Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 11, pp. 1–10.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.11

EXISTENCE OF SOLUTIONS WITH PRESCRIBED FREQUENCY FOR PERTURBED SCHRÖDINGER-BOPP-PODOLSKY SYSTEMS IN BOUNDED DOMAINS

DANILO GREGORIN AFONSO, BRUNO MASCARO

Abstract. In this article, we show that the Schrödinger-Bopp-Podolsky system with Dirichlet boundary conditions in a bounded domain possesses infinitely many solutions of prescribed frequency, for any set of (continuous) boundary conditions, provided that the Schrödinger equation is perturbed with a suitable nonlinearity. Our approach is variational, and our proof is based on a symmetric variant of the Mountain Pass theorem.

1. Introduction

In this article, we analyze the existence of solutions to the so-called Schrödinger-Bopp-Podolsky system,

$$\begin{gathered} -\frac{1}{2}\Delta u + \phi u - g(x,u) = \omega u \quad \text{in } \Omega \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, g is a suitable nonlinearity and $\Delta^2\phi = \Delta(\Delta\phi)$ is the bi-Laplacian operator. We consider Dirichlet boundary conditions, i.e.,

$$\begin{gathered} u = 0 \quad \text{on } \partial\Omega \\ \phi = h_1 \quad \text{on } \partial\Omega \\ \Delta\phi = h_2 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

and assume, for simplicity, that $h_1, h_2 \in C(\partial\Omega)$. We refer to [8] for a discussion of appropriate boundary operators for higher-order elliptic problems in bounded domains.

The system of equations (1.1) models the (stationary) interaction of a charged particle with an electromagnetic field with the ansatz that the wave function is of the form

$$\psi(x,t) = u(x)e^{i\omega t},$$

where u plays the role of the amplitude of the wave and ω is the frequency. To our knowledge, the first variational analysis of this kind of system appeared in [7] (see also [15, 10]) in the case of the whole space $\mathbb{R}^3$. In fact, (1.1) is a refinement of the much more studied Schrödinger-Maxwell system, introduced in [5] (see also [12, 11, 13] and the references therein), where the second equation is just $-\Delta\phi = 4\pi u^2$. Besides the physical motivation, which consists of trying to overcome the so-called infinity problem of classical Maxwell theory (we refer to [7] for more on the physical aspects of the problem), the addition of the bi-Laplacian in the second equation gives rise to many interesting mathematical phenomena also when one considers boundary value problems in bounded domains, see e.g. [1].

2020 *Mathematics Subject Classification.* 35D30, 35J10, 35J20, 35J35, 35J40, 35J58, 35J91, 35Q40.
Key words and phrases. Schrödinger-Bopp-Podolsky systems; Dirichlet boundary conditions; higher-order elliptic problems; variational methods.

Submitted October 30, 2025. Published February 11, 2026.

The main approaches developed to treat (1.1) are variational, and a choice is to be made. One can consider a normalization condition of the type

$$\int_\Omega u^2\,dx = 1,$$

in which case the parameter ω appears as a Lagrange multiplier (as done, e.g., in [5, 13, 1]), or one can perform a parametric analysis on the values of ω for which the free problem has a solution (see [12]).

In this work, we follow the second approach. We are able to show that, provided that we perturb the Schrödinger equation with a suitable nonlinearity, then for any prescribed frequency ω and any set of continuous boundary conditions h_1 and h_2, the system (1.1)-(1.2) possesses infinitely many solutions (u,ϕ) (see Theorem 3.7).

This work is organized as follows. In Section 2 we collect the main notations and definitions, and recall an important result that will be useful in our proofs. Section 3 is devoted to the variational analysis of the perturbed problem and the proof of our main result, Theorem 3.7. We address the question of non-existence for the unperturbed problem in Section 4.

2. Preliminaries

2.1. **Notations and definitions.** Throughout the paper, Ω is a smooth, bounded domain (connected open set). For $1 \le p \le \infty$, $\|\cdot\|_p$ denotes the L^p norm (whether on Ω or $\partial\Omega$ will be clear from the context). As usual, we denote by $H_0^1(\Omega)$ the completion of $C_c^\infty(\Omega)$ with respect to the Sobolev norm $W^{1,2}(\Omega)$. However, we consider $H_0^1(\Omega)$ with the equivalent norm

$$\|u\| := \|\nabla u\|_2, \quad u \in H_0^1(\Omega).$$

Its dual space is denoted by $H^{-1}(\Omega)$.

The eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ (counted with multiplicity) are denoted by λ_k, with $k \in \mathbb{N}$. The corresponding eigenspaces are denoted by H_k.

