Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 10, pp. 1–22.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.10

EXISTENCE AND BOUNDEDNESS OF THE CLASSICAL GLOBAL SOLUTION TO 2D CHEMOTAXIS-FLUID SYSTEM WITH DOUBLE CHEMICAL SIGNALS

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Abstract. This article concerns the chemotaxis-fluid system with double chemical signals,

$$\begin{gathered} n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(n\nabla c) + \xi\nabla\cdot(n\nabla v) + f(n), \quad x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, \quad x\in\Omega,\ t>0,\\ v_t + u\cdot\nabla v = \Delta v - v + n, \quad x\in\Omega,\ t>0,\\ u_t = \Delta u + \nabla P + n\nabla\phi, \quad x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, \quad x\in\Omega,\ t>0, \end{gathered}$$

in a smooth bounded domain $\Omega\subset\mathbb{R}^2$ with no-flux/no-flux/no-flux/no-slip boundary conditions. Here $f(n)$ is a given sub-logistic source function satisfying $f(s)\geq\zeta s$ for small $s\geq 0$, where $\zeta\in\mathbb{R}$, and the growth condition

$$\liminf_{s\to\infty}\big\{-f(s)\frac{\ln s}{s^2}\big\} = \mu\in(0,\infty].$$

We obtain the existence and uniform-in-time boundedness of classical global solutions to the corresponding initial-boundary value problem. We assume that the initial data and the physical coefficients satisfy

$$\Big(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)}+4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)}-\mu\Big)^+ < \frac{1}{2C_{\rm GN}^4 M},$$

where M is a constant,

$$M := \|n_0\|_{L^1(\Omega)} + |\Omega|\inf_{\eta>0}\frac{\sup\{f(s)+\eta s : s>0\}}{\eta},$$

$C_{\rm GN}>0$ is the Gagliardo-Nirenberg constant, and $C_{\rm p}$ the Poincaré constant. This result reveals that the sub-logistic source plays a crucial role in preventing blow-up phenomena within the chemotaxis-fluid system with double chemical signals.

1. Introduction

This article concerns the initial-boundary value problem of the chemotaxis-fluid system with double chemical signals and sub-logistic source, which is formulated as follows:

$$\begin{gathered} n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(n\nabla c) + \xi\nabla\cdot(n\nabla v) + f(n), \quad x\in\Omega, t>0,\\ c_t + u\cdot\nabla c = \Delta c - nc, \quad x\in\Omega, t>0,\\ v_t + u\cdot\nabla v = \Delta v - v + n, \quad x\in\Omega, t>0,\\ u_t = \Delta u + \nabla P + n\nabla\phi, \quad \nabla\cdot u = 0, \quad x\in\Omega, t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0,\ u=0, \quad x\in\partial\Omega, t>0,\\ n(x,0)=n_0(x),\ c(x,0)=c_0(x),\ v(x,0)=v_0(x),\ u(x,0)=u_0(x), \quad x\in\Omega. \end{gathered} \tag{1.1}$$

Here, $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, and ν is the unit outward normal vector to $\partial\Omega$. The unknown function $n(x,t)$ denotes the density of cells, $c(x,t)$ the concentration

2020 *Mathematics Subject Classification.* 35K20, 35K55, 92C17.
Key words and phrases. Keller-Segel system; boundedness; global existence; sub-logistic source.
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Submitted October 21, 2025. Published February 10, 2026.

of oxygen, and $v(x,t)$ the concentration of the chemical signal secreted by the cells. $u(x,t)$ and $P(x,t)$ stand for the fluid velocity field and its associated pressure, respectively. In addition, ϕ is a given potential function satisfying

$$\phi \in C^{1+\kappa}(\Omega) \quad \text{for some } \kappa > 0, \tag{1.2}$$

and $f(n)$ is a sub-logistic source depending n, which satisfies the condition (1.8) below. The physical constants χ and ξ quantify the interaction strengths between the cell and its environment: $\chi > 0$ denotes oxygen as an attractor to the cell, while $\chi < 0$ signifies repulsion; $\xi > 0$ indicates that chemical signals secreted by the cell act as repellents, whereas $\xi < 0$ defines them as attractors.

In this article, we aim to prove the global boundedness of the classical solutions to (1.1) subject to the initial data (n_0, c_0, v_0, u_0) satisfying

$$\begin{gathered} 0 \le n_0 \in C(\bar{\Omega}), \quad 0 \le c_0, \quad v_0 \in W^{1,\infty}(\Omega), \\ u_0 \in D(\mathcal{A}^\alpha) \quad \text{for some } \alpha \in (\frac{1}{2}, 1), \end{gathered} \tag{1.3}$$

where $\mathcal{A}$ is the Stokes operator $\mathcal{A} := -\mathcal{P}\Delta$, with domain

$$D(\mathcal{A}) := W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega).$$

Here, $L^2_\sigma(\Omega)$ represents the space of square-integrable divergence-free vector fields,

$$L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0\},$$

and $\mathcal{P}$ stands for the Helmholtz projection from $L^2(\Omega)$ onto $L^2_\sigma(\Omega)$ (see [32]).

To state our main theorem, we first recall some key developments related to system (1.1) and its variations.

Keller-Segel-type chemotaxis system. Chemotaxis is a biological phenomenon characterized by the oriented movement of cells in response to chemical signals. The movement towards a higher concentration of the chemical substance is termed positive chemotaxis, and the movement towards regions of lower chemical concentration is called negative chemotaxis. The interesting phenomena in diverse biological processes has attracted a lot of attention from the biological and mathematical communities (see [10]). Among these chemotaxis models, the general minimal chemotaxis system

$$\begin{gathered} n_t = \Delta n - \xi \nabla \cdot (n \nabla v) + f(n), \quad x \in \Omega \subset \mathbb{R}^N,\ t > 0, \\ v_t = \Delta v - v + n, \quad x \in \Omega \subset \mathbb{R}^N,\ t > 0, \end{gathered} \tag{1.4}$$

has been extensively studied. For $f(n) = 0$, Keller and Segel [15, 16] first proposed system (1.4) in the 1970s. Since their classical work, the Keller-Segel model (1.4) and its variants have been widely investigated by numerous researchers, with substantial advances made in understanding the global existence and uniform boundedness of solutions. When the space dimension $N = 1$, all the solutions are global and bounded (see [27]). However, if $N = 2$ or $N \ge 3$, the solution to the system (1.4) exhibits a remarkable feature of finite/infinite time blow-up, which arises from the competitive interplay between the diffusion term and the aggregation term $-\nabla \cdot (n \nabla v)$. For $N = 2$, there exists a threshold: if the mass $\int_\Omega n_0$ is less than this threshold, the solution is global and bounded (see [6, 25]); conversely, if $\int_\Omega n_0$ exceeds this threshold, the system may admit a solution that blows up in finite/infinite time (see [11, 31]). For $N \ge 3$, if $\int_\Omega n_0 > 0$, the solutions to the system (1.4) blow up in finite time for some given radially symmetric positive initial data (see [8, 3, 24]). For more relevant results, we may refer to [4, 9, 42] and the references therein. On the other hand, the blow-up phenomena can be ruled out if a suitable logistic source $f(n)$ is introduced. Indeed, taking $f(n) = an - bn^2$ with $a, b \in \mathbb{R}$ in the system (1.4), all the solutions are global and uniformly bounded for arbitrary $b > 0$ on $N = 1, 2$ (see [14, 28]). For $N \ge 3$, Winkler [47] found that the Neumann boundary value problem of the system (1.4) admits a global bounded classical solution provided that $b > b_0$ for some sufficiently large constant $b_0 > 0$. Later, Xiang [51] improved the result in [47] by giving an explicit expression of the low bound as $b_0 = 9\chi/(\sqrt{10} - 2)$ in some non-convex domains. Recently, in the case of $f(n) = an - bn^\alpha$, Winkler [48] established the global existence of generalized solution for arbitrary $\alpha > 1$. For the sub-logistic source $f(n)$ satisfying (1.8) and (1.9), Xiang [50] proved that the sub-logistic source can prevent blow-up in

2D settings, which indicated that the logistic damping is not the weakest damping that ensures boundedness for the model (1.4).

Keller-Segel-(Navier-)Stokes system with production of chemosignal. Numerous cells in natural environments reside in viscous fluids and engage in significant mutual interactions with the surrounding fluid flow. Since fluid motion is governed by the incompressible (Navier-)Stokes equations, Tuval et al. [34] proposed a coupled chemotaxis-fluid model:

$$\begin{gathered} n_t + u\cdot\nabla n = \Delta n - \xi\nabla\cdot(n\nabla v) + f(n), \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\ v_t + u\cdot\nabla v = \Delta v - v + n, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi + g, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\ \nabla\cdot u = 0, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0. \end{gathered} \tag{1.5}$$

This model can be used to describe the dynamics of swimming aerobic bacteria, Bacillus subtilis, in water droplets. The coefficient $\kappa\geq 0$ is related to the intensity of nonlinear fluid convection. Specifically, $\kappa=0$ means that the fluid flows slowly, and the resulting system is called incompressible chemotaxis-Stokes system. If $f(n)=0$ and $g=0$ in (1.5), Li and Xiao [20] proved the global existence of the bounded classical solution provided that the initial mass $\|n_0\|_{L^1(\Omega)} < \frac{\sqrt{6}}{2C_{\rm GN}}$. Taking $f(n)=-n^2$ and $g=0$ in (1.5), Espejo and Suzuki [5] first gave the global existence of weak solution with $\kappa=0$ in the whole space $\mathbb{R}^2$. Jin [13] further derived the existence of strong solutions and analyzed their large-time behavior with $\kappa=1$ in a bounded domain $\Omega\subset\mathbb{R}^2$. In [53], Zhang obtained the existence and uniqueness of weak solutions with $\kappa=1$ in $\mathbb{R}^2$ by employing the Fourier localization technique.

Tao and Winkler [37] proposed a Keller-Segel-Navier-Stokes system of the form (1.5) with logistic damping $f(n)=rn-\mu n^2$ and $g\neq 0$ in a smooth bounded domain $\Omega\subset\mathbb{R}^3$. They proved that this system admits a global bounded classical solution provided that $\mu>23$ and $r\geq 0$. Moreover, they also gave the large-time behavior of the solutions: when $r=0$, $\|n\|_{L^\infty(\Omega)}\to 0$, $\|c\|_{L^\infty(\Omega)}\to 0$ and $\|u\|_{L^\infty(\Omega)}\to 0$ as $t\to 0$. Later, Winkler [49] showed that system (1.5) admits at least one global weak solution under the explicit hypothesis $\mu>\frac{\sqrt{r_+}}{4}$ ($r_+=\max\{0,r\}$). In addition, these solutions were proven to stabilize toward a spatially homogeneous steady state as time tends to infinity. For the 2D case, Tao and Winkler [36] found an interesting phenomenon that logistic damping $f(n)=rn-\mu n^2$ ($\mu>0$ and $r\geq 0$) can prevent the blow-up of the solutions. Furthermore, Jin and Xiang [13] refined the results in [36] by giving how the upper bounds of solutions qualitatively depend on χ and μ. They also derived the large-time asymptotic behavior of the solutions under the assumption $r=0$.

