ we have

$$\begin{aligned} M(1,1,1,1) = \Big(&\mu_1\|u^+\|_4^4 + \beta\|u^+v\|_2^2, \mu_1\|u^-\|_4^4 + \beta\|u^-v\|_2^2, \\ &\mu_2\|v^+\|_4^4 + \beta\|uv^+\|_2^2, \mu_2\|v^-\|_4^4 + \beta\|uv^-\|_2^2\Big). \end{aligned}$$

So $(u,v) \in \mathcal{N}_{\rm nod}$ if and only if $M(1,1,1,1) = W$, which implies $(u,v) \in \mathcal{O}$. □

Lemma 3.4. *Let $q = 2$ and $0 < \beta < \sqrt{\mu_1\mu_2}/2$. Then for every $(u,v) \in \mathcal{O}$ the function θ has a unique maximum point in $(0,\infty)^4$ characterized by the condition $(\sqrt{a}u^+ - \sqrt{b}u^-, \sqrt{c}v^+ - \sqrt{d}v^-) \in \mathcal{N}_{\rm nod}$.*

Proof. Now $\nabla\theta(a,b,c,d) = W - M(a,b,c,d)$, hence for $(u,v) \in \mathcal{O}$ the function θ has an unique critical point $(a,b,c,d) = M^{-1}W \in (0,\infty)^4$. Such critical point is a global maximum by concavity. Furthermore $W - M(a,b,c,d) = 0$ is equivalent to $(\sqrt{a}u^+ - \sqrt{b}u^-, \sqrt{c}v^+ - \sqrt{d}v^-) \in \mathcal{N}_{\rm nod}$. □

In particular, $\mathcal{N}_{\rm nod}$ is nonempty also when $q = 2$ and $0 < \beta < \sqrt{\mu_1\mu_2}/2$.

3.2. **Existence of a least energy nodal solution.** Let

$$c_{\rm nod} = \inf\{\mathcal{I}(u,v) : (u,v) \in E_{\rm nod},\ \mathcal{I}'(u,v) = 0\}, \tag{3.8}$$

$$\gamma_{\rm nod} = \inf\{\mathcal{I}(u,v) : (u,v) \in \mathcal{N}_{\rm nod}\}. \tag{3.9}$$

Of course $\mathcal{N}_{\rm nod}$ contains every sign changing solutions of (1.1), hence $\gamma_{\rm nod} \le c_{\rm nod}$. Here we show that $\gamma_{\rm nod} = c_{\rm nod}$ is attained by a least energy sign changing solution.

Theorem 3.5. *For every $q \in [2, q_N)$, there exists $\beta_0 > 0$ such that for every $\beta < \beta_0$ the infimum $\gamma_{\rm nod}$ is attained by a (componentwise) sign changing function which solves* (1.1)*. In particular $c_{\rm nod} = \gamma_{\rm nod}$.*

By going through the proof, it becomes clear that β_0 can be taken as $\frac{1}{2}\sqrt{\mu_1\mu_2}$ for $q > 2$.

Proof. We begin by noticing that for $(u,v) \in \mathcal{N}_{\rm nod}$ we have

$$\mathcal{I}(u,v) = \frac{q-1}{2q}\Big[\mu_1\|u\|_{2q}^{2q} + \mu_2\|v\|_{2q}^{2q} + 2\beta\|uv\|_q^q\Big] = \frac{q-1}{2q}\big[\|u\|_{E_1}^2 + \|v\|_{E_2}^2\big] > 0. \tag{3.10}$$

hence $\gamma_{\rm nod} \ge 0$.

Step 1: Convergence of a minimizing sequence. Let (u_n, v_n) a minimizing sequence. By standard compactness arguments, up to a subsequence, (u_n, v_n) converges strongly in $(L^{2q}(\Omega))^2$ and weakly in E to some $(u,v) \in E$. In particular $u_n^\pm, v_n^\pm$ converge strongly in $L^{2q}(\Omega)$ to $u^\pm, v^\pm$ and

$$\frac{q-1}{2q}[\mu_1\|u\|_{2q}^{2q} + \mu_2\|v\|_{2q}^{2q} + 2\beta\|uv\|_q^q] = \lim_{n\to\infty}\mathcal{I}(u_n,v_n) = \gamma_{\rm nod}, \tag{3.11}$$

$$\|u^\pm\|_{E_1}^2 \le \lim_{n\to\infty}\|u_n^\pm\|_{E_1}^2 = \mu_1\|u^\pm\|_{2q}^{2q} + \beta\|u^\pm v\|_q^q, \tag{3.12}$$

$$\|v^\pm\|_{E_2}^2 \le \lim_{n\to\infty}\|v_n^\pm\|_{E_2}^2 = \mu_2\|v^\pm\|_{2q}^{2q} + \beta\|uv^\pm\|_q^q. \tag{3.13}$$

Furthermore there exists a constant B not depending by β such that

$$\|u_n^\pm\|_{2q}, \|v_n^\pm\|_{2q} \le B \tag{3.14}$$

for every n. To check (3.14), take w_1, w_2, z_1, z_2 any smooth nonnegative functions with disjoint supports. It is clear that $(w_1 - w_2, z_1 - z_2)$ is in E_0, and also in $\mathcal{O}$ when $q = 2$ and $\beta > 0$. Then, let $w = aw_1 - bw_2$, $z = cz_1 - dz_2$, where a,b,c,d are chosen according to Propositions 3.2, 3.4 in such a way that $(w,z) \in \mathcal{N}_{\rm nod}$. So

$$\gamma_{\rm nod} \le \mathcal{I}(w,z) = \frac{1}{2}\|(w,z)\|_E^2 - \frac{\mu_1}{2q}\|w\|_{2q}^{2q} - \frac{\mu_2}{2q}\|z\|_{2q}^{2q} = B.$$

If $\beta \ge 0$, (3.14) readily follows by (3.11). If $\beta \le 0$, instead, using also (3.10) gives a bound for $\|u^\pm\|_{E_1}$, $\|v^\pm\|_{E_2}$ and (3.14) follows by Sobolev immersion.

Step 2: The limit function belongs to E_0 for every $\beta < \frac{1}{2}\sqrt{\mu_1\mu_2}$ if $q > 2$. If $q = 2$, there exists $\beta_0 \in (0, \frac{1}{2}\sqrt{\mu_1\mu_2}]$ such that the same holds true for every $\beta < \beta_0$.

First we show that $(u, v) \in E_{\rm nod}$: we only check that $u^+ \neq 0$, $u^-, v^\pm \neq 0$ follow similarly. Let C_1 be the best constants of the Sobolev embedding of E_1 into $L^{2q}(\Omega)$. Now

$$\begin{aligned}\frac{1}{C_1}\|u_n^\pm\|_{2q}^2 \le \|u_n^\pm\|_{E_1}^2 &= \mu_1\|u_n^\pm\|_{2q}^{2q} + \beta\|u_n^\pm v_n\|_q^2 q \\ &\le \mu_1\|u_n^\pm\|_{2q}^{2q} + \beta^+\|u_n^\pm\|_{2q}^q\|v_n\|_{2q}^q \quad \text{(by Hölder's inequality)} \\ &\le \|u_n^\pm\|_{2q}^q(\mu_1\|u_n\|_{2q}^q + 2\beta^+ B) \quad \text{(by (3.14))}.\end{aligned}$$

where $\beta^+ = \max(\beta, 0)$. If $q > 2$, then clearly $\|u_n^\pm\|_{2q}$ can not vanish. But also for $q = 2$, there exists $\beta_0 > 0$ such that $\|u_n^\pm\|_4$ is bounded away from zero for $\beta < \beta_0$.

