

A NOTE ON SUFFICIENT CONDITIONS FOR THE REAL JACOBIAN CONJECTURE IN \mathbb{R}^2

XIULI CEN, KANGNONG HU, YUZHOU TIAN

ABSTRACT. The real Jacobian conjecture in \mathbb{R}^2 claims that if $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a polynomial map such that $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$, then F is globally injective. However, it is known that there exists a counterexample on this conjecture. Since then, various sufficient conditions have been proposed to ensure that the real Jacobian conjecture holds. In this article, we generalize a result in [26] to the quasi-homogeneous case. Moreover, we provide several examples to illustrate the relationships between some existing sufficient conditions regarding this conjecture.

1. INTRODUCTION AND MAIN THEOREM

In 1939, Keller [19] proposed the *Jacobian conjecture* in \mathbb{C}^2 : If $F = (f, g) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a polynomial map such that $\det DF(x, y)$ is a non-zero constant for all $(x, y) \in \mathbb{C}^2$, then F is globally injective. See [4, 25, 27, 28] for further progress on this conjecture.

In 1983, Randall [23] raised the *real Jacobian conjecture* in \mathbb{R}^2 : If $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a polynomial map such that $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$, then F is globally injective. As noted in [13, 27], it is beneficial to study the real Jacobian conjecture for understanding the Jacobian conjecture. Regrettably, Pinchuk [22] gave a counterexample to the real Jacobian conjecture in \mathbb{R}^2 . As a result, a natural problem is to find sufficient conditions such that the real Jacobian conjecture holds. Based on algebraic and analytical methods, many results have been obtained, see [5, 7, 10, 14, 16, 18].

Cima et al. [12, 13] established a sufficient condition for the n -dimensional real Jacobian conjecture based on the structure of polynomial maps. When their finding is applied to the two-dimensional case, the result is as follows.

Theorem 1.1 ([13]). *Assume that the polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $F(0, 0) = (0, 0)$ and $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. If there exists a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 , then F is injective.*

Sabatini [24] and Gavrilov [17] presented a global dynamical condition related to the real Jacobian conjecture in \mathbb{R}^2 . Based on their result, a variety of sufficient conditions for this conjecture have been obtained via the qualitative theory of dynamical systems, see [8, 9, 11, 21, 26]. One nice result is described as follows.

Theorem 1.2 ([8]). *Assume that the polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $F(0, 0) = (0, 0)$ and $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. If the higher homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ do not have real linear factors in common, then F is injective.*

Braun et al. [8] also put forward an open problem that whether Theorem 1.1 implies Theorem 1.2. However, they only solved this problem for some special cases. Liu and Tian [20] gave a

2020 *Mathematics Subject Classification*. 34C05, 34C08, 14R15.

Key words and phrases. Real Jacobian conjecture; Newton diagram; Bendixson compactification; quasi-homogeneous.

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Submitted January 15, 2026. Published February 11, 2026.

positive answer and obtained a more general result for quasi-homogeneous type in \mathbb{R}^n . When their result is restricted to the two-dimensional case, the theorem is stated as follows.

Theorem 1.3 ([20]). *Assume that the polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $F(0, 0) = (0, 0)$ and $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Let $H(x, y) = (f^2 + g^2)/2$. If there exists a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous part of vector field $(-\partial_x H, -\partial_y H)$ only vanishes at the origin in \mathbb{R}^2 , then there exists a weight exponent $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2) \in \mathbb{N}_+^2$ such that the higher $\tilde{\mathbf{s}}$ -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 .*

Following Theorems 1.1 and 1.3, a direct corollary can be obtained.

Corollary 1.4. *Assume that the polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $F(0, 0) = (0, 0)$ and $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. If there exists a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin, then F is injective.*

Tian and Cen [26] presented a sufficient condition such that the two-dimensional real Jacobian conjecture holds through the Newton diagram. For the definition of the Newton diagram, see Section 2.

Theorem 1.5 ([26]). *Consider a polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(0, 0) = (0, 0)$ and $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Denote by $b(\mathcal{X})$ the Bendixson compactification of the Hamiltonian vector field $\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x)$, and by $N(b(\mathcal{X}))$ the Newton diagram of $b(\mathcal{X})$. For each bounded edge of type \mathbf{t} in $N(b(\mathcal{X}))$, if its associated Hamiltonian does not have any factor of the form $v^{t_1} - \lambda u^{t_2}$ with $\mathbf{t} = (t_1, t_2) \in \mathbb{N}_+^2$ and $\lambda \in \mathbb{R} \setminus \{0\}$, then F is injective.*

It was also proved that Theorem 1.5 implies Theorem 1.2.

Theorem 1.6 ([26]). *Consider a polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(0, 0) = (0, 0)$ and $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Denote by $b(\mathcal{X})$ the Bendixson compactification of the Hamiltonian vector field $\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x)$, and by $N(b(\mathcal{X}))$ the Newton diagram of $b(\mathcal{X})$. If the higher homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ do not have real linear factors in common, then for each bounded edge of type \mathbf{t} in $N(b(\mathcal{X}))$, its associated Hamiltonian does not have any factor of the form $v^{t_1} - \lambda u^{t_2}$ with $\mathbf{t} = (t_1, t_2) \in \mathbb{N}_+^2$ and $\lambda \in \mathbb{R} \setminus \{0\}$.*

In this article, we generalize Theorem 1.6 to the quasi-homogeneous case.

Theorem 1.7. *Consider a polynomial map $F = (f, g) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $F(0, 0) = (0, 0)$ and $\det DF(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Denote by $b(\mathcal{X})$ the Bendixson compactification of the Hamiltonian vector field $\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x)$, and by $N(b(\mathcal{X}))$ the Newton diagram of $b(\mathcal{X})$. If there exists a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin, then for each bounded edge of type \mathbf{t} in $N(b(\mathcal{X}))$, its associated Hamiltonian does not have any factor of the form $v^{t_1} - \lambda u^{t_2}$ with $\mathbf{t} = (t_1, t_2) \in \mathbb{N}_+^2$ and $\lambda \in \mathbb{R} \setminus \{0\}$.*

Theorem 1.7 tells us that the condition in Theorem 1.5 is weaker than the condition in Corollary 1.4. Now, we are intrigued by the relationships between the sufficient conditions for Theorems 1.1, 1.2, 1.5 and Corollary 1.4. Combining with the existing results, we illustrate the relationships by some examples, as shown in Figure 1.

This article is organized as follows. In Section 2, we provide some preliminary results. In Section 3, we give a proof of Theorem 1.7. Finally, some examples are presented to show the relationships between the sufficient conditions for Theorems 1.1, 1.2, 1.5 and Corollary 1.4 in Section 4.

2. PRELIMINARIES

2.1. Bendixson compactification. Consider a Hamiltonian vector field

$$\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x) := (\Phi(x, y), \Psi(x, y)). \quad (2.1)$$

Let $d_0 = \max\{\deg \Phi(x, y), \deg \Psi(x, y)\}$. We introduce the Bendixson transformation,

$$x = \frac{u}{u^2 + v^2}, \quad y = \frac{v}{u^2 + v^2},$$

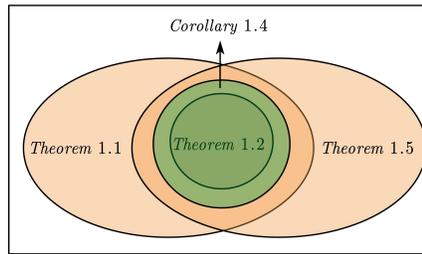


FIGURE 1. Relationships between the sufficient conditions for Theorems 1.1, 1.2, 1.5, and Corollary 1.4.

and denote by $b(\mathcal{X})$ the Bendixson compactification of the Hamiltonian vector field \mathcal{X} given in (2.1). Then the explicit expression for $b(\mathcal{X})$ is

$$\begin{aligned} \dot{u} &= (u^2 + v^2)^{d_0} \left[(v^2 - u^2)\Phi\left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\right) - 2uv\Psi\left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\right) \right], \\ \dot{v} &= (u^2 + v^2)^{d_0} \left[(u^2 - v^2)\Psi\left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\right) - 2uv\Phi\left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\right) \right]. \end{aligned} \tag{2.2}$$

For more details about Bendixson compactification, see [3, 15].