Keller-Segel-(Navier-)Stokes system with consumption of chemosignal. Another chemotaxis-(Navier-)Stokes system, different from the model (1.5), with consumption of chemosignal was proposed and is formulated as

$$\begin{gathered} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(\chi(c)n\nabla c) + f(n), \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - h(c)n, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0,\\ \nabla\cdot u = 0, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0, \end{gathered} \tag{1.6}$$

where the concentration-dependent function $h(c)$ represents the oxygen consumption rate. We first recall some results related to the system (1.6) in the case $f(n)=0$. In [4], it was demonstrated that the chemotaxis-Stokes system (1.6) admits globally defined classical solutions in the whole space $\mathbb{R}^3$, provided that the initial data $\|n_0,c_0,u_0\|_{H^3(\Omega)}$ is suitable small. Liu and Lord [21] continued to establish the global existence of the weak solution of the full chemotaxis-Navier-Stokes system only under certain structural conditions governing the relationship between $\chi(c)$ and $h(c)$ in the whole space $\mathbb{R}^2$. For $h(c)=c$, $\chi(c)=\chi>0$ and $\kappa=0$, the system (1.6) reduces to the chemotaxis-fluid model, which is proposed in [34]. Due to the dissipative structure $-nc$, a new energy functional that incorporates the logarithmic entropy $\int_\Omega n\ln n$ should be introduced.

This constitutes a crucial step in deriving the global dynamical behavior of solutions, and it applies equally to scenarios involving both the Stokes equation and the Navier-Stokes equation in [43, 44, 45, 46]. In a series of seminal works, Winkler investigated the case $\chi(c) = 1$ and $h(c) = c$, establishing the following sharp results: for $N = 2$, the full chemotaxis-Navier-Stokes system admits a unique global classical solution; for $N = 3$, the simplified chemotaxis-Stokes system (with $\kappa = 0$) possesses a global weak solution at least.

For the case $f(n) \neq 0$, $\chi(c) \equiv \chi > 0$ and $h(c) = c$, the model (1.6) has been studied by many authors. The results concerning the case where the source term takes the common logistic form $f(n) = n(1-n)$ are established in [1, 19, 39]. In particular, Baghaeia and Khelghatib [1] investigated the global existence and boundedness of solutions to the chemotaxis model under conditions $\chi > 0$ and

$$\|c_0\|_{L^\infty(\Omega)} \leq \frac{1}{\chi}\sqrt{\frac{d_1}{2(N+1)}}\Big[\pi - 2\arctan\frac{d_1-1}{2}\sqrt{\frac{2(N+1)}{d_1}}\Big].$$

Additionally, Xiang [50] demonstrated that a sub-logistic source is capable of preventing blow-up in two-dimensional settings. Ma [23] proved the global existence and uniform-in-time boundedness of the solutions of the 2D chemotaxis-fluid model (with Neumann initial-boundary conditions), where the source term $f(n)$ has weaker damping source than the standard logistic source.

Chemotaxis-fluid models with double chemical signals. In many natural biological processes, cells are influenced by more than one chemotactic signal, each of which may act as an attractant or a repellent, giving rise to a variety of intricate patterns (see [29]). In light of this, Kozono, Miura and Sugiyama [17] proposed a chemotaxis-(Navier-)Stokes system with two different types of chemical attractants, formulated as

$$\begin{gathered} n_t + u\cdot\nabla n = \Delta n - \chi\nabla\cdot(n\nabla c) + \xi\nabla\cdot(n\nabla v), \quad x\in\Omega\subset\mathbb{R}^N,\ t>0, \\ c_t + u\cdot\nabla c = \Delta c - nc, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0, \\ v_t + u\cdot\nabla v = \Delta v - v + n, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0, \\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0, \\ \nabla\cdot u = 0, \quad x\in\Omega\subset\mathbb{R}^N,\ t>0. \end{gathered} \tag{1.7}$$

This system presents a lot of the significant mathematical challenges due to its nature as a coupling of systems (1.5) and (1.6). In [17], by using the implicit function theorem, Kozono, Miura and Sugiyama constructed the unique global mild solution to the corresponding Cauchy problem in $\mathbb{R}^N$ $(N \geq 2)$ and its asymptotic behavior under some restrictive assumptions on the initial data. In the case where χ and ξ are positive constants, Ren [30] proved the global existence of the chemotaxis-Navier-Stokes system (1.7). Recently, Xie and Xu [52] established that the chemotaxis-Navier-Stokes system (1.7) admits a global classical solution under conditions $\chi > 0$, $\xi < 0$, $\|n_0\|_{L^1(\Omega)} < \min\{\frac{1}{4|\xi|C_{\rm GN}}, \frac{2\pi}{|\xi|}\}$ and $\|c_0\|_{L^\infty(\Omega)} < \frac{1}{\chi}$.

Inspired by the aforementioned works, we examine the chemotaxis-Stokes system with double chemical signals and sub-logistic source as presented in (1.1). We demonstrate that for any given real constant $\chi \in \mathbb{R}$ and $\xi \in \mathbb{R}$, system (1.1) admits a globally unique bounded classical solution. This result indicates that the sub-logistic source can prevent blow-up arising from the chemotaxis-Stokes system. Below, we state our main result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain, $\chi, \xi \in \mathbb{R}$, and let (n_0, c_0, v_0, u_0) satisfy* (1.3)*. Suppose further that the potential function ϕ satisfies* (1.2)*, and the source function $f \in W^{1,\infty}_{loc}(\mathbb{R}^+)$ satisfies $f(s) \geq \zeta s$ for small $s \geq 0$, where $\zeta \in \mathbb{R}$, and the growth condition*

$$\liminf_{s\to\infty}\Big(-f(s)\frac{\ln s}{s^2}\Big) = \mu \in (0,\infty]. \tag{1.8}$$

If it holds that

$$\Big(2 + \chi^2 + \xi^2 + 4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)} - \mu\Big)^+ < \frac{1}{2C_{\rm GN}^4 M}, \tag{1.9}$$

where $C_{\rm GN} > 0$ denotes the Gagliardo-Nirenberg constant and $C_{\rm p}$ the Poincaré constant for Ω, and

$$M := \|n_0\|_{L^1(\Omega)} + |\Omega| \inf_{\eta>0} \frac{\sup\{f(s)+\eta s : s>0\}}{\eta}, \tag{1.10}$$

then the initial-boundary problem (1.1)-(1.3) admits a unique global-in-time solution (n, c, v, u, P) satisfying

$$n, c, v, u \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), P \in C^{1,0}(\bar{\Omega} \times (0,\infty)). \tag{1.11}$$

Moreover, there exists a constant $C > 0$ independent of t such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \text{for all } t > 0. \tag{1.12}$$

Remark 1.2. For sub-logistic sources like $f(n) = n\big(a - \frac{bn}{\ln^\gamma(n+1)}\big)$ with $a \in \mathbb{R}, b > 0, \gamma \in (0,1)$ or $f(n) = n\big(a - \frac{bn}{\ln(\ln(n+e))}\big)$ with $a \in \mathbb{R}, b > 0$, one can easily compute from (1.8) that $\mu = +\infty$ and so (1.9) holds trivially.

To the best of our knowledge, Theorem 1.1 provides the first rigorous mathematical result concerning the double-chemical-signal chemotaxis-Stokes system (1.1) with a sub-logistic source. This theorem also extends the existing results, for instance, those in [23] where $v \equiv 0$ in (1.1), by providing an explicit condition (1.9) that quantifies the relationship between μ (characterizing the sub-logistic source) and the physical coefficients (e.g., χ, ξ).

In contrast to existing results for chemotaxis-Navier-Stokes systems with nonlinear diffusion (see [22]), where Δn is replaced by $\nabla \cdot (D(n)\nabla n)$ and $f(n) = 0$ (cf. (1.1)), our result in Theorem 1.1 does not rely on the signs of χ and ξ (under the condition (1.9)). Furthermore, we establish not only the global existence but also the uniform-in-time boundedness of the classical solution, which constitutes an improvement over the aforementioned works.

This article is organized as follows. In Section 2, we recall some well-known lemmas, facts and inequalities that will be frequently used in the subsequent proofs. In Section 3, we first derive a series of a priori estimates for the local solution (n, c, v, u) (whose existence is guaranteed by Lemma 2.7), and finally establish Theorem 1.1 by applying the continuity method.

2. Preliminaries

In this section, we recall some preliminary lemmas that will be used frequently hereafter. Firstly, we recall the well-known Gagliardo-Nirenberg inequality (see [26]).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $p \ge 1$, $r > 0$.*

(i) *For any $q \in (0, p)$, there exists a positive constant $C_{\rm GN} = C(p, r, \Omega)$ such that*

$$\|\varphi\|_{L^p(\Omega)} \le C_{\rm GN}(\|\nabla\varphi\|^\theta_{L^2(\Omega)}\|\varphi\|^{1-\theta}_{L^q(\Omega)} + \|\varphi\|_{L^r(\Omega)}) \quad \text{for all } \varphi \in H^1(\Omega) \cap L^q(\Omega), \tag{2.1}$$

where $\theta = 1 - \frac{q}{p} \in (0,1)$.

(ii) *For some $q, l, s \ge 1$, $j, m \in \mathbb{N}_0$ and $\theta \in [\frac{j}{m}, 1]$ satisfying*

$$\frac{1}{p} = \frac{j}{2} + \Big(\frac{1}{l} - \frac{m}{2}\Big)\theta + \frac{1-\theta}{q}.$$

Then there is a positive constant C such that

$$\|D^j\varphi\|_{L^p(\Omega)} \le C\big(\|D^m\varphi\|^\theta_{L^l(\Omega)}\|\varphi\|^{1-\theta}_{L^q(\Omega)} + \|\varphi\|_{L^s(\Omega)}\big) \tag{2.2}$$

holds for all $\varphi \in W^{m,l}(\Omega) \cap L^s(\Omega)$.

To apply the integrability condition $|n \ln n| \in L^1(\Omega)$ for deriving further estimates (see the proof of Lemma 3.4), we need the following logarithmic version of the generalized Gagliardo-Nirenberg inequality (see [35]).

