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MULTIPLE NORMALIZED SOLUTIONS FOR FRACTIONAL (p,q)-LAPLACIAN EQUATIONS IN $\mathbb{R}^N$

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Abstract. In this article, we consider the multiplicity results of normalized solutions of the fractional (p,q)-Laplacian equation

$$(-\Delta)_p^s u + (-\Delta)_q^s u = \lambda |u|^{p-2}u + h(\varepsilon x)f(u) + g(u) \quad \text{in } \mathbb{R}^N,$$
$$\int_{\mathbb{R}^N} |u|^p \mathrm{d}x = a^p,$$

where $\varepsilon, a > 0$, $0 < s < 1 < p < q$, $sq < N < \frac{spq(q-1)}{q-p}$ and $\lambda \in \mathbb{R}$ is a unknown Lagrange multiplier, h is a continuous positive function, f and g are also continuous and satisfy some growth conditions. When ε is small enough, we show that the number of normalized solutions is at least the number of global maximum points of h according to the Ekeland's variational principle and the concentration compactness principle.

1. Introduction

This article focuses on the existence of multiple normalized solutions in $X := W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ for the fractional (p,q)-Laplacian equation

$$(-\Delta)_p^s u + (-\Delta)_q^s u = \lambda |u|^{p-2}u + h(\varepsilon x)f(u) + g(u) \quad \text{in } \mathbb{R}^N \tag{1.1}$$

under the restriction

$$\int_{\mathbb{R}^N} |u|^p \mathrm{d}x = a^p, \tag{1.2}$$

where $\varepsilon, a > 0$, $0 < s < 1 < p < q$, $sq < N < \frac{spq(q-1)}{q-p}$ and $\lambda \in \mathbb{R}$ is a Lagrange multiple which is unknown. The fractional p-Laplace operator $(-\Delta)_p^s$ and the fractional q-Laplace operator $(-\Delta)_q^s$ are defined along a smooth function (up to a normalizing constant) $u : \mathbb{R}^N \to \mathbb{R}$ by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+sp}} \mathrm{d}y, \quad x \in \mathbb{R}^N$$

and

$$(-\Delta)_q^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{q-2}(u(x) - u(y))}{|x-y|^{N+sq}} \mathrm{d}y, \quad x \in \mathbb{R}^N,$$

respectively. The continuous functions f, g, h satisfies the following conditions:

- (A1) f is odd and $\lim_{t\to 0} \frac{|f(t)|}{|t|^{m_1-1}} = \alpha_1 > 0$ for some $m_1 \in (p, \min\{p + \frac{sp^2}{N}, m^*\})$, where $m^* = 1 + \frac{Nq(p-1)}{p(N-sq)}$;
- (A2) there exist constants $C_1, C_2 > 0$ and $m_2 \in (p, \min\{q + \frac{spq}{N}, m^*\})$ such that $|f(t)| \le C_1 + C_2|t|^{m_2-1}$, $\forall t \in \mathbb{R}$;
- (A3) for all $t > 0$, $qF(t) \le f(t)t$, $F(t)$ is non-decreasing, where $F(t) = \int_0^t f(s)\mathrm{d}s$.
- (A4) $0 < h_0 = \inf_{x\in\mathbb{R}^N} h(x) \le \max_{x\in\mathbb{R}^N} h(x) := h_{\max}$;
- (A5) $h_\infty := \lim_{|x|\to\infty} h(x) < h_{\max}$;

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(A6) $h^{-1}(h_{\max}) = e_1, e_2, \cdots, e_l$ with $e_1 = 0$ and $e_j \neq e_k$ when $j \neq k$.
(A7) g is odd;
(A8) there exist $q \leq \beta < \min\{q + \frac{spq}{N}, m^*\}$, such that $0 \leq qG(s) \leq g(s)s \leq \beta G(s)$, $\forall s \in \mathbb{R}$.
(A9) $G(t)$ is non-decreasing, where $G(t) = \int_0^t g(s)\mathrm{d}s$.

Because of the restriction (1.2), we are seeking a normalized solution to (1.1), which is equivalent to finding the critical point of the functional

$$
\begin{aligned}
&I_\varepsilon(u) \\
&= \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \frac{1}{q}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N} h(\varepsilon x)F(u)\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x,
\end{aligned}
$$

on the sphere

$$
S(a) := \{u \in X := W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) : |u|_p^p = \int_{x\in\mathbb{R}^N} |u|^p \mathrm{d}x = a^p\}, \tag{1.3}
$$

where $|\cdot|_\tau$ denotes the usual norm on $L^\tau(\mathbb{R}^N)$ for $\tau \in [1, +\infty)$. Here the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ and $W^{s,q}(\mathbb{R}^N)$ are defined for any $p, q > 1$ and $s \in (0,1)$ by

$$
W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} := (\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y)^{1/p} < \infty\}
$$

and

$$
W^{s,q}(\mathbb{R}^N) := \{u \in L^q(\mathbb{R}^N) : [u]_{s,q} := (\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y)^{1/q} < \infty\},
$$

with the norm

$$
\|u\|_{W^{s,p}(\mathbb{R}^N)} = (|u|_p^p + [u]_{s,p}^p)^{1/p}
$$

and

$$
\|u\|_{W^{s,q}(\mathbb{R}^N)} = (|u|_q^q + [u]_{s,q}^q)^{1/q}.
$$

Moreover, we set $\|u\|_X = \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}$. It is well known that $I_\varepsilon \in C^1(X, \mathbb{R})$ and

$$
\begin{aligned}
\langle I'_\varepsilon(u), \varphi\rangle &= \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}(u(x)-u(y))(\varphi(x)-\varphi(y))|}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^N} h(\varepsilon x)f(u)\varphi\mathrm{d}x - \int_{\mathbb{R}^N} g(u)\varphi\mathrm{d}x
\end{aligned}
$$

for all $u, \varphi \in X$.

The study of problems involving fractional Laplace operators has garnered significant attention in recent years. This growing interest stems not only from its natural extension of the classical Laplace operators but also from its capacity to model numerous novel phenomena through its distinctive nonlinear integral structure, see [8, 15, 18]. Maya Chhetri, Petr Girg, and Elliott Hollifield [13] systematically studied a class of nonlinear partial differential equations involving the fractional Laplacian operator of the form

$$
\begin{gathered}
(-\Delta)^s u = \lambda f(x,u) \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\end{gathered}
$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. They established a method of subsolutions and supersolutions without monotone iteration, proving the existence of positive weak solutions for various types of sublinear nonlinearities, including logarithmic-type terms. Significant research efforts have also been devoted to understanding the multiplicity of normalized solutions for fractional Laplace operators, see [23, 24, 29, 38]. In [9], employing the Mountain Pass Lemma, the authors obtained the existence results of the problem

$$
(-\Delta)^{s_1}u(x) + (-\Delta)^{s_2}u(x) + \lambda u(x) + V(x)u(x) = g(u(x)), \quad x \in \mathbb{R}^d,
$$

$$\int_{\mathbb{R}^d} |u(x)|^2 \mathrm{d}x = a,$$

where $0 < s_1 < s_2 < 1$, $2s_1 < d < \frac{2s_1 s_2}{s_2 - s_1}$, $a > 0$. Function g satisfies the following conditions:

(A10) g is continuous and odd;

(A11) there exits $(\alpha, \beta) \in \mathbb{R}_+^2$ satisfying

$$2 + \frac{4s_2}{d} < \alpha < \beta < \frac{2d}{d - 2s_1},$$

such that

$$\alpha G(s) \le g(s)s \le \beta G(s), \quad \forall s \in \mathbb{R}, \quad \text{with } G(s) = \int_0^s g(t)\mathrm{d}t;$$

(A12) let $\tilde{G} : \mathbb{R} \to \mathbb{R}$ and $\tilde{G}(s) = \frac{g(s)s}{2} - G(s)$.; there exists $\tilde{G}'$ and

$$\tilde{G}'(s)s \ge \alpha \tilde{G}(s), \quad \forall s \in \mathbb{R}.$$

In recent years, fractional p-Laplace operators have been extensively studied, see [3, 4, 22, 30, 36]. Additionally, a significant contribution was made by Vincenzo Ambrosio and Teresa Isernia [1], who analyzed the problem

$$\begin{gathered}
\varepsilon^{sp}(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) \quad \text{in } \mathbb{R}^N, \\
u \in W^{s,p}(\mathbb{R}^N), \\
u(x) > 0 \quad x \in \mathbb{R}^N,
\end{gathered}$$

where $\varepsilon > 0$ is a parameter, $s \in (0,1)$, $1 < p < \infty$, $N > sp$. In the subcritical case, the authors established existence, multiplicity and concentration results for positive solutions, with small parameters through variational methods combined with Ljusternik-Schnirelmann theory. In case of critical nonlinearity, they obtained existence and multiplicity of solutions of the equation

$$\begin{gathered}
(-\Delta)_p^s u + V(\varepsilon x)|u|^{p-2}u = f(u) + |u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N, \\
u \in W^{s,p}(\mathbb{R}^N), \\
u(x) > 0 \quad x \in \mathbb{R}^N.
\end{gathered}$$

Several researchers have done a great deal of research in the multiplicity of normalized solutions of fractional Laplace operators, see [27, 31, 39]. In [40], the following equation was considered,

$$\begin{gathered}
(-\Delta)_p^s u + Z(kx)|u|^{p-2}u = \lambda |u|^{p-2}u + [\frac{1}{|x|^{N-q} * |u|^q}]|u|^{q-2}u + \sigma |u|^{p_s^*-2}u, \quad x \in \mathbb{R}^N, \\
\int_{\mathbb{R}^N} |u(x)|^p \mathrm{d}x = a^p,
\end{gathered}$$

where $k > 0$ is small parameter and $Z : \mathbb{R}^N \to [0, \infty)$ is a continuous function. The authors establish the existence of multiple normalized solutions through a combination of minimization techniques, truncation methods, variational approaches, and Lusternik-Schnirelmann category theory. In particular, the fractional p-Laplace operator, which reduces to the standard p-Laplacian when $s = 1$, has been extensively studied, see [7, 19, 25, 12]. Moreover, in the special case that $s = 1$ and $p = 2$, the operator coincides with the classical Laplacian, see [16, 32, 33].