The Sobolev space $W^{2,2}(\Omega)$ is, as usual, denoted by $H^2(\Omega)$. We also consider the functional space

$$\mathcal{H}(\Omega) := H^2(\Omega) \cap H_0^1(\Omega)$$

endowed with the equivalent norm

$$\|\varphi\|_{\mathcal{H}(\Omega)} := \|\Delta\varphi\|_2, \quad \varphi \in \mathcal{H}(\Omega).$$

Recall that a C^1 functional J defined in a Banach space E is said to satisfy the Palais-Smale condition if any sequence $(u_n)_{n\in\mathbb{N}}$ for which $(J(u_n))_{n\in\mathbb{N}}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

In our proofs, we perform some estimates and denote by c_j, $j \in \mathbb{N}$, some positive constants appearing in these estimates and whose exact values are not of interest.

For the nonlinearity $g \in C(\overline{\Omega} \times \mathbb{R})$ appearing in (1.1) we use the following assumptions:

- (A1) g is anti-symmetric: $g(x,-\xi) = -g(x,\xi)$ for all $x \in \overline{\Omega}, \xi \in \mathbb{R}$;
- (A2) g satisfies
$$\lim_{\xi\to 0} \frac{g(x,\xi)}{\xi} = 0 \quad \text{uniformly in } x;$$
- (A3) there exist constants $a_1, a_2 \ge 0$ and $p \in (4,6)$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}$ it holds
$$|g(x,\xi)| \le a_1 + a_2|\xi|^{p-1};$$
- (A4) there exist $r > 0$ and $\mu > 4$ such that for any $\xi \in \mathbb{R}$ with $|\xi| \ge r$ and any $x \in \overline{\Omega}$ it holds
$$0 \le \mu G(x,\xi) \le \xi g(x,\xi),$$
where
$$G(x,\xi) = \int_0^\xi g(x,t)\,dt.$$

2.2. **An abstract critical point theorem.** Here, we recall a version of the Mountain Pass Theorem for functionals that are symmetric with respect to the action of the group $\mathbb{Z}_2 = \{-\mathrm{id}, \mathrm{id}\}$ that will be useful in our proof.

Theorem 2.1 ([14, Theorem 9.12])**.** *Let E be an infinite-dimensional Banach space, $J \in C^1(E)$ be an even functional such that $J(0) = 0$ and J satisfies the Palais-Smale condition. Suppose that $E = V \oplus X$, where V is finite-dimensional and J satisfies*

(1) *There exist constants $\rho, \alpha > 0$ such that*
$$J_{B_\rho \cap X} \geq \alpha;$$
(2) *for each finite dimensional subspace $\widetilde{E}$ of E, there exists an $R = R(\widetilde{E})$ such that $J \leq 0$ in $E \setminus B_{R(\widetilde{E})}$.*

Then I possesses an unbounded sequence of critical values, and therefore there exist infinitely many critical points.

3. Existence of solutions

3.1. **Variational framework.** To perform a variational analysis of the problem, it is convenient to slightly modify our system so as to make all boundary conditions homogeneous. To this aim, we consider the auxiliary problem:
$$\begin{gathered}
-\Delta\chi + \Delta^2\chi = 0 \quad \text{in } \Omega \\
\chi = h_1 \quad \text{on } \partial\Omega \\
\Delta\chi = h_2 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.1}$$

Lemma 3.1. *Let $h_1, h_2 \in C(\Omega)$. Then there exists a weak solution $\chi \in H^2(\Omega)$ to (3.1). Moreover, $\chi \in C^4(\Omega) \cap C(\overline{\Omega})$.*

Proof. Notice that substituting $\theta = \Delta\chi$ we obtain
$$-\Delta\chi + \Delta^2\chi = \Delta\theta - \theta = 0 \quad \text{in } \Omega.$$
Now, by well-known results of linear elliptic equations (see, e.g., [6]), the problem
$$\begin{gathered}
-\Delta\theta + \theta = 0 \quad \text{in } \Omega \\
\theta = h_2 \quad \text{on } \partial\Omega
\end{gathered}$$
admits a (unique) weak solution $\theta \in H^1(\Omega)$. Moreover, standard regularity estimates (see, e.g., [9]) yield $\theta \in C^2(\Omega) \cap C(\overline{\Omega})$. It is also well-known that the problem
$$\begin{gathered}
-\Delta\chi = \theta \quad \text{in } \Omega \\
\chi = h_1 \quad \text{on } \partial\Omega
\end{gathered}$$
admits a unique weak solution $\chi \in H^1(\Omega)$. Moreover, regularity theory yields $\chi \in C^4(\Omega) \cap C(\overline{\Omega})$. The proof is complete, since χ thus found satisfies (3.1). □