Let $f(x, y) = \sum_{i=1}^n f_i(x, y)$, $g(x, y) = \sum_{j=1}^m g_j(x, y)$ and $d = \max\{n, m\}$, where $f_i(x, y)$ and $g_j(x, y)$ are homogeneous polynomials of degree i and j , respectively. From equation (2.2), the expression for $b(\mathcal{X})$ can be written as

$$\begin{aligned} \dot{u} &= \sum_{i=1}^d \sum_{j=1}^d (u^2 + v^2)^{2d-i-j} [(u^2 - v^2)(f_i(u, v)f_{jy}(u, v) + g_i(u, v)g_{jy}(u, v)) \\ &\quad - 2uv(f_i(u, v)f_{jx}(u, v) + g_i(u, v)g_{jx}(u, v))], \\ \dot{v} &= \sum_{i=1}^d \sum_{j=1}^d (u^2 + v^2)^{2d-i-j} [(u^2 - v^2)(f_i(u, v)f_{jx}(u, v) + g_i(u, v)g_{jx}(u, v)) \\ &\quad + 2uv(f_i(u, v)f_{jy}(u, v) + g_i(u, v)g_{jy}(u, v))]. \end{aligned}$$

Note that $f_{jx}(u, v) = f_{ju}(u, v)$, $f_{jy}(u, v) = f_{jv}(u, v)$, $g_{jx}(u, v) = g_{ju}(u, v)$ and $g_{jy}(u, v) = g_{jv}(u, v)$. The vector field $b(\mathcal{X})$ can be rewritten as the sum of its homogeneous components

$$b(\mathcal{X}) = \frac{1}{2} \sum_{i=1}^d \mathcal{F}_{4d+1-2i} + \sum_{1 \leq i < j \leq d} \mathcal{F}_{4d+1-i-j}, \tag{2.3}$$

where

$$\mathcal{F}_{4d+1-i-j} = \begin{pmatrix} (u^2 + v^2)^{2d-i-j} [(u^2 - v^2)\partial_v(f_i f_j + g_i g_j) - 2uv\partial_u(f_i f_j + g_i g_j)] \\ (u^2 + v^2)^{2d-i-j} [(u^2 - v^2)\partial_u(f_i f_j + g_i g_j) + 2uv\partial_v(f_i f_j + g_i g_j)] \end{pmatrix}^T. \tag{2.4}$$

2.2. Newton diagram. Let $\mathbf{s} = (s_1, s_2) \neq \mathbf{0}$ with $s_1, s_2 \in \mathbb{N}$ coprime. We say that a polynomial $R(x, y)$ is quasi-homogeneous of weighted degree k with respect to weight exponent $\mathbf{s} = (s_1, s_2)$ if $R(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^k R(x, y)$ for all $\lambda > 0$. Denote by $\mathcal{P}_k^{\mathbf{s}}$ the vector space of quasi-homogeneous polynomials of weighted degree k with respect to weight exponent $\mathbf{s} = (s_1, s_2)$. If $P_{k+s_1} \in \mathcal{P}_{k+s_1}^{\mathbf{s}}$ and $Q_{k+s_2} \in \mathcal{P}_{k+s_2}^{\mathbf{s}}$, then the vector field $\mathcal{X}_k = (P_{k+s_1}, Q_{k+s_2})$ is called quasi-homogeneous of weighted degree k with respect to weight exponent $\mathbf{s} = (s_1, s_2)$. Denote by $\mathcal{Q}_k^{\mathbf{s}}$ the vector space of the quasi-homogeneous polynomial vector fields of weighted degree k with respect to weight exponent $\mathbf{s} = (s_1, s_2)$.

We introduce the Newton diagram briefly, see [2, 6] for more details. Consider the vector field $\mathcal{X} = (P(x, y), Q(x, y))$. Let $yP(x, y) = \sum a_{ij}x^i y^j$, $xQ(x, y) = \sum b_{ij}x^i y^j$ and $R(x, y) = \sum c_{ij}x^i y^j$. The support of \mathcal{X} is

$$\text{supp}(\mathcal{X}) = \{(i, j) \mid (a_{ij}, b_{ij}) \neq (0, 0)\} \subseteq \mathbb{R}^2.$$

The vector (a_{ij}, b_{ij}) is called the *vector coefficient* corresponding to (i, j) in the support. Similarly, the set

$$\text{supp}(R) = \{(i, j) \mid c_{ij} \neq 0\} \subseteq \mathbb{R}^2$$

is called the *support* of the polynomial $R(x, y)$.

The boundary of the convex hull for the set

$$\cup_{(i,j) \in \text{supp}(\mathcal{X})} ((i, j) + \mathbb{R}_+^2)$$

includes two open rays and a polygonal line that may be just a single point. This polygonal line with the rays that do not lie on a coordinate axis, when they exist, is called the *Newton diagram* of the vector field \mathcal{X} . The Newton diagram of the polynomial $R(x, y)$ can be defined similarly.

The line segments and their endpoints of the polygonal line are the *edges* and *vertices* of the Newton diagram, respectively. A vertex is classified as an *exterior vertex* if it lies on a coordinate axis; otherwise, it is called an *inner vertex*. The *exponent* of a *bounded edge* l in the Newton diagram is defined as a positive rational number t_2/t_1 , which equals the reciprocal of the absolute value of the edge's slope. The pair $t = (t_1, t_2)$ is called the *type* of the edge l .

For each bounded edge of type $\mathbf{t} = (t_1, t_2)$, we can define a quasi-homogeneous Hamiltonian vector field $\mathcal{X}_k = (P_{k+t_1}, Q_{k+t_2}) \in \mathcal{Q}_k^{\mathbf{t}}$. The associated Hamiltonian for this vector field is as follows

$$h_{k+|\mathbf{t}|} = \frac{1}{k + |\mathbf{t}|} (t_1 x Q_{k+t_2} - t_2 y P_{k+t_1}) \in \mathcal{P}_{k+|\mathbf{t}|}^{\mathbf{t}}, \quad |\mathbf{t}| = t_1 + t_2, \tag{2.5}$$

see [1] for more details.

