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PRICING CALLABLE BONDS AND OPTIMAL CALLABLE TIME UNDER THE FRACTIONAL BLACK-SCHOLES MARKET

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Abstract. This article concerns the pricing of callable bonds and the determination of optimal call time under the fractional Black-Scholes model. By employing a discrete approximation of the continuous asset price process, we efficiently estimate the continuation value as well as the optimal callable time, and analyze the path-dependent nature of the asset dynamics under the fractional Black-Scholes model. We ensure the accuracy of the numerical estimation and perform numerical experiments to illustrate the effectiveness of the proposed method.

1. Introduction

The valuation of callable bonds presents a significant challenge in financial engineering due to their embedded option features. The models estimate the bond's value based on possible callable paths, assigning probabilities to different scenarios. This approach renders the bond price highly sensitive to the timing of the call, as early or delayed call can significantly affect the bond's ultimate payoff and, therefore, its market value. The optimal redemption time for a callable bond is influenced by interest rates, time to maturity, and potential call features. This timing directly impacts callable bond pricing models, which aim to incorporate optimal callable strategies to determine fair value.

For investigating callable bonds, Narayanan and Lim [18] explored the optimality of calling corporate zero-coupon bonds for refunding, considering corporate tax effects. Their analysis reveals that refunding is not optimal unless the corporate tax rate exceeds 50%. The study also shows that callable zero-coupon bonds more frequently include restrictive covenants, indicating the call feature offers firms flexible, low-cost options for future recapitalization. Farto and Vázquez [11] focused on a complex financial product==a coupon-bearing callable bond with a compulsory notice period==by addressing the price discontinuities caused by discrete coupons and call features. The bond's value depends on time and a stochastic interest rate, modeled through a Black-Scholes type PDE with appropriate boundary and jump conditions. To solve this, the authors proposed a numerical method combining characteristics-based time discretization with finite element methods for the interest rate variable. The method shows strong performance when tested against known solutions under the Vasicek and Cox-Ingersoll-Ross (CIR) models, and outperforms prior finite volume approaches in the call-with-notice scenario.

Aron [1] estimated a dynamic term structure model directly on treasury coupon bond data, bypassing the need for a zero-coupon yield curve. A linearity generating approach enables fast estimation and separates cross-sectional from time series parameters. The model quantifies the on-the-run and "notes versus bonds" premiums from 1990 to 2017 within a unified, no-arbitrage framework. Díaz et al. [8] compared various zero-coupon yield curve datasets, which differ by asset selection and fitting methods, despite representing the same underlying reality. Using an empirical analysis on callable bond pricing, it finds that results can vary significantly depending on the dataset used, particularly due to differences in volatility inputs. This sensitivity may create

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incentives for financial institutions to selectively choose datasets in their favor, posing a potential moral hazard.

Lin and Zhu [17] addressed the pricing of callable-puttable convertible bonds using an integral equation (IE) approach. The complexity arises from the interplay of call, put, and conversion features, leading to scenarios with one or two moving boundaries depending on time to maturity and contract terms. The analysis identifies critical time points where the structure of the problem changes, requiring different pricing strategies==ranging from handling only the put feature to managing both put and conversion, or reducing to a simpler vanilla bond case. Each scenario necessitates a distinct numerical solution of corresponding IE systems.

Skalicky et al. [20] presented a new method for valuing the risk of bond buybacks by issuers at a specified call price, specifically for privately traded bonds with embedded European or multiple options. Unlike traditional models for marketable callable bonds, this approach addresses the challenges of limited transaction data. Its modular structure enables the incorporation of issuer behavior, alternative investment opportunities, and varying interest rate expectations.

Kusnadi et al. [15] focused on the impact of early redemption and default risk on the value of Indonesian bonds using binomial interest rate tree models. Simulations reveal that early redemption risk generally lowers a bond's present value, while default risk, under certain assumptions, can unexpectedly increase it. The higher bond value under default risk is attributed to an inflated recovery fraction in the first period, driven by data limitations. Dow and Orfanos [9] addressed the limitations of duration and convexity matching in hedging fixed income securities with negative convexity, such as callable bonds and mortgage-backed securities. It introduces the concepts of bond tilt and bond agility to manage residual risk, deriving approximation formulas that incorporate higher-order effects. The proposed methods closely track price-yield dynamics, achieving mean absolute errors below 2.5% across various callable bonds under yield shifts of up to ± 200 basis points.

Hobson et al. [12] examined the callable convertible bond problem under a liquidity constraint represented by Poisson signals. Unlike traditional models, it assumes that when the bondholder and the firm act simultaneously, neither has full priority, instead, a proportion $m \in [0, 1]$ of the bond is converted to equity, while the remainder is called by the firm. This framework extends the special case analyzed by Liang and Sun [16], where the bondholder held full priority ($m = 1$), and provides a comprehensive solution to the problem under this more general setting.

In addition, some scholars use deep learning methods to deal with option and bond pricing. Becker et al. [3] proposed a deep learning method for optimal stopping problems which directly learns the optimal stopping rule from Monte Carlo samples. Accurate results are obtained for the problem of optimal stopping a fractional Brownian motion with short computing times. Becker et al. [4] presented a deep learning-based algorithm to price early exercise options, effectively addressing high-dimensional optimal stopping problems common in derivatives such as American and Bermudan options. The method approximates both the option price and optimal exercise strategy and can be applied to other optimal stopping problems with simulatable stochastic processes. Numerical results, including cases with up to 5000 dimensions, demonstrate the algorithm's accuracy and efficiency compared to existing benchmarks.

Tan et al. [22] introduced Deeppricing model, a data-driven model for pricing convertible bonds using a novel financial time series GAN called FinGAN. FinGAN accurately captures complex features of stock return dynamics, such as fat tails, long-range dependence, and asymmetry, and transitions to a risk neutral distribution, enabling more realistic and flexible pricing. Experiments on the Chinese convertible bond market show that Deeppricing model outperforms traditional models (e.g., Black-Scholes, CEV, GARCH) and other GAN-based methods, especially for higher volatility or equity-like bonds. Deeppricing model yield strong annualized returns, demonstrating their practical value.

The framework of this paper is as follows. Section 2 introduces the pricing problem of callable bonds. We apply deep learning method to calculate the lower bound and construct confidence intervals for the price of callable bonds in Section 3. The results of the numerical simulation are presented in Section 4. Conclusions and future work are drawn in Section 5.