Next, we make the change of variables
$$\varphi = \phi - \chi.$$
Then we write the system in the variables (u, φ) as
$$\begin{gathered}
-\frac{1}{2}\Delta u + (\varphi + \chi)u - g(x, u) = \omega u \quad \text{in } \Omega \\
-\Delta\varphi + \Delta^2\varphi = 4\pi u^2 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega \\
\varphi = 0 \quad \text{on } \partial\Omega \\
\Delta\varphi = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.2}$$

Let us consider the functional $F_\omega : H_0^1(\Omega) \times \mathcal{H}(\Omega) \to \mathbb{R}$ given by

$$\begin{aligned} F_\omega(u,\varphi) &= \frac{1}{4}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega (\varphi+\chi-\omega)u^2\,dx - \int_\Omega G(x,u)\,dx \\ &\quad - \frac{1}{16\pi}\int_\Omega |\nabla\varphi|^2\,dx - \frac{1}{16\pi}\int_\Omega |\Delta\varphi|^2\,dx. \end{aligned}$$

All terms of this functional, except for $\int_\Omega G(x,u)\,dx$, are linear or quadratic forms in the variables u and φ, and thus are of class C^1 (see [4]). Moreover, also $\int_\Omega G(x,u)\,dx$ is of class C^1, due to the subcritical growth assumption (A2) (see, e.g, [14, Appendix B]). Therefore $F_\omega \in C^1(H_0^1(\Omega) \times \mathcal{H}(\Omega))$. Straightforward computations yield the following expressions for the partial derivatives:

$$\frac{\partial F_\omega}{\partial u}(u,\varphi)[v] = \frac{1}{2}\int_\Omega \nabla u \nabla v\,dx + \int_\Omega (\varphi+\chi-\omega)uv\,dx - \int_\Omega g(x,u)v\,dx, \quad v \in H_0^1(\Omega), \tag{3.3}$$

$$\frac{\partial F_\omega}{\partial \varphi}(u,\varphi)[\eta] = \frac{1}{2}\int_\Omega \eta u^2\,dx - \frac{1}{8\pi}\int_\Omega \nabla\varphi\nabla\eta - \frac{1}{8\pi}\int_\Omega \Delta\varphi\Delta\eta\,dx, \quad \eta \in \mathcal{H}(\Omega). \tag{3.4}$$

From the expressions of the partial derivatives, we readily obtain that

Proposition 3.2. *The pair $(u,\varphi) \in H_0^1(\Omega)\times\mathcal{H}(\Omega)$ is a weak solution to (3.2) if and only if (u,φ) is a critical point of F_ω in $H_0^1(\Omega)\times\mathcal{H}(\Omega)$.*

Observe that the functional is strongly indefinite both from above and from below. More precisely, for every fixed pair $(u,\varphi) \in H_0^1(\Omega)\times\mathcal{H}(\Omega)$, we have $F_\omega(tu,\varphi) \to +\infty$ as $t \to +\infty$, because the gradient term "wins" against the L^p-subcritical terms (due to the Sobolev embeddings). Similarly, $F_\omega(u,t\varphi) \to -\infty$ as $t \to +\infty$, because of the quadratic terms with negative sign.

Because of this fact, standard variational methods do not apply directly, as was already noticed in [5]. Indeed, a key idea of the argument presented in [5] (and successfully employed by many other authors) is to perform a suitable "substitution" in the system, thereby reducing the study to an analysis of a functional of a single variable.

The procedure goes as follows. For each $u \in H_0^1(\Omega)$, we consider the unique weak solution $\Phi_u \in \mathcal{H}(\Omega)$ of the problem

$$\begin{gathered} -\Delta\varphi + \Delta^2\varphi = 4\pi u^2 \quad \text{in } \Omega \\ \varphi = 0 \quad \text{on } \partial\Omega \\ \Delta\varphi = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.5}$$

which can be shown to exist by a slight modification of the argument presented in the proof of Lemma 3.1. In this way, we can define a map

$$\Phi : u \in H_0^1(\Omega) \mapsto \Phi(u) = \Phi_u \in \mathcal{H}(\Omega).$$

Observe that the map Φ is implicitly defined by the equation

$$\frac{\partial F_\omega}{\partial \varphi}(u,\varphi) = 0, \quad \varphi \in \mathcal{H}(\Omega), \tag{3.6}$$

which is nothing more than the weak formulation of (3.5).

Observe that Φ is even and $\Phi(0) = 0$. Moreover, we have the following result:

Lemma 3.3. *The map Φ is of class C^1 and bounded. Moreover,*

$$\|\Phi(u)\| \le C\|u\|^2 \tag{3.7}$$

for some $C > 0$.