2.3. Properties of the Newton diagram.

Lemma 2.1 ([26]). *Assume that $R(x, y) \in \mathcal{P}_k^{\mathbf{t}}$ and the vector field $\mathcal{X}_k = (P_{k+t_1}, Q_{k+t_2}) \in \mathcal{Q}_k^{\mathbf{t}}$. Then $\text{supp}(\mathcal{X}_k)$ lies on the straight line $t_1 x + t_2 y = k + |\mathbf{t}|$, and $\text{supp}(R)$ lies on the straight line $t_1 x + t_2 y = k$.*

Lemma 2.2 ([26]). *Consider the vector field $\mathcal{X} = \sum \mathcal{X}_i$ with $\mathcal{X}_i \in \mathcal{Q}_i^{\mathbf{t}}$. Let V_{1i} and V_{2i} be the vertices of the Newton diagram of \mathcal{X}_i . Then the vertices of the Newton diagram of \mathcal{X} are contained in the set $\cup_i \{V_{1i}, V_{2i}\} \subset \text{supp}(\mathcal{X})$.*

Lemma 2.3 ([26]). *Consider the vector field*

$$\mathcal{Y} = ((x^2 - y^2)\partial_y \mathcal{H} - 2xy\partial_x \mathcal{H}, (x^2 - y^2)\partial_x \mathcal{H} + 2xy\partial_y \mathcal{H}),$$

where $\mathcal{H} = \mathcal{H}(x, y)$ is a homogeneous polynomial of degree k in the variables x and y . Then the Newton diagram of the vector field \mathcal{Y} and the polynomial $(x^2 + y^2)\mathcal{H}(x, y)$ have the same vertices.

3. PROOF OF THEOREM 1.7

We first give the following lemma.

Lemma 3.1. *Assume that the higher \mathbf{s} -quasi-homogeneous term of the polynomial $H(x, y)$ with respect to the weight exponent $\mathbf{s} = (s_1, s_2)$ has the form*

$$H^{\mathbf{s}}(x, y) = a_1 x^{k_1} + a_2 y^{k_2} + \sum_{i \geq 1, j \geq 1} c_{ij} x^i y^j,$$

where $a_1 > 0, a_2 > 0, c_{ij} \in \mathbb{R}, k_1 \in \mathbb{N}_+, k_2 \in \mathbb{N}_+$, and the equality $k_1 s_1 = k_2 s_2 = i s_1 + j s_2$ holds. If $s_1 < s_2$, then the higher homogeneous term of $H(x, y)$ is $a_1 x^{k_1}$, while the highest degree term with respect to y in $H(x, y)$ is $a_2 y^{k_2}$, and the inequality $k_2 < i + j < k_1$ holds.

Proof. Since $k_1 s_1 = k_2 s_2$ and $s_1 < s_2$, then $k_1 > k_2$. Let $a_{m-l, l} x^{m-l} y^l$ ($l \geq 1$) be a term of degree m in $H(x, y)$. If $m \geq k_1$, we have $(m-l)s_1 + ls_2 > ms_1 \geq k_1 s_1$. This contradicts the fact that $a_1 x^{k_1}$ is the higher \mathbf{s} -quasi-homogeneous term of $H(x, y)$. Therefore, $m < k_1$ and the higher homogeneous term of $H(x, y)$ can only be $a_1 x^{k_1}$. Similarly, we can verify that the highest degree term with respect to y in $H(x, y)$ is $a_2 y^{k_2}$.

Now, we prove $i+j > k_2$ by contradiction. If $i+j \leq k_2$, then we have $i s_1 + j s_2 < (i+j) s_2 \leq k_2 s_2$. This is in contradiction with $i s_1 + j s_2 = k_2 s_2$. Hence, $k_2 < i+j < k_1$. This completes the proof. \square

Proof of Theorem 1.7. Let $f(x, y)$ and $g(x, y)$ be the polynomials of degree n and m , respectively. Assume that $f(x, y) = \sum_{i=1}^{n_s} f_i^s(x, y)$, $g(x, y) = \sum_{j=1}^{m_s} g_j^s(x, y)$, $d = \max\{n, m\}$ and $d_s = \max\{n_s, m_s\}$, where $f_i^s(x, y)$ and $g_j^s(x, y)$ are quasi-homogeneous polynomials of weighted degree i and j with respect to weight exponent $\mathbf{s} = (s_1, s_2)$, respectively. Then

$$H(x, y) = \frac{f^2(x, y) + g^2(x, y)}{2} = \frac{1}{2} \sum_{j=2}^{2d_s} H_j^s(x, y),$$

where $H_j^s(x, y)$ is a quasi-homogeneous polynomial of weighted degree j with respect to weight exponent $\mathbf{s} = (s_1, s_2)$.

According to the condition of Theorem 1.7, there exist the \mathbf{s} -quasi-homogeneous terms $H_{r_1}^s(x, y)$ and $H_{r_2}^s(x, y)$ in $H(x, y)$ such that $\partial_x H_{r_1}^s$ and $\partial_y H_{r_2}^s$ only vanish at the origin. Without loss of generality, we can assume that $2d_s = r_1 \geq r_2$ and $s_1 < s_2$. Consequently, our proof can be divided into the following two cases.

Case 1: $r_1 = r_2 = 2d_s$. In this case, the higher \mathbf{s} -quasi-homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at $(0, 0)$ if and only if $\partial_x H_{2d_s}^s$ and $\partial_y H_{2d_s}^s$ only vanish at $(0, 0)$. This means that $(0, 0)$ is the unique real zero of $H_{2d_s}^s(x, y)$, see [20, Corollary 1].

Suppose that $H_{2d_s}^s(x, y)$ has the general form

$$H_{2d_s}^s(x, y) = Ax^{k_1} + By^{k_2} + \sum_{i_0 s_1 + j_0 s_2 = 2d_s, i_0, j_0 \geq 1} C_{i_0 j_0} x^{i_0} y^{j_0}, \tag{3.1}$$

where

$$k_1 s_1 = k_2 s_2 = i_0 s_1 + j_0 s_2 = 2d_s. \tag{3.2}$$

Since $H_{2d_s}^s(x, y) = ((f_{d_s}^s)^2 + (g_{d_s}^s)^2) / 2 \geq 0$, we conclude that $H_{2d_s}^s(0, y) = By^{k_2} \geq 0$ and $H_{2d_s}^s(x, 0) = Ax^{k_1} \geq 0$. If either $A = 0$ or $B = 0$, then it would contradict the fact that $(0, 0)$ is the unique real zero of $H_{2d_s}^s(x, y)$. Therefore, $A > 0$, $B > 0$ and k_1 and k_2 are even integers. According to Lemma 3.1, the higher homogeneous term in $H(x, y)$ is Ax^{k_1} . Thus, $k_1 = 2d$, and $H_{2d}(x, y) = Ax^{2d}$. Moreover, let $k_2 = 2d_2$, it follows from the equality (3.2) that $s_1 = d_s/d$, $s_2 = d_s/d_2$, and

$$i_0 d_2 + j_0 d = 2dd_2. \tag{3.3}$$

It is clear that the Hamiltonian $H(x, y)$ has the form

$$H(x, y) = \frac{1}{2} (H_{2d}(x, y) + \dots + H_k(x, y) + \dots + H_{2d_2}(x, y) + \dots + H_2(x, y)) = \frac{1}{2} \sum_{l=2}^{2d} H_l(x, y),$$

where $2d_2 < k < 2d$, and $H_l(x, y)$ is a homogeneous polynomial of degree l for $l = 2, 3, \dots, 2d$. By Lemma 3.1, the degree of each term with respect to y in $H_k(x, y)$ is less than $2d_2$, and the polynomial $H_{2d_2}(x, y)$ takes the form

$$H_{2d_2}(x, y) = By^{2d_2} + \sum_{i+j=2d_2, j < 2d_2} b_{ij} x^i y^j. \tag{3.4}$$

Moreover, the higher \mathbf{s} -quasi-homogeneous terms in $H_{2d_s}^s(x, y)$ cannot appear in all $H_l(x, y)$ with $2 \leq l < 2d_2$.

