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DYNAMICAL PROPERTIES OF CONSTRAINED HEAT FLOW ON HILBERT MANIFOLDS

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Abstract. We study the long-time dynamics of solutions to a nonlinear gradient flow associated with Problem (2.7), where trajectories are constrained to evolve on a manifold. Using energy methods and spectral properties of the Dirichlet Laplacian, we first establish global existence and precompactness of trajectories in the natural energy space. By proving a Łojasiewicz-Simon inequality for the corresponding energy functional, we deduce convergence of all global solutions to stationary equilibria. Moreover, we provide sharp convergence rates: exponential in the case of nondegenerate equilibria, and polynomial otherwise. Finally, we demonstrate the existence of a compact global attractor in $\mathcal{V} \cap \mathcal{M}$ that captures the asymptotic behavior of all bounded trajectories. These results place the problem within the general theory of dissipative gradient systems and give a precise description of its asymptotic dynamics.

1. Introduction

In this paper, we focus on the long-time dynamics of the projected nonlinear heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \pi_u \left(\Delta u - |u|^{2n-2} u \right), \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

posed in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions. Our primary objectives are to establish the convergence of solutions to equilibria, derive precise decay estimates, and prove the existence of a compact global attractor.

The nonlinear term $|u|^{2n-2}u$ arises naturally as the variational derivative of a polynomial potential and appears in a range of diffusion–reaction models. The unit-sphere constraint in $\mathcal{L}^2(\mathcal{O})$ enforces preservation of a global invariant, such as mass, probability normalization, or total intensity, depending on the modeling context. Flows of this type occur, for instance, in normalized diffusion processes, constrained relaxation dynamics, and evolution problems where the state is required to remain on a prescribed manifold. We are mainly interested in investigating the consequences of these structural features, independently of a specific physical realization. Within this line of research, the nonlinear heat flow constrained to the L^2-unit sphere of a Hilbert space has emerged as a natural and analytically tractable model. The projection operator π ensures that the modified vector field $\widetilde{f}(x) = \pi(x)[f(x)]$ is tangent to the manifold, thus preserving invariance. This approach, initiated by Zdzislaw and Javed [3, 18, 19], investigated that the projected nonlinear heat equation admits global solutions, generates a gradient flow on the manifold, and maintains invariance of the constraint.

The projection of parabolic dynamics onto constrained manifolds has been examined in several seminal works. Rybka [26] and Caffarelli–Lin [7] considered the heat equation in $\mathcal{L}^2(\mathcal{O})$ under algebraic constraints of the form

$$\mathcal{M} = \left\{ u \in \mathcal{L}^2(\mathcal{O}) \cap C(\mathcal{O}) : \int_{\mathcal{O}} u^k(x)dx = C_k,\ k = 1, 2, \dots, N \right\},$$

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where $\mathcal{O}$ is a bounded, connected region in $\mathbb{R}^2$. Rybka established the global existence and uniqueness of solutions to the projected heat flow

$$\begin{gathered}\frac{du}{dt} = \Delta u - \sum_{k=1}^{N} \lambda_k u^{k-1} \quad \text{in } \mathcal{O} \subset \mathbb{R}^2,\\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\mathcal{O},\\ u(0,x) = u_0\end{gathered} \tag{1.2}$$

with Lagrange multipliers $\lambda_k(u)$ chosen so that u_t is orthogonal to $\operatorname{Span}\{u^{k-1}\}$. He proved that solutions converge asymptotically to constant states. In related work, Caffarelli and Lin [7] proved global well-posedness of energy-conserving heat flows, and extended their framework to singularly perturbed nonlocal parabolic systems, establishing convergence to weak solutions of limiting constrained flows.

The constrained dynamics paradigm has since been extended to fluid equations and other PDEs. For example, Brzeźniak, Dhariwal, and Mariani [6] studied two-dimensional Navier–Stokes equations under an energy constraint, proving global existence, uniqueness, and convergence to Euler flows as viscosity tends to zero. These works illustrate the broader significance of projecting dissipative PDEs onto constraint manifolds.

This article continues the line of work on parabolic evolutions constrained to manifolds initiated for heat-type flows in [3, 7, 18, 19, 26] and subsequently developed for other dissipative equations. In contrast with the linear heat flow under algebraic constraints treated in [7, 26], we consider a power-type reaction term under the $\mathcal{L}^2$-unit-sphere constraint and homogeneous Dirichlet boundary conditions. This leads to a constrained evolution driven by the Dirichlet Laplacian and a superlinear Nemytskii nonlinearity, where the constraint is enforced through the orthogonal projection onto the tangent bundle of $\mathcal{M}$. Within this framework, the contribution is threefold. First, we derive an explicit projected vector field and verify that the induced semiflow is the gradient flow of the energy restricted to $\mathcal{M}$, so that the energy identity provides a Lyapunov structure on the manifold. Second, we establish a constrained Łojasiewicz-Simon inequality near every stationary point $\varphi \in \mathcal{S}$ in a form adapted to the Riemannian structure of the $\mathcal{L}^2$-sphere; this yields convergence of every global trajectory to a single equilibrium and permits a sharp dichotomy between exponential decay near nondegenerate equilibria and algebraic decay otherwise. Third, combining the resulting asymptotic compactness with the dissipative bounds supplied by the Lyapunov functional, we prove the existence of a compact global attractor in $\mathcal{V} \cap \mathcal{M}$.

The broader context of convergence to equilibrium in parabolic flows is well studied (see [1, 2, 12, 13, 16, 15, 31, 23, 24, 27]). For decay rates we refer to [14, 28], and for attractor theory in dissipative PDEs to [8, 28, 22, 30]. Our results place the projected nonlinear heat equation firmly within this classical framework while highlighting its structural and geometric features.
Outline of the paper. Section 2 collects notations, functional spaces, and preliminary results. Section 3 establishes pre-compactness of solution orbits, characterizes the ω-limit set, and proves a Łojasiewicz-Simon inequality on the manifold. Section 4 derives decay rates for convergence to equilibria, distinguishing exponential and algebraic regimes. Section 5 proves the existence of a global attractor in $\mathcal{M} \cap H^1$. Together, these results provide a mathematically rigorous and comprehensive description of the asymptotic dynamics of the constrained nonlinear heat flow.

2. Functional settings, assumptions and preliminaries

Let us set the notation that will be used throughout this paper.

2.1. **Manifold and projection.** In this paper we work with the unit sphere

$$\mathcal{M} = \{u \in \mathcal{H} : |u|^2_{\mathcal{H}} = 1\}$$

as a smooth submanifold of the Hilbert space $\mathcal{H}$ endowed with inner product $\langle \cdot, \cdot \rangle$. It is well known that $\mathcal{M}$ is a Hilbert (Riemannian) submanifold of $\mathcal{H}$. For any $a \in \mathcal{M}$, the tangent space is

$$T_a\mathcal{M} = \{v \in \mathcal{H} : \langle a, v \rangle = 0\}.$$

Let $\pi_a : \mathcal{H} \longrightarrow T_a\mathcal{M}$ be the orthogonal projection onto $T_a\mathcal{M}$.