Proof. To show that Φ is of class C^1, we use the implicit formulation (3.6). Note that the derivatives of $\frac{\partial F_\omega}{\partial\varphi}$ are given by

$$\begin{aligned} \frac{\partial^2 F_\omega}{\partial\varphi\partial u}(u,\varphi)[\eta,v] &= \int_\Omega u\eta v\,dx, \\ \frac{\partial^2 F_\omega}{\partial\varphi^2}(u,\varphi)[\eta,\zeta] &= -\frac{1}{8\pi}\Big(\int_\Omega \nabla\eta\nabla\zeta\,dx + \int_\Omega \Delta\eta\Delta\zeta\,dx\Big), \end{aligned} \tag{3.8}$$

which are readily seen to be continuous. Therefore Φ is of class C^1 (see, e.g., [3]).

Next, we use (3.5) and the Sobolev embeddings to show that Φ is bounded in $H_0^1(\Omega)$. Indeed, multiplying (3.5) by $\Phi(u)$ and integrating by parts twice we obtain

$$\int_\Omega |\nabla\Phi(u)|^2\,dx + \int_\Omega |\Delta\Phi(u)|^2\,dx = 4\pi\int_\Omega \Phi(u)u^2\,dx$$

Therefore, by invoking well-known Sobolev embeddings, we have

$$\|\Phi(u)\|^2_{\mathcal{H}(\Omega)} \le 4\pi\|\Phi(u)\|_2\|u^2\|_2 \le c_1\|\Phi(u)\|_{\mathcal{H}(\Omega)}\|u\|_4^2 \le c_2\|\Phi(u)\|_{\mathcal{H}(\Omega)}\|u\|^2,$$

which completes the proof. □

With the map Φ at hand, we can define the reduced functional

$$J_\omega : u \in H_0^1(\Omega) \mapsto F_\omega(u,\Phi(u)) \in \mathbb{R},$$

which can be written as

$$\begin{aligned} J_\omega(u) &= \frac14\int_\Omega |\nabla u|^2\,dx + \frac12\int_\Omega (\chi-\omega)u^2\,dx - \int_\Omega G(x,u)\,dx \\ &\quad + \frac{1}{16\pi}\int_\Omega |\nabla\Phi(u)|^2\,dx + \frac{1}{16\pi}\int_\Omega |\Delta\Phi(u)|^2\,dx. \end{aligned} \tag{3.9}$$

By using the chain rule together with (3.6), we obtain

$$\begin{aligned} J'_\omega(u)[v] &= \frac{\partial F_\omega}{\partial u}(u,\Phi(u))[v] + \Big(\frac{\partial F_\omega}{\partial\varphi}(u,\Phi(u))\circ\Phi'(u)\Big)[v] \\ &= \frac12\int_\Omega \nabla u\nabla v\,dx + \int_\Omega (\Phi(u)+\chi-\omega)uv\,dx - \int_\Omega g(x,u)v\,dx. \end{aligned} \tag{3.10}$$

The next result tells us that the problem (3.2) can be studied through the functional J_ω.

Proposition 3.4. *The pair $(u,\varphi) \in H_0^1(\Omega)\times\mathcal{H}(\Omega)$ is a critical point of F_ω if and only if u is a critical point of J_ω and $\varphi = \Phi(u)$.*

Proof. Suppose (u,φ) is a critical point for F_ω. From (3.6), it follows that $\varphi = \Phi(u)$, and then it follows from (3.3) that $J'_\omega(u) = 0$.

Conversely, if $\varphi = \Phi(u)$ then $\frac{\partial F_\omega}{\partial\varphi}(u,\Phi(u)) = 0$ by (3.6), and $\frac{\partial F_\omega}{\partial u}(u,\Phi(u)) = 0$ since $J'_\omega(u) = 0$ (taking into account (3.3) and (3.10)). □

3.2. **Analysis of the reduced functional.** We begin by showing that the reduced functional J_ω defined in (3.9) satisfies the Palais-Smale condition.

Lemma 3.5. *The functional J_ω defined in* (3.9) *satisfies the Palais-Smale condition.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a Palais-Smale sequence, that is,

$$|J(u_n)| \le M \quad \forall n\in\mathbb{N}, \text{ and some } M>0, \tag{3.11}$$

$$J'(u_n) \to 0 \quad \text{in } H^{-1} \text{ as } n\to\infty. \tag{3.12}$$

As usual in this type of argument, the first step is to show that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded. To this aim, we take r as in (A4) and use (3.11) to obtain

$$\begin{aligned} &\frac14\int_\Omega |\nabla u|^2\,dx + \frac12\int_\Omega (\chi-\omega)u^2\,dx + \frac{1}{16\pi}\int_\Omega |\nabla\Phi(u)|^2\,dx + \frac{1}{16\pi}\int_\Omega |\Delta\Phi(u)|^2\,dx \\ &\le M + \int_{\{x\in\Omega\,:\,|u_n(x)|<r\}} |G(x,u_n)|\,dx + \int_{\{x\in\Omega:|u_n(x)|\ge r\}} |G(x,u_n)|\,dx \\ &\le M_1 + \frac1\mu\int_{\{x\in\Omega:\||u_n(x)|\ge r\}} g(x,u_n)u_n\,dx \\ &\le M_2 + \frac1\mu\int_\Omega g(x,u_n)u_n\,dx \quad \forall n\in\mathbb{N}, \end{aligned} \tag{3.13}$$

where M_1 and M_2 are suitable positive constants.