If $2|\beta| < \sqrt{\mu_2\mu_2}$ there is nothing left to prove. Otherwise (3.12) and (3.13) ensure that $(u, v) \in E_0$, thanks to Lemma 3.1.

Step 3: the limit function belongs to $\mathcal{N}_{\rm nod}$ and achieves its minimum. The proof differs if $q = 2$ or $q > 2$, so we split it in two.

Step 4: For $q = 2$, if β is positive it is needed to first check that $(u, v) \in \mathcal{O}$. Assume by contradiction that $M^{-1}W \notin (0, \infty)^4$, to fix idea that its first component is nonpositive. By computations

$$M_1\|u^+\|_{E_1}^2 + M_2\|u^-\|_{E_1}^2 + M_3\|v^+\|_{E_2}^2 + M_4\|v^-\|_{E_2}^2 \le 0, \tag{3.15}$$

where

$$\begin{aligned}M_1 &= \mu_1\mu_2^2\|u^-\|_4^4\|v^+\|_4^4\|v^-\|_4^4 - \mu_2\beta^2\|v^+\|_4^4\|u^-v^-\|_2^4 - \mu_2\beta^2\|v^-\|_4^4\|u^-v^+\|_2^4 \\ &\ge \mu_1\mu_2^2\Big(1 - \frac{2\beta^2}{\mu_1\mu_2}\Big)\|u^-\|_4^4\|v^+\|_4^4\|v^-\|_4^4, \\ M_2 &= \mu_2\beta^2\|v^+\|_4^4\|u^+v^-\|_2^2\|u^-v^-\|_2^2 + \mu_2\beta^2\|v^-\|_4^4\|u^+v^+\|_2^2\|u^-v^+\|_2^2 \ge 0, \\ M_3 &= \beta^3\|u^+v^+\|_2^2\|u^-v^-\|_2^4 - \beta^3\|u^+v^-\|_2^2\|u^-v^+\|_2^2\|u^-v^-\|_2^2 - \mu_1\mu_2\beta\|u^-\|_4^4\|v^-\|_4^4\|u^+v^+\|_2^2 \\ &\ge -\mu_1\mu_2\beta\Big(1 + \frac{\beta^2}{\mu_1\mu_2}\Big)\|u^+\|_4^2\|u^-\|_4^4\|v^+\|_4^2\|v^-\|_4^4, \\ M_4 &= \beta^3\|u^+v^-\|_2^2\|u^-v^+\|_2^4 - \beta^3\|u^+v^+\|_2^2\|u^-v^+\|_2^2\|u^-v^-\|_2^2 - \mu_1\mu_2\beta\|u^-\|_4^4\|v^+\|_4^4\|u^+v^+\|_2^2 \\ &\ge -\mu_1\mu_2\beta\Big(1 + \frac{\beta^2}{\mu_1\mu_2}\Big)\|u^+\|_4^2\|u^-\|_4^4\|v^+\|_4^4\|v^-\|_4^2.\end{aligned}$$

Here we have used repeatedly Hölder inequality. Now (3.15) implies that

$$\begin{aligned}&\mu_2\Big(1 - \frac{2\beta^2}{\mu_1\mu_2}\Big)\|u^-\|_4^4\|v^+\|_4^4\|v^-\|_4^4\|u^+\|_{E_1}^2 \\ &\le \beta\Big(1 + \frac{\beta^2}{\mu_1\mu_2}\Big)\|u^+\|_4^2\|u^-\|_4^4\|v^+\|_4^2\|v^-\|_4^2\left(\|v^-\|_4^2\|v^+\|_{E_2}^2 + \|v^+\|_4^2\|v^-\|_{E_2}^2\right).\end{aligned} \tag{3.16}$$

Concerning the right-hand side of (3.16), using (3.13) and Holder inequality gives

$$\|v^\pm\|_{E_2}^2 \le \mu_2\|v^\pm\|_4^2\Big(\|v^\pm\|_4^2 + \frac{\beta}{\mu_2}\|u\|_4^2\Big) \le \mu_2\Big(1 + \frac{2\beta}{\mu_2}\Big)B\|v^\pm\|_4^2,$$

thanks to (3.14). Eventually estimating from below the left-hand side of (3.16) by means of the Sobolev inequality and simplifying all the repeated terms yields

$$\frac{1}{C_1}\Big(1 - \frac{2\beta^2}{\mu_1\mu_2}\Big) \le 2\beta\Big(1 + \frac{\beta^2}{\mu_1\mu_2}\Big)\Big(1 + \frac{2\beta}{\mu_2}\Big)B. \tag{3.17}$$

It leads to a contradiction, possibly after choosing a smaller β_0.

Afterwards Proposition 3.4 ensures that there exist $a,b,c,d>0$ such that $(\sqrt{a}u^+-\sqrt{b}u^-,\sqrt{c}v^+-\sqrt{d}v^-)\in\mathcal{N}_{\rm nod}$, and as $\nabla\theta(a,b,c,d)=0$ we have $M(a,b,c,d)=W$, that is,

$$\begin{aligned}
&\mu_1 a\|u^+\|_4^4+\beta c\|u^+v^+\|_2^2+\beta d\|u^+v^-\|_2^2\\
&=\|u^+\|_{E_1}^2\\
&\le \mu_1\|u^+\|_4^4+\beta\|u^+v^+\|_2^2+\beta\|u^+v^-\|_2^2\quad\text{(by (3.12))},\\
&\mu_1 b\|u^-\|_4^4+\beta c\|u^-v^+\|_2^2+\beta d\|u^-v^-\|_2^2\\
&=\|u^-\|_{E_1}^2\\
&\le \mu_1\|u^-\|_4^4+\beta\|u^-v^+\|_2^2+\beta\|u^-v^-\|_2^2\quad\text{(by (3.12))},\\
&\mu_2 c\|v^+\|_4^4+\beta a\|u^+v^+\|_2^2+\beta b\|u^-v^+\|_2^2\\
&=\|v^+\|_{E_2}^2\\
&\le \mu_2\|v^+\|_4^4+\beta\|u^+v^+\|_2^2+\beta\|u^-v^+\|_2^2\quad\text{(by (3.13))},\\
&\mu_2 d\|v^-\|_4^4+\beta a\|u^-v^-\|_2^2+\beta d\|u^-v^-\|_2^2\\
&=\|v^-\|_{E_2}^2\\
&\le \mu_2\|v^-\|_4^4+\beta\|u^+v^-\|_2^2+\beta\|u^-v^-\|_2^2\quad\text{(by (3.13))}.
\end{aligned}\tag{3.18}$$