Lemma 2.1. *If $a \in \mathcal{M}$, then for all $v \in \mathcal{H}$,*

$$\pi_a(v) = v - \langle a, v\rangle a.$$

Remark 2.2. Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded C^2 domain and let $n \in [1,\infty)$ be such that the Sobolev embedding

$$\mathcal{H}^{1,2}(\mathcal{O}) \hookrightarrow \mathcal{L}^{2n}(\mathcal{O})$$

holds. Equivalently:

- if $d = 2$, the embedding holds for every finite $2n$;
- if $d \geq 3$, we assume $2n \leq 2^* := \frac{2d}{d-2}$, i.e.

$$\frac{1}{d} \geq \frac{1}{2} - \frac{1}{2n}. \tag{2.1}$$

Throughout we adopt the following spaces:

$$\mathcal{H} = \mathcal{L}^2(\mathcal{O}), \quad \mathcal{V} = \mathcal{H}_0^{1,2}(\mathcal{O}), \quad \mathcal{E} = \mathcal{D}(A),$$

where A is the (negative) Laplace operator with homogeneous Dirichlet boundary conditions,

$$\begin{aligned} \mathcal{D}(A) &= \mathcal{H}_0^{1,2}(\mathcal{O}) \cap \mathcal{H}^{2,2}(\mathcal{O}), \\ Au &= -\Delta u, \quad u \in \mathcal{D}(A). \end{aligned} \tag{2.2}$$

It is classical (see [32, Theorem 4.1.2, p. 79]) that A is self-adjoint and positive on $\mathcal{H}$, that $\mathcal{V} = \mathcal{D}(A^{1/2})$, and

$$\|u\|^2 = |A^{1/2}u|_{\mathcal{H}}^2 = \int_{\mathcal{O}} |\nabla u(x)|^2 dx.$$

Moreover,

$$\mathcal{E} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}' =: \mathcal{H}^{-1}(\mathcal{O}),$$

with all injections continuous and dense.

For each $T \geq 0$ we set

$$\mathcal{X}_T := \mathcal{L}^2(0,T;\mathcal{E}) \cap C([0,T];\mathcal{V}). \tag{2.3}$$

Then $\mathcal{X}_T$ is a Banach space with norm

$$|u|_{\mathcal{X}_T}^2 = \sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |u(t)|_{\mathcal{E}}^2 dt, \quad u \in \mathcal{X}_T. \tag{2.4}$$

Corollary 2.3. *Within the framework of Remark 2.2, for any $u \in \mathcal{E} \cap \mathcal{M}$ we have the identity*

$$\pi_u\left(\Delta u - |u|^{2n-2}u\right) = \Delta u - |u|^{2n-2}u + \left(\|u\|^2 + |u|_{\mathcal{L}^{2n}}^{2n}\right)u. \tag{2.5}$$

Proof. Let $u \in \mathcal{E} \cap \mathcal{M}$. Then $\Delta u, |u|^{2n-2}u \in \mathcal{H}$ by elliptic regularity and the embedding from Remark 2.2. By Lemma 2.1,

$$\begin{aligned} \pi_u\left(\Delta u - |u|^{2n-2}u\right) &= \Delta u - |u|^{2n-2}u - \langle \Delta u - |u|^{2n-2}u, u\rangle u \\ &= \Delta u - |u|^{2n-2}u + \langle \nabla u, \nabla u\rangle u + \langle |u|^{2n-2}u, u\rangle u \\ &= \Delta u - |u|^{2n-2}u + \left(\|u\|^2 + |u|_{\mathcal{L}^{2n}}^{2n}\right)u. \end{aligned} \tag{2.6}$$

Here we used integration by parts with Dirichlet boundary conditions, cf. [5, Corollary 8.10, p. 82], and the identity $\int_{\mathcal{O}} |u|^{2n-2}u \cdot u = \|u\|_{\mathcal{L}^{2n}}^{2n}$. ∎

The next result summarizes the well-posedness and gradient-flow structure for the projected equation; the ensuing energy relation will be used repeatedly.

Theorem 2.4 ([18])**.** *For every $u_0 \in \mathcal{H}_0^{1,2}(\mathcal{O}) \cap \mathcal{M}$, where $\mathcal{M} = \{u \in \mathcal{L}^2(\mathcal{O}) : |u|_{\mathcal{L}^2(\mathcal{O})} = 1\}$, there exists a unique function $u : [0,\infty) \to \mathcal{H}_0^{1,2}(\mathcal{O})$ such that $u \in \mathcal{X}_T$ for every $T > 0$ and*

$$\begin{gathered} \frac{\partial u}{\partial t} = \Delta u - |u|^{2n-2}u + \left(\|u\|^2 + |u|^{2n}_{\mathcal{L}^{2n}}\right) u, \\ u(0) = u_0. \end{gathered} \tag{2.7}$$

Moreover, $u(t) \in \mathcal{M}$ for all $t \geq 0$. In particular, the energy identity holds:

$$\Phi(u(t)) - \Phi(u_0) = -\int_0^t |\frac{du}{dt}(s)|^2_{\mathcal{H}} ds, \quad t \geq 0, \tag{2.8}$$

and

$$\|u(t)\| \leq 2\Phi(u_0), \quad t \geq 0,$$

where $\Phi : \mathcal{V} \to \mathbb{R}$ is defined by

$$\Phi(u) = \frac{1}{2}|\nabla u|^2_{\mathcal{L}^2(\mathcal{O})} + \frac{1}{2n}|u|^{2n}_{\mathcal{L}^{2n}(\mathcal{O})}, \quad n \in \mathbb{N}.$$

Remark 2.5. The existence and uniqueness asserted in Theorem 2.4 are obtained by a standard contraction argument applied to the mild formulation of (2.7). The Dirichlet Laplacian generates an analytic semigroup on $\mathcal{L}^2(\mathcal{O})$, and under the standing restrictions on n and the spatial dimension, the Nemytskii map $u \mapsto |u|^{2n-2}u$ is locally Lipschitz from $\mathcal{H}_0^{1,2}(\mathcal{O})$ into $\mathcal{L}^2(\mathcal{O})$. The additional lower-order term involving $\|u\|^2$ and $|u|^{2n}_{\mathcal{L}^{2n}}$ is smooth on bounded sets and does not affect the contraction property on short time intervals. The invariance of $\mathcal{M}$ follows from the fact that the right-hand side of (2.7) is orthogonal in $\mathcal{L}^2(\mathcal{O})$ to $u(t)$, which implies $\frac{d}{dt}|u(t)|^2_{\mathcal{L}^2} = 0$. Global existence is then obtained by combining the local theory with the energy identity (2.8), which prevents finite-time blow-up.

For convenience we recall some definitions and abstract results on convergence to equilibrium and global attractors from Chapter 6 of [28], which will be used throughout.

Corollary 2.6 ([28])**.** *Suppose that $\mathcal{H}$ is a complete metric space and $S(t)$ is a nonlinear C_0-semigroup on $\mathcal{H}$. Let $x \in \mathcal{H}$. If there exists $t_0 \geq 0$ such that*

$$\{u(t) : t > t_0\} := \cup_{t>t_0} S(t)x$$

is relatively compact in $\mathcal{H}$, then the ω-limit set $\omega(x)$ is a compact, connected, invariant set.