2. Problem formulation and fractional Black-Scholes model

2.1. **Problem formulation.** Zero-coupon callable bond pricing problem can be formulated through an interest rate process r_t, $0 \le t \le T$, defined on $(\Omega, \mathcal{F}, P)$, where P is the risk-neutral probability measure. The set of stopping times considered in this paper is finite. During the life of the bond, the issuer has a finite number of time points $t_0 = 0, t_1, \dots, t_N = T$, to choose whether to call the bond or not. If called at time t_i, $i = 0, 1, 2, \dots, N$, the issuer has the right to call the bond by paying a predetermined price S to bond holder. The payoff functions $p_{t_i}(r_{t_i})$ are some known functions $p_{t_i}(\cdot) : R \to [0, +\infty)$. In this paper, we set the payoff function is $p_{t_i}(t_i) = \min\big(K, F \cdot e^{-\int_0^{t_i} r(s)ds}\big)$. If the payoff at t_i is lower than the continuation value at that time, the issuer early call the bond; otherwise, continue to issue it. The bond values and the continuation value satisfy

$$
\begin{gathered}
V_{t_i} = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}\left[D(t_i, \tau) p_\tau(r_\tau) \mid \mathcal{F}_{t_i}\right] = \mathbb{E}\left[D(t_i, \tau_i^*) p_{\tau_i^*}\left(r_{\tau_i^*}\right) \mid \mathcal{F}_{t_i}\right], \\
C_{t_i} = \mathbb{E}\left[D(t_i, t_i + 1) V_{t_i+1}(r_{t_i+1}) \mid \mathcal{F}_{t_i}\right],
\end{gathered}
$$

where $\mathcal{T}_i = \{t_i, t_{i+1}, \dots, t_N\}$, τ_i^* is the optimal moment to call, $D(t, \tau) = e^{-\int_t^\tau r_u du}$ is the discount factor, r_u is the interest rate process we analyze through the paper. For simplicity, we let $\mathcal{F}_i$ denote $\mathcal{F}_{t_i}$, $p_i(r_i)$ denote $p_{t_i}(r_{t_i})$, V_i denote V_{t_i}, and C_i denote C_{t_i}. Then, the bond values and the continuation values satisfy the dynamic programming equations

$$
\begin{gathered}
V_N = p_N(r_N), \\
V_i = \max\Big\{p_i(r_i), \mathbb{E}\Big[e^{-\int_{t_i}^{t_{i+1}} r_u du} V_{i+1}(r_{i+1}) \mid \mathcal{F}_i\Big]\Big\}, \quad i = 0, 1, \dots, N-1, \\
C_N = 0, \\
C_i = \mathbb{E}\Big[e^{-\int_{t_i}^{t_{i+1}} r_u du} \max\{p_{i+1}(r_{i+1}), C_{i+1}(r_{i+1})\} \mid \mathcal{F}_i\Big], \quad i = 0, 1, \dots, N-1,
\end{gathered}
\tag{2.1}
$$

and the bond values satisfy

$$
V_i = \max\{p_i(r_i), C_i(r_i)\}.
$$

2.2. **Fractional Brownian motion and Wick product.** The fractional Brownian motion with Hurst parameter $H \in (0, 1)$ is a zero-mean Gaussian process $\big(B_t^H\big)_{t \in [0,\infty)}$ with covariance

$$
\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right), \quad s, t \in [0, \infty).
$$

The fractional Brownian motion is self-similar, i.e. B_{at} and $a^H B_t^H$ have the same probability law for all $a > 0$. Compared to Brownian motion, it displays positive correlation property when $\frac{1}{2} < H < 1$ and negative correlation property when $0 < H < \frac{1}{2}$. This special property of fractional Brownian motion allows it to describe models with long-range dependence. The fractional Brownian increment

$$
\Delta B_{t,s}^H = B_t^H - B_s^H, \quad 0 \le s < t \le \infty,
$$

has the following properties

$$
\begin{gathered}
\mathbb{E}\left[\Delta B_{t,s}^H\right] = \mathbb{E}\left[B_t^H - B_s^H\right] = 0, \\
\mathbb{E}\left[\left(\Delta B_{t,s}^H\right)^2\right] = \mathbb{E}\left[\left(B_t^H - B_s^H\right)\left(B_t^H - B_s^H\right)\right] = |t - s|^{2H}, \\
\mathbb{E}\left[\Delta B_{t,s}^H \Delta B_{s,0}^H\right] = \mathbb{E}\left[\left(B_t^H - B_s^H\right)\left(B_s^H - B_0^H\right)\right] = \frac{1}{2}\left[t^{2H} - s^{2H} - (t - s)^{2H}\right].
\end{gathered}
$$

In this paper, we consider the case $\frac{1}{2} < H < 1$. We use the following stochastic integral representation of the fractional Brownian motion

$$
B_t^H = \int_0^t z_H(t, s) dB_s, \quad t \in [0, \infty),
$$

where B_t is a standard Brownian motion, and the deterministic kernel

$$
z_H(t, s) = 1_{\{t \ge s\}} c_H\big(H - \frac{1}{2}\big) s^{\frac{1}{2} - H} \int_s^t u^{H - \frac{1}{2}} (u - s)^{H - \frac{3}{2}} du
$$

with the constant

$$c_H = \sqrt{\frac{2H\Gamma\left(\frac{3}{2} - H\right)}{\Gamma\left(H + \frac{1}{2}\right)\Gamma(2 - 2H)}},$$

where Γ is the Gamma function. Because of the existence of a bijective transfer operator, the filtration generated by B_t^H is also the one generated by B_t through expressing B_t as a Wiener integral with respect to B_t^H.

The integral theory of fractional Brownian motion based on the Wick product is introduced by [10] and further developed by [14] through fractional white noise theory, which proves that the financial markets are arbitrage free and complete when the integrals defined by this method are applied to financial market models. Define the fractional kernel $\phi : \mathbb{R}^2 \to \mathbb{R}$, the norm $\|\cdot\|_\phi^2$ and inner product $\langle \cdot, \cdot \rangle_\phi$ by

$$\begin{gathered}
\phi(s,t) := H(2H-1)|t-s|^{2H-2}, \\
\|f\|_\phi^2 := \int_0^\infty \int_0^\infty f(s)f(t)\phi(s,t)\,ds\,dt, \\
\langle f, g \rangle_\phi := \int_0^\infty \int_0^\infty f(s)g(t)\phi(s,t)\,ds\,dt,
\end{gathered}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are Borel measurable functions, and define

$$\mathbb{L}_\phi^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R}, f \text{ is Borel measurable, } \|f\|_\phi^2 < +\infty \right\}.$$