Then

$$\begin{aligned}
\gamma_{\rm nod}&\le \mathcal{I}(\sqrt{a}u^+-\sqrt{b}u^-,\sqrt{c}v^+-\sqrt{d}v^-)\\
&=\frac14\Big[\mu_1a^2\|u^+\|_4^4+\mu_1b^2\|u^-\|_4^4+\mu_2c^2\|v^+\|_4^4+\mu_2d^2\|v^-\|_4^4\\
&\quad+2\beta ac\|u^+v^+\|_2^2+2\beta ad\|u^+v^-\|_2^2+2\beta bc\|u^-v^+\|_2^2+2\beta bd\|u^-v^-\|_2^2\Big]\\
&\le\frac14\Big[a\mu_1\|u^+\|_4^4+b\mu_1\|u^-\|_4^4+c\mu_2\|v^+\|_4^4+d\mu_2\|v^-\|_4^4+\beta(a+c)\|u^+v^+\|_2^2\\
&\quad+\beta(a+d)\|u^+v^-\|_2^2+\beta(b+c)\|u^-v^+\|_2^2+\beta(b+d)\|u^-v^-\|_2^2\Big]\quad\text{(by (3.18))}\\
&\le\frac14\big(\mu_1\|u\|_4^4+2\beta\|uv\|_2^2+\mu_2\|v\|_4^4\big)\quad\text{(by (3.18))}\\
&=\gamma_{\rm nod}\quad\text{(by (3.11))}.
\end{aligned}$$

It follows that (3.18) are indeed equalities, that is $M(a,b,c,d)=W=M(1,1,1,1)$ and since the matrix M is nondegenerate, it follows that $a=b=c=d=1$. It is thus proved that $(u,v)\in\mathcal{N}_{\rm nod}$ and $\mathcal{I}(u,v)=\gamma_{\rm nod}$.

Step 3: For $q>2$, let

$$\begin{aligned}
\theta_n(a,b,c,d)&=\mathcal{I}(a^{\frac1q}u_n^+-b^{\frac1q}u_n^-,c^{\frac1q}v_n^+-d^{\frac1q}v_n^-)\\
&=\frac12\Big(a^{\frac2q}\|u_n^+\|_{E_1}^2+b^{\frac2q}\|u_n^-\|_{E_1}^2+c^{\frac2q}\|v_n^+\|_{E_2}^2+d^{\frac2q}\|v_n^-\|_{E_2}^2\Big)\\
&\quad-\frac{1}{2q}\Big(\mu_1a^2\|u_n^+\|_{2q}^{2q}+\mu_1b^2\|u_n^-\|_{2q}^{2q}+\mu_2c^2\|v_n^+\|_{2q}^{2q}+\mu_2d^2\|v_n^-\|_{2q}^{2q}\Big)\\
&\quad-\frac{\beta}{q}\Big(ac\|u_n^+v_n^+\|_q^q+ad\|u_n^+v_n^-\|_q^q+bc\|u_n^-v_n^+\|_q^q+bd\|u_n^-v_n^-\|_q^q\Big).
\end{aligned}\tag{3.19}$$

We remark that

$$\begin{aligned}
&\theta_n(a,b,c,d)\\
&=\frac{1}{2q}\Big[\mu_1(qa^{\frac2q}-a^2)\|u_n^+\|_{2q}^{2q}+\mu_1(qb^{\frac2q}-b^2)\|u_n^-\|_{2q}^{2q}+\mu_2(qc^{\frac2q}-c^2)\|v_n^+\|_{2q}^{2q}\\
&\quad+\mu_2(qd^{\frac2q}-d^2)\|v_n^-\|_{2q}^{2q}+(qa^{\frac2q}+qc^{\frac2q}-2\beta ac)\|u_n^+v_n^+\|_q^q+(qa^{\frac2q}+qd^{\frac2q}-2\beta ad)\|u_n^+v_n^-\|_q^q\\
&\quad+(qb^{\frac2q}+qc^{\frac2q}-2\beta bc)\|u_n^-v_n^+\|_q^q+(qb^{\frac2q}+qd^{\frac2q}-2\beta bd)\|u_n^-v_n^-\|_q^q\Big]
\end{aligned}\tag{3.20}$$

because $(u_n,v_n)\in\mathcal{N}_{\rm nod}$. From the representation (3.20) and the strong convergence of u_n,v_n in $L^{2q}(\Omega)$, θ_n converges pointwise to the function $\bar\theta$ defined by substituting $u^\pm$ and $v^\pm$ to $u_n^\pm$ and $v_n^\pm$ in the law (3.20). Notice that

$$\begin{aligned}\nabla\bar\theta=\frac{1}{q}\Big(&a^{\frac{2-q}{q}}(\mu_1\|u^+\|_{2q}^{2q}+\beta\|u^+v\|_q^q)-(a\mu_1\|u^+\|_{2q}^{2q}+c\beta\|u^+v^+\|_q^q+d\beta\|u^+v^-\|_q^q),\\ &b^{\frac{2-q}{q}}(\mu_1\|u^-\|_{2q}^{2q}+\beta\|u^-v\|_q^q)-(b\mu_1\|u^-\|_{2q}^{2q}+c\beta\|u^-v^+\|_q^q+d\beta\|u^-v^-\|_q^q),\\ &c^{\frac{2-q}{q}}(\mu_2\|v^+\|_{2q}^{2q}+\beta\|uv^+\|_q^q)-(c\mu_2\|v^+\|_{2q}^{2q}+a\beta\|u^+v^+\|_q^q+b\beta\|u^-v^+\|_q^q),\\ &d^{\frac{2-q}{q}}(\mu_1\|v^-\|_{2q}^{2q}+\beta\|uv^-\|_q^q)-(d\mu_2\|v^-\|_{2q}^{2q}+a\beta\|u^+v^-\|_q^q+b\beta\|u^-v^-\|_q^q)\Big),\end{aligned}$$

$$D^2\bar\theta=-\frac{q-2}{q^2}\operatorname{diag}\begin{pmatrix}a^{\frac{2(1-q)}{q}}(\mu_1\|u^+\|_{2q}^{2q}+\beta\|u^+v\|_q^q)\\ b^{\frac{2(1-q)}{q}}(\mu_1\|u^-\|_{2q}^{2q}+\beta\|u^-v\|_q^q)\\ c^{\frac{2(1-q)}{q}}(\mu_2\|v^+\|_{2q}^{2q}+\beta\|uv^+\|_q^q)\\ d^{\frac{2(1-q)}{q}}(\mu_2\|v^-\|_{2q}^{2q}+\beta\|uv^-\|_q^q)\end{pmatrix}-\frac{1}{q}M$$

Then $\bar\theta$ is strictly concave and has an unique maximum point at $(1,1,1,1)$, with $\bar\theta(1,1,1,1)=\gamma_{\rm nod}$.

Furthermore from the representation (3.19), the weak convergence of $u_n^\pm,v_n^\pm$ in $L^{2q}(\Omega)$, (3.12) and (3.13), it follows that $\bar\theta(a,b,c,d)\ge\theta(a,b,c,d)=\mathcal{I}(a^{\frac1q}u^+-b^{\frac1q}u^-,c^{\frac1q}v^+-d^{\frac1q}v^-)$. If (a_o,b_o,c_o,d_o) is the maximum point of θ, then

$$\gamma_{\rm nod}\le\mathcal{I}(a_o^{\frac1q}u^-b_o^{\frac1q}u^-,c_o^{\frac1q}v^+-d_o^{\frac1q}v^-)=\theta(a_o,b_o,c_o,d_o)\le\bar\theta(a_o,b_o,c_o,d_o)\le\bar\theta(1,1,1,1)=\gamma_{\rm nod}$$

by (3.11). Therefore $a_o=b_o=c_o=d_o=1$, which yields at once that $(u,v)\in\mathcal{N}_{\rm nod}$ and $\mathcal{I}(u,v)=\gamma_{\rm nod}$.