Theorem 2.7. [28] *Let $\Gamma : \mathbb{R}^N \to \mathbb{R}$ be analytic in a neighborhood of a point $a \in \mathbb{R}^N$. Then there exist $\sigma > 0$ and $\theta \in (0, \frac{1}{2}]$ such that for all $x \in \mathbb{R}^N$,*

$$\|x - a\| < \sigma \implies \|\nabla\Gamma(x)\| \geq |\Gamma(x) - \Gamma(a)|^{1-\theta}.$$

Definition 2.8 ([28])**.** Suppose that $\mathcal{H}$ is a complete metric space, and $S(t)$ is a nonlinear C_0-semigroup on $\mathcal{H}$. A set $\mathcal{A} \subset \mathcal{H}$ is an *attractor* if:

(i) $\mathcal{A}$ is invariant: $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$;
(ii) there exists an open neighborhood $\mathcal{U}$ of $\mathcal{A}$ such that for every $u_0 \in \mathcal{U}$,

$$\operatorname{dist}(S(t)u_0, \mathcal{A}) = \inf_{y\in\mathcal{A}} d\,(S(t)u_0, y) \longrightarrow 0 \quad \text{as } t \to \infty.$$

Definition 2.9 ([28])**.** If $\mathcal{A}$ is a compact attractor and it attracts all bounded subsets of $\mathcal{H}$, then $\mathcal{A}$ is called a *global* (or *universal*) attractor.

Theorem 2.10 ([28])**.** *Suppose that $\mathcal{H}$ is a Banach space and $S(t)$ is a nonlinear C_0-semigroup on $\mathcal{H}$ such that:*

(i) *there exists a bounded absorbing set $\mathcal{B}_0$;*
(ii) *for any bounded set $\mathcal{B}$ there exists $t_0(\mathcal{B}) \geq 0$ with $\cup_{t\geq t_0(\mathcal{B})} S(t)\mathcal{B}$ relatively compact in $\mathcal{H}$.*

Then $\mathcal{A} = \omega(\mathcal{B}_0)$ is a global attractor.

We finally recall the following compactness lemma (see [30, Chapter 3, Lemma 1.2]).

Lemma 2.11 ([30, Lemma III 1.2]). *Let $\mathcal{V}, \mathcal{H}, \mathcal{V}'$ be Hilbert spaces with $\mathcal{V}'$ the dual of $\mathcal{V}$ and*

$$\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}',$$

with dense continuous embeddings. If $u \in \mathcal{L}^2(0,T;\mathcal{V})$ and its weak derivative $u_t \in \mathcal{L}^2(0,T;\mathcal{V}')$, then there exists $\tilde{u} \in \mathcal{L}^2(0,T;\mathcal{V}) \cap C([0,T];\mathcal{V})$ such that $\tilde{u} = u$ a.e., and

$$|u(t)|^2 = |u_0|^2 + 2\int_0^t \langle u'(s), u(s)\rangle ds, \quad \forall t \in [0,T].$$

3. Łojasiewicz-Simon inequality and convergence to equilibrium

In this section we prove convergence to equilibrium for the global solutions constructed in Theorem 2.4. The argument combines two ingredients. On the one hand, the orbit $\{u(t) : t \geq 1\}$ is shown to be relatively compact in $\mathcal{V}$, which implies that the ω-limit set is a nonempty compact subset of $\mathcal{V}$. On the other hand, near each stationary point $\varphi \in \mathcal{S}$ we establish a Łojasiewicz-Simon inequality for the energy restricted to the manifold $\mathcal{M}$, formulated in terms of the projected gradient $\pi_u(\Delta u - |u|^{2n-2}u)$. The convergence theorem then follows from the energy identity (2.8) and the fact that the orbit ultimately remains in a neighborhood where a single Łojasiewicz-Simon inequality applies.

Theorem 3.1. *Let $u_0 \in \mathcal{V} \cap \mathcal{M}$, and let $u(\cdot)$ be the global solution of (2.7) given by Theorem 2.4. Then the orbit $\{u(t) : t \geq 0\}$ is relatively compact in $\mathcal{V}$. Equivalently, for every sequence $t_k \to \infty$ there exist a subsequence (not relabelled) and $u^* \in \mathcal{V}$ such that $u(t_k) \to u^*$ in $\mathcal{V}$.*

Proof. For $\alpha \in (0,1)$ denote by A^α the fractional powers of A and set $\mathcal{D}(A^\alpha)$ as usual; the compactness of the resolvent implies that the embedding $\mathcal{D}(A^\alpha) \hookrightarrow \mathcal{H}$ is compact for every $\alpha > 0$. Moreover, for $\alpha > \frac{1}{2}$ one has the continuous embedding

$$\mathcal{D}(A^\alpha) \hookrightarrow \mathcal{V} = H_0^1(\mathcal{O}), \tag{3.1}$$

and the embedding $\mathcal{D}(A^\alpha) \hookrightarrow \mathcal{V}$ is compact. This follows, for instance, from the spectral representation of A and the characterisation $\mathcal{D}(A^{1/2}) = H_0^1(\mathcal{O})$.

Write (2.7) in the semilinear form

$$u'(t) + Au(t) = F(u(t)), \quad t > 0, \tag{3.2}$$

where

$$F(u) := -|u|^{2n-2}u + \big(\|u\|_{\mathcal{V}}^2 + |u|_{L^{2n}}^{2n}\big)u.$$

Fix $\alpha \in (\frac{1}{2}, 1)$ and $T > 0$. The variation-of-constants formula for (3.2) gives, for all $t \geq T$,

$$u(t) = e^{-(t-T)A}u(T) + \int_T^t e^{-(t-s)A}F(u(s))\,ds. \tag{3.3}$$

Applying A^α and using analyticity of the semigroup yields

$$\|A^\alpha e^{-\tau A}\|_{\mathcal{L}(\mathcal{H})} \leq C_\alpha\,\tau^{-\alpha}, \quad \tau \in (0,1], \tag{3.4}$$

and $\|A^\alpha e^{-\tau A}\|_{\mathcal{L}(\mathcal{H})} \leq C_\alpha$ for $\tau \geq 1$.