By considering the step functions

$$f_m(t) = \sum_i a_i^{(m)} \chi_{[t_i, t_{i+1})}(t)$$

approximating a deterministic $f \in \mathbb{L}_\phi^2(\mathbb{R})$, define

$$\int_0^\infty f(t) dB_t^H = \lim_{m\to\infty} \int_0^\infty f_m(t) dB_t^H = \lim_{m\to\infty} \Big(\sum_i a_i^{(m)} \left(B_{t_{i+1}}^H - B_{t_i}^H \right) \Big).$$

Define the Wick exponentials

$$\varepsilon(f) := \exp\Big(\int_0^\infty f(t) dB_t^H - \frac{1}{2}\|f\|_\phi^2 \Big), \quad f \in \mathbb{L}_\phi^2(\mathbb{R}).$$

The Wick product of two exponentials $\varepsilon(f)$ and $\varepsilon(g)$ is defined as

$$\varepsilon(f) \diamond \varepsilon(g) := \varepsilon(f+g).$$

According to [10, Theorem 3.1], the linear span of $\Big\{\varepsilon(f),\ f \in \mathbb{L}_\phi^2(\mathbb{R})\Big\}$ is a dense set of $L^2(\Omega, \mathcal{F}, P)$. In other words, the random variables in $L^2(\Omega, \mathcal{F}, P)$ can be approximated by Wick exponentials' linear combination and the definition of Wick product can be extended to random variables $X, Y \in L^2$ through the fractional Wiener-Itô chaos expansion (see, [14]). Let $X_s \in L^2(\Omega, \mathcal{F}_s, P)$, $s \in [0,T]$. If the Wick product and the limit exists, the fractional Wick-Itô integral with respect to fractional Brownian motion $\left(B_s^H\right)_{[0,T]}$ can be defined by Wick-Riemann sums such as

$$\int_0^T X_s d^\diamond B_s^H := \lim_{n\to\infty} \sum_{t_i \in \pi_n} X_{t_{i-1}} \diamond \left(B_{t_i}^H - B_{t_{i-1}}^H \right),$$

where $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ with $\max_{t_i \in \pi_n} |t_i - t_{i-1}| \to 0$ for $n \to \infty$.

Based on the work of [21] who approximated fractional Brownian motion by disturbed binary random walks and the discrete Wick product introduced by [13], [5] proposed the discrete Wick product on fractional Brownian motion to avoid the technical difficulties in computing the continuous Wick product.

We define the discrete Wick exponential $\exp^{\diamond_n}(I^n(f^n))$, that is,

$$I^n(f^n) := \frac{1}{\sqrt{n}} \sum_{i=1}^n f_i^n \xi_i^n,$$

$$\exp^{\diamond_n}\left(I^n\left(f^n\right)\right):=\prod_{i=0}^{n}\Big(1+\frac{1}{\sqrt{n}}f_i^n\xi_i^n\Big),$$

where $f^n=(f_i^n,\dots,f_n^n)\in\mathbb{R}^n$, $(\xi_1^n,\dots,\xi_n^n)$ is an n-tuple of independent symmetric Bernoulli random variables with $P_n\left(\xi_i^n=1\right)=P_n\left(\xi_i^n=-1\right)=\frac{1}{2}$, living on a probability space $(\Omega_n,\mathcal{F}_n,P_n)$. Analogously, a discrete Wick product of the discrete Wick exponentials $I^n\left(f^n\right),I^n\left(g^n\right)$ is

$$\exp^{\diamond_n}\left(I^n\left(f^n\right)\right)\diamond_n\exp^{\diamond_n}\left(I^n\left(g^n\right)\right)=\exp^{\diamond_n}\left(I^n\left(f^n+g^n\right)\right).$$

Obviously, $L^2\left(\Omega_n,\mathcal{F}_n,P_n\right)$ is a 2^n-dimensional vector space, and a canonical orthonormal basis of $L^2\left(\Omega_n,\mathcal{F}_n,P_n\right)$ consists of

$$\Xi_A^n:=\prod_{i\in A}\xi_i^n,\quad A\subset\{1,\dots,n\}.$$

Every $X,Y\in L^2\left(\Omega_n,\mathcal{F}_n,P_n\right)$ has a unique expansion in terms of this basis, which is called the Walsh decomposition of X,

$$X=\sum_{A\subset\{1,\dots,n\}}X_A^n\Xi_A^n,\quad Y=\sum_{B\subset\{1,\dots,n\}}Y_B^n\Xi_B^n,$$

where $X_A^n,Y_B^n\in\mathbb{R}$.

Motivated by analogies of the chaos decomposition, the definition of discrete Wick product can be extended to random variables $X,Y\in L^2\left(\Omega_n,\mathcal{F}_n,P_n\right)$ (see [6]),

$$X\diamond_n Y=\sum_{C\subset\{1,\dots,n\}}\Big(\sum\left\{X_A^nY_B^n:A\cap B=\emptyset,A\cup B=C\right\}\Big)\,\Xi_C^n.$$

2.3. **Fractional Black-Scholes model and its discrete method.** In fractional Black-Scholes market, the dynamics of the interest rate r_t satisfies

$$dr_t=\mu r_t dt+\sigma r_t d^{\diamond}B_t^H,\ r_0=s_0,\tag{2.2}$$

where $\mu\in\mathbb{R}$ is the drift rate of the risk interest price process, $\sigma>0$ is the volatility, s_0 is the initial price. According to fractional Itô formula in [10, Theorem 4.3], the solution of the stochastic differential equation (2.2) is

$$r_t=r_0\exp\left(\mu t-\frac{1}{2}\sigma^2t^{2H}+\sigma B_t^H\right).$$

We define a discrete version of the fractional Black-Scholes model by

$$r_i^n=\left(1+\frac{\mu}{n}\right)r_{i-1}^n+\sigma r_{i-1}^n\diamond_n\left(B_{\frac{i}{n}}^{H,n}-B_{\frac{i-1}{n}}^{H,n}\right),\ r_0^n=s_0,\tag{2.3}$$

where

$$B_t^{H,n}:=\sum_{i=1}^{\lfloor nt\rfloor}b_{t,i}^n\frac{1}{\sqrt{n}}\xi_i^n,$$

$$b_{t,i}^n:=n\int_{\frac{i-1}{n}}^{\frac{i}{n}}z_H\Big(\frac{\lfloor nt\rfloor}{n},s\Big)ds,$$

$$z_H(t,s)=\begin{cases}C_Hs^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}, & \text{if } s<t,\\ 0, & \text{otherwise},\end{cases}$$

C_H is a normalization constant depending on H.