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $q > 1$ with $r \in (0, q)$. Then, for each $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\|\varphi\|^q_{L^q(\Omega)} \le \varepsilon\|\nabla\varphi\|^{q-r}_{L^2(\Omega)} \||\varphi|\ln|\varphi|\|^r_{L^r(\Omega)} + C_\varepsilon\|\varphi\|^q_{L^r(\Omega)} + C_\varepsilon \tag{2.3}$$

holds for all $\varphi \in H^1(\Omega) \cap L^r(\Omega)$.

Next, we present two inequalities that are special cases of the trace theorem (see [18]).

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $\varphi \in C^2(\bar{\Omega})$ satisfy the Neumann boundary condition $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$.*

(i) *Then*
$$\frac{\partial}{\partial \nu}|\nabla \varphi|^2 \le \Re |\nabla \varphi|^2, \tag{2.4}$$
where $\Re$ is an upper bound on the curvature of $\partial\Omega$.

(ii) *Moreover, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that*
$$\|\nabla \varphi\|_{L^2(\partial\Omega)} \le \varepsilon \|\Delta \varphi\|_{L^2(\Omega)} + C_\varepsilon \|\varphi\|_{L^1(\Omega)}. \tag{2.5}$$

To derive some preliminary time-independent estimates, we recall an auxiliary lemma on boundedness of solutions to a linear differential inequalities, which was established in [33].

Lemma 2.4. *Let $T > 0$, $\tau \in (0, T)$, $a > 0$ and $b > 0$. Suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous and satisfies*
$$y'(t) + ay(t) \le h(t) \quad \text{for a.e. } t \in (0, T).$$
If the nonnegative function $h \in L^1_{loc}([0, T))$ satisfies
$$\int_t^{t+\tau} h(s)ds \le b \quad \text{for all } t \in [0, T - \tau),$$
the following uniform bound holds for all $t \in (0, T)$
$$y(t) \le \max\big\{y(0) + b, \frac{b}{a\tau} + 2b\big\}.$$

In view of the central importance of semigroups in the following analysis, we recall some key estimates for the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ in the following lemma, the proof of which can be found in [2, 12, 42].

Lemma 2.5. *Let $(e^{t\Delta})_{t>0}$ be the Neumann heat semigroup on a bounded smooth domain $\Omega \subset \mathbb{R}^2$, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary condition. Then, there exist two constants $N_1 > 0$, $N_2 > 0$, and $N_3 > 0$, depending only on Ω, such that:*

(i) *For each $1 \le q \le p \le \infty$, the estimate*
$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le N_1 \Big(1 + t^{-\frac{1}{2} - (\frac{1}{q} - \frac{1}{p})}\Big) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \tag{2.6}$$
holds for all $t > 0$ and for any $w \in L^q(\Omega)$.

(ii) *For each $2 \le p < \infty$, the estimate*
$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le N_2 e^{-\lambda_1 t} \|\nabla w\|_{L^p(\Omega)} \tag{2.7}$$
holds for all $t > 0$ and for any $w \in W^{1,p}(\Omega)$.

(iii) *For each $1 < q \le p \le \infty$, the estimate*
$$\|e^{t\Delta} \nabla \cdot w\|_{L^p(\Omega)} \le N_3 \Big(1 + t^{-\frac{1}{2} - (\frac{1}{q} - \frac{1}{p})}\Big) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \tag{2.8}$$
holds for all $t > 0$ and for any $w \in (W^{1,p}(\Omega))^n$.

Similarly, we need to introduce a semigroup to deal with the velocity u in our system (1.1). To this end, we first recall the Helmholtz projection operator $\mathcal{P}$, which maps the space $L^2(\Omega)$ onto its closed subspace
$$L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0 \quad \text{in } \mathcal{D}'(\Omega)\}.$$
The Stokes operator $\mathcal{A} := -\mathcal{P}\Delta$ with domain $D(\mathcal{A}) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^2_\sigma(\Omega)$ is sectorial in $L^2_\sigma(\Omega)$. Consequently, it generates the analytic Stokes contraction semigroup $(e^{-t\mathcal{A}})_{t\ge 0}$ and admits densely defined fractional powers $\mathcal{A}^\alpha$ for any $\alpha \in (0, 1)$ (see [7, 32]). Among the standard embedding and regularity estimates for $\mathcal{A}$, we highlight the following lemma, whose proof can be derived by a minor modification of the three-dimensional counterpart presented in [41, Lemma 3.1–3.3].

Lemma 2.6. *There exists a constant $\lambda > 0$ such that for all $\eta \geq 0$ and $t \geq 0$, the estimate*

$$\|\mathcal{A}^{\eta} e^{-t\mathcal{A}}\varphi\|_{L^p(\Omega)} \leq C(\eta) t^{-\eta} e^{-\lambda t}\|\varphi\|_{L^p(\Omega)} \tag{2.9}$$

holds for all $\varphi \in L^p_\sigma(\Omega)$.

Finally, we state the local solvability and extendibility criterion for the chemotaxis-growth system (1.1). These results can be rigorously established via an appropriate fixed-point argument combined with standard parabolic regularity theory (see, e.g., [38, 47]). For the sake of brevity, we omit the detailed proof.

Lemma 2.7. *Let $\chi, \xi \in \mathbb{R}$ be any given constants, and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Suppose that the initial data (n_0, c_0, v_0, u_0) satisfy* (1.3)*, the potential ϕ satisfies* (1.2)*, and the source function $f \in W^{1,\infty}_{loc}(\mathbb{R}^+)$. There exists a maximal existence time $T_{\max} \in (0, \infty]$ and a unique classical solution (n, c, v, u, P) to system* (1.1) *such that*

$$\begin{gathered} n \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ c \in \cap_{q>2} C^0([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ v \in \cap_{q>2} C^0([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})), \\ u \in \cap_{\alpha \in (\frac{1}{2}, 1)} C^0([0, T_{\max}); D(A^{\alpha})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}); \mathbb{R}^2), \end{gathered}$$

and $n > 0$, $c > 0$ and $v \geq 0$ in $\bar{\Omega} \times (0, T_{\max})$. Moreover, for all $\alpha \in (\frac{1}{2}, 1)$, the following extendibility alternative holds: either $T_{\max} = +\infty$ or

$$\limsup_{t \to T_{\max}^-} (\|n(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|A^{\alpha} u(\cdot, t)\|_{L^2(\Omega)}) = \infty.$$

3. Proof of Theorem 1.1

In this section, we focus on the existence and boundedness of global solutions to system (1.1). To this end, we shall derive a series of a priori estimates. We first establish the L^1-estimates for n and v. These estimates are particularly crucial due to the strong nonlinearity of the equation governing n; the low integrability of L^1 introduces considerable challenges to the subsequent analysis.

Lemma 3.1. *Under the conditions in Theorem 1.1, we have*

$$\|n(\cdot, t)\|_{L^1(\Omega)} \leq M \quad \text{for all } t \in (0, T_{\max}), \tag{3.1}$$

$$\|v(\cdot, t)\|_{L^1(\Omega)} \leq \|v_0\|_{L^1(\Omega)} + M \quad \text{for all } t \in (0, T_{\max}), \tag{3.2}$$

where the constant M is given in (1.10)*.*

Proof. To prove (3.1), we first analyze the source term $f(n)$. By the definition μ in (1.8), there exists a positive constant $\hat{s} \geq 1$ such that

$$f(s) \leq -\hat{\mu} \frac{s^2}{\ln s} \quad \text{for all } s \geq \hat{s}, \tag{3.3}$$

where $\hat{\mu} = \frac{\mu}{2}$ if $0 < \mu < \infty$ and $\hat{\mu}$ can be chosen as any constant greater than χ if $\mu = \infty$. For any $\eta > 0$, it holds that

$$f(s) + \eta s \leq -\hat{\mu} \frac{s^2}{\ln s} + \eta s \quad \text{for all } s \geq \hat{s}. \tag{3.4}$$

This, together with the fact f is bounded on any finite interval, yields for any $\eta > 0$,

$$M_\eta := \sup\{f(s) + \eta s : s > 0\} < \infty.$$

Integrating $(1.1)_1$ over Ω and using the homogeneous Neumann boundary conditions, we obtain that for any $\eta > 0$,

$$\frac{d}{dt} \int_\Omega n = \int_\Omega f(n) \leq -\eta \int_\Omega n + M_\eta |\Omega|, \tag{3.5}$$

which, together with (1.10), implies that

$$\int_\Omega n \leq \int_\Omega n_0 + \frac{M_\eta}{\eta} |\Omega| \leq M.$$

This is the desired estimate (3.1).

Since $\nabla\cdot u=0$, by integrating $(1.1)_3$ and using (3.1), we have

$$\frac{d}{dt}\int_\Omega v+\int_\Omega v=\int_\Omega n\le M \quad \text{for all } t\in(0,T_{\max}).$$

Solving the above differential inequality directly yields

$$\int_\Omega v\le \|v_0\|_{L^1(\Omega)}+M \quad \text{for all } t\in(0,T_{\max}),$$

which is exactly estimate (3.2). This completes the proof. □

Next, we apply the parabolic smoothing property of the momentum equation $(1.1)_4$ to establish L^p-estimate of u.

Lemma 3.2. *For any $r\in[1,2)$, there exists a positive constant K_1 such that*

$$\|Du(\cdot,t)\|_{L^r(\Omega)}\le K_1 \quad \textit{for all } t\in(0,T_{\max}). \tag{3.6}$$

Furthermore, for each $p\in[1,+\infty)$, there exists a positive constant K_2 such that

$$\|u(\cdot,t)\|_{L^p(\Omega)}\le K_2 \quad \textit{for all } t\in(0,T_{\max}). \tag{3.7}$$

Proof. Based on the conditions (1.2) and the L^1-estimate (3.1) for n, estimate (3.6) is a direct consequence of [40, Lemma 2.5]. Furthermore, by the Sobolev embedding theorem $W^{1,r}(\Omega)\hookrightarrow L^p(\Omega)$ (valid for $r\in[1,2)$ and any $p\in[1,+\infty)$ in two dimensional domains), one easily obtains estimate (3.7). This completes the proof. □

By the non-negativity of n and c, we derive the next L^p-estimate for c.