Step 4: The limit function solves (1.1). It is needed to check that $\mathcal{I}'(u,v)=0$, and one can not use Lagrange multipliers because $\mathcal{N}_{\rm nod}$ is not a differentiable manifold, hence we rely on a deformation argument, see for instance [8]. Assume by contradiction that $\mathcal{I}'(u,v)\ne0$, so there is $\Phi=(\phi,\psi)\in E$ such that $\mathcal{I}'(u,v)\Phi=-2$. For $\varepsilon>0$, let

$$A_\pm=\big(1\pm\varepsilon\|u^+\|_{2q}^{-q}\big)^{\frac{q}{2(q-1)}},\quad B_\pm=\big(1\pm\varepsilon\|u^-\|_{2q}^{-q}\big)^{\frac{q}{2(q-1)}},$$

$C_\pm$ and $D_\pm$ defined accordingly with u replaced by v,

$$Q=[A_-,A_+]\times[B_-,B_+]\times[C_-,C_+]\times[D_-,D_+],$$

$W=(U,V):Q\to E$ defined by the law

$$W=(U,V)=\Big(a^{\frac1q}u^+-b^{\frac1q}u^-,c^{\frac1q}v^+-d^{\frac1q}v^-\Big).$$

By continuity, we can take ε small so that

$$\mathcal{I}'(W(a,b,c,d)+r\Phi)\Phi\le-1\quad\text{for all }(a,b,c,d)\in Q\text{ and }r\in[0,\varepsilon].\tag{3.21}$$

Next, we take $\eta:Q\to\mathbb{R}$ a smooth function such that $0\le\eta\le\varepsilon$, $\eta=0$ on ∂Q and $\eta(1,1,1,1)=\varepsilon$ and define $H:Q\to\mathbb{R}^4$,

$$\begin{aligned}H=\Big(&\mathcal{I}'(W+\eta\Phi)((U+\eta\phi)^+,0),\mathcal{I}'(W+\eta\Phi)((U+\eta\phi)^-,0),\\ &\mathcal{I}'(W+\eta\Phi)(0,(V+\eta\psi)^+),\mathcal{I}'(W+\eta\Phi)(0,(V+\eta\psi)^-)\Big).\end{aligned}$$

We remark that on ∂Q the function η vanishes, so

$$H=(qa\partial_a\theta,-qb\partial_b\theta,qc\partial_c\theta,-qd\partial_d\theta).$$

In particular for $a=A_+$ we have

$$\begin{aligned}&qA_+\partial_a\theta(A_+,b,c,d)\\ &=A_+^{\frac2q}\|u^+\|_{E_1}^2-\mu_1A_+^2\|u^+\|_{2q}^{2q}-\beta A_+c\|u^+v^+\|_q^q-\beta A_+d\|u^+v^-\|_q^q\end{aligned}$$

and because $(u,v) \in \mathcal{N}_{\rm nod}$

$$
\begin{aligned}
&= A_+^{\frac{2}{q}}\Big[\mu_1(1-A_+^{\frac{2(q-1)}{q}})\|u^+\|_{2q}^{2q} - \beta(1-A_+^{\frac{q-2}{q}}c)\|u^+v^+\|_q^q - \beta(1-A_+^{\frac{q-2}{q}}d)\|u^+v^-\|_q^q\Big]\\
&= A_+^{\frac{2}{q}}\Big[-\mu_1\varepsilon\|u^+\|_{2q}^q + \beta(1-A_+^{\frac{q-2}{q}}c)\|u^+v^+\|_q^q + \beta(1-A_+^{\frac{q-2}{q}}d)\|u^+v^-\|_q^q\Big]
\end{aligned}
$$

But

$$
1-A_+^{\frac{q-2}{q}}c \le 1-C_- = 1-\big(1-\varepsilon\|v^+\|_{2q}^{-q}\big)^{\frac{q}{2(q-1)}} \le \varepsilon\|v^+\|_{2q}^{-q}
$$

because $q/2(q-1) \le 1$. Similarly $1-A_+^{\frac{q-2}{q}}d \le \varepsilon\|v^-\|_{2q}^{-q}$ and using Holder inequality we end up with

$$
qA_+\partial_a\theta(A_+,b,c,d) \le \varepsilon A_+^{\frac{2}{q}}\|u^+\|_{2q}^q\,[-\mu_1+2\beta] < 0
$$

for $\beta < \bar\beta$. Similarly one sees that

$$
qB_+\partial_b\theta(a,B_+,c,d),\ qC_+\partial_c\theta(a,b,C_+,d),\ qD_+\partial_d\theta(a,b,c,D_+) < 0
$$

and

$$
qA_-\partial_a\theta(A_-,b,c,d),\ qB_-\partial_b\theta(a,B_-,c,d),\ qC_-\partial_c\theta(a,b,C_-,d),\ qD_-\partial_d\theta(a,b,c,D_-) > 0.
$$

In that way, the classical Miranda Theorem [17] ensures that there is $(a,b,c,d) \in Q$ such that $H(a,b,c,d) = 0$, i.e. $W + \eta\Phi \in \mathcal{N}_{\rm nod}$. Eventually

$$
\mathcal{I}(u,v) = \gamma_{\rm nod} \le \mathcal{I}(W+\eta\Phi) = \mathcal{I}(W) + \int_0^\eta \mathcal{I}'(W+r\Phi)\Phi dr \le \mathcal{I}(W) - \eta
$$

thanks to (3.21). Here we have omitted to specify that W and η are computed at (a,b,c,d) to simplify notations. Besides $\mathcal{I}(W(a,b,c,d)) = \theta(a,b,c,d) \le \theta(1,1,1,1) = \mathcal{I}(u,v) = \gamma_{\rm nod}$, therefore $\eta(a,b,c,d) = 0$ and $\theta(a,b,c,d) = \mathcal{I}(u,v)$. But then the uniqueness of the maximum point for θ in Propositions 3.2, 3.4 implies that $a=b=c=d=1$ and then, in turn, the contradiction $\eta(a,b,c,d) = \varepsilon$. □

3.3. **Further properties of the least energy solutions.** If $\mathcal{A}$ is any subset of E, we introduce the minimax value

$$
m(\mathcal{A}) = \inf\big\{\sup_{a,b,c,d\ge 0} \mathcal{I}(a^{\frac1q}u^+ - b^{\frac1q}u^-, d^{\frac1q}v^+ - d^{\frac1q}v^-) : (u,v) \in \mathcal{A}\big\}.
$$

Taken together, Theorem 3.5 and Lemmas 3.2, 3.4 yield the following characterization of the nodal least energy

$$
c_{\rm nod} = \gamma_{\rm nod} = \begin{cases} m(E_{\rm nod}) & \text{if } q \in (2,q_N) \text{ and } |\beta| < \sqrt{\mu_1\mu_2}/2 \text{ or } q=2 \text{ and } -\sqrt{\mu_1\mu_2}/2 < \beta \le 0,\\ m(E_0) & \text{if } q \in [2,q_N) \text{ and } \beta \le -\sqrt{\mu_1\mu_2}/2,\\ m(\mathcal{O}) & \text{if } q=2 \text{ and } 0<\beta<\beta_0. \end{cases}
$$

Indeed, it is easy to prove the following result.