Recently, research on fractional (p,q)-Laplacian equations has attracted a great deal of attention, see [10, 34, 41]. Vincenzo Ambrosio and Teresa Isernia [2] consider the equation

$$\begin{gathered}
(-\Delta)_p^s u + (-\Delta)_q^s u = \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} + \lambda f(x,u) \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \backslash \Omega,
\end{gathered}$$

where $0 < s < 1$, $1 \le q < p < \frac{N}{s}$, $0 \le \alpha < sp$, and $p_s^*(\alpha) = \frac{p(N-\alpha)}{N-sp}$ is Hardy-Sobolev critical exponent. They use the concentration-compactness principle and the mountain pass lemma to show that infinitely many solutions to the equation exist and that these solutions all tend to zero when λ belongs to a suitable range.

Additionally, Divya Goel, Deepak Kumar and K.Sreenadh [21] studied the problem

$$(-\Delta)_p^{s_1} u + (-\Delta)_q^{s_2} u = \lambda a(x)|u|^{\delta-2}u + b(x)|u|^{r-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $1 < \delta \le q \le p < r \le p_{s_1}^* = \frac{np}{n-s_1 p}$, $0 < s_2 < s_1 < 1$, $n > ps_1$, and $a \in L^{\frac{r}{r-\delta}}(\Omega)$, $b \in L^\infty(\Omega)$ are sign changing functions. They showed the existence and regularity of multiple weak solutions to the convex-concave problem. To the best of our knowledge, the work concerning the normalized solutions for the fractional (p,q)-Laplacian problem in $\mathbb{R}^N$ has not been seen, see [14] for the (p,q)-Laplacian problem, and [37] for the discrete fractional (p,q)-Laplacian problem.

Motivated by the above works, this paper aims to study the existence of multiple normalized solutions for problem (1.1) and (1.2) by employing variational approaches and the concentration-compactness principle from [1, 17]. However, our analysis faces two main difficulties: the first is the nonlinearity of the fractional (p,q)-Laplace operator, and the second is the lack of compactness, since the problem is set on $\mathbb{R}^N$.

The main result of this paper reads as follows.

Theorem 1.1. *Assume* (A1)–(A9) *are satisfied. Then there exists* $\varepsilon_0 > 0$ *such that* (1.1) *and* (1.2) *has at least* l *pairs of weak solutions* (u_j, λ_j) *for* $0 < \varepsilon < \varepsilon_0$*. Moreover,* $\lambda_j < 0$ *and* $I_\varepsilon(u_j) < 0$ *for* $j = 1, 2, \dots, l$.

This article is organized as follows. In Section 2, we prove the compactness theorem for the autonomous case. In Section 3, we use the compactness theorem to study the non-autonomous case. We prove the existence of multiple solutions in Section 4 and Section 5.

2. Autonomous case

Firstly, we consider the existence of the normalized solution $(u, \lambda) \in X \times \mathbb{R}$, of the equation

$$\begin{gathered}(-\Delta)_p^s u + (-\Delta)_q^s u = \lambda |u|^{p-2}u + \mu f(u) + g(u), \\ \int_{\mathbb{R}^N} |u|^p \mathrm{d}x = a^p,\end{gathered} \tag{2.1}$$

where $a, \mu > 0$, $\lambda \in \mathbb{R}$ and f satisfies (A1)–(A3), and g satisfies (A7)–(A9). It is well known that a critical point of the functional

$$J_\mu(u) = \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \frac{1}{q}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y - \mu \int_{\mathbb{R}^N} F(u)\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x$$

is a solution to problem (2.1), which is restricted to the sphere $S(a)$. Next, we will show that problem (2.1) has a normalized solution.

Lemma 2.1. *The functional* J_μ *restricted to* $S(a)$ *is bounded from below.*

Proof. From conditions (A1) and (A2), we deduce the existence of constants $C_1, C_2 > 0$ such that

$$|F(t)| \le C_1 |t|^{m_1} + C_2 |t|^{m_2}, \quad \forall t \in \mathbb{R}.$$

From (A8), there exists a constant C_3 such that

$$|G(t)| \le C_3(|t|^q + |t|^\beta), \quad \forall t \in \mathbb{R}.$$

By the fractional Gagliardo-Nirenberg inequality [28], we have

$$|u|_l \le C[u]_{s,q}^{\xi_l} |u|_p^{1-\xi_l}, \quad \forall u \in W^{s,q}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \tag{2.2}$$

for a positive constant $C>0$, where $\xi_l=\frac{Nq(l-p)}{l[Nq-p(N-sq)]}$, $l\in(p,q_s^*)$, $q_s^*=\frac{Nq}{N-sq}$. Hence,

$$\begin{aligned} J_\mu(u) \geq &\frac{1}{p}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y+\frac{1}{q}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\\ &-CC_1a^{(1-\xi_{m_1})m_1}\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{m_1\xi_{m_1}}{q}}\\ &-CC_2a^{(1-\xi_{m_2})m_2}\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{m_2\xi_{m_2}}{q}}\\ &-CC_3a^{(1-\xi_q)q}\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{q\xi_q}{q}}\\ &-CC_3a^{(1-\xi_\beta)\beta}\Big(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{\beta\xi_\beta}{q}}, \end{aligned} \tag{2.3}$$

as $m_1\in(p,\min\{p+\frac{sp^2}{N},m^*\})$ and $m_2,\ q,\ \beta\in(p,\min\{q+\frac{spq}{N},m^*\})$. Clearly $m_1\xi_{m_1}$, $m_2\xi_{m_2}$, $q\xi_q$, $\beta\xi_\beta<q$. This ensures that J_μ is bounded from below. □

The above lemma ensures that $M_\mu(a):=\inf_{u\in S(a)}J_\mu(u)$ is well-defined.

Lemma 2.2. *Let $\mu,\ a>0$, then $M_\mu(a)<0$.*

Proof. By (A1), we infer that $\lim_{t\to0}\frac{m_1F(t)}{t^{m_1}}=\alpha_1>0$, which implies that, for some $\delta>0$,

$$\frac{m_1F(t)}{t^{m_1}}\geq\frac{\alpha_1}{2} \tag{2.4}$$

for all $t\in(0,\delta]$. From (A8), we can introduce the existence of a constant C_3 such that

$$C_3\min\{|t|^q,|t|^\beta\}\leq G(t)\leq C_3(|t|^q+|t|^\beta),\quad\forall t\in\mathbb{R}. \tag{2.5}$$

Let $0<u_0\in S(a)\cap L^\infty(\mathbb{R}^N)$, we set

$$H(u_0,r)(x)=e^{\frac{Nr}{p}}u_0(e^rx),\quad\forall x\in\mathbb{R}^N,\ \forall r\in\mathbb{R}.$$

It is easy to see that

$$\int_{\mathbb{R}^N}|H(u_0,r)(x)|^p\mathrm{d}x=a^p.$$

Furthermore, a straightforward calculation leads to

$$\begin{aligned} \int_{\mathbb{R}^N}F(H(u_0,r)(x))\mathrm{d}x&=e^{-Nr}\int_{\mathbb{R}^N}F(e^{\frac{Nr}{p}}u_0(x))\mathrm{d}x,\\ \int_{\mathbb{R}^N}G(H(u_0,r)(x))\mathrm{d}x&=e^{-Nr}\int_{\mathbb{R}^N}G(e^{\frac{Nr}{p}}u_0(x))\mathrm{d}x. \end{aligned}$$

Then, for $r<0$ and $|r|$ big enough, we have

$$0<e^{\frac{Nr}{p}}u_0(x)\leq\delta,\quad\forall x\in\mathbb{R}^N.$$

Moreover, by (2.4), we obtain

$$\int_{\mathbb{R}^N}F(H(u_0,r)(x))\mathrm{d}x\geq\frac{\alpha_1}{2m_1}e^{\frac{Nr(m_1-p)}{p}}\int_{\mathbb{R}^N}|u_0(x)|^{m_1}\mathrm{d}x,$$

and by (2.5), we have

$$\int_{\mathbb{R}^N}G(H(u_0,r)(x))\mathrm{d}x\geq e^{-Nr}C_3\min\{e^{\frac{Nrq}{p}}\int_{\mathbb{R}}|u_0(x)|^q\mathrm{d}x,e^{\frac{Nr\beta}{p}}\int_{\mathbb{R}}|u_0(x)|^\beta\mathrm{d}x\}.$$

Note that

$$\begin{aligned} \iint_{\mathbb{R}^{2N}}\frac{|H(u_0,r)(x)-H(u_0,r)(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y&=e^{Nr}\iint_{\mathbb{R}^{2N}}\frac{|u_0(e^rx)-u_0(e^ry)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\ &=e^{rsp}\iint_{\mathbb{R}^{2N}}\frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \end{aligned}$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{|H(u_0,r)(x)-H(u_0,r)(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y = e^{Nr}\iint_{\mathbb{R}^{2N}} \frac{|u_0(e^r x)-u_0(e^r y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y = e^{rsq}\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y.$$