Lemma 3.3. *Let $1\le p<\infty$. Then*

$$\|c(\cdot,t)\|_{L^p(\Omega)}\le\|c_0\|_{L^p(\Omega)} \quad \textit{for all } t\in(0,T_{\max}). \tag{3.8}$$

In particular, it also holds that

$$\|c(\cdot,t)\|_{L^\infty(\Omega)}\le\|c_0\|_{L^\infty(\Omega)} \quad \textit{for all } t\in(0,T_{\max}). \tag{3.9}$$

Proof. For any $p\ge1$, we multiply $(1.1)_2$ by c^{p-1} and integrate the resulting equation over Ω. Using integration by parts and the homogeneous Neumann boundary condition, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega c^p&=-\int_\Omega c^{p-1}u\cdot\nabla c+\int_\Omega c^{p-1}\Delta c-\int_\Omega nc^p\\&=\frac{1}{p}\int_\Omega c^p\nabla\cdot u-(p-1)\int_\Omega c^{p-2}|\nabla c|^2-\int_\Omega nc^p.\end{aligned} \tag{3.10}$$

Recalling that $n>0$, $c>0$ and the incompressibility condition $\nabla\cdot u=0$ in $\Omega\times(0,T_{\max})$, we deduce from (3.10) that

$$\frac{d}{dt}\int_\Omega c^p\le 0.$$

Integrating the above inequality with respect to t immediately yields estimate (3.8). Furthermore, letting $p\to\infty$ in (3.8), this is the desired estimate (3.9). The proof of Lemma 3.3 is complete. □

Lemma 3.4. *There exists a constant $K_3>0$ such that for all $t\in(0,T_{\max})$*

$$\|n\ln n(\cdot,t)\|_{L^1(\Omega)}+\|\nabla c(\cdot,t)\|_{L^2(\Omega)}+\|\nabla v(\cdot,t)\|_{L^2(\Omega)}+\|u(\cdot,t)\|_{L^2(\Omega)}\le K_3. \tag{3.11}$$

Moreover, there exists a constant K_4 such

$$\int_t^{t+t_1}\int_\Omega\Big(|\nabla n^{1/2}|^2+|\Delta c|^2+|\Delta v|^2+|\nabla u|^2\Big)\le K_4 \quad \forall t\in(0,T_{\max}-t_1), \tag{3.12}$$

where $t_1:=\min\big\{1,\frac16T_{\max}\big\}$.

Proof. We multiply $(1.1)_1$ by $\ln n + 1$ and integrate the resulting equation over Ω to see that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n\ln n + 4\int_\Omega |\nabla n^{1/2}|^2 \\
&= \int_\Omega u\cdot\nabla(n\ln n) + \chi\int_\Omega \nabla n\nabla c - \xi\int_\Omega \nabla n\nabla v + \int_\Omega (\ln n+1)f(n) \\
&= -\chi\int_\Omega n\Delta c + \xi\int_\Omega n\Delta v + \int_\Omega (\ln n+1)f(n) \\
&\le \frac{1}{4}\int_\Omega \big(|\Delta c|^2 + |\Delta v|^2\big) + (\chi^2+\xi^2)\int_\Omega n^2 + \int_\Omega (\ln n+1)\, f(n),
\end{aligned} \tag{3.13}$$

where we used Young's inequality and the fact $\nabla\cdot u = 0$ in $\Omega\times(0,T_{\max})$. Following an analogous approach to the derivation of (3.13), we multiply $(1.1)_2$ by $-2\Delta c$ and integrate the resulting equation by parts over Ω. This, together with (3.9), gives

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla c|^2 + 2\int_\Omega |\Delta c|^2 &= 2\int_\Omega nc\Delta c + 2\int_\Omega u\cdot\nabla c\Delta c \\
&\le \frac{1}{4}\int_\Omega |\Delta c|^2 + 4\|c_0\|^2_{L^\infty(\Omega)}\int_\Omega n^2 + 2\int_\Omega u\cdot\nabla c\Delta c.
\end{aligned} \tag{3.14}$$

To proceed further, we need to bound the last term on the right hand of (3.14). Using the incompressibility condition $\nabla\cdot u = 0$, the fact $u\cdot\nabla c = \nabla\cdot(uc)$, and the boundary condition, we obtain after using integration over Ω and Young's inequality that

$$\begin{aligned}
2\int_\Omega u\cdot\nabla c\Delta c &= 2\sum_{i,j}\int_\Omega \partial_j(u_jc)\partial_{ii}c = 2\sum_{i,j}\int_\Omega \partial_i(u_jc)\partial_{ij}c \\
&= 2\sum_{i,j}\int_\Omega c\partial_i u_j\partial_{ij}c + 2\sum_{i,j}\int_\Omega u_j\partial_i c\partial_{ij}c \\
&= 2\int_\Omega cD^2c : \nabla u - \int_\Omega \nabla\cdot u|\nabla c|^2 + \int_{\partial\Omega} |\nabla c|^2 u\cdot\nu \\
&= 2\int_\Omega cD^2c : \nabla u \le 2\|c_0\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega |D^2c|^2.
\end{aligned} \tag{3.15}$$

Next, we establish a priori bound for $\int_\Omega |D^2c|^2$ in terms of $\int_\Omega |\Delta c|^2$. Using integration by parts over Ω, (2.4), (2.5) and (3.9), we obtain, for all $\varepsilon > 0$ and some C_1, $C_2 > 0$, that

$$\begin{aligned}
\frac{1}{2}\int_\Omega |D^2c|^2 &= \frac{1}{2}\int_\Omega |\Delta c|^2 + \frac{1}{4}\int_{\partial\Omega} \frac{\partial}{\partial\nu}|\nabla c|^2 \\
&\le \frac{1}{2}\int_\Omega |\Delta c|^2 + C_1\int_{\partial\Omega} |\nabla c|^2 \\
&\le \frac{1}{2}\int_\Omega |\Delta c|^2 + C_1\varepsilon\int_\Omega |\Delta c|^2 + C_\varepsilon\int_\Omega c^2 \\
&\le (\frac{1}{2} + C_1\varepsilon)\int_\Omega |\Delta c|^2 + C_2.
\end{aligned} \tag{3.16}$$

Putting (3.14), (3.15) and (3.16) together, and taking ε satisfying $C_1\varepsilon = \frac{1}{4}$, we obtain

$$\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 \le 4\|c_0\|^2_{L^\infty(\Omega)}\int_\Omega n^2 + 2\|c_0\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 + C_2. \tag{3.17}$$

We multiply $(1.1)_3$ by $2\Delta v$ in L^2 and using integration by parts and Young's inequality to deduce that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla v|^2 + 2\int_\Omega |\nabla v|^2 + 2\int_\Omega |\Delta v|^2 &= -2\int_\Omega n\Delta v + 2\int_\Omega \Delta v(u\cdot\nabla v) \\
&\le \int_\Omega |\Delta v|^2 + \int_\Omega n^2 + 2\int_\Omega \Delta v(u\cdot\nabla v).
\end{aligned} \tag{3.18}$$

Using (2.2), (3.2), (3.6) and Young's inequality, we have

$$\begin{aligned} 2\int_\Omega \Delta v(u\cdot\nabla v) &= -2\sum_{i,j=1}^{2}\int_\Omega \partial_i u^j \partial_i v \partial_j v + \int_\Omega |\nabla v|^2 \nabla\cdot u \\ &\le C_3\|\nabla u\|_{L^{\frac54}(\Omega)}\|\nabla v\|^2_{L^{10}(\Omega)} \\ &\le C_4\Big(\|v\|^{\frac{2}{15}}_{L^1(\Omega)}\|\Delta v\|^{\frac{28}{15}}_{L^2(\Omega)} + \|v\|^2_{L^1(\Omega)}\Big) \\ &\le \frac12\|\Delta v\|^2_{L^2(\Omega)} + C_5. \end{aligned} \tag{3.19}$$

Then, substituting the above inequality into (3.18), yields

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 + 2\int_\Omega |\nabla v|^2 + \frac12\int_\Omega |\Delta v|^2 \le \int_\Omega n^2 + C_6. \tag{3.20}$$

Next, we multiply $(1.1)_3$ by $2v$ and use integration by parts and Young's inequality to deduce the differential inequality

$$\frac{d}{dt}\int_\Omega v^2 + 2\int_\Omega |\nabla v|^2 + 2\int_\Omega v^2 = 2\int_\Omega nv \le \int_\Omega n^2 + \int_\Omega v^2. \tag{3.21}$$

Putting (3.21) and (3.20) together leads to

$$\frac{d}{dt}\int_\Omega (v^2+|\nabla v|^2) + \int_\Omega \Big(\frac12|\Delta v|^2 + 4|\nabla v|^2 + v^2\Big) \le 2\int_\Omega n^2 + C_6. \tag{3.22}$$

Again, we multiply $(1.1)_4$ by u in L^2 and use integration by parts and Young's inequality to deduce the differential inequality

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^2 + 2\int_\Omega |\nabla u|^2 &= 2\int_\Omega n\nabla\phi\cdot u \le 2\|\nabla\phi\|_{L^\infty}\|u\|_{L^2}\|n\|_{L^2} \\ &\le \int_\Omega |\nabla u|^2 + C_{\rm p}^2\|\nabla\phi\|^2_{L^\infty(\Omega)}\int_\Omega n^2. \end{aligned} \tag{3.23}$$

where we used Poincaré inequality to estimate $\|u\|_{L^2}$ as

$$\|\omega\|_{L^2} \le C_{\rm p}\|\nabla\omega\|_{L^2} \text{for } \omega\in W_0^{1,2}(\Omega),$$

where $C_{\rm p}$ is the Poincaré constant.

Then, we multiply (3.23) by $4\|c_0\|^2_{L^\infty(\Omega)}$ and sum the resulting inequality with (3.13), (3.17) and (3.22). This combination yields the key differential inequality

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \Big(n\ln n - n + 1 + |\nabla c|^2 + |\nabla v|^2 + 4\|c_0\|^2_{L^\infty(\Omega)}u^2 + v^2\Big) \\ &+ \int_\Omega \Big(\frac14|\Delta v|^2 + 4|\nabla v|^2 + v^2\Big) + 4\int_\Omega |\nabla n^{1/2}|^2 + \frac34\int_\Omega |\Delta c|^2 + 2\|c_0\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 \\ &\le \Big(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)}\Big)\int_\Omega n^2 + \int_\Omega f(n)\ln n + C_7, \end{aligned} \tag{3.24}$$

where (3.5) is also used. By (1.8), for any $\varepsilon\in(0,\mu)$ there exists a constant $\hat s_0 \ge 1$ such that

$$f(s) \le -(\mu-\varepsilon)\frac{s^2}{\ln s} \text{ for all } s > \hat s_0. \tag{3.25}$$

Then we have

$$\begin{aligned} &(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)})s^2 + f(s)\ln s \\ &\le (2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)})s^2 - (\mu-\varepsilon)s^2 \\ &\le \Big(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)} - \mu\Big)^+ s^2 + \varepsilon s^2 \quad \text{for all } s \ge \hat s_0. \end{aligned}$$

This, together with (3.24), leads to

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(n\ln n - n + 1 + |\nabla c|^2 + |\nabla v|^2 + 4\|c_0\|^2_{L^\infty(\Omega)}u^2 + v^2\Big)\\
&+\int_\Omega \Big(\frac{1}{4}|\Delta v|^2 + 4|\nabla v|^2 + v^2\Big) + 4\int_\Omega |\nabla n^{1/2}|^2 + \frac{3}{4}\int_\Omega |\Delta c|^2 + 2\|c_0\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2\\
&\le \int_{\{n\le \hat{s}_0\}}\Big[(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)})n^2 + f(n)\ln n\Big]\\
&\quad + \int_{\{n>\hat{s}_0\}}\Big[\Big(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)}\Big)n^2 + f(n)\ln n\Big] + C_7\\
&\le \Big[\Big(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)} - \mu\Big)^+ + \varepsilon\Big]\int_\Omega n^2 + C_8,
\end{aligned} \tag{3.26}$$

where we have used the assumption $f(s) \ge \zeta s$ for small $s \ge 0$, where $\zeta \in \mathbb{R}$, to deduce that

$$\int_{\{n\le \hat{s}_0\}} f(n)\ln n \le C_9,$$

for some positive constant C_9.