On the other hand, since

$$
\begin{aligned}
|J'_\omega(u_n)[u_n]| &= \Big|\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega (\Phi(u)+\chi-\omega)u_n^2\,dx - \int_\Omega g(x,u_n)u_n\,dx\Big| \\
&\ge \Big|\Big|\int_\Omega g(x,u_n)u_n\,dx\Big| - \Big|\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega (\Phi(u_n)+\chi-\omega)u_n^2\,dx\Big|\Big| \\
&\ge \int_\Omega g(x,u_n)u_n\,dx - \Big|\frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega (\Phi(u_n)+\chi-\omega)u_n^2\,dx\Big|,
\end{aligned}
$$

from (3.12) we obtain that there exists some positive constant M_3 such that $|J'_\omega(u_n)[u_n]| \le M_3\|u_n\|$, and therefore

$$
\begin{aligned}
\int_\Omega g(x,u_n)u_n\,dx &\le M_3\|u_n\| + \frac{1}{2}\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega (\Phi(u_n)+\chi-\omega)u_n^2\,dx \\
&= M_3\|u_n\| + \frac{1}{2}\|u_n\|^2 + \int_\Omega \chi u_n^2\,dx - \omega\|u_n\|_2^2 + \frac{1}{4\pi}\|\Phi(u_n)\|^2.
\end{aligned}
$$

Substituting this into (3.13), we obtain

$$
\begin{aligned}
&\frac{1}{4}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega (\chi-\omega)u^2\,dx + \frac{1}{16\pi}\int_\Omega |\nabla\Phi(u)|^2\,dx + \frac{1}{16\pi}\int_\Omega |\Delta\Phi(u)|^2\,dx \\
&\le M_2 + \frac{1}{\mu}\Big(M_3\|u_n\| + \frac{1}{2}\|u_n\|^2 + \int_\Omega \chi u_n^2\,dx - \omega\|u_n\|_2^2 + \frac{1}{4\pi}\|\Phi(u_n)\|^2\Big) \\
&\le M_2 + \frac{1}{\mu}\Big(M_3\|u_n\| + \frac{1}{2}\|u_n\|^2 + (\|\chi\|_\infty-\omega)\|u_n\|_2^2 + \frac{1}{4\pi}\|\Phi(u_n)\|^2\Big) \quad \forall n\in\mathbb{N}.
\end{aligned}
$$

Since $\mu > 4$, we have

$$
\frac{1}{4\pi\mu}\|\Phi(u_n)\|^2 - \frac{1}{16\pi}\Big(\int_\Omega |\nabla\Phi(u_n)|^2\,dx + \int_\Omega |\Delta\Phi(u_n)|^2\,dx\Big) < 0,
$$

and therefore

$$
\frac{\mu-2}{4\mu}(\|u_n\|^2 - (\|\chi\|_\infty-\omega)\|u_n\|_2^2) \le M_2 + \frac{M_3}{\mu}\|u_n\| \quad \forall n\in\mathbb{N}. \tag{3.14}
$$

If $\|\chi\|_\infty - \omega \le 0$, we readily obtain that $(u_n)_{n\in\mathbb{N}}$ is bounded. If, instead, it holds $\|\chi\|_\infty - \omega > 0$, then from (3.14) we infer that

$$
\begin{aligned}
\|u_n\|_2^2 &\ge \frac{c_3}{\|\chi\|_\infty-\omega}\Big(\frac{\mu-2}{4\mu}\|u_n\|^2 - \frac{M_3}{\mu}\|u_n\| - M_2\Big) \\
&\ge c_4\|u_n\|^2 - c_5\|u_n\| - c_6 \quad \forall n\in\mathbb{N}.
\end{aligned}
$$