Corollary 3.6. *We have $c_{\rm nod} = \gamma_{\rm nod} = m(E_{\rm nod})$ for every $\beta < \sqrt{\mu_1\mu_2}/2$ if $q \in (2,q_N)$, or for every $\beta \le 0$ if $q=2$.*

Proof. It is only needed to address negative values of β, and it is already known that $c_{\rm nod} = m(E_0) \ge m(E_{\rm nod})$. Let (u_n,v_n) be a sequence in $E_{\rm nod}$ and define $\theta_n(a,b,c,d) = \mathcal{I}(a^{\frac1q}u_n^+ - b^{\frac1q}u_n^-, d^{\frac1q}v_n^+ - d^{\frac1q}v_n^-)$. If $\sup_{[0,\infty)^4}\theta_n \to m(E_{\rm nod})$, in particular θ_n must be bounded and recalling the representation (3.2) one see that the matrix $M(u_n,v_n)$ must be positive defined. Hence $(u_n,v_n) \in E_0$, so that $m(E_{\rm nod}) \ge m(E_0)$ and the proof is thereby complete. □

Both the Morse index and the number of nodal zones of least energy nodal solutions can be computed, though $\mathcal{N}_{\rm nod}$ is not a manifold in E, relying on the arguments in [4]. We sketch the proof for the sake of completeness.

Proposition 3.7. *Any least energy nodal solution has Morse index 4, and both its components have exactly two nodal zones, meaning that the supports of $u^\pm$ and $v^\pm$ are connected.*

Proof. Let $G : E_{\rm nod} \to \mathbb{R}^4$,

$$G(u,v) = \begin{pmatrix} \mathcal{I}'(u,v)\,(u^+,0) \\ \mathcal{I}'(u,v)\,(u^-,0) \\ \mathcal{I}'(u,v)\,(0,v^+) \\ \mathcal{I}'(u,v)\,(0,v^-) \end{pmatrix},$$

and $H := E \cap \big(H^2(\Omega)\big)^2$. Now $\mathcal{N}_{\rm nod} \cap H = \{(u,v) \in E_{\rm nod} \cap H : G(u,v) = 0\}$ and G is a C^1 function on H with

$$\begin{aligned} &G'(u,v)(\phi,\psi) \\ &= \begin{pmatrix} \int_{\{u>0\}} -\Delta u\phi + \nabla u \nabla\phi + 2\lambda_1 u\phi - q\int_\Omega \big(2\mu_1|u|^{2q-2} + \beta|u|^{q-2}|v|^q\big)\,u^+\phi + \beta|uv|^{q-2}uu^+v\psi \\ \int_{\{u<0\}} -\Delta u\phi + \nabla u \nabla\phi + 2\lambda_1 u\phi - q\int_\Omega \big(2\mu_1|u|^{2q-2} + \beta|u|^{q-2}|v|^q\big)\,u^-\phi + \beta|uv|^{q-2}uu^-v\psi \\ \int_{\{v>0\}} -\Delta v\psi + \nabla v \nabla\psi + 2\lambda_2 v\psi - q\int_\Omega \beta|uv|^{q-2}uvv^+\phi + \big(2\mu_2|v|^{2q-2} + \beta|u|^q|v|^{q-2}\big)\,v^+\psi \\ \int_{\{v<0\}} -\Delta v\psi + \nabla v \nabla\psi + 2\lambda_2 v\psi - q\int_\Omega \beta|uv|^{q-2}uvv^-\phi + \big(2\mu_2|v|^{2q-2} + \beta|u|^q|v|^{q-2}\big)\,v^-\psi \end{pmatrix}, \end{aligned}$$

see [4, Lemma 3.1].

Furthermore for $(u,v) \in \mathcal{N}_{\rm nod} \cap H$, $G'(u,v)$ is a surjective operator from H to $\mathbb{R}^4$. Indeed

$$\begin{aligned} &\big(G'(u,v)(u^+,v), G'(u,v)(u^-,v), G'(u,v)(u,v^+), G'(u,v)(u,v^-)\big) \\ &= -2(q-1)\begin{pmatrix} \|u^+\|^2_{E_1} & 0 & 0 & 0 \\ 0 & \|u^-\|^2_{E_1} & 0 & 0 \\ 0 & 0 & \|v^+\|^2_{E_2} & 0 \\ 0 & 0 & 0 & \|v^-\|^2_{E_2}, \end{pmatrix} \end{aligned}$$

and $u^\pm, v^\pm$ can be approximated by functions of $H^2(\Omega)$. In this way $\mathcal{N}_{\rm nod} \cap H$ is a C^1 manifold of codimension 4 in H.

If (u,v) is a least energy nodal solution, then $(u,v) \in \mathcal{N}_{\rm nod} \cap H$ by elliptic regularity, and by minimality the quadratic form associated to $\mathcal{I}''(u,v)$ is nonnegative on T, the tangent space of $\mathcal{N}_{\rm nod} \cap H$ at (u,v). Since T has codimension 4 in H and H is dense in E, it follows that the Morse index is at most 4.

On the other hand, one can see that the Morse index is at least 4 Proposition **??**. by showing that the same quadratic form is negative on the 4-dimensional space spanned by $(u^\pm,0)$ and $(0,v^\pm)$. For every $(a,b,c,d) \in \mathbb{R}^4 \setminus \{0\}$ we have

$$\begin{aligned} &\langle \mathcal{I}''(au^+ + bu^-, cv^+ + dv^-), (au^+ + bu^-, cv^+ + dv^-)\rangle \\ &= a^2\Big(\|u^+\|^2_{E_1} - (2q-1)\mu_1\|u^+\|^{2q}_{2q} - \beta(q-1)\|u^+v\|^q_q\Big) \\ &\quad + b^2\Big(\|u^-\|^2_{E_1} - (2q-1)\mu_1\|u^-\|^{2q}_{2q} - \beta(q-1)\|u^-v\|^q_q\Big) \\ &\quad + c^2\Big(\|v^+\|^2_{E_2} - (2q-1)\mu_2\|v^+\|^{2q}_{2q} - \beta(q-1)\|uv^+\|^q_q\Big) \\ &\quad + d^2\Big(\|v^-\|^2_{E_2} - (2q-1)\mu_2\|v^-\|^{2q}_{2q} - \beta(q-1)\|uv^-\|^q_q\Big) \\ &\quad - 2\beta q\big(ac\|u^+v^+\|^q_q + ad\|u^+v^-\|^q_q + bc\|u^-v^+\|^q_q + bd\|u^-v^-\|^q_q\big) \end{aligned}$$

and since (u,v) solves (1.1)

$$\begin{aligned} &= -2(q-1)\big(a^2\|u^+\|^2_{E_1} + b^2\|u^-\|^2_{E_1} + c^2\|v^+\|^2_{E_2} + d^2\|v^-\|^2_{E_2}\big) \\ &\quad + \beta q\big((a-c)^2\|u^+v^+\|^q_q + (a-d)^2\|u^+v^-\|^q_q + (b-c)^2\|u^-v^+\|^q_q + (b-d)^2\|u^-v^-\|^q_q\big). \end{aligned}$$

When $\beta \le 0$ this quantity certainly is negative.