The energy identity (2.8) implies that $t \mapsto \Phi(u(t))$ is non-increasing, hence $\sup_{t\geq 0}\Phi(u(t)) \leq \Phi(u_0)$. In particular,

$$\sup_{t\geq 0}\|u(t)\|_{\mathcal{V}} < \infty, \tag{3.5}$$

since $\Phi(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2$. The constraint $u(t) \in \mathcal{M}$ gives $|u(t)|_{\mathcal{H}} = 1$ for all $t \geq 0$. Under the standing restriction on n and the spatial dimension ensuring $\mathcal{V} \hookrightarrow L^{2n}(\mathcal{O})$, the bound (3.5) yields

$$\sup_{t\geq 0}|u(t)|_{L^{2n}} < \infty. \tag{3.6}$$

Consequently, each term in $F(u(t))$ is uniformly bounded in $\mathcal{H}$ for $t \geq 0$. Indeed, the Nemytskii map $u \mapsto |u|^{2n-2}u$ maps $L^{2n}(\mathcal{O})$ into $L^2(\mathcal{O})$, and (3.6) shows that $|\,|u(t)|^{2n-2}u(t)|_{\mathcal{H}}$ is bounded uniformly in time. The remaining lower-order term is bounded in $\mathcal{H}$ as well because $\|u(t)\|_{\mathcal{V}}$ and $|u(t)|_{L^{2n}}$ are uniformly bounded and $|u(t)|_{\mathcal{H}} = 1$. Hence there exists $M > 0$ such that

$$\sup_{t\geq 0} |F(u(t))|_{\mathcal{H}} \leq M. \tag{3.7}$$

Fix any sequence $t_k \to \infty$. Choose $T \geq 1$ so that $t_k \geq T+1$ for all k large, and write (3.3) at $t = t_k$:

$$u(t_k) = e^{-(t_k-T)A}u(T) + \int_T^{t_k} e^{-(t_k-s)A}F(u(s))\,ds. \tag{3.8}$$

Applying A^α and estimating with (3.4) and (3.7) yields

$$\begin{aligned}\|A^\alpha u(t_k)\|_{\mathcal{H}} &\leq \|A^\alpha e^{-(t_k-T)A}u(T)\|_{\mathcal{H}} + \int_T^{t_k} \|A^\alpha e^{-(t_k-s)A}F(u(s))\|_{\mathcal{H}}\,ds \\ &\leq C_\alpha \|u(T)\|_{\mathcal{H}} + C_\alpha M \int_0^{t_k-T} \tau^{-\alpha}\,d\tau.\end{aligned}$$

Since $\alpha < 1$, the integral is finite and bounded uniformly in k:

$$\int_0^{t_k-T} \tau^{-\alpha}\,d\tau \leq \int_0^\infty \tau^{-\alpha}\mathbf{1}_{(0,1]}(\tau)\,d\tau + \int_1^\infty \tau^{-\alpha}\,d\tau < \infty.$$

Thus $\{u(t_k)\}$ is bounded in $\mathcal{D}(A^\alpha)$.

Because A has compact resolvent, the embedding $\mathcal{D}(A^\alpha) \hookrightarrow \mathcal{V}$ is compact for every $\alpha > \frac{1}{2}$ by (3.1). Therefore, the bounded sequence $\{u(t_k)\}$ admits a subsequence converging in $\mathcal{V}$. This proves the relative compactness of the orbit in $\mathcal{V}$. □

Corollary 3.2. *The ω-limit set $\omega(u_0) = \bigcap_{r\geq 1} \overline{\{u(t) : t \geq r\}}^{\mathcal{V}}$ exists and is compact in $\mathcal{V}$.*

Proof. From last corollary, $\{u(t) : t \geq r\}$ is pre-compact in $\mathcal{V}$ for all $r \geq 1$. Since closure of pre-compact is also pre-compact so $\overline{\{u(t) : t \geq r\}}$ is also pre-compact for all $r \geq 1$. Further $\overline{\{u(t) : t \geq r\}}$ is closed and hence complete in $\mathcal{V}$ norm therefore $\overline{\{u(t) : t \geq r\}}$ being pre-compact and complete, it follows that $\overline{\{u(t) : t \geq r\}}$ is compact for all $r \geq 1$, in $\mathcal{V}$. Thus $\omega(u_0)$ being decreasing intersection of non-empty compact sets in $\mathcal{V}$, is non-empty and compact in $\mathcal{V}$. □

Łojasiewicz-Simon inequality on $\mathcal{M}$. Let $\mathcal{S}$ denote the set of stationary points of (2.7) on $\mathcal{M}$ (critical points of Φ under the constraint $|u|_{\mathcal{H}} = 1$). The next result is a constrained Łojasiewicz-Simon inequality; the proof follows the abstract LS theory (see, e.g., [28] and the references therein) and the argument of Jendoubi [21], adapted to the present manifold setting.

Theorem 3.3 (Łojasiewicz-Simon inequality). *Let $\varphi \in \mathcal{V} \cap \mathcal{M}$ be a stationary point of* (2.7). *Then there exist constants $\sigma > 0$, $\theta \in (0, \frac{1}{2}]$, and $C > 0$ such that*

$$|\Phi(u) - \Phi(\varphi)|^{1-\theta} \leq C\|\nabla_{\mathcal{M}}\Phi(u)\|_{\mathcal{H}}, \quad u \in \mathcal{V} \cap \mathcal{M}, \|u - \varphi\|_{\mathcal{V}} < \sigma, \tag{3.9}$$

where $\nabla_{\mathcal{M}}\Phi(u)$ denotes the $\mathcal{H}$-gradient of Φ restricted to the manifold $\mathcal{M}$.

Proof. The manifold is

$$\mathcal{M} = \{u \in \mathcal{H} : \ G(u) = 0\}, \quad G(u) := \frac{1}{2}\big(|u|_{\mathcal{H}}^2 - 1\big),$$

and G is a real-analytic mapping $\mathcal{H} \to \mathbb{R}$. Its derivative satisfies $G'(u)h = (u, h)_{\mathcal{H}}$, hence $G'(\varphi) \neq 0$ and the regular value theorem yields that $\mathcal{M}$ is a real-analytic embedded submanifold of $\mathcal{H}$ in a neighbourhood of φ. The tangent space is

$$T_\varphi\mathcal{M} = \ker G'(\varphi) = \{h \in \mathcal{H} : \ (h, \varphi)_{\mathcal{H}} = 0\}.$$

Write $P_\varphi : \mathcal{H} \to T_\varphi\mathcal{M}$ for the orthogonal projection $P_\varphi h = h - (h, \varphi)_{\mathcal{H}}\varphi$.

A convenient analytic chart is obtained by the normalisation map. Let

$$U := \{h \in T_\varphi\mathcal{M} : \ |h|_{\mathcal{H}} < \tfrac{1}{2}\}, \quad \Psi(h) := \frac{\varphi + h}{|\varphi + h|_{\mathcal{H}}}, \quad h \in U.$$

Since $|\varphi|_{\mathcal{H}} = 1$ and $h \perp \varphi$, one has $|\varphi + h|^2_{\mathcal{H}} = 1 + |h|^2_{\mathcal{H}}$, so the denominator is bounded away from 0 on U. The map $h \mapsto (1 + |h|^2_{\mathcal{H}})^{-1/2}$ is real-analytic on $\{|h|_{\mathcal{H}} < 1\}$, and therefore $\Psi : U \to \mathcal{H}$ is real-analytic. Moreover, $\Psi(U) \subset \mathcal{M}$ by construction, $\Psi(0) = \varphi$, and the differential satisfies $D\Psi(0) = \mathrm{Id}$ on $T_\varphi\mathcal{M}$. In particular, Ψ is a real-analytic immersion.