Using (2.1), (3.1), (3.8) and Young's inequality, we obtain

$$\|\nabla c(t)\|^2_{L^2(\Omega)} \le C_{10}\|\Delta c(t)\|_{L^2(\Omega)}\|c(t)\|_{L^2(\Omega)} + C_{10}\|c(t)\|^2_{L^2(\Omega)} \le \frac{1}{2}\|\Delta c(t)\|^2_{L^2(\Omega)} + C_{11}, \tag{3.27}$$

and

$$\begin{aligned}
\int_\Omega n^2 = \|n^{1/2}\|^4_{L^4(\Omega)} &\le C_{\rm GN}^4(\|\nabla n^{1/2}\|^{1/2}_{L^2(\Omega)}\|n^{1/2}\|^{1/2}_{L^2(\Omega)} + \|n^{1/2}\|_{L^2(\Omega)})^4\\
&\le 8C_{\rm GN}^4(M\|\nabla n^{1/2}\|^2_{L^2(\Omega)} + M^2),
\end{aligned} \tag{3.28}$$

because $(a+b)^4 \le 8(a^4+b^4)$ for all $a, b \ge 0$. Next, noticing that

$$n\ln n \le \varepsilon n^2 + L_\varepsilon, \quad L_\varepsilon = \sup\big\{s\ln s - \varepsilon s^2 : s > 0\big\} < \infty,$$

we conclude from (3.28) that for any $\varepsilon > 0$

$$\int_\Omega n\ln n \le 8\varepsilon M C_{\rm GN}^4\int_\Omega |\nabla n^{\frac{1}{2}}|^2 + C_{12}. \tag{3.29}$$

Then, summing (3.26), (3.27), (3.28) and (3.29), and using Poincaré inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(n\ln n - n + 1 + |\nabla c|^2 + |\nabla v|^2 + v^2 + 4\|c_0\|^2_{L^\infty(\Omega)}u^2\Big) + \int_\Omega (n\ln n - n + 1)\\
&+ \int_\Omega |\nabla c|^2 + \frac{1}{4}\int_\Omega |\Delta c|^2 + \int_\Omega \Big(\frac{1}{4}|\Delta v|^2 + 4|\nabla v|^2 + v^2\Big)\\
&+ \frac{\|c_0\|^2_{L^\infty(\Omega)}}{C_{\rm p}^2}\int_\Omega |u|^2 + \|c_0\|^2_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 + 4\Big(1 - 2C_{\rm GN}^4 M\Big[\Big(2+\chi^2+\xi^2\\
&+ 4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)} - \mu\Big)^+ + 2\varepsilon\Big]\Big)\int_\Omega |\nabla n^{1/2}|^2\\
&\le C_{13}.
\end{aligned} \tag{3.30}$$

Thanks to condition (1.9), we can fix a positive constant ε in (3.30) by setting

$$\varepsilon = \frac{1}{4}\min\Big\{\mu, \frac{1}{2MC_{\rm GN}^4} - \Big(2+\chi^2+\xi^2+4\|c_0\|^2_{L^\infty(\Omega)} + 4C_{\rm p}^2\|c_0\|^2_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^\infty(\Omega)} - \mu\Big)^+\Big\} > 0,$$

where the positivity of ε is guaranteed by (1.9). This, together with (3.30), yields that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big(n\ln n - n + 1 + |\nabla c|^2 + |\nabla v|^2 + v^2 + 4\|c_0\|^2_{L^\infty(\Omega)}u^2\Big)\\
&+ \min\big\{1, \frac{1}{4C_{\rm p}^2}\big\}\int_\Omega \Big(n\ln n - n + 1 + |\nabla c|^2 + |\nabla v|^2 + v^2 + 4\|c_0\|^2_{L^\infty(\Omega)}u^2\Big) \le C_{13}.
\end{aligned}$$

Then, applying Lemma 2.4 to above inequality leads to

$$\int_\Omega \Big(n\ln n - n + 1 + |\nabla c|^2 + |\nabla v|^2 + v^2 + 4\|c_0\|^2_{L^\infty(\Omega)} u^2\Big) \le C_{14}.$$

This, together with the fact $-s\ln s \le e^{-1}$ for $s>0$ and (3.1), immediately leads to (3.11).

By (3.11), integrating (3.30) with respect to time leads to

$$\int_t^{t+t_1}\int_\Omega \Big(|\nabla n^{1/2}|^2 + |\Delta c|^2 + |\Delta v|^2 + |\nabla u|^2\Big) \le C_{15} \quad \text{for all } t\in(0,T_{\max}-t_1). \tag{3.31}$$

This is the estimate given in (3.12). The proof is complete. □

Now, we are in a position to improve the integrability of n by the above lemma.

Lemma 3.5. *There exist two positive constants K_5 and K_6 such that*

$$\int_\Omega \big(n^2(\cdot,t) + |\nabla c(\cdot,t)|^4 + |\nabla v(\cdot,t)|^4\big) \le K_5 \quad \textit{for all } t\in(0,T_{\max}), \tag{3.32}$$

$$\int_t^{t+t_1}\int_\Omega \big(|\nabla n|^2 + |\nabla|\nabla c|^2|^2 + |\nabla|\nabla v|^2|^2\big) \le K_6 \quad \textit{for all } t\in(0,T_{\max}-t_1), \tag{3.33}$$

where $t_1 := \min\{1, \frac{1}{6}T_{\max}\}$.

Proof. We multiply $(1.1)_1$ by $2n$, then integrate by parts over Ω and apply Young's inequality to deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega n^2 + 2\int_\Omega |\nabla n|^2 \\
&= 2\chi\int_\Omega n\nabla n\cdot\nabla c - 2\xi\int_\Omega n\nabla n\cdot\nabla v + 2\int_\Omega nf(n) \\
&\le \int_\Omega |\nabla n|^2 + \varepsilon^{-1/2}\int_\Omega n^3 + C_1\varepsilon\int_\Omega |\nabla c|^6 + C_2\varepsilon\int_\Omega|\nabla v|^6 + 2\int_\Omega nf(n).
\end{aligned} \tag{3.34}$$

After applying the gradient operator ∇ to $(1.1)_3$ and taking the L^2-inner product of the resulting equation with $4|\nabla v|^2\nabla v$, we obtain

$$\begin{aligned}
\frac14\frac{d}{dt}\int_\Omega|\nabla v|^4 &= \int_\Omega |\nabla v|^2\nabla v\cdot\nabla\big(\Delta v - v + n - u\cdot\nabla v\big) \\
&= \frac12\int_\Omega |\nabla v|^2\Delta|\nabla v|^2 - \int_\Omega|\nabla v|^2|D^2v|^2 - \int_\Omega|\nabla v|^4 \\
&\quad - \int_\Omega n\nabla\cdot\big(|\nabla v|^2\nabla v\big) + \int_\Omega (u\cdot\nabla v)\nabla\cdot\big(|\nabla v|^2\nabla v\big) \\
&= -\frac12\int_\Omega \big|\nabla|\nabla v|^2\big|^2 - \int_\Omega|\nabla v|^4 - \int_\Omega |\nabla v|^2|D^2 v|^2 \\
&\quad + \frac12\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} - \int_\Omega n|\nabla v|^2\Delta v - \int_\Omega n\nabla v\cdot\nabla|\nabla v|^2 \\
&\quad + \int_\Omega (u\cdot\nabla v)|\nabla v|^2\Delta v + \int_\Omega (u\cdot\nabla v)\nabla v\cdot\nabla|\nabla v|^2 \\
&:= -\frac12\int_\Omega\big|\nabla|\nabla v|^2\big|^2 - \int_\Omega|\nabla v|^4 - \int_\Omega|\nabla v|^2|D^2v|^2 + \sum_{i=1}^5 I_i,
\end{aligned} \tag{3.35}$$

where the fact $2\nabla v\cdot\nabla\Delta v = \Delta|\nabla v|^2 - 2|D^2v|^2$ is used. Next, we proceed to estimate each term on the right hand of (3.35). By (2.4), (2.5), and (3.11), we obtain

$$\begin{aligned}
I_1 &\le \frac12\int_{\partial\Omega}|\nabla v|^4 \\
&\le \frac1{16}\int_\Omega|\nabla|\nabla v|^2|^2 + C_3\Big(\int_\Omega|\nabla v|^2\Big)^2
\end{aligned}$$

$$\leq \frac{1}{16}\int_\Omega |\nabla|\nabla v|^2|^2 + C_4.$$

Next, by the facts $|\Delta v| \leq \sqrt{2}|D^2 v|$, $|\nabla|\nabla v|| \leq |D^2 v|$, together with Hölder inequality and Young's inequality, we obtain

$$\begin{aligned}
I_2 + I_3 &= -\int_\Omega n|\nabla v|^2\Delta v - \int_\Omega n\nabla v\cdot\nabla|\nabla v|^2\\
&\leq C_5\left(\||\nabla v||D^2v|\|_{L^2(\Omega)} + \|\nabla|\nabla v|^2\|_{L^2(\Omega)}\right)\|n\|_{L^3(\Omega)}\|\nabla v\|_{L^6(\Omega)}\\
&\leq C_6\left(\||\nabla v||D^2v|\|_{L^2(\Omega)} + \|\nabla|\nabla v|^2\|_{L^2(\Omega)}\right)\|n\|_{L^3(\Omega)}\left(\|\nabla|\nabla v|^2\|_{L^2(\Omega)} + 1\right)^{\frac{1}{3}}\\
&\leq \frac{1}{2}\int_\Omega |\nabla v|^2|D^2v|^2 + \frac{1}{16}\int_\Omega |\nabla|\nabla v|^2|^2 + C_7\int_\Omega n^3 + C_8,
\end{aligned}$$