The following theorem shows that $r_{\lfloor nt\rfloor}^n$ obtained by the discrete version (2.3) weakly converges to r_t.

Lemma 2.1 ([6, Theorem 1.4])**.** *Suppose $\mu,s_0\in\mathbb{R},\sigma>0$. Then $\widetilde{r}_t^n:=r_{\lfloor nt\rfloor}^n$, the piecewise constant interpolation of the discrete Wick difference equation* (2.3)*, converges weakly to the interest rate r in the fractional Black-Scholes model, i.e. the solution of the SDE* (2.2)*, in the Skorokhod space $D([0,1],\mathbb{R})$.*

Equation (2.3) shows that recursively functions $r_i^n(x_1,\dots,x_i)$ can be constructed such that $r_i^n = r_i^n(\xi_1^n,\dots,\xi_i^n)$. Applying [6, Lemma 1.3], the discrete Wick difference equation (2.3) can be reformulated as

$$r_i^n = r_{i-1}^n\left(1+\frac{\mu}{n}+\sigma\left(B^{H,n}_{\frac{i}{n}} - B^{H,n}_{\frac{i-1}{n}}\right)\right) - \sigma\frac{1}{n}\sum_{j=1}^{i-1}\left(b^n_{\frac{i}{n},j} - b^n_{\frac{i-1}{n},j}\right)D_j^n r_{i-1}^n, \tag{2.4}$$

where

$$D_i^n r_{i-1}^n = \frac{r_{i-1}^n\left(\xi_1^n,\dots,\xi_{i-1}^n,1,\dots,\xi_n^n\right) - r_{i-1}^n\left(\xi_1^n,\dots,\xi_{i-1}^n,-1,\dots,\xi_n^n\right)}{2/\sqrt{n}} \tag{2.5}$$

is discrete Malliavin derivative with respect to the increments $\frac{1}{\sqrt{n}}\xi_i, i=1,\dots,n$,

$$r_i^n = r_{i-1}^n\left(1+\frac{\mu}{n}+\sigma\left(B^{H,n}_{\frac{i}{n}} - B^{H,n}_{\frac{i-1}{n}}\right)\right).$$

3. Main results

3.1. **Lower bound and confident intervals.** Below are the detailed steps to estimate the lower bound of an interest rate under fractional Black-Scholes model.

Step 1. Simulate n-tuple of independent symmetric Bernoulli random variables

$$\begin{gathered}
\xi_0(k_1) = \left(\xi_0^1(k_1),\dots,\xi_0^n(k_1)\right), \ k_1 = 1,\dots,m,\\
\xi_1(k_1,k_2) = \left(\xi_1^1(k_1,k_2),\dots,\xi_1^n(k_1,k_2)\right), \ k_2 = 1,\dots,m,\\
\vdots\\
\xi_{N-1}(k_1,\dots,k_{N-1}) = \left(\xi_{N-1}^1(k_1,\dots,k_{N-1}),\dots,\xi_{N-1}^n(k_1,\dots,k_{N-1})\right),\\
k_{N-1} = 1,\dots,m,
\end{gathered}$$

with

$$P\left(\xi_i^j(\cdot)=1\right) = P\left(\xi_i^j(\cdot)=-1\right) = \frac{1}{2}, \ i=0,\dots,N-1, \quad j=1,\dots,n.$$

Step 2. For each epoch i and j, compute $b_{i,p}^{j,q}$ by

$$b_{i,p}^{j,q} = n\int_{t_p+\frac{q-1}{n}}^{t_p+\frac{q}{n}} z_H\Big(t_i+\frac{jT}{nN},s\Big)ds, \quad p=0,\dots,i, \ q=1,\dots,n.$$

Step 3. For $i=0$, compute $B_0^j(k_1)$ and $r_0^j(k_1)$ by

$$\begin{gathered}
B_0^j(k_1) = \sum_{q=1}^{j} b_{0,0}^{j,q}\frac{1}{\sqrt{n}}\xi_0^q(k_1),\\
r_0^j(k_1) = r_0^{j-1}(k_1)\Big(1+\frac{\mu T}{nN}+\sigma\left(B_0^j(k_1)-B_0^{j-1}(k_1)\right)\Big) - \frac{\sigma T}{nN}\sum_{q=1}^{j-1}\left(b_{0,0}^{j,q}-b_{0,0}^{j,q-1}\right)D_q^n r_0^{j-1}(k_1),\\
B_0^0(k_1)=0, \quad r_0^0(k_1)=s_0.
\end{gathered}$$

Step 4. For $1 \le i \le N-1$, compute $B_i^j(k_0,\dots,k_i)$ and $r_i^j(k_0,\dots,k_i)$ by

$$\begin{gathered}
B_i^j(k_0,\dots,k_i) = \sum_{p=0}^{i-1}\sum_{q=1}^{n} b_{i,p}^{j,q}\frac{\xi_p^q(k_0,\dots,k_p)}{\sqrt{(i+1)n}} + \sum_{q=1}^{j} b_{i,i}^{j,q}\frac{\xi_i^q(k_0,\dots,k_i)}{\sqrt{(i+1)n}},\\
r_i^j(k_0,\dots,k_i) = r_i^{j-1}(k_0,\dots,k_i)\Big(1+\frac{\mu T}{nN}+\sigma\Big(B_i^j(k_0,\dots,k_i)-B_i^{j-1}(k_0,\dots,k_i)\Big)\Big)\\
-\frac{\sigma T}{nN}\sum_{q=1}^{j-1}\Big(b_{i,i}^{j,q}-b_{i,i}^{j,q-1}\Big)D_{i-1}^q r_i^{j-1}(k_0,\dots,k_i),\\
B_i^0(k_0,\dots,k_i) = B_{i-1}^n(k_0,\dots,k_{i-1}),\\
r_i^0(k_0,\dots,k_i) = r_{i-1}^n(k_0,\dots,k_{i-1}).
\end{gathered} \tag{3.1}$$

For $i=N$, let

$$r_N^0(k_0,\dots,k_{i-1}) = r_{N-1}^n(k_0,\dots,k_{i-1}).$$

Step 5. For each path and epoch, compute the call value of the zero-coupon bond,

$$h_i\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big) = p_{t_{i-1}}\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big)$$