To see that Ψ parametrises a full neighbourhood of φ in $\mathcal{M}$, consider the analytic map

$$\Xi : \mathcal{M} \cap B_{\mathcal{H}}(\varphi, \tfrac{1}{2}) \to T_\varphi\mathcal{M}, \quad \Xi(u) := P_\varphi(u - \varphi).$$

If $u \in \mathcal{M}$, then $|u|_{\mathcal{H}} = |\varphi|_{\mathcal{H}} = 1$ and the identity $|u-\varphi|^2_{\mathcal{H}} = 2\big(1-(u,\varphi)_{\mathcal{H}}\big)$ implies that $(u,\varphi)_{\mathcal{H}} > 0$ whenever u is sufficiently close to φ. For such u set $\alpha(u) := (u,\varphi)_{\mathcal{H}} \in (0,1]$ and $h := \Xi(u) \in T_\varphi\mathcal{M}$. Then $u = \alpha(u)\varphi + h$ with $h \perp \varphi$ and $|u|^2_{\mathcal{H}} = \alpha(u)^2 + |h|^2_{\mathcal{H}} = 1$, so $\alpha(u) = \sqrt{1 - |h|^2_{\mathcal{H}}}$. Hence

$$u = \sqrt{1 - |h|^2_{\mathcal{H}}}\,\varphi + h = \frac{\varphi + \frac{h}{\sqrt{1-|h|^2_{\mathcal{H}}}}}{\big|\varphi + \frac{h}{\sqrt{1-|h|^2_{\mathcal{H}}}}\big|_{\mathcal{H}}} = \Psi\Big(\frac{h}{\sqrt{1-|h|^2_{\mathcal{H}}}}\Big).$$

For u close to φ, $|h|_{\mathcal{H}}$ is small and $h \mapsto h/\sqrt{1-|h|^2_{\mathcal{H}}}$ maps a neighbourhood of 0 in $T_\varphi\mathcal{M}$ into U real-analytically. This shows that Ψ is a real-analytic bijection from a neighbourhood of 0 in $T_\varphi\mathcal{M}$ onto a neighbourhood of φ in $\mathcal{M}$, with analytic inverse, hence a real-analytic chart.

We define the reduced functional on the Hilbert space $T_\varphi\mathcal{M}$ by

$$\mathcal{F}(h) := \Phi(\Psi(h)), \quad h \in U.$$

Since Φ is real-analytic on $\mathcal{V}$ and $\Psi(h) \in \mathcal{V}$ for h small (because $\Psi(h)$ differs from φ by a small $\mathcal{H}$–perturbation and $\varphi \in \mathcal{V}$), the map $\mathcal{F}$ is real-analytic on U as a function on $T_\varphi\mathcal{M}$. The point $h = 0$ is critical: by the chain rule,

$$D\mathcal{F}(0)k = \langle \Phi'(\varphi), D\Psi(0)k\rangle = \langle \Phi'(\varphi), k\rangle, \quad k \in T_\varphi\mathcal{M},$$

and stationarity of φ means precisely that $\Phi'(\varphi)$ annihilates $T_\varphi\mathcal{M}$, so $D\mathcal{F}(0) = 0$.

Let $\nabla\mathcal{F}(h)$ denote the gradient of $\mathcal{F}$ in the Hilbert space $T_\varphi\mathcal{M}$ with the inherited $\mathcal{H}$ inner product. For $k \in T_\varphi\mathcal{M}$,

$$(\nabla\mathcal{F}(h), k)_{\mathcal{H}} = D\mathcal{F}(h)k = \big(\nabla_{\mathcal{H}}\Phi(\Psi(h)), D\Psi(h)k\big)_{\mathcal{H}}.$$

Set $u = \Psi(h)$. Since Ψ takes values in $\mathcal{M}$, one has $\operatorname{Ran} D\Psi(h) \subset T_u\mathcal{M}$. Write π_u for the orthogonal projection $\mathcal{H} \to T_u\mathcal{M}$. Then

$$\big(\nabla_{\mathcal{H}}\Phi(u), D\Psi(h)k\big)_{\mathcal{H}} = \big(\pi_u\nabla_{\mathcal{H}}\Phi(u), D\Psi(h)k\big)_{\mathcal{H}}.$$

The operator $D\Psi(h) : T_\varphi\mathcal{M} \to T_u\mathcal{M}$ is a bounded isomorphism for h sufficiently small, because $D\Psi(0) = \mathrm{Id}$ and $D\Psi$ depends continuously on h. Consequently the operator

$$\mathcal{A}(h) := \big(D\Psi(h)\big)^* : T_u\mathcal{M} \to T_\varphi\mathcal{M}$$

is bounded and invertible for h small, and

$$\nabla\mathcal{F}(h) = \mathcal{A}(h)\big(\pi_u\nabla_{\mathcal{H}}\Phi(u)\big). \tag{3.10}$$

In particular, since $\mathcal{A}(h)$ varies continuously and is invertible near 0, there exist $c_1, c_2 > 0$ and a neighbourhood $U_0 \subset U$ of 0 such that for all $h \in U_0$,

$$c_1\|\pi_{\Psi(h)}\nabla_{\mathcal{H}}\Phi(\Psi(h))\|_{\mathcal{H}} \le \|\nabla\mathcal{F}(h)\|_{\mathcal{H}} \le c_2\|\pi_{\Psi(h)}\nabla_{\mathcal{H}}\Phi(\Psi(h))\|_{\mathcal{H}}. \tag{3.11}$$

Let $L := D(\nabla\mathcal{F})(0)$ be the linearization of the gradient map at 0. A direct differentiation of (3.10) at $h = 0$, using $D\Psi(0) = \mathrm{Id}$ and the fact that φ is a critical point, shows that L is the restriction of the second variation of Φ to $T_\varphi\mathcal{M}$. In the present setting, L is a self-adjoint operator on $T_\varphi\mathcal{M}$ with compact resolvent, inherited from the elliptic part of Φ through the Dirichlet Laplacian. This is the spectral hypothesis required in the Łojasiewicz-Simon theorem [27].

The classical Łojasiewicz-Simon inequality applied to the real-analytic functional $\mathcal{F}$ at the critical point 0 therefore yields constants $\rho > 0$, $\theta \in (0, \frac{1}{2}]$, and $C_0 > 0$ such that

$$|\mathcal{F}(h) - \mathcal{F}(0)|^{1-\theta} \le C_0\|\nabla\mathcal{F}(h)\|_{\mathcal{H}}, \quad h \in T_\varphi\mathcal{M},\ |h|_{\mathcal{H}} < \rho. \tag{3.12}$$

Returning to $u = \Psi(h)$, one has $\mathcal{F}(h) = \Phi(u)$ and $\mathcal{F}(0) = \Phi(\varphi)$. Combining (3.12) with the right-hand inequality in (3.11) gives

$$|\Phi(u) - \Phi(\varphi)|^{1-\theta} \le C_0 \|\nabla \mathcal{F}(h)\|_{\mathcal{H}} \le C_0 c_2 \, \|\pi_u \nabla_{\mathcal{H}} \Phi(u)\|_{\mathcal{H}}.$$

Since Ψ is a local diffeomorphism between neighbourhoods of 0 and φ, smallness of h is equivalent to smallness of $u - \varphi$ in $\mathcal{H}$, and hence, on bounded sets, also in $\mathcal{V}$ by elliptic regularity for the resolvent of A. Shrinking the neighbourhood if necessary yields (3.9) with $\sigma > 0$ and $C := C_0 c_2$. □

Convergence to equilibrium. We now combine the LS inequality with the energy identity to obtain convergence of trajectories.