So,

$$\begin{aligned} J_\mu(H(u_0,r)) \le{}& \frac{e^{rps}}{p}\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &+ \frac{e^{rps}}{q}\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y \\ &- \frac{\mu\alpha_1}{2m_1}e^{\frac{Nr(m_1-p)}{p}}\int_{\mathbb{R}^N}|u_0(x)|^{m_1}\mathrm{d}x \\ &- e^{-Nr}C_3\min\{e^{\frac{Nrq}{p}}\int_{\mathbb{R}}|u_0(x)|^q\mathrm{d}x, e^{\frac{Nr\beta}{p}}\int_{\mathbb{R}}|u_0(x)|^\beta\mathrm{d}x\}. \end{aligned}$$

Since $0<s<1$, $m_1\in(p,\min\{p+\frac{sp^2}{N},m^*\})$ and $m_2,\alpha,\beta\in(p,\min\{q+\frac{spq}{N},m^*\})$, r is less than 0 and $|r|$ is large enough, we have

$$\begin{aligned} &\frac{e^{rps}}{p}\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \frac{e^{rps}}{q}\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y \\ &- \frac{\mu\alpha_1}{2m_1}e^{\frac{Nr(m_1-p)}{p}}\int_{\mathbb{R}^N}|u_0(x)|^{m_1}\mathrm{d}x - e^{-Nr}C_3\min\{e^{\frac{Nr\alpha}{p}}\int_{\mathbb{R}}|u_0(x)|^\alpha\mathrm{d}x, e^{\frac{Nr\beta}{p}}\int_{\mathbb{R}}|u_0(x)|^\beta\mathrm{d}x\} \\ &= A_r<0. \end{aligned}$$

Then $J_\mu(H(u_0,r))\le A_r<0$. Hence $M_\mu(a)<0$. □

Lemma 2.3. *If $\mu>0, a>0$, then*
- (i) *$a\mapsto M_\mu(a)$ is a continuous mapping;*
- (ii) *if $a_1\in(0,a)$ and $a_2=(a^p-a_1^p)^{1/p}$, then $M_\mu(a)<M_\mu(a_1)+M_\mu(a_2)$.*

Proof. (i) Let $a>0$ and $\{a_n\}\subset(0,+\infty)$ such that $a_n\to a$. From the definition of M_μ, there exists $u_n\in S(a_n)$ such that $M_\mu(a_n)\le J_\mu(u_n)<M_\mu(a_n)+\frac{1}{n}$ for every $n\in\mathbb{N}^+$. By Lemma 2.2, $M_\mu(a_n)<0$. Moreover, from (2.3), $\{u_n\}$ is bounded in X.

Now consider $v_n:=\frac{a}{a_n}u_n\in S(a)$. From the boundedness of $\{u_n\}$ and $a_n\to a$, we have

$$\begin{aligned} M_\mu(a) &\le J_\mu(v_n) \\ &= \frac{1}{p}(\frac{a^p}{a_n^p}-1)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \frac{1}{q}(\frac{a^q}{a_n^q}-1)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N}(\mu F(u_n)-\mu F(\frac{a}{a_n}u_n))\mathrm{d}x + \int_{\mathbb{R}^N}(G(u_n)-G(\frac{a}{a_n}u_n))\mathrm{d}x + J_\mu(u_n) \\ &= J_\mu(u_n)+o_n(1). \end{aligned}$$

Let $n\to+\infty$, we can get $M_\mu(a)\le\liminf_{n\to+\infty}M_\mu(a_n)$. Let $\{w_n\}$ be a bounded minimizing sequence of $M_\mu(a)$ and $z_n:=\frac{a_n}{a}w_n\in S(a)$, repeat the above process we have

$$M_\mu(a_n)\le J_\mu(z_n)=J_\mu(w_n)+o_n(1) \implies \limsup_{n\to+\infty}M_\mu(a_n)\le M_\mu(a),$$

so we have $M_\mu(a_n)\to M_\mu(a)$.

(ii) For any fixed $a_1\in(0,a)$, we prove that

$$M_\mu(\theta a_1)<\theta^p M_\mu(a_1),\quad \forall\theta>1. \tag{2.6}$$

Indeed, if $\{u_n\}\in S(a_1)$ be a minimizing sequence for $M_\mu(a_1)$, then $u_n(\theta^{-\frac{p}{N}}x)\in S(\theta a_1)$. Since $\theta>1$ and $\frac{p(N-sq)}{N}<\frac{p(N-sp)}{N}<p$, we have

$$M_\mu(\theta a_1)-\theta^p J_\mu(u_n)\le J_\mu(u_n(\theta^{-\frac{p}{N}}x))-\theta^p J_\mu(u_n)$$

$$
\begin{aligned}
&= \frac{\theta^{\frac{p(N-sp)}{N}} - \theta^p}{p} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\
&\quad + \frac{\theta^{\frac{p(N-sq)}{N}} - \theta^p}{q} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y \le 0.
\end{aligned}
$$

As a consequence, $M_\mu(\theta a_1) \le \theta^p M_\mu(a_1)$.

If $M_\mu(\theta a_1) = \theta^p M_\mu(a_1)$, then $[u_n]_{s,p}^p \to 0$ and $[u_n]_{s,q}^q \to 0$ as $n \to +\infty$, which indicate that $\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to 0$ and $\int_{\mathbb{R}^N} G(u_n)\mathrm{d}x \to 0$ by inequality (2.2). Hence

$$
\begin{aligned}
0 &> M_\mu(a_1) \\
&= \lim_{n\to+\infty} J_\mu(u_n) \\
&= \frac{1}{p}\lim_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \frac{1}{q}\lim_{n\to+\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y \\
&\quad - \lim_{n\to+\infty} \int_{\mathbb{R}^N} \mu F(u_n)\mathrm{d}x - \lim_{n\to+\infty} \int_{\mathbb{R}^N} G(u_n)\mathrm{d}x = 0,
\end{aligned}
$$

which is a contradiction. So we have $M_\mu(\theta a_1) < \theta^p M_\mu(a_1)$. Perform the same steps again, we can get

$$
M_\mu(\theta a_2) < \theta^p M_\mu(a_2), \quad \forall \theta > 1. \tag{2.7}
$$

Finally, applying (2.6) with $\theta = \frac{a}{a_1} > 1$ and (2.7) with $\theta = \frac{a}{a_2} > 1$ respectively, we obtain

$$
M_\mu(a) = \frac{a_1^p}{a^p} M_\mu(\frac{a}{a_1} a_1) + \frac{a_2^p}{a^p} M_\mu(\frac{a}{a_2} a_2) < M_\mu(a_1) + M_\mu(a_2).
$$

□

Lemma 2.4. *Suppose that $\{u_n\} \subset S(a)$ is a minimizing sequence of $M_\mu(a)$. Then for some subsequence either*

(i) *$\{u_n\}$ is strongly convergent, or*
(ii) *there exists a sequence $v_n(\cdot) = u_n(\cdot + y_n)$ with $|y_n| \to +\infty$ and $(y_n) \subset \mathbb{R}^N$, which is strongly convergent to a function $v \in S(a)$ with $J_\mu(v) = M_\mu(a)$.*

Proof. By Lemma 2.1 it is easy to see that $\{u_n\}$ is bounded. Then there exists a subsequence $u_n \rightharpoonup u$ in X, which is still denoted as itself. Assume $u \neq 0$ and $|u|_p = b$, we can infer that $b \in (0,a]$. For the case of $b \in (0,a)$, by fractional Brézis-Lieb lemma [1], we have

$$
[u_n - u]_{s,p}^p = [u_n]_{s,p}^p - [u]_{s,p}^p + o_n(1)
$$

and

$$
[u_n - u]_{s,q}^q = [u_n]_{s,q}^q - [u]_{s,q}^q + o_n(1).
$$

Furthermore, under the assumption of f and g, we obtain

$$
\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^N} F(u)\mathrm{d}x + \int_{\mathbb{R}^N} F(u_n - u)\mathrm{d}x + o_n(1)
$$

and

$$
\int_{\mathbb{R}^N} G(u_n)\mathrm{d}x = \int_{\mathbb{R}^N} G(u)\mathrm{d}x + \int_{\mathbb{R}^N} G(u_n - u)\mathrm{d}x + o_n(1).
$$

Let $v_n = u_n - u$ and $|v_n|_p = d_n \to d$, we have $a^p = b^p + d^p$ and $d_n \in (0,a)$ for n big enough. So,

$$
M_\mu(a) + o_n(1) = J_\mu(u_n) = J_\mu(u) + J_\mu(v_n) + o_n(1) \ge M_\mu(d_n) + M_\mu(b) + o_n(1).
$$

By the continuity of $a \mapsto M_\mu(a)$, we obtain

$$
M_\mu(a) \ge M_\mu(d) + M_\mu(b).
$$

This contradicts the conclusion of Lemma 2.3 (ii), where $a^p = b^p + d^p$. Hence $|u|_p = a$.