where we have used the estimate

$$\begin{aligned}
\int_\Omega |\nabla v|^6 &= \left\||\nabla v|^2\right\|^3_{L^3(\Omega)} \leq C_9\left\|\nabla|\nabla v|^2\right\|^2_{L^2(\Omega)}\left\||\nabla v|^2\right\|_{L^1(\Omega)} + C_{10}\left\||\nabla v|^2\right\|^3_{L^1(\Omega)}\\
&\leq C_{11}\int_\Omega |\nabla|\nabla v|^2|^2 + C_{12},
\end{aligned} \tag{3.36}$$

because of (2.1) and (3.11). Similarly, using (3.7) and (3.36), we derive I_4 and I_5 as

$$\begin{aligned}
I_4 &= \int_\Omega (u\cdot\nabla v)|\nabla v|^2\Delta v\\
&\leq \sqrt{2}\int_\Omega |u\cdot\nabla v|\,|\nabla v|^2\,|D^2v|\\
&\leq C_{13}\||\nabla v||D^2v|\|_{L^2(\Omega)}\|\nabla v\|^2_{L^6(\Omega)}\|u\|_{L^6(\Omega)}\\
&\leq C_{14}\||\nabla v||D^2v|\|_{L^2(\Omega)}\left(\|\nabla|\nabla v|^2\|_{L^2(\Omega)} + 1\right)^{\frac{2}{3}}\\
&\leq \frac{1}{4}\||\nabla v||D^2v|\|^2_{L^2(\Omega)} + \frac{1}{16}\|\nabla|\nabla v|^2\|^2_{L^2(\Omega)} + C_{15},
\end{aligned}$$

and

$$\begin{aligned}
I_5 = \int_\Omega (u\cdot\nabla v)\nabla v\cdot\nabla|\nabla v|^2 &\leq C_{16}\|\nabla|\nabla v|^2\|_{L^2(\Omega)}\|\nabla v\|^2_{L^6(\Omega)}\|u\|_{L^6(\Omega)}\\
&\leq \frac{1}{16}\|\nabla|\nabla v|^2\|^2_{L^2(\Omega)} + C_{17}.
\end{aligned}$$

Substituting the above estimates for I_1–I_5 into (3.35), we obtain

$$\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla|\nabla v|^2|^2 + \int_\Omega |\nabla v|^2|D^2v|^2 + 4\int_\Omega |\nabla v|^4 \leq C_{18}\int_\Omega n^3 + C_{19}. \tag{3.37}$$

Applying the gradient operator ∇ to $(1.1)_2$ and multiplying the resulting equations by $4|\nabla c|^2\nabla c$, we obtain after integrating by parts over Ω that for all $t \in (0, T_{\max})$

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |\nabla c|^4 &= 4\int_\Omega |\nabla c|^2\nabla c\cdot\nabla(\Delta c - nc - u\cdot\nabla c)\\
&= -4\int_\Omega |\nabla c|^2|D^2c|^2 + 2\int_\Omega |\nabla c|^2\Delta|\nabla c|^2 + 4\int_\Omega cn(\nabla|\nabla c|^2\nabla c + |\nabla c|^2\Delta c)\\
&\quad - 4\int_\Omega |\nabla c|^2\nabla c\cdot\nabla(u\cdot\nabla c)\\
&:= -4\int_\Omega |\nabla c|^2|D^2c|^2 + \sum_{i=1}^3 J_i,
\end{aligned} \tag{3.38}$$

where we have used the fact

$$2\nabla v\cdot\nabla\Delta v = \Delta|\nabla v|^2 - 2|D^2v|^2,$$

$$-4\int_\Omega |\nabla c|^2\nabla c\cdot\nabla(nc) = 4\int_\Omega cn(\nabla|\nabla c|^2\nabla c + |\nabla c|^2\Delta c).$$

Firstly, using integration by parts and the same calculation procedure as that for I_1, we obtain

$$J_1 = 2\int_\Omega |\nabla c|^2\Delta|\nabla c|^2 = -2\int_\Omega |\nabla|\nabla c|^2|^2 + 2\int_{\partial\Omega}|\nabla c|^2\frac{\partial|\nabla c|^2}{\partial\nu} \le -\frac{7}{4}\int_\Omega |\nabla|\nabla c|^2|^2 + C_{20}.$$

Using that $|\Delta c| \le \sqrt{2}|D^2c|$, Hölder inequality, (2.1), (2.3), (3.1), (3.9) and (3.11), we obtain

$$\begin{aligned}
J_2 &= 4\int_\Omega cn(\nabla|\nabla c|^2\cdot\nabla c + |\nabla c|^2\Delta c)\\
&\le 4\|c\|_{L^\infty(\Omega)}\|n\|_{L^3(\Omega)}\|\nabla c\|_{L^6(\Omega)}(\|\nabla|\nabla c|^2\|_{L^2(\Omega)} + \||\nabla c|\Delta c\|_{L^2(\Omega)})\\
&\le \frac{1}{4}\int_\Omega \big(|\nabla|\nabla c|^2|^2 + |\nabla c|^2|D^2c|^2\big) + \varepsilon^{-1/2}\int_\Omega n^3 + C_{21}\varepsilon\int_\Omega |\nabla c|^6 + C_{22}.
\end{aligned}$$

Using (3.11) and applying the same calculation procedure as that for I_4 and I_5, we obtain

$$\begin{aligned}
J_3 &= -4\int_\Omega |\nabla c|^2\nabla c\cdot\nabla(u\cdot\nabla c)\\
&= 4\int_\Omega (u\cdot\nabla c)|\nabla c|^2\Delta c + 4\int_\Omega (u\cdot\nabla c)\nabla c\cdot\nabla|\nabla c|^2\\
&\le \int_\Omega |\nabla c|^2\,|D^2c|^2 + \frac{1}{2}\int_\Omega |\nabla|\nabla c|^2|^2 + C_{23}.
\end{aligned}$$

Then, substituting J_1–J_3 into (3.38), we obtain

$$\frac{d}{dt}\int_\Omega |\nabla c|^4 + \int_\Omega |\nabla|\nabla c|^2|^2 + \frac{11}{4}\int_\Omega |\nabla c|^2|D^2c|^2 \le \varepsilon^{-1/2}\int_\Omega n^3 + C_{24}\varepsilon\int_\Omega |\nabla c|^6 + C_{25}. \tag{3.39}$$

Putting (3.37), (3.34), and (3.39) together leads to

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n^2 + |\nabla v|^4 + |\nabla c|^4) + \int_\Omega |\nabla n|^2 + \int_\Omega |\nabla|\nabla v|^2|^2\\
&+ \int_\Omega |\nabla v|^2|D^2v|^2 + 4\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla|\nabla c|^2|^2 + \int_\Omega |\nabla c|^2|D^2c|^2\\
&\le (2\varepsilon^{-1/2} + C_{18})\int_\Omega n^3 + C_{26}\varepsilon\int_\Omega |\nabla c|^6 + C_{27}\varepsilon\int_\Omega |\nabla v|^6 + 2\int_\Omega nf(n) + C_{28}.
\end{aligned}\tag{3.40}$$

On the other hand, using (3.1), (2.3) and (3.11), for any $\varepsilon \in (0,\infty)$ we obtain

$$\begin{aligned}
\int_\Omega n^3 &\le \varepsilon\|\nabla n\|^2_{L^2(\Omega)}\|n\ln n\|_{L^1(\Omega)} + C_\varepsilon\|n\|^3_{L^1(\Omega)} + C_\varepsilon\\
&\le \varepsilon C_{29}\|\nabla n\|^2_{L^2(\Omega)} + C_{30}.
\end{aligned}\tag{3.41}$$

Using (3.3) and that f is bounded on finite interval $[0,\hat{s}]$, there exists a constant C_{31} such that

$$s^lf(s) \le \begin{cases} -\hat{\mu}\frac{s^{l+2}}{\ln s} \le 0 & s \ge \hat{s},\\ s^lf(s) \le C_{31} & s < \hat{s},\end{cases}$$

with $l \in \{1,2\}$. This, together with (3.11), yields that

$$\int_\Omega n^lf(n) \le C_{32}. \tag{3.42}$$

Then, substituting (3.41) and (3.42) ($l = 1$) into (3.40) and taking ε suitable small lead to

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (n^2 + |\nabla v|^4 + |\nabla c|^4) + \frac{1}{2}\int_\Omega |\nabla n|^2 + \frac{1}{2}\int_\Omega |\nabla|\nabla v|^2|^2\\
&+ \int_\Omega |\nabla v|^2|D^2v|^2 + 4\int_\Omega |\nabla v|^4 + \frac{1}{2}\int_\Omega |\nabla|\nabla c|^2|^2 + \int_\Omega |\nabla c|^2|D^2c|^2 \le C_{33},
\end{aligned}\tag{3.43}$$

where we also used the estimate (3.36) to deal with $\|\nabla v\|_{L^6}$ and $\|\nabla c\|_{L^6}$. Using Poincaré inequality, (3.1), and (3.11), we obtain

$$\| |\nabla c|^2\|_{L^2(\Omega)}^2 \le C_{\rm p}^2 \|\nabla |\nabla c|^2\|_{L^2(\Omega)}^2 + C_{34},$$
$$\|n\|_{L^2(\Omega)}^2 \le C_{\rm p}^2 \|\nabla n\|_{L^2(\Omega)}^2 + C_{35}.$$

Substituting the above two inequalities into (3.43), we obtain that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \big(n^2 + |\nabla v|^4 + |\nabla c|^4\big) \\
&+ \min\Big\{1, \frac{1}{4C_{\rm p}^2}\Big\} \int_\Omega \big(n^2 + |\nabla v|^4 + |\nabla c|^4 + |\nabla v|^2|D^2 v|^2 + |\nabla c|^2 |D^2 c|^2\big) \\
&+ \frac14 \int_\Omega \big(|\nabla n|^2 + |\nabla |\nabla c|^2|^2 + |\nabla |\nabla v|^2|^2\big) \le C_{36}.
\end{aligned} \tag{3.44}$$

Applying Lemma 2.4 to the above inequality immediately yields the desired uniform boundedness estimate (3.32). Using the estimate (3.32), and integrating the inequality (3.44) with respect to time lead to the second desired estimate (3.33) in this lemma. Now, we complete the proof of this lemma. □