Theorem 3.4 (Convergence to a single equilibrium)**.** *Let u be the global solution of* (2.7) *with initial datum $u_0 \in \mathcal{V} \cap \mathcal{M}$. Then there exists $u^\infty \in \mathcal{V} \cap \mathcal{M}$ such that*

$$\lim_{t\to\infty} \|u(t) - u^\infty\|_{\mathcal{V}} = 0.$$

Moreover, u^∞ is a stationary point of the constrained dynamics, in the sense that

$$\pi_{u^\infty}\big(\Delta u^\infty - |u^\infty|^{2n-2} u^\infty\big) = 0, \tag{3.13}$$

equivalently,

$$(\Delta u^\infty - |u^\infty|^{2n-2} u^\infty, \ v)_{\mathcal{H}} = 0 \quad \textit{for every } v \in T_{u^\infty}\mathcal{M}.$$

Proof. The energy identity (2.8) implies that $t \mapsto \Phi(u(t))$ is non-increasing and bounded from below; hence there exists $\Phi_\infty \in \mathbb{R}$ such that

$$\Phi(u(t)) \longrightarrow \Phi_\infty \quad \text{as } t \to \infty. \tag{3.14}$$

Moreover, (2.8) yields

$$\int_0^\infty \|\partial_t u(t)\|_{\mathcal{H}}^2 \, dt < \infty. \tag{3.15}$$

By the precompactness result (Theorem 3.1 or its analogue in the manuscript), the orbit $\{u(t) : t \ge 0\}$ is relatively compact in $\mathcal{V}$. Consequently, the ω-limit set

$$\omega(u_0) := \big\{\varphi \in \mathcal{V} \cap \mathcal{M} : \exists\, t_k \to \infty \text{ with } u(t_k) \to \varphi \text{ in } \mathcal{V}\big\}$$

is nonempty and compact in $\mathcal{V}$, and the semiflow invariance gives $S(t)\omega(u_0) = \omega(u_0)$ for every $t \ge 0$. The convergence (3.14) and the continuity of Φ on bounded subsets of $\mathcal{V}$ imply that

$$\Phi(\varphi) = \Phi_\infty \quad \text{for every } \varphi \in \omega(u_0). \tag{3.16}$$

Fix $\varphi \in \omega(u_0)$ and choose $t_k \to \infty$ such that $u(t_k) \to \varphi$ in $\mathcal{V}$. From (3.15) there exists a further subsequence (not relabelled) such that $\partial_t u(t_k) \to 0$ in $\mathcal{H}$. Passing to the limit in (2.7) at times t_k uses the strong convergence in $\mathcal{V}$ (hence in $\mathcal{H}$) and the continuity of the Nemytskii map $w \mapsto |w|^{2n-2} w$ from $\mathcal{V}$ into $\mathcal{H}$ under the standing restrictions on n and the spatial dimension. One obtains

$$0 = \lim_{k\to\infty} \partial_t u(t_k) = \Delta\varphi - |\varphi|^{2n-2}\varphi + \big(\|\varphi\|^2 + |\varphi|_{L^{2n}}^{2n}\big)\varphi \quad \text{in } \mathcal{H},$$

and since $\varphi \in \mathcal{M}$ the last term is a Lagrange multiplier in the normal direction. Equivalently,

$$\pi_\varphi\big(\Delta\varphi - |\varphi|^{2n-2}\varphi\big) = 0,$$

so every $\varphi \in \omega(u_0)$ is a stationary point of the constrained flow.

The convergence of the full trajectory follows from the Łojasiewicz-Simon inequality at points of $\omega(u_0)$. For each $\varphi \in \omega(u_0)$, Theorem 3.3 provides constants $\sigma_\varphi > 0$, $\theta_\varphi \in (0, \frac{1}{2}]$, and $C_\varphi > 0$ such that

$$\|\nabla_{\mathcal{M}} \Phi(w)\|_{\mathcal{H}} = \|\pi_w(\Delta w - |w|^{2n-2} w)\|_{\mathcal{H}} \ge C_\varphi \, |\Phi(w) - \Phi(\varphi)|^{1-\theta_\varphi} \tag{3.17}$$

for every $w \in \mathcal{V} \cap \mathcal{M}$ with $\|w - \varphi\|_{\mathcal{V}} < \sigma_\varphi$.

Since $\omega(u_0)$ is compact, finitely many such neighborhoods $B_{\mathcal{V}}(\varphi_j, \sigma_{\varphi_j})$ cover $\omega(u_0)$. Set

$$\Lambda := \cup_{j=1}^N B_{\mathcal{V}}(\varphi_j, \sigma_{\varphi_j}).$$

The defining property of $\omega(u_0)$ implies $\operatorname{dist}_{\mathcal{V}}(u(t), \omega(u_0)) \to 0$ as $t \to \infty$, hence there exists $T_0 > 0$ such that

$$u(t) \in \Lambda \quad \text{for every } t \geq T_0. \tag{3.18}$$

Since the energy functional $t \mapsto \Phi(u(t))$ is strictly decreasing unless u is stationary, since (2.8) gives $\frac{d}{dt}\Phi(u(t)) = -\|\partial_t u(t)\|_{\mathcal{H}}^2$. In combination with (3.16), this excludes repeated transitions among distinct LS neighborhoods at arbitrarily large times: if $u(t)$ visits two distinct neighborhoods centered at equilibria with different energy levels, then $\Phi(u(t))$ must decrease by a fixed positive amount during each such transition, contradicting the convergence (3.14). Consequently, there exist $\varphi_* \in \omega(u_0)$ and $T_1 \geq T_0$ (cf. [12, 21]) such that

$$\|u(t) - \varphi_*\|_{\mathcal{V}} < \sigma_{\varphi_*} \quad \text{for every } t \geq T_1. \tag{3.19}$$

Set $u^\infty := \varphi_*$. Since $\Phi(u^\infty) = \Phi_\infty$, the LS inequality (3.17) becomes, for all $t \geq T_1$,

$$\|\partial_t u(t)\|_{\mathcal{H}} = \|\nabla_{\mathcal{M}}\Phi(u(t))\|_{\mathcal{H}} \geq C_* \left(\Phi(u(t)) - \Phi_\infty\right)^{1-\theta_*}, \tag{3.20}$$

with constants $C_* > 0$ and $\theta_* := \theta_{u^\infty} \in (0, \frac{1}{2}]$.