Combining with $|u_n|_p = |u|_p = a$ and $u_n \rightharpoonup u$ in $L^p(\mathbb{R}^N)$, we obtain

$$
u_n \to u \quad \text{in} \quad L^p(\mathbb{R}^N). \tag{2.8}
$$

Combining inequality (2.2), (A1) and (A2), we have

$$\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x \to \int_{\mathbb{R}^N} F(u)\mathrm{d}x. \tag{2.9}$$

Similarly,

$$\int_{\mathbb{R}^N} G(u_n)\mathrm{d}x \to \int_{\mathbb{R}^N} G(u)\mathrm{d}x. \tag{2.10}$$

So

$$\begin{aligned} M_\mu(a) &= J_\mu(u_n) + o_n(1) = J_\mu(u) + J_\mu(v_n) + o_n(1) \\ &\geq \frac{1}{p}[v_n]_{s,p}^p + \frac{1}{q}[v_n]_{s,q}^q + M_\mu(a) + o_n(1). \end{aligned}$$

This implies that $[v_n]_{s,p}^p, [v_n]_{s,q}^q \leq o_n(1)$. So we have $v_n \to 0$ in X, which means $u_n \to u$ in X.

Assume $u = 0$, i.e. $u_n \rightharpoonup 0$ in X. Then, for some $\zeta_1, r_1 > 0$ and $y_n \subset \mathbb{R}^N$, we have

$$\int_{B_{r_1}(y_n)} |u_n|^p \mathrm{d}x \geq \zeta_1, \quad \forall y_n \in \mathbb{R}^N. \tag{2.11}$$

Otherwise by [1, Lemma 2.1] we have $u_n \to 0$ in $L^k(\mathbb{R}^N)$ for all $k \in (p, p_s^*) \cup (q, q_s^*)$. Let $r_1 \in (p, q_s^*)$, and taking $p_1 \in (p, p_s^*)$ with $p < r_1$ and $q_1 \in (q, q_s^*)$ with $q_1 > r_1$. Now let $t \in (0,1)$ be such that $r_1 = tp_1 + (1-t)q_1$, by the Hölder inequality we can write

$$\int_{\mathbb{R}^N} |u_n|^{r_1}\mathrm{d}x = \int_{\mathbb{R}^N} |u_n|^{tp_1}|u_n|^{(1-t)q_1}\mathrm{d}x \leq \Big(\int_{\mathbb{R}^N} |u_n|^{p_1}\mathrm{d}x\Big)^t \Big(\int_{\mathbb{R}^N} |u_n|^{q_1}\mathrm{d}x\Big)^{(1-t)} \to 0.$$

Hence $u_n \to 0$ in $L^{r_1}(\mathbb{R}^N)$, $\forall r_1 \in (p, q_s^*)$, which implies $F(u_n) \to 0$ and $G(u_n) \to 0$ in $L^1(\mathbb{R}^N)$. However, this contradicts the fact that

$$0 > M_\mu(a) + o_n(1) = J_\mu(u_n) \geq -\int_{\mathbb{R}^N} F(u_n)\mathrm{d}x - \int_{\mathbb{R}^N} G(u_n)\mathrm{d}x.$$

Hence (2.11) holds. Combining with $u = 0$, the inequality (2.11) and Sobolev embedding, we deduce that $\{y_n\}$ is unbounded. Consider $v_n(\cdot) = u_n(x + y_n)$, it is easy to verify that $\{v_n\}$ is also a minimizing sequence of $M_\mu(a)$ and $\{v_n\} \subset S(a)$. As a result, it holds $v_n \rightharpoonup v$ in X, where $v \in X\backslash\{0\}$. By an argument analogous to that used above, we can deduce $v_n \to v$ in X. □

Lemma 2.5. *Assume* (A1)–(A3), (A7)–(A9) *hold with* $\mu > 0$. *Then, problem* (2.1) *has a positive radial solution* u *and* $\lambda < 0$.

Proof. According to the definition of $J_\mu(u)$, we have $J_\mu(|u|) \leq J_\mu(u)$. In addition, we can also get $|u| \in S(a)$. Then, we deduce that

$$M_\mu(a) = J_\mu(u) \geq J_\mu(|u|) \geq M_\mu(a),$$

and we obtain $J_\mu(|u|) = M_\mu(a)$. Thus, instead of $|u|$, we can use u. If u^* is the fractional Schwarz's Symmetrization of u ([6, Section 9.2]), we obtain

$$\begin{aligned} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y &\geq \iint_{\mathbb{R}^{2N}} \frac{|u^*(x)-u^*(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y, \\ \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y &\geq \iint_{\mathbb{R}^{2N}} \frac{|u^*(x)-u^*(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y, \end{aligned}$$

and by [26, section 3.3],

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^p\mathrm{d}x &= \int_{\mathbb{R}^N} |u^*|^p\mathrm{d}x, \\ \int_{\mathbb{R}^N} G(u)\mathrm{d}x &= \int_{\mathbb{R}^N} G(u^*)\mathrm{d}x, \\ \int_{\mathbb{R}^N} \mu F(u)\mathrm{d}x &= \int_{\mathbb{R}^N} \mu F(u^*)\mathrm{d}x. \end{aligned}$$

It is easy to prove that $u^* \in S(a)$ and $M_\mu(a) = J_\mu(u^*)$. Thus, we replace u by u^*.

From [20], $u \in C^{\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0,1)$. Now we show that $u(x)$ is positive for all $x \in \mathbb{R}^N$. Assume by contradiction that there exists some $x_0 \in \mathbb{R}^N$ satisfying $u(x_0) = 0$, and there exists $x_1 \in \mathbb{R}^N$ satisfying $u(x_1) > 0$ since $u \neq 0$. Thus, we can find a ball of sufficiently large radius $R_1 > 0$ such that $x_0, x_1 \in B_{R_1/2}(0)$. Then, combining this with the weak Harnack inequality [11], we deduce that there exists a constant $C_8 > 0$ such that

$$\sup_{y \in B_{R_1/2}(0)} u(y) \le C_8 \inf_{y \in B_{R_1/2}(0)} u(y),$$

which contradicts the fact that

$$\sup_{y \in B_{R_1/2}(0)} u(y) \ge u(x_1) > 0 \quad \text{and} \quad \inf_{y \in B_{R_1/2}(0)} u(y) = u(x_0) = 0.$$

This implies that there is a positive radial solution u.

Next, we show that $\lambda < 0$. By Lemma 2.2 we can assume that there exists a bounded minimizing sequence $\{u_n\} \in S(a)$. Hence, we can obtain that there exists a constant $\lambda \in \mathbb{R}$ such that

$$J'_\mu(u) = \lambda \Psi'(u) \quad \text{in} \quad X', \tag{2.12}$$

where $\Psi(u) := \int_{\mathbb{R}^N} |u|^p \mathrm{d}x$. Then, from (2.12),

$$(-\Delta)_p^s u + (-\Delta)_q^s u = \lambda |u|^{p-2} u + \mu f(u) + g(u), \quad x \in \mathbb{R}^N$$

and

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y - \lambda \int_{\mathbb{R}^N} |u|^p \mathrm{d}x \\
&- \mu \int_{\mathbb{R}^N} f(u) u \mathrm{d}x - \int_{\mathbb{R}^N} g(u) u \mathrm{d}x = 0.
\end{aligned}$$

By (A3), (A8), and that $M_\mu(a) = J_\mu(u) < 0$, we have

$$\begin{aligned}
0 &> J_\mu(u) - \frac{1}{q}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y - \lambda \int_{\mathbb{R}^N} |u|^p \mathrm{d}x \\
&\quad - \mu \int_{\mathbb{R}^N} f(u) u \mathrm{d}x - \int_{\mathbb{R}^N} g(u) u \mathrm{d}x\Big) \\
&= \Big(\frac{1}{p} - \frac{1}{q}\Big) \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \frac{\lambda}{q} \int_{\mathbb{R}^N} |u|^p \mathrm{d}x + \frac{\mu}{q} \int_{\mathbb{R}^N} f(u) u \mathrm{d}x - \mu \int_{\mathbb{R}^N} F(u) \mathrm{d}x \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^N} g(u) u \mathrm{d}x - \int_{\mathbb{R}^N} G(u) \mathrm{d}x \\
&\ge \frac{1}{q} \int_{\mathbb{R}^N} \lambda |u|^p \mathrm{d}x,
\end{aligned}$$

this implies that $\lambda < 0$. □

From Lemma 2.5, we obtain the following lemma.

Lemma 2.6. *Fix $a > 0$ and let $0 < \mu_1 < \mu_2$. Then, $M_{\mu_2(a)} < M_{\mu_1}(a) < 0$.*

3. Non-autonomous case

We will state some properties of the functional I_ε, which is restricted to $S(a)$. Firstly, we define $I_{\max}, I_\infty : X \to \mathbb{R}$ as

$$\begin{aligned}
I_{\max}(u) = &\frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \frac{1}{q} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y \\
&- \int_{\mathbb{R}^N} h_{\max} F(u) \mathrm{d}x - \int_{\mathbb{R}^N} G(u) \mathrm{d}x
\end{aligned}$$

and

$$I_\infty(u) = \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \frac{1}{q} \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y$$

$$- \int_{\mathbb{R}^N} h_\infty F(u)\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x.$$

Moreover, by Lemma 2.2 we know that

$$M_\infty(a) = \inf_{u\in S(a)} I_\infty(u), \quad M_\varepsilon(a) = \inf_{u\in S(a)} I_\varepsilon(u), \quad M_{\max}(a) = \inf_{u\in S(a)} I_{\max}(u).$$

Then, by Lemma 2.6 and $h_\infty < h_{\max}$, we can immediately obtain

$$M_{\max}(a) < M_\infty(a) < 0. \tag{3.1}$$

Now, we fix $0 < \rho_1 = \frac{1}{p}(M_\infty(a) - M_{\max}(a))$.