When $\beta \in (0, \sqrt{\mu_1\mu_2}/2)$, instead, using again that (u,v) solves (1.1) we write

$$\begin{aligned} &\langle \mathcal{I}''(au^+ + bu^-, cv^+ + dv^-), (au^+ + bu^-, cv^+ + dv^-)\rangle \\ &= -2(q-1)\int_\Omega \mu_1 a^2|u|^{2q} + \mu_1 b^2|u^-|^{2q} + \mu_2 c^2|v^+|^{2q} + \mu_2 d^2|v^-|^{2q} \end{aligned}$$

$$
\begin{aligned}
&- (q-2)\beta \int_\Omega (a^2+c^2)|u^+v^+|^q + (a^2+d^2)|u^+v^-|^q + (b^2+c^2)|u^-v^+|^q + (b^2+d^2)|u^-v^-|^q \\
&- 2\beta q \int_\Omega ac|u^+v^+|^q + ad|u^+v^-|^q + bc|u^-v^+|^q + bd|u^-v^-|^q \\
\le & -(q-1)\int_\Omega \big(\sqrt{\mu_1}|a||u^+|^q - \sqrt{\mu_2}|c||v^+|^q\big)^2 + \big(\sqrt{\mu_1}|a||u^+|^q - \sqrt{\mu_2}|d||v^-|^q\big)^2 \\
&- (q-1)\int_\Omega \big(\sqrt{\mu_1}|b||u^-|^q - \sqrt{\mu_2}|c||v^+|^q\big)^2 + \big(\sqrt{\mu_1}|b||u^-|^q - \sqrt{\mu_2}|d||v^-|^q\big)^2 \\
&- (q-2)\beta \int_\Omega (|a|-|c|)^2|u^+v^+|^q + (|a|-|d|)^2|u^+v^-|^q + (|b|-|c|)^2|u^-v^+|^q + (|b|-|d|)^2|u^-v^-|^q \\
&+ 2\left(2\beta - (q-1)\sqrt{\mu_1\mu_2}\right)\int_\Omega |ac||u^+v^+|^q + |ad||u^+v^-|^q + |bc||u^-v^+|^q + |bd||u^-v^-|^q < 0.
\end{aligned}
$$

As for the number of nodal regions of u and v, assume that the support of u^+ splits into two sets A_1 and A_2 which are the closure of open disjoint sets. Then letting u_1^+ and u_2^+ the restrictions of u to the sets A_1 and A_2, respectively, and repeating the previous computations one sees that the quadratic form $\mathcal{I}''(u,v)$ is negative on the 5-dimensional space spanned by $(u_1^+,0)$, $(u_2^+,0)$, $(u^-,0)$ and $(0,v^\pm)$, contradicting the fact that the Morse index is 4. □

3.4. **Semi-nodal solutions.** Least energy semi-nodal solutions can be produced in a very similar way. We seek for a solution (u,v) with u sign-changing and v sign-definite, which attains the infimum

$$c_{\rm sn} = \inf\{\mathcal{I}(u,v) : (u,v)\in E,\ u^\pm \neq 0,\ |v|>0,\ \mathcal{I}'(u,v)=0\},$$

and works on the Nehari type set

$$\mathcal{N}_{\rm sn} := \{(u,v)\in E\ : u^\pm, v\neq 0,\ \mathcal{I}'(u,v)(u^\pm,0) = \mathcal{I}'(u,v)(0,v) = 0\}. \tag{3.22}$$

Precisely we characterize the seminodal least energy solution by the constrained minimization problem

$$\gamma_{\rm sn} = \inf\{\mathcal{I}(u,v) : (u,v)\in\mathcal{N}_{\rm sn}\}.$$

The converse case (u sign-definite, v sign-changing) can be handled with the obvious adjustments.

Notice that $\mathcal{N}_{\rm nod}\subset\mathcal{N}_{\rm sn}$, therefore establishing that $\gamma_{\rm sn}$ is attained does not suffice to ensure the existence of a semi-nodal solution.

We mimic the reasoning of Section 3 and introduce the auxiliary function

$$\theta(a,b,c) = \mathcal{I}(a^{\frac1q}u^+ - b^{\frac1q}u^-, c^{\frac1q}v),$$

and the sets

$$
\begin{gathered}
E_0 = \{(u,v)\in E : M(u,v) \text{ is positive defined}\},\\
\mathcal{O} = \{(u,v)\in E_0 : M^{-1}W\in(0,\infty)^3\},
\end{gathered}
$$

where now M is the matrix

$$
M = \begin{pmatrix} \mu_1\|u^+\|_{2q}^{2q} & 0 & \beta\|u^+v\|_q^q \\ 0 & \mu_1\|u^-\|_{2q}^{2q} & \beta\|u^-v\|_q^q \\ \beta\|u^+v\|_q^q & \beta\|u^-v\|_q^q & \mu_2\|v\|_{2q}^{2q}\end{pmatrix}.
$$

The arguments of Lemmas 3.1–3.4 give the following result.

Lemma 3.8. *For $|\beta|<\sqrt{\mu_1\mu_2/2}$, $E_0 = \{(u,v)\in E : u^\pm, v\neq 0\}$. For $\beta\le 0$, E_0 contains any $(u,v)\in E$ such that*

$$\mu_1\|u^\pm\|_{2q}^{2q} + \beta\|u^\pm v\|_q^q > 0,\quad \mu_2\|v\|_{2q}^{2q} + \beta\|uv\|_q^q > 0,$$

and in particular $\mathcal{N}_{\rm sn}\subset E_0$.

For $q=2$ and $0<\beta<\sqrt{\mu_1\mu_2/2}$, $\mathcal{O}$ is nonempty and $\mathcal{N}_{\rm sn}\subset\mathcal{O}$.

For $q\in(2,q_N)$ and $\beta<\sqrt{\mu_1\mu_2/2}$, or $q=2$ and $\beta\le 0$, if $(u,v)\in E_0$ then the function θ has a unique maximum point in $(0,\infty)^3$ characterized by the condition $(a^{\frac1q}u^+ - b^{\frac1q}u^-, c^{\frac1q}v)\in\mathcal{N}_{\rm sn}$

If $q=2$ and $0<\beta<\sqrt{\mu_1\mu_2/2}$, the same holds true provided that $(u,v)\in\mathcal{O}$.

Next, repeating the reasoning of Theorem 3.5 one shows the following proposition.

Proposition 3.9. *Let $q\in(2,q_N)$, for every $\beta<\sqrt{\mu_1\mu_2/2}$ the infimum $\gamma_{\rm sn}$ is attained by a function which solves (1.1). If $q=2$, there exists $\beta_0>0$ such that the same holds true for every $\beta<\beta_0$.*

The solution we just constructed is not necessarily semi-nodal, because also v could change sign. Next statement ensures that it is not the case.

Proposition 3.10. *If $(u,v)\in\mathcal{N}_{\rm sn}$ realizes the minimum $c_{\rm sn}$, then it has Morse index 3, u has exactly two nodal zones and v has fixed sign.*

Proof. Letting $H:=E\cap\big(H^2(\Omega)\big)^2$ and using the function $G:E_{\rm sn}\to\mathbb{R}^3$,

$$G(u,v)=\begin{pmatrix}\mathcal{I}'(u,v)\,(u^+,0)\\ \mathcal{I}'(u,v)\,(u^-,0)\\ \mathcal{I}'(u,v)\,(0,v)\end{pmatrix},$$

one sees that $N_{\rm sn}\cap H$ is a C^1 manifold of codimension 3 in H. So the same arguments of Proposition 3.7 yield that (u,v) has Morse index 3. Furthermore $u^\pm,v\neq0$ by assumption. If v changes sign or u has more than two nodal zones, it follows that the Morse index is at least 4, and it ends the proof. □

Because $\mathcal{N}_{\rm nod}\subset\mathcal{N}_{\rm sn}\subset\mathcal{N}\subset\mathcal{N}_{\rm full}$, a noteworthy consequence of this line of reasoning is that all these inclusions are strict and we have the statement.

Theorem 3.11. $c_{\rm full}<c<c_{\rm sn}<c_{\rm nod}$.