Step 6. For each path, compute the continuation value C_i^* of the bond, defined as the present value of the expected one-period-ahead bond value,

$$C_i^*\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big) = \frac{1}{m}\sum_{l=1}^{m} e^{-r_iT/N}V_{i+1}^*\left(r_i^n(k_0,\dots,k_{i-1},l)\right),$$

where $\Delta t_i = t_{i+1}-t_i$ and V_i^* is defined in Step 8 below,

$$\begin{gathered}
V_N^*\big(r_{N-1}^n(k_0,\dots,k_{N-1})\big) = h_N\big(r_{N-1}^n(k_0,\dots,k_{N-1})\big),\\
C_N^*\big(r_{N-1}^n(k_0,\dots,k_{N-1})\big) = 0.
\end{gathered}$$

Step 7. For each path, define the tentative redeem-or-issue indicator variable $x_i(k_0,\dots,k_{i-1})$,

$$x_i(k_0,\dots,k_{i-1}) = \begin{cases} 1, & \text{if } h_i\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big) \ge C_i^*\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big),\\ 0, & \text{if } h_i\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big) < C_i^*\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big).\end{cases}$$

Step 8. For each path k, define the current value $V_i^*(k_0,\dots,k_{i-1})$ of the bond

$$V_i^*(k_0,\dots,k_{i-1}) = \begin{cases} h_i(k_0,\dots,k_{i-1}) & \text{if } x_i(k_0,\dots,k_{i-1}) = 1,\\ C_i^*(k_0,\dots,k_{i-1}) & \text{if } x_i(k_0,\dots,k_{i-1}) = 0.\end{cases}$$

Step 9. If $i>0$, then set $t_i = t_{i-1}$ and return to Step 6, otherwise stop the iterations and compute the redeem-or-issue indicator variable $y_i(k_0,\dots,k_{i-1})$ and optimal stopping time $\hat\tau(k_0,\dots,k_{N-1})$

$$y_i(k_0,\dots,k_{i-1}) = \begin{cases} 1 & \text{if } x_i(k_0,\dots,k_{i-1}) = 1 \text{ and } x_j(k_0,\dots,k_{j-1}) = 0 \text{ for all } j<i,\\ 0 & \text{otherwise.}\end{cases}$$

$$\hat\tau(k_0,\dots,k_{N-1}) = \begin{cases} i & \text{if } y_i(k_0,\dots,k_{i-1}) = 1,\\ N & \text{otherwise.}\end{cases}$$

Step 10. The price of the zero-coupon callable bond is estimated by

$$\hat V = \frac{1}{m^N}\sum_{k_0=0}^{m}\sum_{k_1=0}^{m}\cdots\sum_{k_N-1=0}^{m}\sum_{i=0}^{N} e^{-r_{i-1}t_i} y_i(k_0,\dots,k_{i-1})\, h_i\big(r_{i-1}^n(k_0,\dots,k_{i-1})\big). \tag{3.2}$$

For establishing profs of convergence, we use the following assumptions.

(A1) For $i \in \{0, \dots, N-1\}$, we have
$$E[h_i^2] < \infty, \text{ and } E[\big(r_{i-1}^n\big)^2] < \infty.$$
(A2) $P[C_i = h_i] = 0$ for $i = 1, \dots, N-1$.

Let $F_i\,(x \mid \mathcal{F}_i)$ be the conditional probability of $V_{i+1}\left(r_{i+1}^0\right)$ given $\xi_0, \dots, \xi_{i-1}$, that is,
$$F_i\,(x \mid \mathcal{F}_i) = P[V_{i+1}\,(r_i^n) \le x \mid \xi_0, \dots, \xi_{i-1}], \quad i = 0, 1, \dots, N-1.$$
Let the estimator for $F_i\,(x \mid \mathcal{F}_i)$ be defined by
$$\hat{F}_i^m\,(x \mid \mathcal{F}_i) = \frac{N\,(x, \xi_0, \dots, \xi_{i-1})}{m},$$
where $N\,(x, \xi_0, \dots, \xi_{i-1})$ is the number of path index such that
$$r_{i+1}^0\,(k_0, \dots, k_{i-1}, l) \le x.$$

Lemma 3.1. *For $i = 0, 1, \dots, N-1$, we have that*
$$\lim_{m\to\infty} P\Big(\sup_{-\infty<x<\infty} \big|\hat{F}_i^m\,(x \mid \mathcal{F}_i) - F_i\,(x \mid \mathcal{F}_i)\big| = 0\Big) = 1.$$

Proof. The proof is based on the Glivenko-Cantelli theorem. For any positive integer c, take $x_{c,k}$ to be the smallest x that satisfies
$$F_i\,(x-0 \mid \mathcal{F}_i) = F_i\,(x \mid \mathcal{F}_i) \le \frac{k}{c} \le F_i\,(x+0 \mid \mathcal{F}_i)\,, \quad k = 1, 2, \dots, c.$$
By the Law of Large Numbers and (2.4),
$$P\Big(\lim_{m\to\infty} \hat{F}_i^m(x_{c,k} \mid \mathcal{F}_i) = F_i(x_{c,k} \mid \mathcal{F}_i)\Big) = 1,$$
$$P\Big(\lim_{m\to\infty} \hat{F}_i^m\,(x_{c,k}+0 \mid \mathcal{F}_i) = F_i\,(x_{c,k}+0 \mid \mathcal{F}_i)\Big) = 1.$$
Let
$$\begin{aligned}
A_k^c &= \Big\{\lim_{m\to\infty} \hat{F}_i^m\,(x_{c,k} \mid \mathcal{F}_i) = F_i\,(x_{c,k} \mid \mathcal{F}_i)\Big\} \\
&= \Big\{\lim_{m\to\infty} \Big|\hat{F}_i^m\,(x_{c,k} \mid \mathcal{F}_i) - F_i\,(x_{c,k} \mid \mathcal{F}_i)\Big| = 0\Big\}\,, \\
B_k^c &= \Big\{\lim_{m\to\infty} \hat{F}_i^m\,(x_{c,k}+0 \mid \mathcal{F}_i) = F_i\,(x_{c,k}+0 \mid \mathcal{F}_i)\Big\} \\
&= \Big\{\lim_{m\to\infty} \Big|\hat{F}_i^m\,(x_{c,k}+0 \mid \mathcal{F}_i) - F_i\,(x_{c,k}+0 \mid \mathcal{F}_i)\Big| = 0\Big\}\,,
\end{aligned}$$
$$A^c = \cap_{k=1}^c \left(A_k^c \cap B_k^c\right) = \Big\{\lim_{m\to\infty} \max_{1\le k\le c} \Big\{\max\Big(\Big|\hat{F}_i^m\,(x_{c,k} \mid \mathcal{F}_i) - F_i\,(x_{c,k} \mid \mathcal{F}_i)\Big|,$$
$$\Big|\hat{F}_i^m\,(x_{c,k}+0 \mid \mathcal{F}_i) - F_i\,(x_{c,k}+0 \mid \mathcal{F}_i)\Big|\Big)\Big\} = 0\Big\}\,,$$
$$A = \cap_{c=1}^\infty A^c.$$
Obviously, $P\left(A_k^c\right) = P\left(B_k^c\right) = 1$, and
$$P\left(\overline{A^c}\right) = P\Big(\cup_{k=1}^c \left(\overline{A_k^c} \cup \overline{B_k^c}\right)\Big) \le \sum_{k=1}^c \left(P\left(\overline{A_k^c}\right) + P\left(\overline{B_k^c}\right)\right) = 0.$$
Thus,
$$P(\overline{A}) = P\Big(\cup_{c=1}^\infty \overline{A^c}\Big) = P\Big(\lim_{n\to\infty} \cup_{c=1}^n \overline{A^c}\Big) \le \lim_{n\to\infty} \sum_{c=1}^n P\left(\overline{A^c}\right) = 0.$$
For $k = 0, 1, 2, \dots, c$,
$$\hat{F}_i^m\,(x \mid \mathcal{F}_i) - F_i\,(x \mid \mathcal{F}_i) \le \hat{F}_i^m\,(x_{c,k+1} \mid \mathcal{F}_i) - F_i\,(x_{c,k}+0 \mid \mathcal{F}_i) \tag{3.3}$$
$$\le \max_{1\le k\le c} \Big|\hat{F}_i^m\,(x_{c,k} \mid \mathcal{F}_i) - F_i\,(x_{c,k} \mid \mathcal{F}_i)\Big| + \frac{1}{c} \tag{3.4}$$
$$F_i\,(x \mid \mathcal{F}_i) - \hat{F}_i^m\,(x \mid \mathcal{F}_i) \le F_i\,(x_{c,k+1} \mid \mathcal{F}_i) - \hat{F}_i^m\,(x_{c,k}+0 \mid \mathcal{F}_i) \tag{3.5}$$