Lemma 3.6. *Let $\alpha \in (\frac12, 1)$. Then there exist two positive constants K_7 and K_8 such that*

$$\|\mathcal{A}^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le K_7 \quad \textit{for all } t \in (0, T_{\max}), \tag{3.45}$$

and consequently

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le K_8 \quad \textit{for all } t \in (0, T_{\max}). \tag{3.46}$$

Proof. First of all, we derive the variation-of-constants formula for u from the equation (1.1) as

$$u(x,t) = e^{-t\mathcal{A}} u_0 + \int_0^t e^{-(t-s)\mathcal{A}} \mathcal{P}(n\nabla\phi(\cdot,s)) ds \quad \text{for all } t \in (0, T_{\max}).$$

Applying the operator $\mathcal{A}^\alpha$ to the above identity and taking the L^2-norm, one obtains

$$\|\mathcal{A}^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le \|\mathcal{A}^\alpha e^{-t\mathcal{A}} u_0\|_{L^2(\Omega)} + \int_0^t \|\mathcal{A}^\alpha e^{-(t-s)\mathcal{A}} \mathcal{P}(n\nabla\phi(\cdot,s))\|_{L^2(\Omega)} ds. \tag{3.47}$$

Next, we shall use the properties of Stokes operator (see Lemma 2.6) to deal with the two terms on the right hand of (3.47). Firstly, using (2.9) and the fact $u_0 \in D(\mathcal{A}^\alpha)$, there exists positive constant C_1 such that

$$\|\mathcal{A}^\alpha e^{-t\mathcal{A}} u_0\|_{L^2(\Omega)} \le \|e^{-t\mathcal{A}} \mathcal{A}^\alpha u_0\|_{L^2(\Omega)} \le \|\mathcal{A}^\alpha u_0\|_{L^2(\Omega)} \le C_1 \quad \text{for all } t \in (0, T_{\max}). \tag{3.48}$$

Since the Helmholtz projection $\mathcal{P}$ is a linear operator from $L^2(\Omega)$ to $L^2_\sigma(\Omega)$, we obtain

$$\begin{aligned}
\int_0^t \|\mathcal{A}^\alpha e^{-(t-s)\mathcal{A}} \mathcal{P}(n(\cdot,s)\nabla\phi)\|_{L^2(\Omega)} ds &\le C_2 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \|n\nabla\phi\|_{L^2(\Omega)} ds \\
&\le C_2 \|n\|_{L^2(\Omega)} \|\nabla\phi\|_{L^\infty(\Omega)} \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} ds \\
&\le C_3,
\end{aligned} \tag{3.49}$$

because of the inequality (2.9), (3.32), the boundedness of $\nabla\phi$, and $\alpha \in (\frac12, 1)$.

Substituting (3.48) and (3.49) into (3.47), we conclude that there exists a positive constant C_4 such that

$$\|\mathcal{A}^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C_4 \quad \text{for all } t \in (0, T_{\max}),$$

which is exactly the estimate (3.45).

Finally, since the domain $D(\mathcal{A}^\alpha)$ is continuously embedded into $L^\infty(\Omega)$ for $\frac12 < \alpha < 2$ (see e.g. [32]), the estimate (3.45) immediately implies that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_5 \quad \text{for all } t \in (0, T_{\max}).$$

This is (3.46). The proof of Lemma is complete. □

Lemma 3.7. *For each $2 < q < \infty$, there exists a constant $K_9 > 0$ such that*

$$\|\nabla c(\cdot,t)\|_{L^q(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le K_9 \quad \text{for all } t \in (0, T_{\max}), \tag{3.50}$$

Proof. We first derive the variation-of-constants formulas for $(1.1)_2$ and $(1.1)_3$, respectively, which read

$$c(\cdot,t) = e^{t\Delta}c_0 - \int_0^t e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla c(\cdot,s) + n(\cdot,s)c(\cdot,s)]ds \quad \text{for all } t \in (0, T_{\max}), \tag{3.51}$$

$$v(\cdot,t) = e^{t\Delta}v_0 - \int_0^t e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla v(\cdot,s) + v(\cdot,s) - n(\cdot,s)]ds \quad \text{for all } t \in (0, T_{\max}). \tag{3.52}$$

Next, we estimate the L^q-norm of ∇c. By applying the gradient operator ∇ to (3.51) and taking the L^q-norm, together with (2.6) and (2.7), we obtain for any $2 < q < \infty$ that

$$\begin{aligned}
&\|\nabla c(\cdot,t)\|_{L^q(\Omega)}\\
&\le \|\nabla e^{t\Delta}c_0\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla c(\cdot,s) + n(\cdot,s)c(\cdot,s)]\|_{L^q(\Omega)}ds\\
&\le C_1\|\nabla c_0\|_{L^q(\Omega)} + C_2\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_1(t-s)}\|u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^2(\Omega)}ds\\
&\quad + C_3\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_2(t-s)}\|n(\cdot,s)c(\cdot,s)\|_{L^2(\Omega)}ds\\
&\le C_4 + C_2\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_1(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla c(\cdot,s)\|_{L^2(\Omega)}ds\\
&\quad + C_3\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_2(t-s)}\|c(\cdot,s)\|_{L^\infty(\Omega)}\|n(\cdot,s)\|_{L^2(\Omega)}ds \quad \text{for all } t \in (0, T_{\max}),
\end{aligned}$$

where C_i $(i = 1,2,3,4)$ and λ_j $(j = 1,2)$ are some positive constants. Using (3.9), (3.11), (3.32) and (3.46), one obtains that

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^q(\Omega)} &\le C_4 + C_5\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_1(t-s)}ds\\
&\quad + C_6\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_2(t-s)}ds\\
&\le C_7 \quad \text{for all } t \in (0, T_{\max}).
\end{aligned} \tag{3.53}$$

By a similar argument as that applied to the c-equation (3.51), we deduce from (3.52) for all $t \in (0, T_{\max})$

$$\begin{aligned}
&\|\nabla v(\cdot,t)\|_{L^q(\Omega)}\\
&\le \|\nabla e^{t\Delta}v_0\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla v(\cdot,s) + v(\cdot,s) - n(\cdot,s)]\|_{L^q(\Omega)}ds\\
&\le C_8\|\nabla v_0\|_{L^q(\Omega)} + C_9\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_3(t-s)}\|u(\cdot,s)\cdot\nabla v(\cdot,s)\|_{L^2(\Omega)}ds\\
&\quad + C_{10}\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_4(t-s)}(\|v(\cdot,s)\|_{L^2(\Omega)} + \|n(\cdot,s)\|_{L^2(\Omega)})ds\\
&\le C_{11} + C_9\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_3(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla v(\cdot,s)\|_{L^2(\Omega)}ds\\
&\quad + C_{10}\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_4(t-s)}(\|v(\cdot,s)\|_{L^2(\Omega)} + \|n(\cdot,s)\|_{L^2(\Omega)})ds,
\end{aligned}$$

where C_i $(i = 8, 9, 10, 11)$ and λ_j $(j = 3, 4)$ are some positive constants. Using (3.11), (3.32) and (3.46), one obtains that

$$\begin{aligned} \|\nabla v(\cdot,t)\|_{L^q(\Omega)} &\le C_{11} + C_{12}\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_3(t-s)}ds \\ &\quad + C_{13}\int_0^t (1+(t-s)^{-1+\frac{1}{q}})e^{-\lambda_4(t-s)}ds \\ &\le C_{14} \quad \text{for all } t \in (0, T_{\max}). \end{aligned} \tag{3.54}$$

Then (3.50) follows from (3.53) and (3.54). This completes the proof. □

Lemma 3.8. *There exists a constant $K_{10} > 0$ such that*

$$\|n(\cdot,t)\|_{L^3(\Omega)} \le K_{10} \quad \textit{for all } t \in (0, T_{\max}). \tag{3.55}$$

Proof. We multiply both sides of $(1.1)_1$ by $3n^2$ and integrate the resulting equation by parts to deduce that

$$\begin{aligned} \frac{d}{dt}\int_\Omega n^3 &= 3\int_\Omega \nabla\cdot(\nabla n - \chi n\nabla c + \xi n\nabla v)n^2 + 3\int_\Omega n^2 f(n) \\ &\le -6\int_\Omega n|\nabla n|^2 + 6\chi\int_\Omega n^2\nabla n\cdot\nabla c - 6\xi\int_\Omega n^2\nabla n\cdot\nabla v + C_1 \\ &\le -2\int_\Omega n|\nabla n|^2 + 2\int_\Omega |n|^4 + C_2\int_\Omega |\nabla c|^8 + C_3\int_\Omega |\nabla v|^8 + C_1 \\ &\le -2\int_\Omega n|\nabla n|^2 + 2\int_\Omega |n|^4 + C_4 \quad \text{for all } t \in (0, T_{\max}), \end{aligned} \tag{3.56}$$

where we have used Young's inequality, (3.50) and (3.42) (taking $l = 2$). Using (3.32), we turn to estimate the second term on the right hand of (3.56) as

$$\begin{aligned} \|n\|^4_{L^4(\Omega)} &= \|n^{3/2}\|^{8/3}_{L^{8/3}(\Omega)} \\ &\le C_5\Big(\|\nabla n^{3/2}\|^{4/3}_{L^2(\Omega)}\|n^{3/2}\|^{4/3}_{L^{4/3}(\Omega)} + \|n^{3/2}\|^{8/3}_{L^{4/3}(\Omega)}\Big) \\ &\le \frac{1}{4}\int_\Omega n|\nabla n|^2 + C_6. \end{aligned} \tag{3.57}$$

Substituting (3.57) into (3.56) and using the fact that $\|n\|^3_{L^3} \le \|n\|^4_{L^4} + |\Omega|$, we obtain

$$\frac{d}{dt}\int_\Omega n^3 + \int_\Omega n^3 \le C_7 \quad \text{for all } t \in (0, T_{\max}). \tag{3.58}$$

Applying Lemma 2.4 to (3.58) immediately yields

$$\|n(\cdot,t)\|_{L^3(\Omega)} \le C_8 \quad \text{for all } t \in (0, T_{\max}). \tag{3.59}$$

This completes the proof. □

Lemma 3.9. *There exists a constant $K_{11} > 0$ such that*

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le K_{11} \quad \textit{for all } t \in (t_1, T_{\max}), \tag{3.60}$$

where $t_1 := \min\big\{1, \frac{T_{\max}}{6}\big\}$.