Now, should the sequence $(u_n)_{n\in\mathbb{N}}$ be unbounded in $H_0^1(\Omega)$, we would obtain $\|u_n\|_2^2 \to +\infty$ as $n\to\infty$. However, from assumption (A4) we deduce that there exist constants $b_1, b_2 > 0$ such that for any $x\in\overline{\Omega}$ and $\xi\in\mathbb{R}$ it holds

$$
G(x,\xi) \ge b_1|\xi|^\mu - b_2.
$$

But then, since Φ is bounded, we have

$$
\begin{aligned}
J_\omega(u_n) &\le \frac{1}{4}\|u_n\|^2 + \frac{1}{2}(\|\chi\|_\infty-\omega)\|u_n\|_2^2 - b_1\int_\Omega \|u_n\|^\mu\,dx + b_2|\Omega| + \frac{1}{16\pi}\|\Phi(u_n)\|^2 \\
&\le c_7\|u_n\|^2 + c_8\|u_n\|_2^2 - b_1\int_\Omega |u_n|^\mu\,dx + b_2|\Omega| + c_9\|u_n\|_2^2 \\
&\to -\infty \quad \text{as } n\to\infty,
\end{aligned}
$$

since $\mu > 4$. This contradicts (3.11), and therefore we conclude that $(u_n)_{n\in\mathbb{N}}$ is bounded also in case $\|\chi\|_\infty - \omega > 0$.

Since $(u_n)_{n\in\mathbb{N}}$ is bounded, then there exists $u\in H_0^1(\Omega)$ such that

$$
u_n \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega) \text{ as } n\to\infty.
$$

We now proceed to show that the convergence is, in fact, strong in $H_0^1(\Omega)$.

To this aim, we apply (3.10) and obtain

$$-\frac{1}{2}\Delta u_n = J'_\omega(u_n) - \Phi(u_n)u_n + (\chi - \omega)u_n + g(x, u_n), \tag{3.15}$$

where this equality is understood in $H^{-1}(\Omega)$. Now, since the resolvent operator $(-\Delta)^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$ is compact, to conclude our proof it suffices to show that the right-hand side in (3.15) is bounded.

By assumption, $J'_\omega(u_n) \to 0$ as $n \to \infty$, so $\{J'_\omega(u_n)\}_{n\in\mathbb{N}}$ is bounded. Since g is continuous and has subcritical growth, the Nemitski operator

$$u \in H^1_0(\Omega) \mapsto g(x, u) \in H^{-1}(\Omega)$$

is compact (see, for example, [2, 3]) and therefore, since $(u_n)_{n\in\mathbb{N}}$ is bounded, also $\{g(x, u_n)\}_{n\in\mathbb{N}}$ converges in $H^{-1}(\Omega)$. Furthermore, since $H^1_0(\Omega)$ is reflexive, then $H^{-1}(\Omega)$ is reflexive. Since $(\chi - \omega)u_n \to (\chi - \omega)u$ in the weak-* topology of $H^{-1}(\Omega)$, then $\{(\chi - \omega)u_n\}_{n\in\mathbb{N}}$ is bounded in $H^{-1}(\Omega)$. Finally, Hölder's inequality together with (3.7) yields

$$\|\Phi(u_n)u_n\|^{3/2}_{3/2} \le \|\Phi(u_n)\|^{3/2}_3 \|u_n\|^{3/2}_3 \le c_{10}\|\Phi(u_n)\|^{3/2}\|u_n\|^{3/2}_3 \le c_{11}\|u_n\|^3\|u_n\|^{3/2}_3. \tag{3.16}$$

Hence $\{\Phi(u_n)u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{3/2}(\Omega)$, which is continuously embedded into $H^{-1}(\Omega)$, because $H^1_0(\Omega)$ is continuously embedded into $L^3(\Omega)$.

Hence all terms in the right-hand side of (3.15) are bounded, and since $(-\Delta)^{-1}$ is compact, it follows that the sequence $(u_n)_{n\in\mathbb{N}}$ converges strongly in $H^1_0(\Omega)$. □

Proposition 3.6. *For each $\omega \in \mathbb{R}$, the functional J_ω defined in* (3.9) *has infinitely many critical points in $H^1_0(\Omega)$.*

Proof. Our aim is to apply Theorem 2.1 to the functional J_ω. Since J_ω is even (because so is Φ), $J_\omega(0) = 0$ and J_ω satisfies the Palais-Smale condition (Lemma 3.5), it remains only to prove that the geometrical conditions of Theorem 2.1 hold.

We begin by recalling from the proof of Lemma 3.5 that

$$J_\omega(u) \le \frac{1}{4}\|u\|^2 + \frac{1}{2}(\|\chi\|_\infty - \omega)\|u\|^2_2 - b_1\int_\Omega \|u\|^\mu\,dx + b_2|\Omega| + \frac{1}{16\pi}\|\Phi(u)\|^2 \to -\infty$$

as $\|u\| \to +\infty$, since $\mu > 4$. Therefore, condition *(ii)* in Theorem 2.1 is satisfied.