$$\leq \frac{1}{c} + \max_{1\leq k\leq c}\Big|F_i\left(x_{c,k}+0 \mid \mathcal{F}_i\right) - \hat{F}_i^m\left(x_{c,k}+0 \mid \mathcal{F}_i\right)\Big|. \tag{3.6}$$

This completes the proof by letting c and m approach infinity. ◻

We define

$$\widetilde{C}_i^m\left(r_{i-1}^n\left(k_0,\dots,k_{i-1}\right)\right) = \frac{e^{-rT/N}}{m}\sum_{l=1}^{m} V_{i+1}\left(r_i^n\left(k_0,\dots,k_{i-1},l\right)\right).$$

Lemma 3.2. *Assume that the prices of asset are bounded almost surely. Then we have*

$$\lim_{m\to\infty} E\left[\left|\widetilde{C}_i^m\left(r_{i-1}^n\right) - C_i\left(r_{i-1}^n\right)\right|\right] = 0$$

for $i=0,1,\dots,N-1$.

Proof. From (2.1), obviously, $\widetilde{C}_i^m\left(r_{i-1}^n\right)$ and $C_i\left(r_{i-1}^n\right)$ are all bounded for $m=1,2,\dots,$ $i=1,\dots,N$. We define

$$M = \max\left\{\max_i C_i\left(r_{i-1}^n\right),\ \max_{i,m}\widetilde{C}_i^m\left(r_{i-1}^n\right)\right\}.$$

From the definition of $\widetilde{C}_i^m$,

$$\begin{aligned}
E\left[\left|\widetilde{C}_i^m\left(r_{i-1}^n\right) - C_i\left(r_{i-1}^n\right)\right| \mid \mathcal{F}_i\right] &= E\Big[\Big|\int_0^M x d\hat{F}_i^m\left(x \mid \mathcal{F}_i\right) - \int_0^M x dF_i\left(x\mid\mathcal{F}_i\right)\Big|\Big] \\
&\leq M\int_0^M E\Big[\Big|\hat{F}_i^m\left(x\mid\mathcal{F}_i\right) - F_i\left(x\mid\mathcal{F}_i\right)\Big|\Big]dx.
\end{aligned}$$

By Lemma 3.1,

$$\lim_{m\to\infty} E\left[\left|\hat{F}_i^m\left(x\mid\mathcal{F}_i\right) - F_i\left(x\mid\mathcal{F}_i\right)\right|\right] = 0.$$

Hence, by dominated convergence theorem, we complete the proof. ◻

Lemma 3.3. *Under assumptions of Lemma 3.2, for* $i=0,1,\dots,N-1$,

$$\lim_{m\to\infty} E\left[\left|C_i^*\left(r_{i-1}^n\right) - \widetilde{C}_i^m\left(r_{i-1}^n\right)\right|\right] = 0, \tag{3.7}$$

and therefore,

$$\lim_{m\to\infty} E\left[\left|C_i^*\left(r_{i-1}^n\right) - C_i\left(r_{i-1}^n\right)\right|\right] = 0. \tag{3.8}$$

Proof. The proof uses a recursive method. For $i=N-1$, $C_N^*\left(r_{N-1}^n\right) = \widetilde{C}_N^m\left(r_{N-1}^n\right) = C_N\left(r_{N-1}^n\right) = 0$,

$$\begin{aligned}
C_{N-1}^*\left(r_{N-2}^n\right) &= \frac{e^{-rT/N}}{m}\sum_{l=1}^{m} V_N^*\left(r_{N-1}^n\left(k_0,\dots,l\right)\right) \\
&= \frac{e^{-rT/N}}{m}\sum_{l=1}^{m}\max\left\{h_N\left(r_{N-1}^n\left(k_0,\dots,l\right)\right),\ C_N^*\left(r_{N-1}^n\left(k_0,\dots,l\right)\right)\right\} \\
&= \frac{e^{-rT/N}}{m}\sum_{l=1}^{m}\max\left\{h_N\left(r_{N-1}^n\left(k_0,\dots,l\right)\right),\ C_N\left(r_{N-1}^n\left(k_0,\dots,l\right)\right)\right\} \\
&= \widetilde{C}_{N-1}^m\left(r_{N-2}^n\right).
\end{aligned}$$