Lemma 3.1. $\lim_{\varepsilon\to 0^+} M_\varepsilon(a) \le M_{\max}(a)$, *and there exists* $\varepsilon_0 > 0$ *such that* $M_\varepsilon(a) < M_\infty(a)$ *for all* $0 < \varepsilon < \varepsilon_0$.

Proof. Let $u_0 \in S(a)$ satisfying $I_{\max}(u_0) = M_{\max}(a)$. A simple calculation leads to

$$\begin{aligned} M_\varepsilon(a) &\le I_\varepsilon(u_0) \\ &= \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \frac{1}{q}\iint_{\mathbb{R}^{2N}} \frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^N} h(\varepsilon x)F(u_0)\mathrm{d}x - \int_{\mathbb{R}^N} G(u_0)\mathrm{d}x. \end{aligned}$$

Letting $\varepsilon \to 0^+$ and applying (A6) we can obtain

$$\limsup_{\varepsilon\to 0^+} M_\varepsilon(a) \le \lim_{\varepsilon\to 0^+} I_\varepsilon(u_0) = I_{\max}(u_0) = M_{\max}(a).$$

According to (3.1), we have $M_\varepsilon(a) < M_\infty(a)$ for ε small enough. □

The following two lemmas will be used to prove the $(PS)_c$ condition for I_ε at certain levels.

Lemma 3.2. *Assume that* $\{u_n\} \subset S(a)$ *is a minimizing sequence with* $I_\varepsilon(u_n) \to c$ *and* $c < M_{\max}(a) + \rho_1 < M_\infty(a)$. *If* $u_n \rightharpoonup u$ *in* X, *then* $u \neq 0$.

Proof. Firstly, we assume that the conclusion is not true, i.e. $u \equiv 0$. Then, we obtain

$$c = M_\varepsilon(a) = I_\varepsilon(u_n) + o_n(1) = I_\infty(u_n) + \int_{\mathbb{R}^N} (h_\infty - h(\varepsilon x))F(u_n)\mathrm{d}x + o_n(1).$$

From (A5), for any arbitrary number $\zeta_2 > 0$, there exists some constant $R_2 > 0$ such that

$$h_\infty \ge h(x) - \zeta_2, \quad |x| > R_2.$$

Then

$$\begin{aligned} c &= I_\varepsilon(u_n) + o_n(1) \\ &\ge I_\infty(u_n) + \int_{B_{R_2/\varepsilon}(0)} (h_\infty - h(\varepsilon x))F(u_n)\mathrm{d}x - \zeta_2 \int_{B^c_{R_2/\varepsilon}(0)} F(u_n)\mathrm{d}x + o_n(1). \end{aligned}$$

Since $\{u_n\}$ is bounded in X, then for some constant $C_9 > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(u_n)\mathrm{d}x &\le C_1 a^{(1-\xi_{m_1})m_1} \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{m_1\xi_{m_1}}{q}} \\ &\quad + C_2 a^{(1-\xi_{m_2})m_2} \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\Big)^{\frac{m_2\xi_{m_2}}{q}} \\ &\le C_9. \end{aligned}$$

From the fact that $u_n \to 0$ in $L^{k_1}(B_{R/\varepsilon}(0))$ when $k_1 \in [1, q_s^*)$, we have

$$c = I_\varepsilon(u_n) + o_n(1) \ge I_\infty(u_n) - \zeta_2 C_9 + o_n(1) \ge M_\infty(a) - \zeta_2 C_9 + o_n(1).$$

This together with small enough $\zeta_2 > 0$, we obtain $c \ge M_\infty(a)$. This contradicts $c < M_{\max}(a) + \rho_1 < M_\infty(a)$. Therefore, we can obtain $u \neq 0$. □

Lemma 3.3. *Assume that* $\{u_n\}\subset S(a)$ *is a* $(PS)_c$ *sequence of* I_ε *satisfying* $u_n \rightharpoonup u_\varepsilon$ *in* X *where* $c<M_{\max}(a)+\rho_1<M_\infty(a)$, *that is, as* $n\to+\infty$,

$$I_\varepsilon(u_n)\to c \quad \text{and} \quad \|I_\varepsilon|'_{S(a)}(u_n)\|_{X'}\to 0.$$

Then

$$\liminf_{n\to+\infty}|u_n-u_\varepsilon|_p^p\geq \beta_1,$$

where $u_n \not\to u_\varepsilon$ *in* X *and* $\beta_1>0$ *independent of* $\varepsilon\in(0,\varepsilon_0)$.

Proof. Firstly, define the functional $\Gamma(u)=\frac{1}{p}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x$. We can see that $S(a)=\Gamma^{-1}(a^p/p)$. According to [35, Proposition 5.12], there exists $\{\lambda_n\}\subset\mathbb{R}$ such that

$$\|I'_\varepsilon(u_n)-\lambda_n\Gamma'(u_n)\|_{X'}\to 0$$

as $n\to+\infty$. Since I_ε is bounded from below and $I_{\max}$ is forced, it follows that $\{u_n\}$ is bounded in X. This ensures that $\{\lambda_n\}$ is bounded, up to a subsequence, we may assume $\lambda_n\to\lambda_\varepsilon$ as $n\to+\infty$. Now we prove that

$$I'_\varepsilon(u_\varepsilon)-\lambda_\varepsilon\Gamma'(u_\varepsilon)=0 \quad \text{in } X', \tag{3.2}$$

$$\|I'_\varepsilon(v_n)-\lambda_\varepsilon\Gamma'(v_n)\|_{X'}\to 0 \quad \text{as } n\to+\infty, \tag{3.3}$$

where $v_n=u_n-u_\varepsilon$. Since $\|I'_\varepsilon(u_n)-\lambda_n\Gamma'(u_n)\|_{X'}\to 0$, to prove (3.2), it is sufficient to prove that for all $\phi\in X$,

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\ &\to\iint_{\mathbb{R}^{2N}}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^{p-2}(u_\varepsilon(x)-u_\varepsilon(y))(\phi(x)-\phi(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{q-2}(u_n(x)-u_n(y))(\phi(x)-\phi(y))}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y\\ &\to\iint_{\mathbb{R}^{2N}}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^{q-2}(u_\varepsilon(x)-u_\varepsilon(y))(\phi(x)-\phi(y))}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{3.5}$$

$$\int_{\mathbb{R}^N}h(\varepsilon x)f(u_n)\phi\mathrm{d}x\to\int_{\mathbb{R}^N}h(\varepsilon x)f(u_\varepsilon)\phi\mathrm{d}x, \tag{3.6}$$

$$\int_{\mathbb{R}^N}g(u_n)\phi\mathrm{d}x\to\int_{\mathbb{R}^N}g(u_\varepsilon)\phi\mathrm{d}x, \tag{3.7}$$

$$\lambda_n\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\phi\mathrm{d}x\to\lambda_\varepsilon\int_{\mathbb{R}^N}|u_\varepsilon|^{p-2}u_\varepsilon\phi\mathrm{d}x. \tag{3.8}$$

We first prove (3.4) holds using the technique in [5]. From $u_n\rightharpoonup u_\varepsilon$ in X, $u_n\to u_\varepsilon$ in $L^t_{\rm loc}(\mathbb{R}^N)$ for all $t\in[1,q_s^*)$. For any $\phi\in C_c^\infty(\mathbb{R}^N)$, define

$$\Phi(x,y)=\frac{\phi(x)-\phi(y)}{|x-y|^{\frac{N+sp}{p}}}\in L^p(\mathbb{R}^{2N}).$$

We also define

$$U_n(x,y)=\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^{\frac{N+sp}{p'}}}\in L^{p'}(\mathbb{R}^{2N}),$$

$$U_\varepsilon(x,y)=\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^{p-2}(u_\varepsilon(x)-u_\varepsilon(y))}{|x-y|^{\frac{N+sp}{p'}}}\in L^{p'}(\mathbb{R}^{2N}),$$

where $p'=\frac{p}{p-1}$. It is easy to see that $\{U_n\}$ is bounded in $L^{p'}(\mathbb{R}^{2N})$. Since $L^{p'}(\mathbb{R}^{2N})$ is reflexive, there exists a subsequence, still denoted by $\{U_n\}$, such that $U_n\rightharpoonup U_\varepsilon$ in $L^{p'}(\mathbb{R}^{2N})$, hence

$$\iint_{\mathbb{R}^{2N}}U_n(x,y)\Phi(x,y)\mathrm{d}x\mathrm{d}y\to\iint_{\mathbb{R}^{2N}}U_\varepsilon(x,y)\Phi(x,y)\mathrm{d}x\mathrm{d}y.$$

Consequently, we obtain (3.4). Using the same technique it can be shown that (3.5) holds.