Now, suppose that

$$\|\chi\|_\infty + \omega < \frac{1}{2}\lambda_1, \tag{3.17}$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$ (see Section 2). In this case, we can take $V = \{0\}$ and $X = H^1_0(\Omega)$. Indeed, J_ω has a strict local minimum at $u = 0$. This can be proven as follows. From (A2) and (A3) we deduce that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}$ it holds

$$|G(x, \xi)| \le \frac{\varepsilon}{2}\xi^2 + C_\varepsilon|\xi|^p,$$

for $p \in (4, 6)$.

Let us fix a number c such that $\|\chi\|_\infty + \omega < c < \frac{1}{2}\lambda_1$. Since $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$, by Poincaré inequality we obtain

$$\begin{aligned} J_\omega(u) &\ge \frac{1}{4}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\|\chi\|_\infty\int_\Omega u^2\,dx - \frac{1}{2}\omega\int_\Omega u^2\,dx - \int_\Omega G(x, u)\,dx \\ &\ge \frac{1}{4}(\|u\|^2 - 2(\|\chi\|_\infty + \omega)\|u\|^2_2) - \frac{\varepsilon}{2}\|u\|^2_2 - C_\varepsilon\|u\|^p_p \\ &> \frac{1}{4}\frac{\lambda_1 - c}{\lambda_1}\|u\|^2 - \frac{\varepsilon}{2\lambda_1}\|u\|^2 - C'_\varepsilon\|u\|^p \quad \forall u \in H^1_0(\Omega), \end{aligned}$$

which is positive for $\|u\| = \rho$ if ρ is sufficiently small. So condition (i) of Theorem 2.1 is satisfied in the case when (3.17) holds.

Next, we assume that

$$\frac{1}{2}\lambda_1 \le \|\chi\|_\infty + \omega \tag{3.18}$$

and let $c \in \big(\frac{1}{2}\lambda_1, \|\chi\|_\infty + \omega\big)$. Set

$$k_\omega := \min\{k \in \mathbb{N} : \|\chi\|_\infty + \omega < \lambda_k\},$$

and consider the following splitting

$$H_0^1(\Omega) = V_k \oplus X,$$

where

$$V_k = \oplus_{k=1}^{k_\omega - 1} H_k, \quad X = V_k^\perp = \overline{\oplus_{k=k_\omega}^\infty H_k}.$$

Then, since

$$\|v\|^2 \geq \lambda_{k_\omega} \|v\|^2 \quad \forall v \in X,$$

(by the variational characterization of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$), we have

$$\begin{aligned} J_\omega(u) &\geq \frac{1}{4}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2}\|\chi\|_\infty \int_\Omega u^2\,dx - \frac{1}{2}\omega\int_\Omega u^2\,dx - \int_\Omega G(x,u)\,dx \\ &\geq \frac{1}{4}(\|u\|^2 - 2(\|\chi\|_\infty + \omega)\|u\|_2^2) - \frac{\varepsilon}{2}\|u\|_2^2 - C_\varepsilon\|u\|_p^p \\ &> \frac{1}{4}\frac{\lambda_{k_\omega} - c}{\lambda_{k_\omega}}\|u\|^2 - \frac{\varepsilon}{2\lambda_1}\|u\|^2 - C'_\varepsilon\|u\|^p \quad \forall u \in X, \end{aligned}$$

so that J_ω is positive in small spheres of X.

We can therefore apply Theorem 2.1 to conclude the proof. □

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^3$ be a smooth, bounded domain. If $g \in C(\overline{\Omega} \times \mathbb{R})$ satisfies* (A1)–(A4), *then for every triple $(h_1, h_2, \omega) \in C(\partial\Omega) \times C(\partial\Omega) \times \mathbb{R}$ there exist infinitely many weak solutions $(u, \phi) \in H_0^1(\Omega) \times H^2(\Omega)$ to the problem*

$$\begin{gathered} -\frac{1}{2}\Delta u + \phi u - g(x,u) = \omega u \quad \text{in } \Omega \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \\ \phi = h_1 \quad \text{on } \partial\Omega \\ \Delta\phi = h_2 \quad \text{on } \partial\Omega. \end{gathered}$$

Proof. The statement of the theorem follows by combining Propositions 3.2, 3.6, and 3.6. □

Remark 3.8. The physical meaning of Theorem 3.7 is that, as happens in the case of Maxwell electrodynamics ([12]), if the Schrödinger equation is perturbed with a suitable nonlinearity, then for any prescribed frequency ω there exist (infinitely many) stationary solutions with that frequency.

4. Non-existence of solutions for the unperturbed problem

Lemma 4.1. *Let $h_1, h_2 \in C(\partial\Omega)$ be such that $h_1 - h_2 \geq 0$ on $\partial\Omega$. If $\phi \in H^2(\Omega)$ satisfies*

$$\begin{gathered} -\Delta\phi + \Delta^2\phi \geq 0 \quad \text{in } \Omega \\ \phi = h_1 \quad \text{on } \partial\Omega \\ \Delta\phi = h_2 \quad \text{on } \partial\Omega, \end{gathered}$$

in the weak sense, then $\phi > 0$ in Ω.