4. Symmetry breaking and more solutions with pre-assigned symmetries

Henceforth we take that $\Omega=B_R(0)$ is a ball. The constrained minimization considered in the previous sections may be restricted to the subspace consisting of radially symmetric functions

$$E_{\rm rad}:=\{(u,v)\in E: u(x)=u(|x|),\, v(x)=v(|x|) \text{ for all } x\in\Omega\},$$

and produces a pure vector, a nodal and a seminodal radial solutions, which have respectively *radial Morse index* equal to 2, 3 and 4. By radial Morse index of a solution (u,v) we mean the maximal dimension of a subspace of $E_{\rm rad}$ where the quadratic form related to $\mathcal{I}''(u,v)$ is negative. Observe that the radial Morse index can be less than the Morse index.

It remains to determine whether this provides genuinely new solutions or rather a symmetry property of the least energy solutions. One can find an answer, and observe symmetry breaking, by computing the exact Morse index of radial solutions. The issue is by itself interesting and deserves a detailed study. Here we argue by perturbation and present an estimate in the case β close to zero.

Proposition 4.1. *Let $N=2,3$ and $2\le q<q_N$. There exist $b_1,b_2>0$ such that*

1. *if $|\beta|<b_1$, a radial sign-changing least energy solution has Morse index greater or equal to $2N+2$.*
2. *if $|\beta|<b_2$, a radial seminodal least energy solution has Morse index greater or equal to $N+2$.*

Comparing these estimates with Propositions 3.7 and 3.10 proves the symmetry breaking stated by Theorem 1.1 .

Before proving Proposition 4.1, we check the uniform convergence of solutions when β vanishes. To this end, we take λ_i, μ_i and q as fixed, and emphasize the dependence by β by writing $\mathcal{I}_\beta$ for the functional introduced in (2.1) and (u_β,v_β) for a solution of (1.1). Recall that when $\beta=0$ system (1.1) decouples and gives rise to two Lane-Emden problems

$$\begin{gathered}-\Delta u+\lambda_1u=\mu_1|u|^{2q-2}u\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega,\end{gathered}\tag{4.1}$$

and

$$\begin{gathered} -\Delta v+\lambda_2 v=\mu_2|v|^{2q-2}v \quad \text{in } \Omega,\\ v=0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.2}$$

Lemma 4.2. *For $\beta\neq 0$, let (u_β,v_β) a nodal (respectively, semi-nodal) least energy radial solution to (1.1). There are u_0 and v_0, nontrivial radial solutions to (4.1) and (4.2), such that $u_\beta\to u_0$ and $v_\beta\to v_0$ uniformly as $\beta\to 0$, up to an extracted sequence. Furthermore if (u_β,v_β) are nodal (resp., semi-nodal), the same holds for u_0 and v_0.*

Proof. We take that for every $\beta\neq 0$, (u_β,v_β) is a radial sign-changing least energy solution. By Proposition 4.3 we know that $(u_\beta,v_\beta)\in\mathcal{N}_{\rm nod}\cap E_{\rm rad}$ and

$$\mathcal{I}_\beta(u_\beta^{\rm nod},v_\beta^{\rm nod})=\min\left\{\mathcal{I}_\beta(u,v):(u,v)\in\mathcal{N}_{\rm nod}\cap E_{\rm rad}\right\}.$$

The arguments used in the proof of Theorem 3.5, to obtain (3.14), proves that there exist C not depending by β such that

$$\|u_\beta\|_{2q},\|v_\beta\|_{2q}\le C, \tag{4.3}$$

and since $(u_\beta,v_\beta)\in\mathcal{N}$ it follows readily that

$$\|u_\beta\|_{E_1},\|v_\beta\|_{E_2}\le C \tag{4.4}$$

for $|\beta|<b$. Therefore by standard arguments u_β and v_β converge weakly in H_0^1, strongly in $L^{2q}(\Omega)$ and pointwise a.e. to radial solutions u_0 and v_0 of (4.1), (4.2). Reasoning as in the step 2 of the proof of Theorem 3.5 one deduces from (4.3) that u_0 v_0 are sign-changing (in particular, non-trivial).

If (u_β,v_β) are semi-nodal, let us say that u_β are sign-changing and $v_\beta>0$ on Ω, the same reasoning yields that u_0 changes sign and v_0 is not identically zero. Eventually $v_0\ge 0$ by the pointwise convergence, and Hopf's boundary Lemma assures that $v_0>0$ on Ω.

It remains to see that the convergence is uniform, indeed. First we point out that radial weak solutions to (1.1) are indeed classical and integrating the equation in (1.1) gives

$$u_\beta'(r)=-\frac{1}{r^{N-1}}\int_0^r\rho^{N-1}\left(\mu_1|u_\beta|^{2q-2}+\beta|u_\beta|^{q-2}|v_\beta|^q-\lambda_1\right)u_\beta d\rho, \tag{4.5}$$

$$v_\beta'(r)=-\frac{1}{r^{N-1}}\int_0^r\rho^{N-1}\left(\mu_2|v_\beta|^{2q-2}+\beta|u_\beta|^{q}|v_\beta|^{q-2}-\lambda_2\right)v_\beta d\rho, \tag{4.6}$$

$$u_\beta(r)=-\int_r^R u_\beta'(\rho)d\rho,\quad v_\beta(r)=-\int_r^R v_\beta'(\rho)d\rho. \tag{4.7}$$

Estimating (4.5) and (4.6) by Holder's inequality and (4.3) gives

$$|u_\beta'(r)|,|v_\beta'(r)|\le Cr^{1-N}\Big(r^{\frac{N}{2q}}+r^{N\frac{2q-1}{2q}}\Big). \tag{4.8}$$

If $N=2$, inserting (4.8) inside (4.7) yields $|u_\beta(r)|,|v_\beta(r)|\le C$, which in turn, together with (4.5) and (4.6), ensures that $|u_\beta'(r)|,|v_\beta'(r)|\le C$ and concludes the proof.

If $N=3$, instead, it is needed to start a bootstrap argument by the so called Radial Lemma due to Strauss [21]: if w is any radial function in $H_0^1(\Omega)$, then

$$|w(r)|\le\frac{\|\nabla w\|_2}{\sqrt{N-2}}\,r^{-\frac{N-2}{2}}. \tag{4.9}$$

To simplify notations, we write $p=2q-1$. Estimating (4.5) and (4.6) by (4.9) and (4.4) gives

$$|u_\beta'(r)|,|v_\beta'(r)|\le C\Big(1+r^{1-p\frac{N-2}{2}}\Big). \tag{4.10}$$

If $1-p\frac{N-2}{2}>-1$, the proof ends as for $N=2$. Also when $1-p\frac{N-2}{2}=-1$, that is $p=\frac{4}{N-2}$, by plugging (4.10) into (4.7) we obtain

$$|u_\beta(r)|,|v_\beta(r)|\le C\left(1+\log r\right),$$

and going back to (4.5) and (4.6), we ave

$$|u'_\beta(r)|,|v'_\beta(r)|\le C\Big(1+r^{1-N}\int_0^r\rho^{N-1}|\log\rho|^{\frac{4}{N-2}}d\rho\Big)\le C,$$

which ensures uniform convergence. Otherwise if $1-p\frac{N-2}{2}<-1$, we put (4.10) into (4.7) and obtain

$$|u_\beta(r)|,|v_\beta(r)|\le C\left(1+r^{2-p\frac{N-2}{2}}\right), \tag{4.11}$$

which in turn implies

$$|u'_\beta(r)|,|v'_\beta(r)|\le C\left(1+r^{1+2p-p^2\frac{N-2}{2}}\right).$$

If $1+2p-p^2\frac{N-2}{2}\ge -1$ the proof ends like in the previous step, otherwise we iterate the argument and end up with

$$|u'_\beta(r)|,|v'_\beta(r)|\le C\left(1+r^{1+2p+2p^2-p^3\frac{N-2}{2}}\right).$$