Next we prove that (3.6) holds. By the condition of h we know that h is bounded, then we only need to prove

$$\int_{\mathbb{R}^N}(f(u_n)-f(u_\varepsilon))\phi \mathrm{d}x \to 0. \tag{3.9}$$

From conditions (A1) and (A2) we know that

$$|f(t)| \le C_1|t|^{m_1-1}+C_2|t|^{m_2-1},$$

then there exists a constant C_{10} such that

$$|f(u_n)-f(u_\varepsilon)| \le C_{10}(|u_n|^{m_1-1}+|u_n|^{m_2-1}+|u_\varepsilon|^{m_1-1}+|u_\varepsilon|^{m_2-1}).$$

To prove (3.9), we divide the interval of integration into two parts, i.e. for a sufficiently large $R_3>0$ independent of N,

$$\int_{\mathbb{R}^N}(f(u_n)-f(u_\varepsilon))\phi \mathrm{d}x = \int_{B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi \mathrm{d}x + \int_{\mathbb{R}^N\setminus B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi \mathrm{d}x.$$

(i) For integrals within B_{R_3}. Using Hölder's inequality we have

$$\begin{aligned}
|\int_{B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi \mathrm{d}x| &\le \int_{B_{R_3}}|f(u_n)-f(u_\varepsilon)||\phi|\mathrm{d}x\\
&\le \Big(\int_{B_{R_3}}|f(u_n)-f(u)|^{(q_s^*)'}\Big)^{\frac{1}{(q_s*)'}}\Big(\int_{B_{R_3}}|\phi|^{q_s^*}\Big)^{\frac{1}{q_s^*}},
\end{aligned}$$

where $\frac{1}{q_s^*}+\frac{1}{(q_s^*)'}=1$. Because of $u_n \rightharpoonup u$ in $X_{R_3}=W^{s,p}(B_{R_3})\cap W^{s,q}(B_{R_3})$, and $X_{R_3}\hookrightarrow\hookrightarrow L^{k_2}(B_{R_3})$, where $k_2\in[1,q_s^*)$, it follows that $u_n\to u$ in $L^{k_2}(B_{R_3})$. Also by the condition (A2) and the continuity of f it follows that $f\in C(L^{k_2}(B_{R_3}),L^{(q_s^*)'}(B_{R_3}))$, and $\int_{B_{R_3}}|f(u_n)-f(u_\varepsilon)|^{(q_s^*)'}\mathrm{d}x\to 0$, hence

$$\int_{B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi \mathrm{d}x \to 0.$$

(ii) For integrals outside the B_{R_3}. By Hölder's inequality we have

$$\begin{aligned}
|\int_{\mathbb{R}^N\setminus B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi \mathrm{d}x| &\le \Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|f(u_n)-f(u_\varepsilon)|^{p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}\\
&\le C_{10}\Big[\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_n|^{(m_1-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}\\
&\quad+\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_\varepsilon|^{(m_1-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}\\
&\quad+\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_n|^{(m_2-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}\\
&\quad+\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_\varepsilon|^{(m_2-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}\Big],
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. By (A1) we have $p<(m_1-1)p'<q_s^*$, then $\{u_n\}$ is bounded in $L^{(m_1-1)p'}(\mathbb{R}^N)$. Therefore, there exists a large enough $R_3>0$, such that

$$\int_{\mathbb{R}^N\setminus B_{R_3}}|u_n|^{(m_1-1)p'}\mathrm{d}x<\varepsilon.$$

From the definition of ϕ it follows that ϕ is bounded in X, i.e. there exists a constant $M>0$ such that $|\phi|_p<M$. Hence,

$$\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_n|^{(m_1-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}<\varepsilon M.$$

For the same reason,

$$\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_\varepsilon|^{(m_1-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}<\varepsilon M.$$

By (A2) and repeat the above steps, we have

$$\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_n|^{(m_2-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}<\varepsilon M,$$
$$\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|u_\varepsilon|^{(m_2-1)p'}\mathrm{d}x\Big)^{1/p'}\Big(\int_{\mathbb{R}^N\setminus B_{R_3}}|\phi|^p\mathrm{d}x\Big)^{1/p}<\varepsilon M.$$

Hence, $|\int_{\mathbb{R}^N\setminus B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi\mathrm{d}x|<4\varepsilon M$. By the arbitrariness of ε, we have

$$\int_{\mathbb{R}^N}(f(u_n)-f(u_\varepsilon))\phi\mathrm{d}x=\int_{B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi\mathrm{d}x+\int_{\mathbb{R}^N\setminus B_{R_3}}(f(u_n)-f(u_\varepsilon))\phi\mathrm{d}x\to 0.$$

Summarizing, we have (3.6). Carry out the same method once more, we can obtain (3.7).

To prove (3.8), let $A(u_n)=|u_n|^{p-2}u_n$. It follows from the boundness of $\{u_n\}$ in X, there exists a constant C_{11} such that

$$|A(u_n)|_{L^{p'}}=||u_n|^{p-2}u_n|_{L^{p'}}=|u_n|_{L^p}^{p-1}\le C_{11}^{p-1}.$$

Then for each $\varepsilon_1>0$ there exists a compact support function ψ_1 such that for every $\phi\in X$ with $|\phi-\psi_1|_{L^p}<\varepsilon_1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}(A(u_n)-A(u_\varepsilon))\phi\mathrm{d}x&\le|A(u_n)-A(u_\varepsilon)|_{L^{p'}}|\phi-\psi_1|_{L^p}+\int_{\mathbb{R}^N}(A(u_n)-A(u_\varepsilon))\psi_1\mathrm{d}x\\&<\varepsilon_1C_{11}^{p-1}+\int_{\mathbb{R}^N}(A(u_n)-A(u_\varepsilon))\psi_1\mathrm{d}x.\end{aligned}$$

By the arbitrariness of ε_1, we have

$$\int_{\mathbb{R}^N}(A(u_n)-A(u_\varepsilon))\phi\mathrm{d}x\to 0.$$

Hence (3.8) holds.

From (A3), we have

$$\begin{aligned}0>M_{\max}(a)+\rho_1>c&=\liminf_{n\to+\infty}I_\varepsilon(u_n)\\&=\liminf_{n\to+\infty}(I_\varepsilon(u_n)-\frac{1}{q}\langle I_\varepsilon'(u_n),u_n\rangle+\frac{1}{q}\lambda_na^p)\\&\ge\frac{1}{q}\lambda_\varepsilon a^p,\end{aligned}$$

this implies

$$\limsup_{\varepsilon\to 0}\lambda_\varepsilon\le\frac{q(M_{\max}(a)+\rho_1)}{a^p}<0.$$

Then, there exists a constant λ^* independent of ε that satisfies $\lambda_\varepsilon<\lambda^*<0$, so

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y+\iint_{\mathbb{R}^{2N}}\frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y-\lambda_\varepsilon\int_{\mathbb{R}^N}|v_n|^p\mathrm{d}x\\&=\int_{\mathbb{R}^N}h(\varepsilon x)f(v_n)v_n\mathrm{d}x+\int_{\mathbb{R}^N}g(v_n)v_n\mathrm{d}x+o_n(1)\end{aligned}$$

and

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y+\iint_{\mathbb{R}^{2N}}\frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y-\lambda^*\int_{\mathbb{R}^N}|v_n|^p\mathrm{d}x\\&\le\int_{\mathbb{R}^N}h(\varepsilon x)f(v_n)v_n\mathrm{d}x+\int_{\mathbb{R}^N}g(v_n)v_n\mathrm{d}x+o_n(1).\end{aligned}$$

According to (A1) and (A2), we have $|f(t)| < C_1|t|^{m_1-1} + C_1|t|^{m_2-1}$ for all $t \in \mathbb{R}$. Then

$$\begin{aligned}\int_{\mathbb{R}^N} f(v_n)v_n\mathrm{d}x &\le \int_{\mathbb{R}^N} |f(v_n)||v_n|\mathrm{d}x\\ &\le \int_{\mathbb{R}^N} (C_1|v_n|^{m_1-1} + C_2|v_n|^{m_2-1})|v_n|\mathrm{d}x\\ &\le C_1\int_{\mathbb{R}^N} |v_n|^{m_1}\mathrm{d}x + C_2\int_{\mathbb{R}^N} |v_n|^{m_2}\mathrm{d}x\,.\end{aligned}$$

From (A8), we have $|g(t)| \le C_3(|t|^{q-1} + |t|^{\beta-1})$ for all $t \in \mathbb{R}$, hence

$$\int_{\mathbb{R}^N} |g(v_n)v_n|\mathrm{d}x \le C_3\int_{\mathbb{R}^N} |v_n|^q + |v_n|^\beta\mathrm{d}x = C_3\int_{\mathbb{R}^N} |v_n|^q\mathrm{d}x + C_3\int_{\mathbb{R}^N} |v_n|^\beta\mathrm{d}x.$$

So, we have

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y + C_0\int_{\mathbb{R}^N} |v_n|^p\mathrm{d}x\\ &\le h_{\max}\int_{\mathbb{R}^N} f(v_n)v_n\mathrm{d}x + \int_{\mathbb{R}^N} g(v_n)v_n\mathrm{d}x\\ &\le C_1h_{\max}\int_{\mathbb{R}^N} |v_n|^{m_1}\mathrm{d}x + C_2h_{\max}\int_{\mathbb{R}^N} |v_n|^{m_2}\mathrm{d}x + C_3\int_{\mathbb{R}^N} |v_n|^q\mathrm{d}x + C_3\int_{\mathbb{R}^N} |v_n|^\beta\mathrm{d}x + o_n(1)\end{aligned}$$

for some constant $C_0 = -\lambda^* > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$. Since $v_n \not\to 0$ in X, we can assume that $\liminf_{n\to+\infty}\|v_n\|_X > C^* > 0$. Thus, for some constant $C_{12} > 0$, by (2.2), we infer that

$$\begin{aligned}C_{12} &\le \liminf_{n\to+\infty}(|v_n|_{m_1}^{m_1} + |v_n|_{m_2}^{m_2} + |v_n|_q^q + |v_n|_\beta^\beta)\\ &\le \liminf_{n\to+\infty}C_{13}([v_n]_{s,q}^{m_1\xi_{m_1}}|v_n|_p^{m_1(1-\xi_{m_1})} + [v_n]_{s,q}^{m_2\xi_{m_2}}|v_n|_p^{m_2(1-\xi_{m_2})}\\ &\quad + [v_n]_{s,q}^{q\xi_q}|v_n|_p^{q(1-\xi_q)} + [v_n]_{s,q}^{\beta\xi_\beta}|v_n|_p^{\beta(1-\xi_\beta)})\\ &\le \liminf_{n\to+\infty} C_{13}K_1(|v_n|_p^{m_1(1-\xi_{m_1})} + |v_n|_p^{m_2(1-\xi_{m_2})} + |v_n|_p^{q(1-\xi_q)} + |v_n|_p^{\beta(1-\xi_\beta)})\end{aligned} \tag{3.10}$$

where $K_1 > 0$ is independent of $\varepsilon \in (0, \varepsilon_0)$ with $[v_n]_{s,q} \le K_1$ for all $n \in \mathbb{N}$. Since $m_1 \in (p, \min\{p+\frac{sp^2}{N}, m^*\})$, $m_2, q, \beta \in (p, \min\{q+\frac{spq}{N}, m^*\})$, then $m_1(1-\xi_{m_1}),\ m_2(1-\xi_{m_2}),\ q(1-\xi_q),\ \beta(1-\xi_\beta) < p$, hence

$$\liminf_{n\to+\infty}|v_n|_p^p \ge C_{15} \tag{3.11}$$

for a constant $C_{15} > 0$. This complete the proof. □

Next, we consider $0 < \rho < \min\{\frac{1}{p}, \frac{\beta_1}{a^p}\}(M_\infty(a) - M_{\max}(a))$.