By equation (2.2), the expression of $b(\mathcal{X})$ is given by

$$\begin{aligned} \dot{u} &= \frac{1}{2} \sum_{l=2}^{2d} (u^2 + v^2)^{2d-l} [(u^2 - v^2)H_{l,v}(u, v) - 2uvH_{l,u}(u, v)], \\ \dot{v} &= \frac{1}{2} \sum_{l=2}^{2d} (u^2 + v^2)^{2d-l} [(u^2 - v^2)H_{l,u}(u, v) + 2uvH_{l,v}(u, v)], \end{aligned} \tag{3.5}$$

where $H_{l,u}(u, v)$ and $H_{l,v}(u, v)$ represent the partial derivatives of $H_l(u, v)$ with respect to u and v , respectively. Let

$$\mathcal{F}_{4d+1-l} = \begin{pmatrix} (u^2 + v^2)^{2d-l}[(u^2 - v^2)H_{l,v}(u, v) - 2uvH_{l,u}(u, v)] \\ (u^2 + v^2)^{2d-l}[(u^2 - v^2)H_{l,u}(u, v) + 2uvH_{l,v}(u, v)] \end{pmatrix}^T \in \mathcal{Q}_{4d+1-l}^{(1,1)} \tag{3.6}$$

for $l = 2, 3, \dots, 2d$. Then

$$\text{supp}(b(\mathcal{X})) = \cup_{l=2}^{2d} \text{supp}(\mathcal{F}_{4d+1-l}).$$

By Lemma 2.1, the support of \mathcal{F}_{4d+1-l} lies on the straight line $\mathcal{L}_{4d+1-l} : x + y = 4d + 2 - l$ for $l = 2, 3, \dots, 2d$. Additionally, from Lemma 2.3, we know that the Newton diagrams of \mathcal{F}_{4d+1-l} and $(u^2 + v^2)^{2d+1-l}H_l(u, v)$ have the same vertices for $l = 2, 3, \dots, 2d$.

Recall that $H_{2d}(x, y) = Ax^{2d}$. Thus, for $l = 2d$, we have $(u^2 + v^2)H_{2d}(u, v) = Au^{2d}(u^2 + v^2)$. Consequently, the vertices of the Newton diagram of \mathcal{F}_{2d+1} are

$$V_{1,2d} = (2(d + 1), 0) \quad \text{and} \quad V_{2,2d} = (2d, 2).$$

For $l = 2d_2$, from the expression of $H_{2d_2}(x, y)$ given by (3.4), the Newton diagram of $\mathcal{F}_{4d+1-2d_2}$ has an exterior vertex

$$V_{2,2d_2} = (0, 2(2d - d_2 + 1)).$$

Note that $V_{2,2d}$ and $V_{2,2d_2}$ lie on the straight line

$$\mathcal{L}_0 : (2d - d_2)x + dy = 2d(2d - d_2 + 1). \tag{3.7}$$

For $2d_2 < l < 2d$, two cases are considered.

(i) If $H_l(x, y)$ includes the higher \mathbf{s} -quasi-homogeneous term $C_{i_0j_0}x^{i_0}y^{j_0}$, then it can be expressed as

$$H_l(x, y) = C_{i_0j_0}x^{i_0}y^{j_0} + \sum_{i+j=l, j < j_0} h_{ij}x^i y^j,$$

where $i_0 = l - j_0$. Consequently, one of the vertices of the Newton diagram of \mathcal{F}_{4d+1-l} is given by

$$V_{2,l} = (i_0, 2(2d + 1 - l) + j_0).$$

Recall that $i_0d_2 + j_0d = 2dd_2$, see (3.3). It is easy to verify that

$$(2d - d_2)i_0 + d(2(2d + 1 - l) + j_0) = 2d(2d + 1) - (i_0d_2 + j_0d) = 2d(2d - d_2 + 1).$$

This implies that the vertex $V_{2,l}$ lies on the line \mathcal{L}_0 given by (3.7).

(ii) If $H_l(x, y)$ does not contain the higher \mathbf{s} -quasi-homogeneous term $C_{i_0j_0}x^{i_0}y^{j_0}$, then

$$H_l(x, y) = \sum_{i+j=l, j < j_0} h_{ij}x^i y^j.$$

Let j^* be the maximum degree of $H_l(x, y)$ with respect to y and $i^* = l - j^*$. Therefore, $i^*d_2 + j^*d < 2dd_2$. As a result, one of the vertices of the Newton diagram of \mathcal{F}_{4d+1-l} is

$$V_{2,l} = (i^*, 2(2d + 1 - l) + j^*).$$

Note that

$$(2d - d_2)i^* + d(2(2d + 1 - l) + j^*) = 2d(2d + 1) - (i^*d_2 + j^*d) > 2d(2d - d_2 + 1).$$

This shows that the vertex $V_{2,l}$ is located to the right of the line \mathcal{L}_0 .

Owing to the above analysis, we obtain the configuration of the support for the vector field $b(\mathcal{X})$, see Figure 2(a). By Lemma 2.2, the Newton diagram of $b(\mathcal{X})$ has two exterior vertices $V_{1,2d}$ and $V_{2,2d_2}$, and an inner vertex $V_{2,2d}$, and it includes two edges of type $(2d - d_2, d)$ and $(1, 1)$, as illustrated in Figure 2(b). The vector fields associated to the vertices $V_{1,2d}, V_{2,2d}$ and $V_{2,2d_2}$ are $(0, Adu^{2d+1})$, $(-2Adu^{2d}v, -Adu^{2d-1}v^2)$ and $(-Bd_2v^{4d-2d_2+1}, 0)$, respectively.

For the vector field $b(\mathcal{X})$, the lowest-degree homogeneous terms with weight exponent $(1, 1)$ are

$$\frac{1}{2}\mathcal{F}_{2d+1} = \frac{1}{2} \begin{pmatrix} (u^2 - v^2)(H_{2d,v}(u, v) - 2uvH_{2d,u}(u, v)) \\ (u^2 - v^2)(H_{2d,u}(u, v) + 2uvH_{2d,v}(u, v)) \end{pmatrix}^T = \begin{pmatrix} -2Adu^{2d}v \\ Ad(u^2 - v^2)u^{2d-1} \end{pmatrix}^T.$$

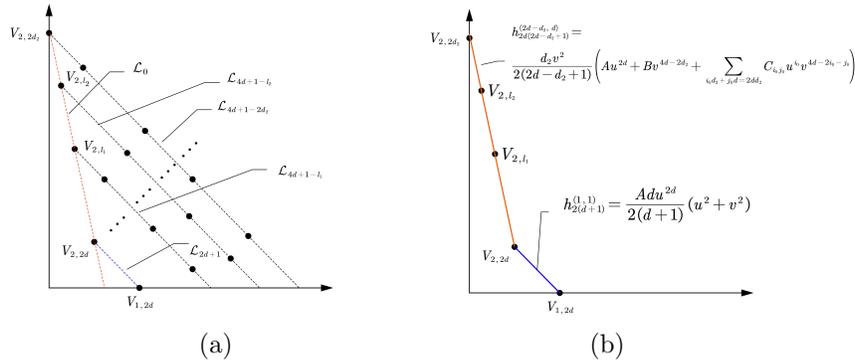


FIGURE 2. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

By (2.5), the Hamiltonian associated with the edge of type $(1, 1)$ is

$$h_{2(d+1)}^{(1,1)} = \frac{Adu^{2d}}{2(d+1)}(u^2 + v^2).$$

It is clear that $h_{2(d+1)}^{(1,1)}$ does not contain any factor of the form $v - \lambda u$ with $\lambda \in \mathbb{R} \setminus \{0\}$.