In that way, after n steps we get

$$|u'_\beta(r)|,|v'_\beta(r)|\le C\left(1+r^{-1+\gamma_n}\right),$$

for

$$\gamma_n=2\sum_{k=0}^n p^k-p^{n+1}\frac{N-2}{2}=\Big(\frac{2}{p-1}-\frac{N-2}{2}\Big)p^{n+1}-\frac{2}{p-1},$$

and the proof is concluded if $\gamma_n\ge 0$ after a finite number of steps. But this must hold because $\frac{2}{p-1}=\frac{1}{q-1}>\frac{N-2}{2}$. □

Proof of Proposition 4.1. We prove in full detail part (1). Using the same notations of Lemma 4.2, it is know by [1, Theorem 1.1] that u_0 (resp., v_0) has Morse index greater or equal to $N+1$, as a solution to (4.1) (resp., (4.2)). This means that there are $N+1$ linearly independent functions $\phi_i\in H^1_0(\Omega)$ that solve

$$-\Delta\phi_i+\lambda_1\phi_i=(2q-1)\mu_1|u_0|^{2q-2}\phi_i+\nu_i\phi_i \tag{4.12}$$

in Ω, with $\nu_1\le\nu_2\dots\le\nu_{N+1}<0$ and $N+1$ linearly independent functions $\psi_i\in H^1_0(\Omega)$ that solve

$$-\Delta\psi_i+\lambda_2\psi_i=(2q-1)\mu_2|v_0|^{2q-2}\psi_i+\sigma_i\psi_i \tag{4.13}$$

in Ω, with $\sigma_1\le\sigma_2\dots\le\sigma_{N+1}<0$.

Let W_1,W_2 be the $(N+1)$–dimensional subspaces of $H^1_0(\Omega)$ spanned by $\phi_1,\dots\phi_{N+1}$ and $\psi_1,\dots\psi_{N+1}$, respectively.

The claim follows by checking that, if $|\beta|$ is small, then the quadratic form related to $\mathcal{I}''_\beta(u_\beta,v_\beta)$ is negative on $W_1\times W_2$. Recall that by (4.12) and (4.13) we have

$$\|\phi\|_{E_1}\le(2q-1)\mu_1\int_\Omega|u_0|^{2q-2}\phi^2+\nu_{N+1}\|\phi\|_2^2\quad\text{for every }\phi\in W_1,$$
$$\|\psi\|_{E_2}\le(2q-1)\mu_2\int_\Omega|v_0|^{2q-2}\psi^2+\sigma_{N+1}\|\psi\|_2^2\quad\text{for every }\psi\in W_2.$$

Hence

$$\begin{aligned}
&\langle\mathcal{I}''_\beta(u_\beta,v_\beta)(\phi,\psi),(\phi,\psi)\rangle\\
&=\|\phi\|_{E_1}^2+\|\psi\|_{E_1}^2-(2q-1)\mu_1\int_\Omega|u_\beta|^{2q-2}\phi^2-(2q-1)\mu_2\int_\Omega|v_\beta|^{2q-2}\psi^2\\
&\quad-(q-1)\beta\int_\Omega\big(|u_\beta|^{q-2}|v_\beta|^q\phi^2+|u_\beta|^q|v_\beta|^{q-2}\psi^2\big)-2q\beta\int_\Omega|u_\beta v_\beta|^{q-2}u_\beta v_\beta\phi\psi\\
&\le\nu_{N+1}\|\phi\|_2^2+\sigma_{N+1}\|\psi\|_2^2+(2q-1)\mu_1\int_\Omega\left(|u_0|^{2q-2}-|u_\beta|^{2q-2}\right)\phi^2\\
&\quad+(2q-1)\mu_2\int_\Omega\left(|v_0|^{2q-2}-|v_\beta|^{2q-2}\right)\psi^2-(q-1)\beta\int_\Omega\big(|u_\beta|^{q-2}|v_\beta|^q\phi^2+|u_\beta|^q|v_\beta|^{q-2}\psi^2\big)
\end{aligned}$$

$$- 2q\beta \int_\Omega |u_\beta v_\beta|^{q-2} u_\beta v_\beta \phi\psi$$

and thanks to the uniform convergence in Lemma 4.2

$$\leq (\max\{\mu_{N+1}, \sigma_{N+1}\} + o(1)) \left(\|\phi\|_2^2 + \|\psi\|_2^2\right)$$

where $o(1) \to 0$ as $\beta \to 0$. □

4.1. **Other type of symmetries and multiplicity of solutions.** Here $\Omega \subset \mathbb{R}^2$ is a disc. We write r and θ for the usual polar coordinates, and for every $k \in \mathbb{Z}_+$ we introduce the set of k-symmetric functions

$$H_k := \{(u, v) \in \left(H_0^1(\Omega)\right)^2 : \ u \text{ and } v \text{ are even and } \frac{2\pi}{k}\text{-periodic w.r.t. } \theta\}.$$

For a *k-symmetric* solution, we denote by m_k its k Morse index, i.e. the maximal dimension of a subspace of H_k where the second derivative of $\mathcal{I}$ gives rise to a negative defined quadratic form. By constrained minimization in the symmetric Nehari set $\mathcal{N}_{\text{nod}} \cap H_k$ it is straightforward to check the following.

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^2$ be a disc, $q \geq 2$, then there exist $\bar\beta_k > 0$, such that for $\beta < \beta_k$ the problem* (1.1) *has a (component-wise) sign-changing k-symmetric solution which has the least energy among all sign-changing and k-symmetric solutions and has k Morse index 4.*

Relying on the convergence in Lemma 4.2 and the results about Lane-Emden equation in [12] we can estimate the k-Morse index when β is close to zero, thus proving Theorem 1.2.

Proof of Theorem 1.2. If a sign-changing k-symmetric least energy solution to (1.1) is radial, then it has to be a sign-changing radial ground state, because the set of component-wise radial functions is contained in H_k. We prove this cannot happen by showing that the k Morse index of a sign-changing radial ground state is at least 6, provided that q is large and β is close to 0. To this aim, we denote by (u_β, v_β) a sign-changing radial solution. For $\beta \to 0$, Lemma 4.2 states that u_β and v_β converge uniformly to u_0 and v_0, which are sign-changing radial solutions to (4.1) and (4.2), respectively. It is known by [12, Proposition 6.5] that the k Morse index of both u_0 and v_0 (as solutions of (4.1) and (4.2), respectively) is not less than 3 when q is large. Hence reasoning as in the proof of Proposition 4.1 one sees that the quadratic form related to $\mathcal{I}_\beta''(u_\beta, v_\beta)$ is negative on a subset of H_k of dimension 6, if β is close to 0. □

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Anna Lisa Amadori

Dipartimento di Scienze e Tecnologie, Università di Napoli "Parthenope", Centro Direzionale di Napoli, Isola C4, 80143 Napoli, Italy

Email address: `annalisa.amadori@uniparthenope.it`