Proof. We first derive the variation-of-constants formulas for $(1.1)_2$ and $(1.1)_3$, respectively, which read

$$c(\cdot,t) = e^{(t-t_1)\Delta}c(\cdot,t_1) - \int_{t_1}^t e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla c(\cdot,s) + n(\cdot,s)c(\cdot,s)]ds \quad \forall t \in (t_1, T_{\max}), \tag{3.61}$$

$$v(\cdot,t) = e^{(t-t_1)\Delta}v(\cdot,t_1) - \int_{t_1}^t e^{(t-s)\Delta}[u(\cdot,s)\cdot\nabla v(\cdot,s) + v(\cdot,s) - n(\cdot,s)]ds \quad \forall t \in (t_1, T_{\max}). \tag{3.62}$$

By applying the gradient operator ∇ to (3.61) and taking the L^∞-norm, together with and using (2.6), (3.9), (3.46), (3.50) and (3.55), we obtain

$$\begin{aligned}
\|\nabla c(t)\|_{L^\infty(\Omega)} &\le \|\nabla e^{(t-t_1)\Delta} c(\cdot,t_1)\|_{L^\infty(\Omega)} + \int_{t_1}^t \|\nabla e^{(t-s)\Delta}(u\cdot\nabla c)(s)\|_{L^\infty(\Omega)} ds \\
&\quad + \int_{t_1}^t \|\nabla e^{(t-s)\Delta}(nc)(s)\|_{L^\infty(\Omega)} ds \\
&\le C_1(1+t^{-\frac{3}{2}})\|c(\cdot,t_1)\|_{L^1(\Omega)} + C_2\int_{t_1}^t (1+(t-s)^{-\frac{3}{4}})e^{-\lambda_5(t-s)}\|(u\cdot\nabla c)(s)\|_{L^4(\Omega)} ds \\
&\quad + C_3\int_{t_1}^t (1+(t-s)^{-\frac{5}{6}})e^{-\lambda_6(t-s)}\|(nc)(s)\|_{L^3(\Omega)} ds \\
&\le C_4 + C_2\int_{t_1}^t (1+(t-s)^{-\frac{3}{4}})e^{-\lambda_5(t-s)}\|u(s)\|_{L^\infty(\Omega)}\|\nabla c(s)\|_{L^4(\Omega)} ds \\
&+ C_3\int_{t_1}^t (1+(t-s)^{-\frac{5}{6}})e^{-\lambda_6(t-s)}\|n(s)\|_{L^3(\Omega)}\|c(s)\|_{L^\infty(\Omega)} ds \\
&\le C_5 \quad \text{for all } t\in(t_1,T_{\max}).
\end{aligned}$$

By a similar argument as that applied to (3.61) and using Poincaré inequality, we deduce from (3.62) that

$$\begin{aligned}
&\|\nabla v(t)\|_{L^\infty(\Omega)} \\
&\le \|\nabla e^{(t-t_1)\Delta} v(\cdot,t_1)\|_{L^\infty(\Omega)} + \int_{t_1}^t \|\nabla e^{(t-s)\Delta}(u\cdot\nabla v)(s)\|_{L^\infty(\Omega)} ds \\
&\quad + \int_{t_1}^t \|\nabla e^{(t-s)\Delta} v(s)\|_{L^\infty(\Omega)} ds + \int_{t_1}^t \|\nabla e^{(t-s)\Delta} n(s)\|_{L^\infty(\Omega)} ds \\
&\le C_6(1+t^{-\frac{3}{2}})\|v(\cdot,t_1)\|_{L^1(\Omega)} + C_7\int_{t_1}^t (1+(t-s)^{-\frac{3}{4}})e^{-\lambda_7(t-s)}\|(u\cdot\nabla v)(s)\|_{L^4(\Omega)} ds \\
&\quad + C_8\int_{t_1}^t (1+(t-s)^{-\frac{5}{6}})e^{-\lambda_8(t-s)}\|v(s)\|_{L^3(\Omega)} ds \\
&\quad + C_8\int_{t_1}^t (1+(t-s)^{-\frac{5}{6}})e^{-\lambda_8(t-s)}\|n(s)\|_{L^3(\Omega)} ds \\
&\le C_9 + C_7\int_{t_1}^t (1+(t-s)^{-\frac{3}{4}})e^{-\lambda_7(t-s)}\|u(s)\|_{L^\infty(\Omega)}\|\nabla v(s)\|_{L^4(\Omega)} ds \\
&\quad + C_{10}\int_{t_1}^t (1+(t-s)^{-\frac{5}{6}})e^{-\lambda_8(t-s)}\|\nabla v(s)\|_{L^3(\Omega)} ds + C_{11} \le C_{12} \quad \text{for all } t\in(t_1,T_{\max}).
\end{aligned} \tag{3.63}$$

Combining (3.63) and (3.63), we obtain

$$\|\nabla c\|_{L^\infty(\Omega)} + \|\nabla v\|_{L^\infty(\Omega)} \le C_{13} \quad \text{for all } t\in(t_1,T_{\max}).$$

This, together with (3.1) and (3.2), leads to (3.60). Now, we complete the proof . □

Lemma 3.10. *There exists a constant $K_{12}>0$ such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le K_{12} \quad \textit{for all } t\in(0,T_{\max}). \tag{3.64}$$

Proof. By the fact that $\nabla\cdot u\equiv 0$, we derive the variation-of-constants formula for n from $(1.1)_1$. For each $t\in(0,T_{\max})$, this formula reads

$$\begin{aligned} n(\cdot,t) &= e^{(t-t_0)\Delta}n(\cdot,t_0)-\chi\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(n(\cdot,s)\nabla c(\cdot,s))ds\\ &\quad+\xi\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(n(\cdot,s)\nabla v(\cdot,s))ds\\ &\quad-\int_{t_1}^t e^{(t-s)\Delta}\nabla\cdot(n(\cdot,s)u(\cdot,s))ds+\int_{t_0}^t e^{(t-s)\Delta}f(n(\cdot,s))ds\\ &:= n_1(\cdot,t)+n_2(\cdot,t)+n_3(\cdot,t)+n_4(\cdot,t)+n_5(\cdot,t), \end{aligned} \tag{3.65}$$

where $t_0:=(t-1)^+$. Next, we proceed to estimate each of the five terms on the right-hand side of (3.65) one by one. First, by the maximum principle for the Neumann heat semigroup, we can estimate

$$\|n_1(\cdot,t)\|_{L^\infty(\Omega)}\le\|n_0\|_{L^\infty(\Omega)}\quad\text{for all }\ t\in(0,1],$$

whereas if $t>1$, then standard $L^p\to L^q$ estimates for the Neumann heat semigroup provide $C_1>0$ such that

$$\|n_1(\cdot,t)\|_{L^\infty(\Omega)}\le C_1(t-t_0)^{-1}\|n(\cdot,t_0)\|_{L^1(\Omega)}\le C_1\|n\|_{L^1(\Omega)}\le C_1M\quad\text{for all }\ t\in(1,T_{\max}],$$

holds because of (3.1). Thus, we obtain for any $t\in(0,T_{\max})$

$$\|n_1(\cdot,t)\|_{L^\infty(\Omega)}\le C_2. \tag{3.66}$$

Then, using (3.46), (3.55), (3.60) and Hölder inequality, one applies the known smoothing properties of the semigroup (2.8) to obtain for any $t\in(0,T_{\max})$ that

$$\begin{aligned} \|n_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_3\int_{t_0}^t(1+(t-s)^{-\frac56})e^{-\lambda_9(t-s)}\|n(\cdot,s)\nabla c(\cdot,s)\|_{L^3(\Omega)}ds\\ &\le C_3\int_{t_0}^t(1+(t-s)^{-\frac56})e^{-\lambda_9(t-s)}\|n(\cdot,s)\|_{L^3(\Omega)}\|\nabla c(\cdot,s)\|_{L^\infty(\Omega)}ds\le C_4, \end{aligned} \tag{3.67}$$

$$\begin{aligned} \|n_3(\cdot,t)\|_{L^\infty(\Omega)} &\le C_5\int_{t_0}^t(1+(t-s)^{-\frac56})e^{-\lambda_{10}(t-s)}\|n(\cdot,s)\nabla v(\cdot,s)\|_{L^3(\Omega)}ds\\ &\le C_5\int_{t_0}^t(1+(t-s)^{-\frac56})e^{-\lambda_{10}(t-s)}\|n(\cdot,s)\|_{L^3(\Omega)}\|\nabla v(\cdot,s)\|_{L^\infty(\Omega)}ds\le C_6, \end{aligned} \tag{3.68}$$

and

$$\begin{aligned} \|n_4(\cdot,t)\|_{L^\infty(\Omega)} &\le C_7\int_{t_0}^t(1+(t-s)^{-\frac56})e^{-\lambda_{11}(t-s)}\|n(\cdot,s)u(\cdot,s)\|_{L^3(\Omega)}ds\\ &\le C_7\int_{t_0}^t(1+(t-s)^{-\frac56})e^{-\lambda_{11}(t-s)}\|n(\cdot,s)\|_{L^3(\Omega)}\|u(\cdot,s)\|_{L^\infty(\Omega)}ds\le C_8, \end{aligned} \tag{3.69}$$

where $\lambda_j>0$ $(j=9,10,11)$ denotes the first nonzero eigenvalue of $-\Delta$ under homogeneous Neumann boundary condition. Finally, by virtue of the assumptions on the sub-logistic term $f(s)$, it is easy to check that there exists a constant C_9 suchu $f(s)\le C_9$ for all $s\in\mathbb{R}^+$. Using the maximum principle once more, we have for any $t\in(0,T_{\max})$

$$n_5(\cdot,t)\le\int_{t_0}^t e^{(t-s)\Delta}C_9ds=C_9(t-t_0)\le C_9. \tag{3.70}$$

Combination with (3.66)–(3.70), we obtain for any $t\in(0,T_{\max})$

$$\|n(\cdot,t)\|_{L^\infty(\Omega)}=\sup_{x\in\Omega}n(x,t)\le\sum_{i=1}^5\sup_{x\in\Omega}n_i(x,t)\le C_{10}.$$

This is the desired estimate (3.64). The proof is complete. □

With the above estimates in hand, we are in a position to show Theorem 1.1.

Proof of Theorem 1.1. Combining (3.45), (3.60) and (3.64), we can obtain the time independent boundedness for $\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}$. By the extendibility alternative in Lemma 2.7, we thus infer that $T^* = \infty$, i.e., the solution (n, c, v, u, P) is global in time. Furthermore, in view of (3.46), (3.60) and (3.64), we directly verify that the boundedness assertion in (1.12) holds true. This completes the proof. □

Acknowledgements. The authors are grateful to the anonymous reviewers for their carefully reading and valuable suggestions which greatly improved this article. This work is partially supported by the Shandong Provincial Natural Science Foundation (No. ZR2022JQ06) as well as by the National Natural Science Foundation of China (No. 11601215, No. 12226347), and by the Natural Science Foundation of Liaoning Province (No. 2023-MS-137).

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