The lowest-degree quasi-homogeneous terms with weight exponent $(2d - d_2, d)$ for the vector field $b(\mathcal{X})$ are

$$\begin{pmatrix} -2Adu^{2d}v - \frac{1}{2} \sum_{i_0 d_2 + j_0 d = 2dd_2} C_{i_0 j_0} (j_0 + 2i_0) u^{i_0} v^{4d-2i_0-j_0+1} - Bd_2 v^{4d-2d_2+1} \\ -Adu^{2d-1}v^2 - \frac{1}{2} \sum_{i_0 d_2 + j_0 d = 2dd_2} i_0 C_{i_0 j_0} u^{i_0-1} v^{4d-2i_0-j_0+2} \end{pmatrix}^T.$$

By (2.5), the Hamiltonian associated with the edge of type $(2d - d_2, d)$ is

$$h_{2d(2d-d_2+1)}^{(2d-d_2,d)} = \frac{d_2 v^2}{2(2d - d_2 + 1)} \left(Au^{2d} + Bv^{4d-2d_2} + \sum_{i_0 d_2 + j_0 d = 2dd_2} C_{i_0 j_0} u^{i_0} v^{4d-2i_0-j_0} \right).$$

Note that $\mathbf{s} = (s_1, s_2) = (d_2, d)$ and

$$H_{2d_s}^{\mathbf{s}}(x, y) = Ax^{2d} + By^{2d_2} + \sum_{i_0 d_2 + j_0 d = 2dd_2, i_0, j_0 \geq 1} C_{i_0 j_0} x^{i_0} y^{j_0} \quad (\text{see (3.1)})$$

only vanishes at the origin. Thus the Hamiltonian

$$h_{2d(2d-d_2+1)}^{(2d-d_2,d)} = \frac{d_2 v^2}{2(2d - d_2 + 1)} H_{2d_s}^{\mathbf{s}} \left(u, v^{\frac{2d-d_2}{d_2}} \right)$$

has no factor of the form $v^{2d-d_2} - \lambda u^d$ with $\lambda \neq 0$. Otherwise, the higher \mathbf{s} -quasi-homogeneous term $H_{2d_s}^{\mathbf{s}}(x, y)$ will vanish at $y^{d_2} - \lambda x^d = 0$.

Case 2: $r_1 = 2d_s > r_2$. In this case,

$$H(x, y) = \frac{1}{2} (H_{2d_s}^{\mathbf{s}}(x) + \cdots + H_k^{\mathbf{s}}(x) + \cdots + H_{r_2+1}^{\mathbf{s}}(x) + H_{r_2}^{\mathbf{s}}(x, y) + \cdots + H_2^{\mathbf{s}}(x, y)), \quad (3.8)$$

where $H_k^{\mathbf{s}}(x) = A_k x^{a_k}$, $k = r_2 + 1, \dots, 2d_s$, and $H_k^{\mathbf{s}}(x, y)$, $k = 2, 3, \dots, r_2$ are quasi-homogeneous polynomials of weighted degree k with weight exponent $\mathbf{s} = (s_1, s_2)$. The polynomial $H_{r_2}^{\mathbf{s}}(x, y)$ has the form

$$H_{r_2}^{\mathbf{s}}(x, y) = By^b + \sum_{i s_1 + j s_2 = r_2} b_{ij} x^i y^j + A_{r_2} x^{a_{r_2}},$$

where $b_{ij} \in \mathbb{R}$ and $A_{r_2} \in \mathbb{R}$. It follows from

$$H_{2d_s}^{\mathbf{s}}(x) = \frac{1}{2} \left((f_{d_s}^{\mathbf{s}})^2 + (g_{d_s}^{\mathbf{s}})^2 \right) = \frac{1}{2} A_{2d_s} x^{a_{2d_s}} \geq 0$$

that $A_{2d_s} > 0$ and $a_{2d_s} = 2d$. By Lemma 3.1, it is easy to verify that the term By^b is the highest degree term in $H(x, y)$ with respect to y . Therefore,

$$\frac{f^2(0, y) + g^2(0, y)}{2} = H(0, y) = \frac{1}{2}(By^b + \dots + H_2^s(0, y)) \geq 0.$$

This implies that $B \geq 0$ and b is even. Thus, $r_2 = bs_2$ is also even. Since $H_{2d_s}^s(x)$ and $H_{r_2}^s(x, y)$ only vanish at $(0, 0)$, we have $B > 0$. For convenience, we denote $a_{r_2} = d_1$ and $b = 2d_2$.

By Lemma 3.1, $H(x, y)$ can be arranged in descending homogeneous order as follows

$$\begin{aligned} H(x, y) &= \frac{1}{2} (H_{2d}(x) + \dots + H_{a_k}(x) + \dots + H_{d_1}(x) + \dots + H_{2d_2}(x, y) + \dots + H_2(x, y)) \\ &= \frac{1}{2} (A_{2d_s}x^{2d} + \dots + A_kx^{a_k} + \dots + A_{r_2}x^{d_1} + \dots + By^{2d_2} + \dots + H_2(x, y)). \end{aligned}$$

Similar to the computation in Case 1, we have that $(u^2 + v^2)^{2d+1-l} H_l(u, v) = (u^2 + v^2)^{2d+1-l} A_{ls_1} u^l$, and the vertices of the Newton diagram of \mathcal{F}_{4d+1-l} are

$$V_{1,l} = (2(2d + 1) - l, 0) \quad \text{and} \quad V_{2,l} = (l, 2(2d + 1 - l))$$

for $l = d_1, d_1 + 1, \dots, 2d$. Note that $V_{2,l}$ lies on the straight line $L_1 : 2x + y = 2(2d + 1)$.

For $l = 2d_2$, the Newton diagram of $\mathcal{F}_{4d+1-2d_2}$ has an exterior vertex

$$V_{2,2d_2} = (0, 2(2d - d_2 + 1)).$$

Note that the segment $\overline{V_{2,d_1}V_{2,2d_2}}$ lies on the straight line $L_2 : 2(d_1 - d_2)x + d_1y = 2d_1(2d + 1 - d_2)$, while the segment $\overline{V_{2,2d}V_{2,2d_2}}$ lies on the straight line $L_3 : (2d - d_2)x + dy = 2d(2d - d_2 + 1)$. Moreover, the slope of L_3 is larger than that of L_1 owing to $-(2d - d_2)/d = -2 + d_2/d > -2$. For $2d_2 < l < d_1$, our analysis from Case 1 shows that the vertices of the Newton diagram of \mathcal{F}_{4d+1-l} are either on the line L_2 or to the right of the line L_2 . Thus, we obtain the configuration of the support for the vector field $b(\mathcal{X})$, as shown in Figure 3(a). And the Newton diagram of $b(\mathcal{X})$ includes two exterior vertices $V_{1,2d}$ and $V_{2,2d_2}$, an inner vertex $V_{2,2d}$, and two edges of type $(2d - d_2, d)$ and $(1, 1)$, as illustrated in Figure 3(b).