Define $e(t) := \Phi(u(t)) - \Phi_\infty \geq 0$. Then $e(t) \to 0$ and, by (2.8),

$$\dot{e}(t) = -\|\partial_t u(t)\|_{\mathcal{H}}^2 \quad \text{for a.e. } t \geq 0. \tag{3.21}$$

Combining (3.20) with (3.21) yields, for a.e. $t \geq T_1$,

$$-\dot{e}(t) = \|\partial_t u(t)\|_{\mathcal{H}}^2 \geq C_*^2\, e(t)^{2(1-\theta_*)}.$$

This differential inequality implies that $\partial_t u \in L^1(T_1, \infty; \mathcal{H})$. Indeed, consider $f(t) := e(t)^{\theta_*}$. Then f is absolutely continuous on $[T_1, \infty)$ and, using (3.21) and (3.20),

$$-\dot{f}(t) = \theta_* e(t)^{\theta_*-1}\|\partial_t u(t)\|_{\mathcal{H}}^2 \geq \theta_* C_*\, \|\partial_t u(t)\|_{\mathcal{H}} \quad \text{for a.e. } t \geq T_1.$$

Integrating from t to $+\infty$ and using $e(t) \to 0$ gives

$$\int_t^\infty \|\partial_s u(s)\|_{\mathcal{H}}\, ds \leq \frac{1}{\theta_* C_*}\, e(t)^{\theta_*} \quad \text{for every } t \geq T_1. \tag{3.22}$$

Therefore $u(t)$ is Cauchy in $\mathcal{H}$ as $t \to \infty$ and converges in $\mathcal{H}$ to some limit $\tilde{u}^\infty \in \mathcal{H}$. Since $u(t) \in \mathcal{M}$ for all t and $\mathcal{M}$ is closed in $\mathcal{H}$, one has $\tilde{u}^\infty \in \mathcal{M}$.

To identify the limit, note that any sequence $t_k \to \infty$ has a subsequence along which $u(t_k) \to \psi$ in $\mathcal{V}$ for some $\psi \in \omega(u_0)$. On the other hand, (3.22) implies $u(t_k) \to \tilde{u}^\infty$ in $\mathcal{H}$ along the full sequence, hence $\psi = \tilde{u}^\infty$. Thus $\tilde{u}^\infty \in \omega(u_0)$, and in particular $\tilde{u}^\infty = u^\infty$ because (3.19) forces $\omega(u_0)$ to be contained in $B_{\mathcal{V}}(u^\infty, \sigma_{u^\infty})$. Consequently,

$$u(t) \longrightarrow u^\infty \quad \text{in } \mathcal{H}.$$

Finally, the convergence holds in $\mathcal{V}$. Writing the equation for $w(t) := u(t) - u^\infty$, one has

$$-\Delta w(t) = \mathcal{N}(u(t)) - \mathcal{N}(u^\infty) - \partial_t u(t), \quad w(t)|_{\partial\mathcal{O}} = 0,$$

where $\mathcal{N}(u) := -|u|^{2n-2}u + (\|u\|^2 + |u|_{L^{2n}}^{2n})u$ maps bounded subsets of $\mathcal{V}$ into $\mathcal{H}$ and is locally Lipschitz there. Elliptic regularity for the Dirichlet Laplacian yields a constant $C > 0$ such that

$$\|w(t)\|_{\mathcal{V}} \leq C\Big(\|\mathcal{N}(u(t)) - \mathcal{N}(u^\infty)\|_{\mathcal{H}} + \|\partial_t u(t)\|_{\mathcal{H}}\Big).$$

Since $\|u(t) - u^\infty\|_{\mathcal{H}} \to 0$, the local Lipschitz property gives $\|\mathcal{N}(u(t)) - \mathcal{N}(u^\infty)\|_{\mathcal{H}} \to 0$, and (3.15) together with (3.20) implies $\|\partial_t u(t)\|_{\mathcal{H}} \to 0$. Hence $\|u(t) - u^\infty\|_{\mathcal{V}} \to 0$, completing the proof. The stationarity condition (3.13) was proved above for all points of $\omega(u_0)$. □

4. Rate of decay to equilibrium

In this section the convergence furnished by Theorem 3.4 is quantified in terms of the Łojasiewicz-Simon exponent from Theorem 3.3. The resulting decay rates are expressed first for the energy gap and then transferred to $\mathcal{H}$ and $\mathcal{V}$ by the finite-length estimate (3.22) and elliptic regularity.

Theorem 4.1 (Energy and norm decay rates)**.** *Let u be the global solution of (2.7) with $u_0 \in \mathcal{V}\cap\mathcal{M}$, and let $u^\infty \in \mathcal{V}\cap\mathcal{M}$ be the equilibrium from Theorem 3.4. Set $e(t) := \Phi(u(t)) - \Phi(u^\infty) \geq 0$. Then there exist $T_0 \geq 0$ and $C_0 > 0$ such that for all $t \geq T_0$,*

$$\dot{e}(t) \leq -C_0\, e(t)^{2(1-\theta)}, \tag{4.1}$$

where $\theta \in (0, \frac{1}{2}]$ is the Łojasiewicz-Simon exponent at u^∞. In particular:

(i) *If $\theta = \frac{1}{2}$, then there exist $C, \mu > 0$ such that*

$$e(t) \leq Ce^{-\mu t}, \quad \|u(t) - u^\infty\|_{\mathcal{H}} + \|u(t) - u^\infty\|_{\mathcal{V}} \leq Ce^{-\mu t}, \quad t \geq T_0.$$

(ii) *If $\theta \in (0, \frac{1}{2})$, then there exists $C > 0$ such that*

$$e(t) \leq C(1+t)^{-\frac{1}{1-2\theta}}, \quad \|u(t) - u^\infty\|_{\mathcal{H}} + \|u(t) - u^\infty\|_{\mathcal{V}} \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad t \geq T_0.$$

Proof. By Theorem 3.4, there exists $T_0 > 0$ such that $u(t)$ remains in an LS neighborhood of u^∞ for all $t \geq T_0$. The Łojasiewicz-Simon inequality from Theorem 3.3 yields constants $C_* > 0$ and $\theta \in (0, \frac{1}{2}]$ such that

$$\|\partial_t u(t)\|_{\mathcal{H}} = \|\nabla_{\mathcal{M}} \Phi(u(t))\|_{\mathcal{H}} \geq C_*\, e(t)^{1-\theta} \quad \text{for all } t \geq T_0.$$

On the other hand, the energy identity gives $\dot{e}(t) = -\|\partial_t u(t)\|_{\mathcal{H}}^2$ for a.e. $t \geq 0$, hence for a.e. $t \geq T_0$,

$$\dot{e}(t) = -\|\partial_t u(t)\|_{\mathcal{H}}^2 \leq -C_*^2\, e(t)^{2(1-\theta)}.$$

This is (4.1) with $C_0 = C_*^2$.

If $\theta = 1/2$, the differential inequality reduces to $\dot{e} \leq -C_0 e$ and therefore $e(t) \leq e(T_0)e^{-C_0(t-T_0)}$ for $t \geq T_0$. If $\theta \in (0, \frac{1}{2})$, integration yields

$$e(t) \leq \Big(e(T_0)^{-(1-2\theta)} + C_0(1-2\theta)(t - T_0)\Big)^{-\frac{1}{1-2\theta}} \leq C(1+t)^{-\frac{1}{1-2\theta}}.$$

To pass from the energy gap to norm decay in $\mathcal{H}$, use the finite-length estimate (3.22) (with $\theta_* = \theta$), which gives

$$\|u(t) - u^\infty\|_{\mathcal{H}} \leq \int_t^\infty \|\partial_s u(s)\|_{\mathcal{H}}\, ds \leq C\, e(t)^\theta, \quad t \geq T_0.$$

Substituting the decay of $e(t)$ yields the stated rates in $\mathcal{H}$.