Proof. Notice that

$$-\Delta(\phi - \Delta\phi) \geq 0 \quad \text{in } \Omega,$$

so that the maximum principle yields

$$\phi - \Delta\phi \geq 0.$$

Another application of the maximum principle yields the claim. □

Proposition 4.2. *If $h_1 - h_2 \geq 0$ and $\omega < \frac{\lambda_1}{2}$, then there are no solutions to the problem*

$$
\begin{gathered}
-\frac{1}{2}\Delta u + \phi u = \omega u \quad \text{in } \Omega \\
-\Delta\phi + \Delta^2\phi = 4\pi u^2 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega \\
\phi = h_1 \quad \text{on } \partial\Omega \\
\Delta\phi = h_2 \quad \text{on } \partial\Omega.
\end{gathered}
$$

Proof. By Lemma 4.1, $\phi > 0$ in Ω. Multiplying the first equation by u and integrating by parts, we obtain

$$\omega \int_\Omega u^2\,dx > \frac{1}{2}\int_\Omega |\nabla u|^2\,dx,$$

which, by the variational characterization of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$, can only hold if $\omega > \frac{\lambda_1}{2}$. □

Acknowledgements. This work was partially funded by the European Union - NextGenerationEU within the framework of PNRR Mission 4 - Component 2 - Investment 1.1 under the Italian Ministry of University, and by the Research (MUR) program PRIN 2022 - Grant number 2022BCFHN2 - Advanced theoretical aspects in PDEs and their applications - CUP: H53D23001960006. This research was also partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilitá e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

References

[1] Danilo Gregorin Afonso, Gaetano Siciliano; *Normalized solutions to a Schrödinger-Bopp-Podolsky system under Neumann boundary conditions*, Commun. Contemp. Math. (2021).
[2] Antonio Ambrosetti, Andrea Malchiodi; *Nonlinear analysis and semilinear elliptic problems*, Cambridge University Press, 2007.
[3] Antonio Ambrosetti, Giovanni Prodi; *A primer of nonlinear analysis*, Cambridge University Press, 1993.
[4] Marino Badiale, Enrico Serra; *Semilinear elliptic equations for beginners*, Springer, 2011.
[5] Vieri Benci, Donato Fortunato; *An eigenvalue problem for the Schrödinger-Maxwell equations*, T opol. Method. Nonl. An. (1998).
[6] Haim Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.
[7] Pietro d'Avenia, Gaetano Siciliano; *Nonlinear Schrödinger equation in the Bopp-Podolsky electrodynamics: solutions in the electrostatic case*, J. Differential Equations (2019).
[8] Filippo Gazzola, Hans-Christoph Grunau, Guido Sweers; *Polyharmonic boundary value problems*, Springer, 2010.
[9] David Gilbarg, Neil S. Trudinger; *Elliptic partial differential equations of second order*, Springer, 2001.
[10] Bruno Mascaro, Gaetano Siciliano; *Positive solutions for a Schrd̈inger-Bopp-Podolsky system in* $\mathbb{R}^3$, Communications in Mathematics **31** (2022).
[11] Lorenzo Pisani, Gaetano Siciliano; *Neumann condition in the Schrödinger-Maxwell system*, Topol. Method. Nonl. An. **29** (2007), 251–264.
[12] Lorenzo Pisani, Gaetano Siciliano; *Note on a Schrödinger-Poisson system in a bounded domain*, Appl. Math. Lett. (2007).
[13] Lorenzo Pisani, Gaetano Siciliano; *Constrained Schrödinger-Poisson System with Non-Constant Interaction*, Commun. Contemp. Math. **15** (2013), no. 1.
[14] Paul H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*, American Mathematical Society, 1986.
[15] Humberto Ramos Quoirin, Gaetano Siciliano, Kaye Silva; *Critical points with prescribed energy for a class of functionals depending on a parameter: existence, multiplicity and bifurcation results*, Nonlinearity **37** (2024), no. 6, 065010.

Danilo Gregorin Afonso
Università San Raffaele Roma, Via di Val Cannuta 247, 00166, Roma, Italy
Email address: danilo.afonso@uniroma5.it

Bruno Mascaro
Faculdade de Computação e Informática, Universidade Presbiteriana Mackenzie, R. da Consolação 930, São Paulo, Brasil.
Instituto Mauá de Tecnologia, Praça Mauá 1, São Caetano - SP, Brasil
Email address: bruno.mascaro@mackenzie.br