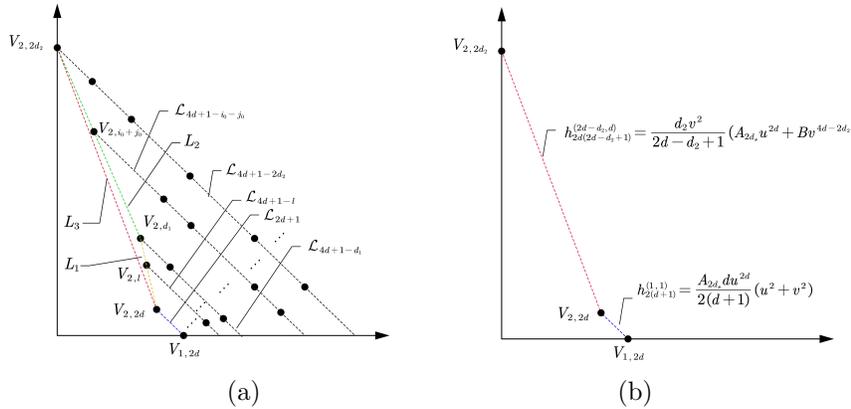


FIGURE 3. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

Based on Case 1, we can directly conclude that the Hamiltonian associated with the edge of type $(1, 1)$ is

$$h_{2(d+1)}^{(1,1)} = \frac{A_{2d_s} du^{2d}}{2(d+1)} (u^2 + v^2).$$

And the Hamiltonian associated with the edge of type $(2d - d_2, d)$ is

$$h_{2d(2d-d_2+1)}^{(2d-d_2,d)} = \frac{d_2 v^2}{2(2d - d_2 + 1)} (A_{2d_s} u^{2d} + B v^{4d-2d_2}).$$

It follows that the Hamiltonians $h_{2d+2}^{(1,1)}$ and $h_{2d(2d-d_2+1)}^{(2d-d_2,d)}$ have no factor of the form $v^{t_1} - \lambda u^{t_2}$ with $\lambda \neq 0$.

For $s_1 > s_2$, similar results can be obtained only by interchanging x and y . This completes the proof. \square

4. RELATIONSHIPS BETWEEN THE EXISTING THEOREMS

From the known results, we conclude that

- (i) Theorem 1.1 implies Theorem 1.2 and Corollary 1.4 by Theorem 1.3,
- (ii) Theorem 1.5 implies Theorem 1.2 by Theorem 1.6,
- (iii) Theorem 1.5 implies Corollary 1.4 by Theorem 1.7,
- (iv) Corollary 1.4 implies Theorem 1.2.

However, we cannot determine the relationships between Theorems 1.1 and 1.5. The following examples will illustrate the relationships between Theorems 1.1, 1.2, 1.5 and Corollary 1.4. We can see from the examples that Theorems 1.1 and 1.5 have their own strengths. Nevertheless, there still exist examples which cannot be solved by both theorems.

In the next example, Theorems 1.1, 1.2 and Corollary 1.4 do not work. The polynomial map F is injective by Theorem 1.5.

Example 4.1. Consider the polynomial map $F = (f, g) = (y + y^3, x + xy^2)$ with $\det DF(x, y) = -1 - 4y^2 - 3y^4 < 0$.

Obviously, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 . Therefore, Theorem 1.1 does not work.

Consider the Hamiltonian vector field

$$\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x) = (-y - 2x^2y - 4y^3 - 2x^2y^3 - 3y^5, x + 2xy^2 + xy^4).$$

The higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ have the real linear common factor y with multiplicity 3. Consequently, Theorem 1.2 also does not apply. Moreover, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin in \mathbb{R}^2 . It follows that Corollary 1.4 is invalid.

The Newton diagram of the Bendixson compactified vector field $b(\mathcal{X})$ is shown in Figure 4. There exist two bounded edges in the Newton diagram, and by (2.5) the associated Hamiltonians are $h_8^{(1,1)} = \frac{3}{8}v^4(u^2 + v^2)^2$ and $h_{12}^{(1,2)} = \frac{1}{12}u^4(u^4 + v^2)^2$, respectively. Hence, F is injective by Theorem 1.5.

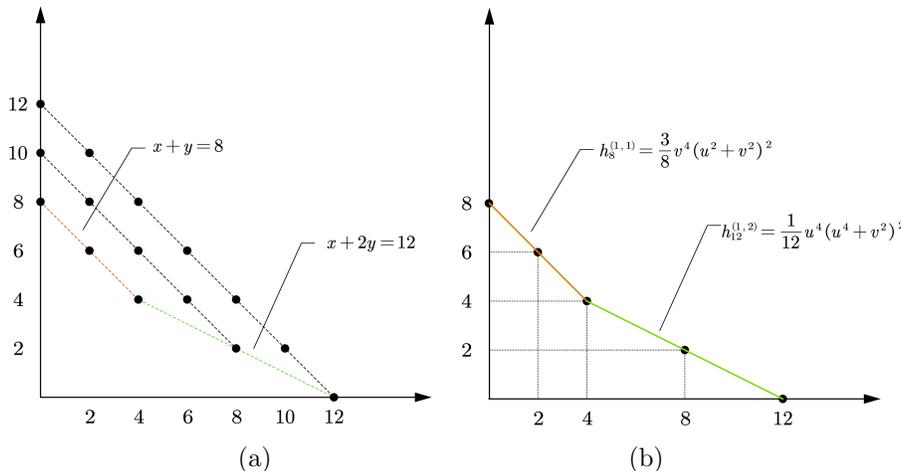


FIGURE 4. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

In the next example, Theorems 1.1, 1.2 and Corollary 1.4 do not work. The polynomial map F is injective by Theorem 1.5. Moreover, the Newton diagram in this example is more complex than that in Example 4.1.

Example 4.2. Consider the polynomial map $F = (f, g) = (x + x^3y^2, y)$ with $\det DF(x, y) = 1 + 3x^2y^2 > 0$.

Obviously, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 . Therefore, Theorem 1.1 does not work.

Consider the Hamiltonian vector field

$$\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x) = (-y - 2x^4y - 2x^6y^3, x + 4x^3y^2 + 3x^5y^4).$$

The higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ have the real linear common factor x, y with multiplicity 5, 3, respectively. Consequently, Theorem 1.2 also does not apply. Moreover, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin in \mathbb{R}^2 . It follows that Corollary 1.4 is invalid.

The Newton diagram of the Bendixson compactified vector field $b(\mathcal{X})$ is shown in Figure 5. There exist three bounded edges in the Newton diagram, and by (2.5) the associated Hamiltonians are computed as follows.

$$h_{60}^{(7,3)} = \frac{1}{20}v^6(u^6 + v^{14}), \quad h_{12}^{(1,1)} = \frac{5}{12}u^6v^4(u^2 + v^2), \quad h_{20}^{(1,3)} = \frac{1}{20}u^8(u^6 + v^2)^2.$$

Hence, F is injective by Theorem 1.5.

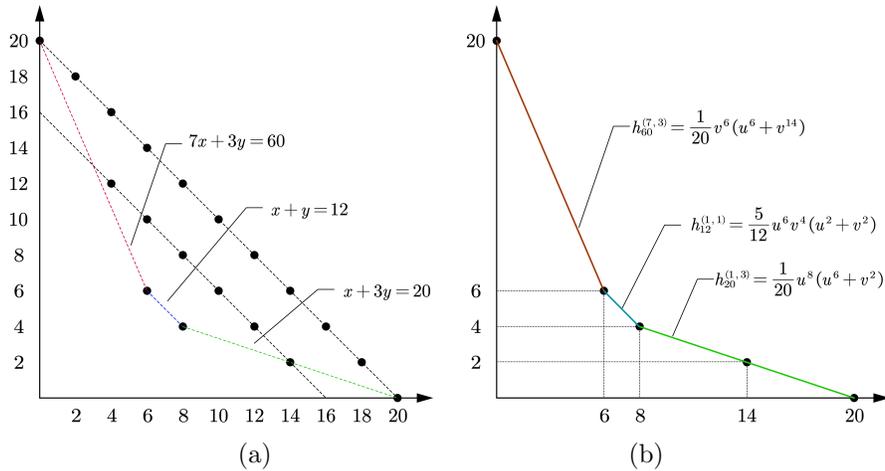


FIGURE 5. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

In the next example, Theorems 1.2, 1.5 and Corollary 1.4 do not work. The polynomial map F is injective by Theorem 1.1.