Finally, elliptic regularity for the Dirichlet Laplacian and the local Lipschitz property of the nonlinearity on bounded subsets of $\mathcal{V}$ give

$$\|u(t) - u^\infty\|_{\mathcal{V}} \leq C\Big(\|\mathcal{N}(u(t)) - \mathcal{N}(u^\infty)\|_{\mathcal{H}} + \|\partial_t u(t)\|_{\mathcal{H}}\Big), \quad t \geq T_0,$$

with $\mathcal{N}(u) = -|u|^{2n-2}u + (\|u\|^2 + |u|_{L^{2n}}^{2n})u$. Since $\|u(t) - u^\infty\|_{\mathcal{H}} \to 0$, one has $\|\mathcal{N}(u(t)) - \mathcal{N}(u^\infty)\|_{\mathcal{H}} \leq C\|u(t) - u^\infty\|_{\mathcal{H}}$ for t large, and $\|\partial_t u(t)\|_{\mathcal{H}} \leq Ce(t)^{1-\theta}$ by the LS inequality. Therefore $\|u(t) - u^\infty\|_{\mathcal{V}}$ decays with the same exponential or algebraic rate as $\|u(t) - u^\infty\|_{\mathcal{H}}$, concluding the proof. □

5. Existence of global attractor

We now establish the existence of a global attractor for the dynamical system generated by problem (2.7).

Theorem 5.1. *The semiflow $\{S(t)\}_{t\geq 0}$ generated by problem (2.7) on $\mathcal{V} \cap \mathcal{M}$ possesses a global attractor $\mathcal{A}$. More precisely, there exists a bounded absorbing set $B_0 \subset \mathcal{V} \cap \mathcal{M}$ such that*

$$\mathcal{A} := \omega(B_0)$$

is compact in $\mathcal{V}$, invariant under $S(t)$, and attracts every bounded subset of $\mathcal{V} \cap \mathcal{M}$.

Proof. Let $u_0 \in \mathcal{V} \cap \mathcal{M}$ and denote by $u(t) = S(t)u_0$ the corresponding global solution of (2.7). By Theorem 2.4, the solution exists for all $t \geq 0$, remains in $\mathcal{M}$, and satisfies the energy identity

$$\Phi(u(t)) + \int_0^t \|\partial_s u(s)\|_{\mathcal{H}}^2 \, ds = \Phi(u_0).$$

In particular, $\Phi(u(t)) \leq \Phi(u_0)$ for all $t \geq 0$, and since $\Phi(u) \geq \frac{1}{2}\|u\|_{\mathcal{V}}^2$ one obtains,

$$\|u(t)\|_{\mathcal{V}} \leq C\,\Phi(u_0) \quad \text{for all } t \geq 0, \tag{5.1}$$

with a constant C independent of t.

Let $B \subset \mathcal{V} \cap \mathcal{M}$ be bounded. Then $\sup_{u_0 \in B} \Phi(u_0) < \infty$, and (5.1) implies the existence of $R(B) > 0$ such that

$$\|S(t)u_0\|_{\mathcal{V}} \leq R(B) \quad \text{for all } t \geq 0,\ u_0 \in B.$$

Fix $R > 0$ and set $B_0 := \{v \in \mathcal{V} \cap \mathcal{M} : \ \|v\|_{\mathcal{V}} \leq R\}$. Given a bounded set $B \subset \mathcal{V} \cap \mathcal{M}$, choose $R := \sup_{u_0 \in B} R(B)$. Then $S(t)B \subset B_0$ for all $t \geq 0$, and in particular B_0 is absorbing. .

Asymptotic compactness follows from the precompactness of trajectories: by Theorem 3.1, for any bounded sequence $(u_{0,k}) \subset \mathcal{V} \cap \mathcal{M}$ and any sequence $t_k \to \infty$, the sequence $\{S(t_k)u_{0,k}\}$ is relatively compact in $\mathcal{V}$. Equivalently, the semiflow $\{S(t)\}_{t \geq 0}$ is asymptotically compact on $\mathcal{V}$.

The semiflow is continuous on $\mathcal{V} \cap \mathcal{M}$, possesses a bounded absorbing set, and is asymptotically compact. Therefore, by the abstract theory of dissipative dynamical systems (see, for example, [29, Theorem I.1.1] or [10, Theorem 2.4]), the set

$$\mathcal{A} := \omega(B_0)$$

is a global attractor for $\{S(t)\}_{t \geq 0}$. It is compact in $\mathcal{V}$, invariant under the semiflow, and attracts all bounded subsets of $\mathcal{V} \cap \mathcal{M}$. □

Remark 5.2. The semiflow generated by (2.7) is gradient in the sense that Φ is a strict Lyapunov functional. In particular, every complete bounded trajectory has α- and ω-limit sets contained in the set $\mathcal{S}$ of equilibria. If, in addition, the semiflow is sufficiently smooth on $\mathcal{V}$ so that the local unstable manifolds $W^u(\varphi)$ are well defined for $\varphi \in \mathcal{S}$, then the attractor satisfies

$$\mathcal{A} = \overline{\cup_{\varphi \in \mathcal{S}} W^u(\varphi)},$$

see, for example, [29, Chapter IV].

Corollary 5.3 (Finite-dimensionality of the attractor)**.** *Assume that there exists $t_* > 0$ such that $S(t_*)$ maps bounded subsets of $\mathcal{V} \cap \mathcal{M}$ into bounded subsets of $\mathcal{E}$, and that $S(t_*)$ is Lipschitz on $\mathcal{A}$ in the $\mathcal{V}$–metric. Then the global attractor $\mathcal{A}$ has finite fractal dimension in $\mathcal{V}$.*

Proof. Under these hypotheses, the standard finite-dimensionality theory for dissipative semiflows applies; see, for example, [29, Chapters VII–VIII]. □

6. Conclusion

This article studied the long–time dynamics of a nonlinear heat flow constrained to evolve on a smooth manifold in a Hilbert space. The analysis was carried out within a variational framework, exploiting the gradient structure of the equation and the analyticity of the associated energy functional.

It was shown that every global solution converges in the natural energy space to a stationary solution of the constrained problem. The convergence mechanism is governed by the Łojasiewicz-Simon inequality, which yields quantitative decay estimates for the energy gap and for the solution itself. In particular, the convergence is exponential in the presence of a nondegenerate equilibrium, while in the degenerate case it occurs at an algebraic rate determined by the Łojasiewicz-Simon exponent.

The semiflow generated by the equation was further shown to be dissipative and asymptotically compact, which ensures the existence of a compact global attractor in $\mathcal{V} \cap \mathcal{M}$. This attractor captures all bounded asymptotic dynamics and consists entirely of equilibria and trajectories connecting them. Under additional smoothing and regularity assumptions on the semiflow, standard results from attractor theory imply that the attractor has finite fractal dimension.

These results place the constrained heat flow within the general class of analytic gradient systems, extending classical convergence and attractor theory to a setting with nonlinear constraints. The framework developed here is flexible and may be applied to other constrained dissipative evolutions arising in geometric analysis and nonlinear diffusion.

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