Lemma 3.4. *Assume that $0 < \varepsilon < \varepsilon_0$ and $c < M_{\max}(a) + \rho$. Then, I_ε limited to $S(a)$ satisfies the $(PS)_c$ condition.*

Proof. By Lemma 2.2 $\{u_n\}$ is bounded. Let $\{u_n\} \subset S(a)$ be $(PS)_c$ sequence of I_ε with $u_n \rightharpoonup u_\varepsilon$, where $u_\varepsilon \ne 0$ by Lemma 3.2 and $c < M_{\max}(a) + \rho$. Set $v_n = u_n - u_\varepsilon$. If $v_n \to 0$ in X, then the proof is complete. If $v_n \not\to 0$ in X and $|u_\varepsilon|_p = b$, by Lemma 3.3, we obtain

$$\liminf_{n\to+\infty}|v_n|_p^p \ge \beta_1 \tag{3.12}$$

for some $\beta_1 > 0$ which is independent of $\varepsilon \in (0, \varepsilon_0)$.

Let $|v_n|_p = d_n \to d \ge {\beta_1}^{1/p}$, by Brézis-Lieb lemma [35], we have $a^p = b^p + d^p$. From $d_n \in (0, a)$ for n large enough, we have

$$c + o_n(1) = I_\varepsilon(u_n) = I_\varepsilon(v_n) + I_\varepsilon(u_\varepsilon) + o_n(1) \ge M_\infty(d_n) + M_{\max}(b) + o_n(1).$$

By Lemma 2.3 (i) and (2.5), letting $n \to +\infty$, we have

$$M_{\max}(a) + \rho > c \ge M_\infty(d) + M_{\max}(b) \ge \frac{d^p}{a^p}M_\infty(a) + \frac{b^p}{a^p}M_{\max}(a).$$

Then
$$\rho \geq \frac{d^p}{a^p}(M_\infty(a) - M_{\max}(a)) \geq \frac{\beta_1}{a^p}(M_\infty(a) - M_{\max}(a)).$$
This contradicts $\rho < \frac{\beta_1}{a^p}(M_\infty(a) - M_{\max}(a))$. Hence, $v_n \to 0$ holds in X, i.e. $u_n \to u_\varepsilon$ in X. This implies that $u_\varepsilon \in S(a)$ and
$$(-\Delta)_p^s u_\varepsilon + (-\Delta)_q^s u_\varepsilon = \lambda_\varepsilon |u_\varepsilon|^{p-2} u_\varepsilon + h(\varepsilon x) f(u_\varepsilon) + g(u_\varepsilon), \quad x \in \mathbb{R}^N. \qquad \square$$

4. Multiplicity result

In the following, we will discuss some technical issues. Let $\rho_0,\ r_0 > 0,\ e_j$ defined in (A6), satisfy

- $\overline{B_{\rho_0}(e_i)} \cap \overline{B_{\rho_0}(e_j)} = \emptyset$ for $i \neq j$ and$i,j \in \{1,\dots,l\}$;
- $\cup_{i=1}^l B_{\rho_0}(e_i) \subset B_{r_0}(0)$;
- $K_{\frac{\rho_0}{2}} = \bigcup_{i=1}^l \overline{B_{\frac{\rho_0}{2}}(e_i)}$.

We set $\kappa : \mathbb{R}^N \to \mathbb{R}^N$ with
$$\kappa(x) = \begin{cases} x, & \text{if } |x| \leq r_0, \\ r_0 \frac{x}{|x|}, & \text{if } |x| > r_0. \end{cases}$$
Now, we consider the function $G_\varepsilon : X\backslash\{0\} \to \mathbb{R}^N$ defined by
$$G_\varepsilon(u) := \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon x)|u|^p \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^p \mathrm{d}x}.$$
Then, by the next two theorems, we can obtain the existence of the (PS) sequence I_ε restricted to $S(a)$.

Lemma 4.1. *Reducing ε_0 if necessary, there exists a positive constant $\delta_0 < \rho$ such that*
$$G_\varepsilon(u) \in K_{\frac{\rho_0}{2}}, \quad \forall \varepsilon \in (0, \varepsilon_0),$$
where $u \in S(a)$ and $I_\varepsilon(u) \leq M_{\max}(a) + \delta_0$.

Proof. We assume that the conclusion is false, and hence there exist $\delta_n \to 0, u_n \in S(a)$ and $\varepsilon_n \to 0$, such that
$$I_{\varepsilon_n}(u_n) \leq M_{\max}(a) + \delta_n$$
and $G_{\varepsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}$. Firstly, we have
$$M_{\max}(a) \leq I_{\max}(u_n) \leq I_{\varepsilon_n}(u_n) \leq M_{\max}(a) + \delta_n,$$
then
$$I_{\max}(u_n) \to M_{\max}(a), \quad \text{as } n \to \infty.$$
We will analyze the following two cases by means of Lemma 2.4.

(i) $u_n \to u$ in X, where $u \in S(a)$. According to Lebesgue convergence theorem, we can deduce that
$$G_{\varepsilon_n}(u_n) = \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon_n x)|u_n|^p \mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^p \mathrm{d}x} \to \frac{\int_{\mathbb{R}^N} \kappa(0)|u|^p \mathrm{d}x}{\int_{\mathbb{R}^N} |u|^p \mathrm{d}x} = 0 \in K_{\frac{\rho_0}{2}},$$
which contradicts to $G_{\varepsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}$ for n large.

(ii) There exists a sequence $v_n(\cdot) = u_n(\cdot + y_n)$ with $|y_n| \to +\infty$ and $\{y_n\} \subset \mathbb{R}^N$ which converges to $v \in S(a)$ in X. Then, we can also study the following two cases:

When $|\varepsilon_n y_n| \to +\infty$, We can infer that
$$\begin{aligned} I_{\varepsilon_n}(u_n) &= \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \frac{1}{q} \iint_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^N} h(\varepsilon_n x + \varepsilon_n y_n) F(v_n) \mathrm{d}x - \int_{\mathbb{R}^N} G(v_n) \mathrm{d}x \\ &\to I_\infty(v). \end{aligned}$$

Since $I_{\varepsilon_n}(u_n) \le M_{\max}(a) + \delta_n$, it holds

$$M_{\max}(a) \ge I_\infty(v) \ge M_\infty(a),$$

which contradicts to (3.1).

When $\varepsilon_n y_n \to y$ for some $y \in \mathbb{R}^N$, we have

$$\begin{aligned} I_{\varepsilon_n}(u_n) &= \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \frac{1}{q}\iint_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^N} h(\varepsilon_n x + \varepsilon_n y_n)F(v_n)\mathrm{d}x - \int_{\mathbb{R}^N} G(v_n)\mathrm{d}x \\ &\to I_{h(y)}(v), \end{aligned}$$

then we have

$$M_{h(y)}(a) \le M_{\max}(a). \tag{4.1}$$

Since $h(y) < h_{\max}$, Lemma 2.6 implies that $M_{h(y)}(a) > M_{\max}(a)$, which contradicts (4.1). Therefore $h(y) = h_{\max}$ holds, i.e. $y = e_i$ for some $i = 1, \dots, l$. Then we have

$$\begin{aligned} G_{\varepsilon_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon_n x)|u_n|^p\mathrm{d}x}{\int_{\mathbb{R}^N} |u_n|^p\mathrm{d}x} = \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon_n x + \varepsilon_n y_n)|v_n|^p\mathrm{d}x}{\int_{\mathbb{R}^N} |v_n|^p\mathrm{d}x} \\ &\to \frac{\int_{\mathbb{R}^N} \kappa(y)|v|^p\mathrm{d}x}{\int_{\mathbb{R}^N} |v|^p\mathrm{d}x} = e_i \in K_{\frac{\rho_0}{2}}, \end{aligned}$$

which contradicts to $G_{\varepsilon_n}(u_n) \notin K_{\frac{\rho_0}{2}}$ for n large. □

Next, we introduce some symbols:

- $\theta^i_\varepsilon := \{u \in S(a) : |G_\varepsilon(u) - e_i| \le \rho_0\}$,
- $\partial\theta^i_\varepsilon := \{u \in S(a) : |G_\varepsilon(u) - e_i| = \rho_0\}$,
- $\eta^i_\varepsilon := \inf_{u\in\theta^i_\varepsilon} I_\varepsilon(u)$,
- $\widetilde{\eta}^i_\varepsilon := \inf_{u\in\partial\theta^i_\varepsilon} I_\varepsilon(u)$.