Example 4.3. Consider the polynomial map $F = (f, g) = (x^3 + y^3 + x, y)$ with $\det DF(x, y) = 1 + 3x^2 > 0$.

It is easy to verify that for any weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ with $s_1 > s_2$, the higher \mathbf{s} -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 . Hence, F is injective by Theorem 1.1.

Consider the Hamiltonian vector field

$$\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x) = \begin{pmatrix} -y - 3xy^2 - 3y^2(x + y)(x^2 - xy + y^2) \\ x + 4x^3 + y^3 + 3x^2(x + y)(x^2 - xy + y^2) \end{pmatrix}^T.$$

The higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ have the real linear common factor $x + y$. Therefore, Theorem 1.2 does not work. Moreover, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin in \mathbb{R}^2 . It follows that Corollary 1.4 is invalid.

The Newton diagram of the Bendixson compactified vector field $b(\mathcal{X})$ is shown in Figure 6. There exists only one bounded edge in the Newton diagram, and by (2.5) the associated Hamiltonian is

$$h_{\mathbf{s}}^{(1,1)} = \frac{3}{8}(u + v)^2(u^2 + v^2)(u^2 - uv + v^2)^2.$$

Obviously, it has a factor $v + u$. Consequently, Theorem 1.5 also does not apply.

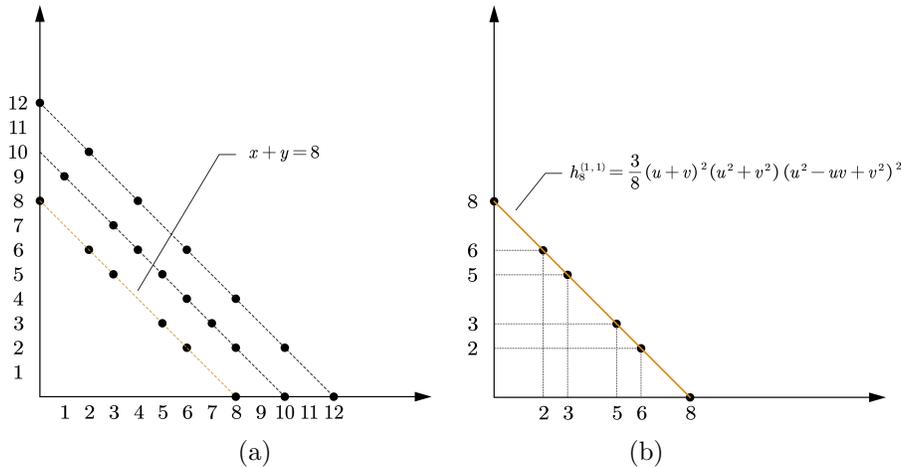


FIGURE 6. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

In the next example, Theorem 1.2 and Corollary 1.4 do not work. The polynomial map F is injective by Theorems 1.1 and 1.5.

Example 4.4. Consider the polynomial map $F = (f, g) = (x^7 + x^3y^2 + xy^2 + x, y)$ with $\det DF(x, y) = 1 + 7x^6 + y^2 + 3x^2y^2 > 0$.

It is easy to verify that for any weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ with $s_1 > s_2$, the higher \mathbf{s} -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 . Hence, F is injective by Theorem 1.1.

Consider the Hamiltonian vector field

$$\begin{aligned} \mathcal{X} &= (-ff_y - gg_y, ff_x + gg_x) \\ &= \begin{pmatrix} -y - 2x^2y - 2x^4y - 2x^2y^3 - 4x^4y^3 - 2x^8y - 2x^6y^3 - 2x^{10}y \\ x + 2xy^2 + 4x^3y^2 + xy^4 + 4x^3y^4 + 8x^7 + 3x^5y^4 + 8x^7y^2 + 10x^9y^2 + 7x^{13} \end{pmatrix}^T. \end{aligned}$$

The higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ have the real linear common factor x with multiplicity 10. Consequently, Theorem 1.2 also does not apply. Moreover, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin in \mathbb{R}^2 . It follows that Corollary 1.4 is invalid.

The Newton diagram of the Bendixson compactified vector field $b(\mathcal{X})$ is shown in Figure 7. There exist four bounded edges in the Newton diagram, and by (2.5) the associated Hamiltonians are as follows.

$$\begin{aligned} h_{28}^{(3,1)} &= \frac{1}{28}v^{22}(u^2 + v^6), & h_{26}^{(2,1)} &= \frac{1}{13}u^2v^{14}(u^2 + v^4)^2, \\ h_{46}^{(3,2)} &= \frac{7}{46}u^6v^2(u^4 + v^6)^2, & h_{16}^{(1,1)} &= \frac{7}{16}u^{14}(u^2 + v^2). \end{aligned}$$

Hence, F is injective by Theorem 1.5.

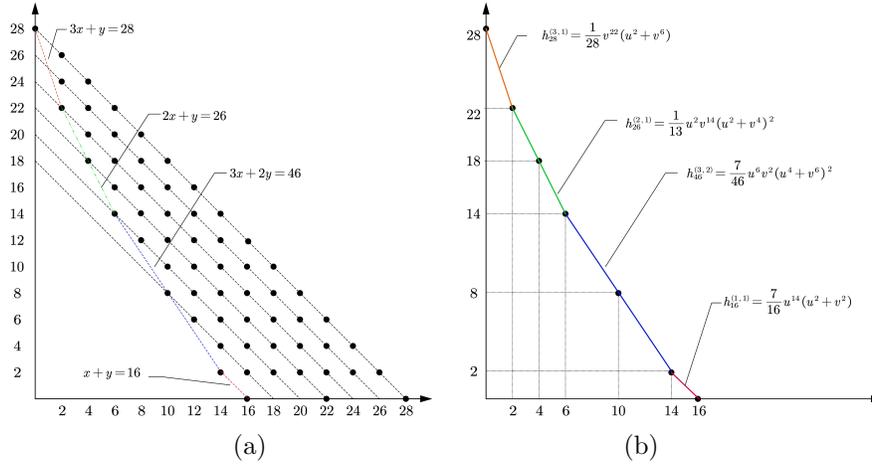


FIGURE 7. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

In the next example, Theorem 1.2 does not work. The polynomial map F is injective by Theorems 1.1, 1.5 and Corollary 1.4.

Example 4.5. Consider the polynomial map $F = (f, g) = (x + x^3, y + x^2)$ with $\det DF(x, y) = 1 + 3x^2 > 0$.

It is easy to verify that for any weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ with $s_2 \geq 2s_1$, the higher \mathbf{s} -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 . Hence, F is injective by Theorem 1.1.

Consider the Hamiltonian vector field

$$\mathcal{X} = (-ff_y - gg_y, ff_x + gg_x) = (-y - x^2, x + 2xy + 6x^3 + 3x^5).$$

The higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ have the real linear common factor x with multiplicity 2. Therefore, Theorem 1.2 does not work.

However, for any weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ with $2s_1 \leq s_2 \leq 4s_1$, the higher \mathbf{s} -quasi-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin in \mathbb{R}^2 . Hence, F is injective by Corollary 1.4.