Hence, (3.7) and (3.8) hold for $i=N-1$. For $i=N-2$,

$$\begin{aligned}
&C_{N-1}^*\left(r_{N-2}^n\left(k_0,\dots,l\right)\right), \quad l=1,\dots,m, \\
&C_{N-1}\left(r_{N-2}^n\left(k_0,\dots,l\right)\right), \quad l=1,\dots,m,
\end{aligned}$$

are independent and identically distributed, respectively. Moreover,

$$E\left[\left|C_{N-2}^*\left(r_{N-3}^n\right) - \widetilde{C}_{N-2}^m\left(r_{N-3}^n\right)\right| \mid \mathcal{F}_{N-2}\right]$$

$$
\begin{aligned}
&= E\Big[\Big|\frac{e^{-rT/N}}{m}\sum_{l=1}^{m}\max\left\{h_{N-1}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right),\ C_{N-1}^{*}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right)\right\}\\
&\quad -\frac{e^{-rT/N}}{m}\sum_{l=1}^{m}\max\left\{h_{N-1}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right),\ C_{N-1}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right)\right\}\Big| \mid \mathcal{F}_{N-2}\Big]\\
&\le \frac{e^{-rT/N}}{m}\sum_{l=1}^{m}E\left[\left|\max\left\{h_{N-1}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right),\ C_{N-1}^{*}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right)\right\}\right.\right.\\
&\quad \left.\left.-\max\left\{h_{N-1}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right),\ C_{N-1}\left(r_{N-2}^{n}\left(k_0,\dots,l\right)\right)\right\}\right| \mid \mathcal{F}_{N-2}\right]\\
&= e^{-rT/N}E\left[\left|\max\left\{h_{N-1}\left(r_{N-2}^{n}\right),\ C_{N-1}^{*}\left(r_{N-2}^{n}\right)\right\}\right.\right.\\
&\quad \left.\left.-\max\left\{h_{N-1}\left(r_{N-2}^{n}\right),\ C_{N-1}\left(r_{N-2}^{n}\right)\right\}\right| \mid \mathcal{F}_{N-2}\right]\\
&\le e^{-rT/N}E\left[\left|C_{N-1}^{*}\left(r_{N-2}^{n}\right)-C_{N-1}\left(r_{N-2}^{n}\right)\right| \mid \mathcal{F}_{N-2}\right],
\end{aligned}
$$

Taking conditional expectation, we have that

$$
E\left[\left|C_{N-2}^{*}\left(r_{N-3}^{n}\right)-\widetilde{C}_{N-2}^{m}\left(r_{N-3}^{n}\right)\right|\right] \le e^{-rT/N}E\left[\left|C_{N-1}^{*}\left(r_{N-2}^{n}\right)-C_{N-1}\left(r_{N-2}^{n}\right)\right|\right]
$$

Hence, (3.7) and (3.8) hold for $i=N-2$. The proof is completed by recurrence. □

Lemma 3.4. *Under Assumption* (A1)*, without imposing boundedness condition, we have that*

$$
\lim_{m\to\infty}E\left[\left|C_i^{*}\left(r_{i-1}^{n}\right)-C_i\left(r_{i-1}^{n}\right)\right|\right]=0,\quad i=0,1,\dots,N-1.
$$

Proof. Define the event

$$
\mathcal{F}_M=\left\{\left|h_i\left(\hat{r}_{t_i}\right)\right|\le M, \|\hat{r}_{t_i}\|_2\le M, i=1,\dots,N\right\},\quad M>0.
$$

From Assumption (A1),

$$
\lim_{M\to\infty}P\left[\mathcal{F}_M^c\right]=0,\quad \max_i E\left[C_i\left(r_{i-1}^{n}\right)^2\right]<\infty,\quad \max_{i,m}E\left[\widetilde{C}_i^{m}\left(r_{i-1}^{n}\right)^2\right].
$$

Hence,

$$
\begin{aligned}
&E\left[\left|C_i^{*}\left(r_{i-1}^{n}\right)-C_i\left(r_{i-1}^{n}\right)\right|\right]\\
&=P\left[\mathcal{F}_M\right]E\left[\left|C_i^{*}\left(r_{i-1}^{n}\right)-C_i\left(r_{i-1}^{n}\right)\right|\mathcal{F}_M\right]+E\left[\left|C_i^{*}\left(r_{i-1}^{n}\right)-C_i\left(r_{i-1}^{n}\right)\right|I\left(\mathcal{F}_M^c\right)\right]\\
&\le P\left[\mathcal{F}_M\right]E\left[\left|C_i^{*}\left(r_{i-1}^{n}\right)-C_i\left(r_{i-1}^{n}\right)\right|\mathcal{F}_M\right]+\left(E\left[\left|C_i^{*}\left(r_{i-1}^{n}\right)-C_i\left(r_{i-1}^{n}\right)\right|^2\right]\right)^{1/2}P\left[\mathcal{F}_M^c\right]^{1/2}.
\end{aligned}
$$

Let $M\to\infty$, and the Lemma 3.4 is proved by Lemma 3.3. □

Lemma 3.5. *Under Assumptions* (A1) *and* (A2)*, we have that*

$$
\lim_{m\to\infty}P\left[\hat{\tau}_m\neq\tau\right]=0.
$$

Proof. The proof is based on the work of Broadie and Glasserman [7].

$$
\begin{aligned}
P\left[\hat{\tau}_m\neq\tau\right]&=P\left[\hat{\tau}_m<\tau\right]+P\left[\hat{\tau}_m>\tau\right]\\
&=P\left\{\exists i,\ \text{s.t. } C_i^{*}\left(r_{i-1}^{n}\right)\le h_i\left(r_{i-1}^{n}\right)<C_i\left(r_{i-1}^{n}\right)\right\}\\
&\quad +P\left\{\exists i,\ \text{s.t. } C_i^{*}\left(r_{i-1}^{n}\right)>h_i\left(r_{i-1}^{n}\right)\ge C_i\left(r_{i-1}^{n}\right)\right\}\\
&\le\sum_{i=0}^{N-1}P\left\{C_i^{*}\left(r_{i-1}^{n}\right)\le h_i\left(r_{i-1}^{n}\right)<C_i\left(r_{i-1}^{n}\right)\right\}\\
&\quad +\sum_{i=0}^{N-1}P\left\{C_i^{*}\left(r_{i-1}^{n}\right)>h_i\left(r_{i-1}^{n}\right)\ge C_i\left(r_{i-1}^{n}\right)\right\}.
\end{aligned}
$$

From (A1), for any $\epsilon>0$, there exists $\gamma>0$ such that

$$
P\left[\left|h_i-C_i\right|\ge\gamma\right]\le\epsilon.
$$

Thus,

$$P\left[\hat{\tau}_m \neq \tau\right] \leq N\epsilon + \sum_{i=0}^{N-1} P\left[\left|C_i^*\left(r_{i-1}^n\right) - C_i\left(r_{i-1}^n\right)\right| \geq \gamma\right].$$