Lemma 4.2. *Let $0 < \delta_0 < \rho < \min\{\frac{1}{p}, \frac{\beta}{a^p}\}(M_\infty(a) - M_{\max}(a))$. Then*

$$\eta^i_\varepsilon < M_{\max}(a) + \rho \quad \textit{and} \quad \eta^i_\varepsilon < \widetilde{\eta}^i_\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Proof. According to Lemmas 2.4 and 2.5, we set

$$M_{\max}(a) = I_{\max}(u), \quad I'_{\max}(u) = 0,$$

where $u \in S(a)$. Let $u^i_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ be $u^i_\varepsilon = u(x - e_i/\varepsilon)$ for $1 \le i \le l$. By direct calculation we have

$$\begin{aligned} I_\varepsilon(u^i_\varepsilon(x)) &= \frac{1}{p}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \frac{1}{q}\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^q}{|x-y|^{N+sq}}\mathrm{d}x\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^N} h(\varepsilon x + e_i)F(u)\mathrm{d}x - \int_{\mathbb{R}^N} G(u)\mathrm{d}x, \end{aligned}$$

which implies that

$$\limsup_{\varepsilon\to 0} I_\varepsilon(u^i_\varepsilon(x)) \le I_{\max}(u) = M_{\max}(a). \tag{4.2}$$

If $\varepsilon \to 0^+$, then

$$G_\varepsilon(u^i_\varepsilon) = \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon x)|u^i_\varepsilon|^p\mathrm{d}x}{\int_{\mathbb{R}^N} |u^i_\varepsilon|^p\mathrm{d}x} = \frac{\int_{\mathbb{R}^N} \kappa(\varepsilon x + e_i)|u|^p\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^p\mathrm{d}x} \to e_i.$$

We can deduce that $u^i_\varepsilon \in \theta^i_\varepsilon$ when ε is small enough. Moreover, according to (4.2),

$$I_\varepsilon(u^i_\varepsilon(x)) \le M_{\max}(a) + \frac{\delta_0}{4}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Hence, reduce ε_0 if necessary,

$$\eta^i_\varepsilon \le M_{\max}(a) + \frac{\delta_0}{4}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Then

$$\eta_\varepsilon^i \le M_{\max}(a) + \rho, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

If there exists $u \in \partial\theta_\varepsilon^i$, i.e.

$$u \in S(a) \quad \text{and} \quad |G_\varepsilon(u) - e_i| = \rho_0 > \frac{\rho_0}{2},$$

then $G_\varepsilon(u) \notin K_{\frac{\rho_0}{2}}$. In conjunction with Lemma 4.1, we obtain

$$I_\varepsilon(u) > M_{\max}(a) + \frac{\delta_0}{2}, \quad \forall u \in \partial\theta_\varepsilon^i,\ \forall \varepsilon \in (0, \varepsilon_0),$$

and so,

$$\widetilde{\eta}_\varepsilon^i > M_{\max}(a) + \frac{\delta_0}{2}, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

From this, it can be seen that $\eta_\varepsilon^i < \widetilde{\eta}_\varepsilon^i$ for all $\varepsilon \in (0, \varepsilon_0)$. □

5. Proof of Theorem 1.1

According to Ekeland's variational principle, we there exists a sequence $\{u_n^i\} \subset \theta_\varepsilon^i$ $(\subset S(a))$ such that

$$I_\varepsilon(u_n^i) \to \eta_\varepsilon^i$$

and

$$I_\varepsilon(v) - I_\varepsilon(u_n^i) \ge -\frac{1}{n}\|v - u_n^i\|, \quad \forall v \in \theta_\varepsilon^i \quad \text{with} \quad v \ne u_n^i$$

for each $i \in \{1, \dots, l\}$.

Then, getting $u_n^i \in \theta_\varepsilon^i \backslash \partial\theta_\varepsilon^i$ for sufficiently large n by Lemma 4.2. Given $v \in T_{u_n^i} S(a) = \{\omega \in X : \int_{\mathbb{R}^N} |u_n^i|^{p-2} u_n^i \omega \mathrm{d}x = 0\}$, We can define the path $\sigma : (-\xi_1, \xi_1) \to S(a)$ with

$$\sigma(t) = a\frac{(u_n^i + tv)}{|u_n^i + tv|_p},$$

where $\xi_1 > 0$. It is clear that $\sigma \in C^1((-\xi_1, \xi_1), S(a))$, and

$$\sigma(t) \in \theta_\varepsilon^i \backslash \partial\theta_\varepsilon^i, \quad \forall t \in (-\xi_1, \xi_1), \quad \sigma(0) = u_n^i \quad \text{and} \quad \sigma'(0) = v.$$

Then we have

$$I_\varepsilon(\sigma(t)) - I_\varepsilon(u_n^i) \ge -\frac{1}{n}\|\sigma(t) - u_n^i\|$$

for $t \in (-\xi_1, \xi_1)$, this means that

$$\begin{aligned}
\frac{I_\varepsilon(\sigma(t)) - I_\varepsilon(\sigma(0))}{t} &= \frac{I_\varepsilon(\sigma(t)) - I_\varepsilon(u_n^i)}{t} \\
&\ge -\frac{1}{n}\Big\|\frac{\sigma(t) - u_n^i}{t}\Big\| \\
&= -\frac{1}{n}\Big\|\frac{\sigma(t) - \sigma(0)}{t}\Big\|, \quad \forall t \in (0, \xi_1).
\end{aligned}$$

Taking the limit of $t \to 0^+$, we obtain

$$\langle I_\varepsilon'(u_n^i), v\rangle \ge -\frac{1}{n}\|v\|.$$

Then, instead of v, we can use $-v$ to derive

$$\sup\{|\langle I_\varepsilon'(u_n^i), v\rangle| : \|v\| \le 1\} \le \frac{1}{n},$$

which implies that

$$I_\varepsilon(u_n^i) \to \eta_\varepsilon^i \quad \text{and} \quad \|I_\varepsilon|_{S(a)}'(u_n)\| \to 0 \quad \text{as} \quad n \to \infty,$$

which means $\{u_n^i\} \subset S(a)$ is a $(PS)_{\eta_\varepsilon^i}$ sequence of I_ε. Combining Lemma 3.4 and $\eta_\varepsilon^i < M_{\max}(a) + \rho$, we can deduce that there exists u^i such that $u_n^i \to u^i$ in X,

$$u^i \in \theta_\varepsilon^i,\ I_\varepsilon(u^i) = \eta_\varepsilon^i \quad \text{and} \quad I_\varepsilon|_{S(a)}'(u^i) = 0.$$

According to our assumptions, we have

$$G_\varepsilon(u^i) \in \overline{B_{\rho_0}(e_i)}, \quad G_\varepsilon(u^j) \in \overline{B_{\rho_0}(e_j)}$$

and

$$\overline{B_{\rho_0}(e_i)} \cap \overline{B_{\rho_0}(e_j)} = \emptyset \quad \text{for } i \neq j \text{ and } i, j \in \{1, \dots, l\},$$

which means $u^i \neq u^j$ for $i \neq j$ while $1 \leq i, j \leq l$. Thus, for any $\varepsilon \in (0, \varepsilon_0)$, I_ε has at least l nontrivial critical points (u^i, λ_i), i.e.

$$(-\Delta)_p^s u^i + (-\Delta)_q^s u^i = \lambda_i |u^i|^{p-2} u^i + h(\varepsilon x) f(u^i) + g(u^i), \quad \forall i \in \{1, 2, \dots, l\},$$

which means

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}} \frac{|u^i(x) - u^i(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|u^i(x) - u^i(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N} \lambda_i |u^i|^p \mathrm{d}x \\
&- \int_{\mathbb{R}^N} h(\varepsilon x) f(u^i) u^i \mathrm{d}x - \int_{\mathbb{R}^N} g(u^i) u^i \mathrm{d}x = 0.
\end{aligned}$$

Combining with $I_\varepsilon(u^i) < 0$, we have

$$\begin{aligned}
0 &> I_\varepsilon(u^i) - \frac{1}{q}\Big(\iint_{\mathbb{R}^{2N}} \frac{|u^i(x) - u^i(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{|u^i(x) - u^i(y)|^q}{|x-y|^{N+sq}} \mathrm{d}x\mathrm{d}y - \int_{\mathbb{R}^N} \lambda_i |u^i|^p \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} \mu f(u^i) u^i \mathrm{d}x - \int_{\mathbb{R}^N} g(u^i) u^i \mathrm{d}x\Big) \\
&= \Big(\frac{1}{p} - \frac{1}{q}\Big) \iint_{\mathbb{R}^{2N}} \frac{|u^i(x) - u^i(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y + \frac{1}{q} \int_{\mathbb{R}^N} \lambda_i |u^i|^p \mathrm{d}x \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^N} h(\varepsilon x) f(u^i) u^i \mathrm{d}x - \int_{\mathbb{R}^N} h(\varepsilon x) F(u^i) \mathrm{d}x \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^N} g(u^i) u^i \mathrm{d}x - \int_{\mathbb{R}^N} G(u^i) \mathrm{d}x \\
&\geq \frac{1}{q} \int_{\mathbb{R}^N} \lambda_i |u^i|^p \mathrm{d}x,
\end{aligned}$$

which implies $\lambda_i < 0$. This proves the expected result.

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