The Newton diagram of the Bendixson compactified vector field $b(\mathcal{X})$ is shown in Figure 8. There exist two bounded edges in the Newton diagram, and by (2.5) the associated Hamiltonians are

$$h_{36}^{(5,3)} = \frac{v^2}{8}(u^6 + v^{10}), \quad h_8^{(1,1)} = \frac{3u^6}{8}(u^2 + v^2),$$

respectively. Hence, F is injective by Theorem 1.5.

In the next example, Theorems 1.1, 1.2, 1.5 and Corollary 1.4 do not work. The polynomial map F is injective by [18].

Example 4.6 ([21]). Consider the polynomial map $F = (f, g) = (x^3 + y^3 + x, x^3 + y^3 + y)$ with $\det DF(x, y) = 1 + 3(x^2 + y^2) > 0$.

Obviously, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous part of F only vanishes at the origin in \mathbb{R}^2 . Therefore, Theorem 1.1 does not work.

Consider the Hamiltonian vector field

$$\begin{aligned} \mathcal{X} &= (-ff_y - gg_y, ff_x + gg_x) \\ &= \begin{pmatrix} -y - 3xy^2 - x^3 - 4y^3 - 6x^3y^2 - 6y^5 \\ x + 4x^3 + 3x^2y + y^3 + 6x^2y^3 + 6x^5 \end{pmatrix}^T \end{aligned}$$

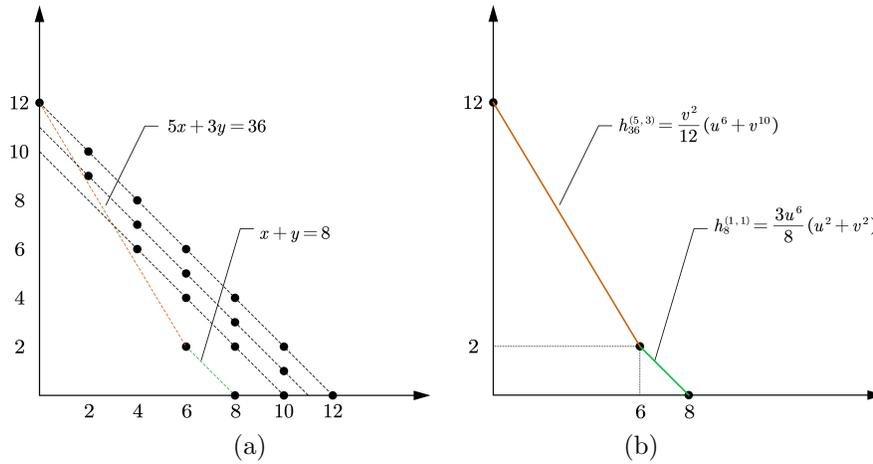


FIGURE 8. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

$$= \begin{pmatrix} -y - x^3 - 3xy^2 - 4y^3 - 6y^2(x+y)(x^2 - xy + y^2) \\ x + 4x^3 + 3x^2y + y^3 + 6x^2(x+y)(x^2 - xy + y^2) \end{pmatrix}^T.$$

The higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ have the real linear common factor $x + y$. Consequently, Theorem 1.2 also does not apply. Moreover, there does not exist a weight exponent $\mathbf{s} = (s_1, s_2) \in \mathbb{N}_+^2$ such that the higher \mathbf{s} -quasi-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ only vanish at the origin in \mathbb{R}^2 . It follows that Corollary 1.4 is invalid.

The Newton diagram of the Bendixson compactified vector field $b(\mathcal{X})$ is shown in Figure 8. There exists only one bounded edge in the Newton diagram, and by (2.5) the associated Hamiltonian is

$$h_8^{(1,1)} = \frac{3}{4}(u+v)^2(u^2+v^2)(u^2-uv+v^2)^2.$$

Note that it has a factor $v + u$, therefore Theorem 1.5 also does not work. Although all these methods have failed, it is injective by [18].

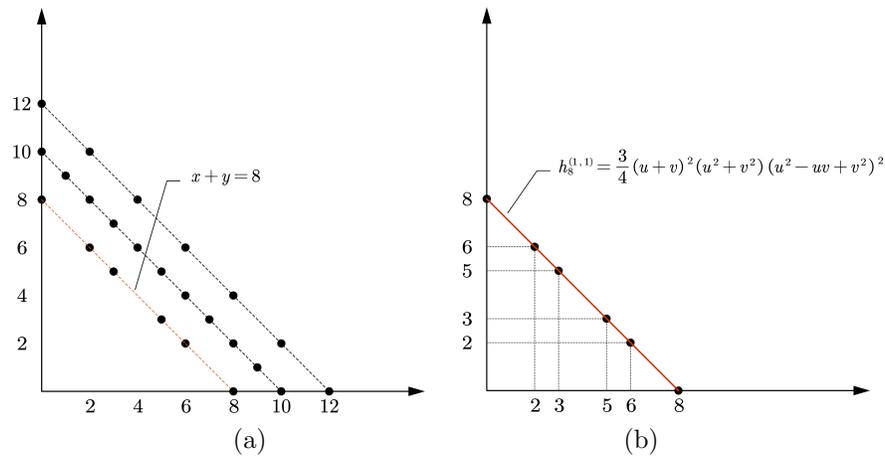


FIGURE 9. (a) Configuration of the support of $b(\mathcal{X})$, (b) Newton diagram for $b(\mathcal{X})$.

To summarize the findings above, the relationships between Theorem 1.1, Theorem 1.2, Theorem 1.5, and Corollary 1.4 are presented in Figure 10. Owing to Example 4.6, additional sufficient conditions remain to be discovered.

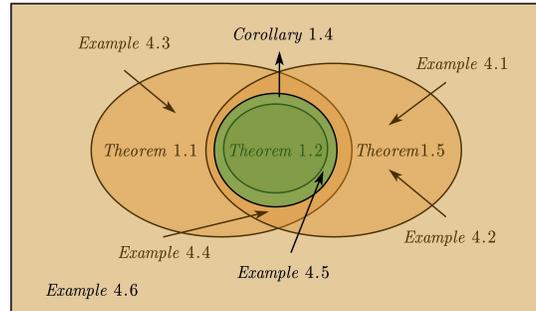


FIGURE 10. Relationships between the sufficient conditions for Theorems 1.1, 1.2, 1.5, and Corollary 1.4.

Acknowledgements. This research was supported by the National Natural Science Foundation of China (No. 12371182). X. Cen and K. Hu were supported by the Hunan Provincial Natural Science Foundation of China (No. 2024JJ4048) and by the Science and Technology Innovation Program of Hunan Province (No. 2024RC3002). Y. Tian was supported by the National Natural Science Foundation of China (No. 12401205), by the Guangdong Basic and Applied Basic Research Foundation (No. 2023A1515110430), by the Science and Technology Planning Project of Guangzhou (No. 2025A04J3404), by the National Funded Postdoctoral Researcher Program (No. GZC20230970) and by the Fundamental Research Funds for the Central Universities (No. 21624347).

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XIULI CEN

SCHOOL OF MATHEMATICS AND STATISTICS, HNP-LAMA, CENTRAL SOUTH UNIVERSITY, CHANGSHA, HUNAN 410083, CHINA

Email address: `cenxiuli2010@163.com`

KANGNONG HU

SCHOOL OF MATHEMATICS AND STATISTICS, HNP-LAMA, CENTRAL SOUTH UNIVERSITY, CHANGSHA, HUNAN 410083, CHINA

Email address: `hukangnong2000@csu.edu.cn`

YUZHOU TIAN

DEPARTMENT OF MATHEMATICS, JINAN UNIVERSITY, GUANGZHOU, GUANGDONG 510632, CHINA

Email address: `tianyuzhou2016@163.com`