By Lemma 3.4, it can be proved that $\lim_{m\to\infty} P\left[\hat{\tau}_n \neq \tau\right] = 0$, since ϵ is an arbitrary positive number. □

Theorem 3.6. *Under Assumptions* (A1) *and* (A2)*, we have that*

$$\lim_{m\to\infty} E\left[\left|\hat{V} - V_0\right|\right] = 0.$$

Proof. From (A1), by the Law of Large Numbers and dominated convergence theorem,

$$\lim_{m\to\infty} E\Big[\Big|\frac{1}{m^N}\sum_{k_0=0}^{m}\sum_{k_1=0}^{m}\cdots\sum_{k_N-1=0}^{m} e^{-r\tau}h_\tau\left(r_{\tau-1}^n\right) - V_0\left(r_0\right)\Big|\Big] = 0.$$

Therefore, all we need to prove is that

$$\lim_{m\to\infty} E\Big[\Big|\frac{1}{m^N}\sum_{k_0=0}^{m}\sum_{k_1=0}^{m}\cdots\sum_{k_N-1=0}^{m} e^{-r\tau}h_\tau\left(r_{\tau-1}^n\right) - \hat{V}\left(r_0\right)\Big|\Big] = 0.$$

According to (3.2), $\hat{V}$ can be rewritten as

$$\hat{V} = \frac{1}{m^N}\sum_{k_0=0}^{m}\sum_{k_1=0}^{m}\cdots\sum_{k_N-1=0}^{m} e^{-r\hat{\tau}}h_{\hat{\tau}}\left(r_{\hat{\tau}-1}^n\right).$$

Hence,

$$\begin{aligned}
&E\Big[\Big|\frac{1}{m^N}\sum_{k_0=0}^{m}\sum_{k_1=0}^{m}\cdots\sum_{k_N-1=0}^{m} e^{-r\tau}h_\tau\left(r_{\tau-1}^n\right) - \hat{V}\left(r_0\right)\Big|\Big]\\
&= E\Big[\Big|\frac{1}{m^N}\sum_{k_0=0}^{m}\sum_{k_1=0}^{m}\cdots\sum_{k_N-1=0}^{m}\left(e^{-r\tau}h_\tau\left(r_{\tau-1}^n\right) - e^{-r\hat{\tau}}h_{\hat{\tau}}\left(r_{\hat{\tau}-1}^n\right)\right)I\left(\hat{\tau}\neq\tau\right)\Big|\Big]\\
&\leq E\left[\left|\left(e^{-r\tau}h_\tau\left(r_{\tau-1}^n\right) - e^{-r\hat{\tau}}h_{\hat{\tau}}\left(r_{\hat{\tau}-1}^n\right)\right)I\left(\hat{\tau}\neq\tau\right)\right|\right]\\
&\leq \left(E\left[\left(e^{-r\tau}h_\tau\left(r_{\tau-1}^n\right) - e^{-r\hat{\tau}}h_{\hat{\tau}}\left(r_{\hat{\tau}-1}^n\right)\right)^2\right]\right)^{1/2}\left(P\left(\hat{\tau}\neq\tau\right)\right)^{1/2}.
\end{aligned}$$

Letting $m\to\infty$, the theorem is proved by Lemma 3.5. □

Remark 3.7. Based on the dual formulation proposed by Rogers [19], we can solve the upper bound estimate for zero-coupon bonds. Since calculating the upper bound is more complex, we will address it in future research by providing the point estimate of the bound along with its 95% confidence interval. For now, we only present the expression for the point estimate. The point estimate of V_0 is $\frac{\hat{L}+\hat{U}}{2}$.

4. Numerical simulations

Reducing the dimensionality parameter n amplifies volatility and upward drift in simulated interest rates (r_t), increasing the likelihood of early redemption. Bonds are called sooner, shortening their effective duration and reducing present values, thereby depressing prices. Thus, spatial discretization directly impacts pricing through its effect on rate dynamics.

The parameter m, governing the number of independent Bernoulli paths, exerts a modest influence. Smaller m values slightly elevate prices by concentrating the distribution of the random variable B, reducing extreme rate paths and thus redemption events. However, this effect is secondary to the Monte Carlo sample size, which primarily determines pricing precision.

Lower volatility (σ) stabilizes interest rate paths, reducing both rate spikes that trigger early redemption and deep troughs that amplify discounting. High-volatility regimes $(\sigma = 0.4)$ shorten bond durations and lower prices, whereas low-volatility environments $(\sigma = 0.1)$ allow bonds to remain outstanding longer, increasing valuations.

TABLE 1. Parameter settings, mean of lower bound estimate, 95% confidence interval and time elapsed.

Para.\ Case	(a)	(b)	(c)	(d)	(e)	(f)
n	256	256	256	256	128	256
m	10000	10000	10000	10000	10000	1000
μ	0.1	0.1	0.1	0.1	0.1	0.1
σ	0.4	0.4	0.4	0.1	0.4	0.4
T	1	1	1	1	1	1
N	4	8	4	4	4	4
r_0	0.02	0.02	0.05	0.02	0.02	0.02
$\hat{L}$	74.6926	74.0934	74.0590	91.8860	74.4922	74.7357
95% CI	[74.6849, 74.7004]	[74.0872, 74.0996]	[74.0520, 74.0661]	[91.8833, 91.8888]	[74.4850, 74.4993]	[74.7287, 74.7427]
t_L(h)	3.57	10.47	3.23	3.22	0.82	3.22

Increasing the epoch N improves the detection of redemption opportunities but substantially raises computational costs. Finer discretization leads to more frequent call optimization, truncating cash flows and exerting downward pressure on bond prices. Empirical comparisons confirm that higher N values lower prices while extending run times.

The lower bound of bond price decreases (increases) monotonically with higher (lower) initial interest rates. In a high-interest rate environment, issuers are more likely to meet the redemption threshold conditions. In real markets, high interest rates are typically associated with declines in bond prices.

5. Conclusions

This paper addresses the pricing of callable bonds and the determination of optimal callable time within the framework of the fractional Black-Scholes model introduced by [3]. To handle the path-dependence of the asset dynamics under this model, we discretize the continuous asset price process and develop a method for estimating both the continuation value and the optimal callable time. We establish convergence results to ensure the reliability of the estimation, and numerical experiments are conducted to demonstrate the practical effectiveness of the proposed approach.

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