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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR FRACTIONAL p-LAPLACIAN KIRCHHOFF TYPE PROBLEMS

SHUWEN HE, SHIQING ZHANG

Abstract. In this article we study the fractional p-Laplacian Kirchhoff type problem in $\mathbb{R}^N$,

$$\Big(a+b\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)(-\Delta)_p^s u+\lambda V(x)|u|^{p-2}u=f(x,u)+g(x,u),$$

where $s\in(0,1), 2\le p<\infty, N>sp$, $a,b,\lambda>0$ are parameters. Under suitable assumptions on V,f and g, if b is sufficiently small and λ is large enough, we show that the existence of at least two different nontrivial solutions by combining the variational methods and the truncation technique. At the same time, we explore the asymptotic behavior of solutions as $b\to 0$ and $\lambda\to\infty$. We also obtain the nonexistence of nontrivial solutions when a is large enough.

1. Introduction

The aim of this article is to study the fractional p-Laplacian Kirchhoff type problems with steep potential well and concave-convex nonlinearities,

$$\begin{gathered}\Big(a+b\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)(-\Delta)_p^s u+\lambda V(x)|u|^{p-2}u=f(x,u)+g(x,u)\quad\text{in }\mathbb{R}^N,\\ u\in W^{s,p}(\mathbb{R}^N),\end{gathered}\tag{1.1}$$

where $s\in(0,1), 2\le p<\infty, N>sp, a,b,\lambda>0$ are real parameters, V,f and g are continuous functions, $(-\Delta)_p^s$ is the fractional p-Laplacian which (up to normalization factors) is the nonlinear nonlocal operator defined on smooth functions by

$$(-\Delta)_p^s u(x)=2\lim_{\epsilon\to 0}\int_{\mathbb{R}^N\setminus B_\epsilon(x)}\frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)]}{|x-y|^{N+sp}}dy\quad\text{in }\mathbb{R}^N,$$

where $u\in C_0^\infty(\mathbb{R}^N)$, $B_\epsilon(x):=\{y\in\mathbb{R}^N:|y-x|<\epsilon\}$. Nonlocal fractional operators come from many different contexts, such as finance, quantum mechanics, game theory and so on, see for example [5, 14, 15, 18] and the references therein. One of the main reasons of considering (1.1) is related to the equation

$$\rho u_{tt}-\Big(\frac{p_0}{h}+\frac{E}{2L}\int_0^L|u_x|^2dx\Big)u_{xx}=w(x,u),$$

which was proposed by Kirchhoff in [13] as an extension of the classical D'Alembert wave equations for free vibrations of elastic strings. This model takes into account the changes in length of the string produced by transverse vibrations. For more mathematical and physical background on the Kirchhoff type problems, we refer the readers to [1, 4] and the references therein.

In recent years, when parameters $a,b,\lambda>0$, V and f satisfy some appropriate conditions, the general Kirchhoff type problem

$$-\Big(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)\Delta u+\lambda V(x)u=f(x,u)\quad\text{in }\mathbb{R}^N,$$

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$$u \in H^1(\mathbb{R}^N)$$

has been widely studied by many researchers, they obtained the existence, concentration, and multiplicity of solutions to this equation through variational methods and analytical techniques see [10, 16, 23, 27, 30, 34] and the references therein.

More recently, Xiang et al. [31] investigated the nonlocal fractional p-Laplacian Kirchhoff type problem

$$\begin{gathered} M\Big(\int\int_{\mathbb{R}^{2N}} |u(x)-u(y)|^p K(x-y)\,dx\,dy\Big)\mathcal{L}_K^p u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.2}$$

where M is a continuous function, $\mathcal{L}_K^p$ is a nonlocal operator with singular kernel K. When Carathéodory function f is sublinear growth or superlinear growth, the authors obtained the existence nontrivial solutions separately. Additionally, when $p = 2$, Fiscella and Valdinoci [8] established the existence of nontrivial solutions to problem (1.2) with a critical growth term. In particular, a typical example for K is given by singular kernel $K(x-y) = |x-y|^{-(N+sp)}$ and $V(x)|u|^{p-2}u, g(x)$ are added to (1.2) defined on the unbounded domain $\mathbb{R}^N$. In this case, problem (1.2) becomes

$$M\Big(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + g(x) \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

If $g(x) \in L^{\frac{p}{p-1}}(\mathbb{R}^N)$ is a nonzero perturbation term and V and f satisfy the following assumptions

(1) $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$ for the positive constant V_0;
(2) there exists $R > 0$ such that $\lim_{|y|\to\infty} |\{x \in B_R(y) : V(x) \leq d\}| = 0$ for every $d > 0$, where $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^N$;
(3) there is $\mu > p$ such that
$$\mu \int_0^t f(x,\tau)d\tau \leq f(x,t)t \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Pucci et al. [19] proved that problem (1.3) admits at least two nontrivial solutions. Subsequently, Torres Ledesma [26] also obtained the existence of two nontrivial solutions to problem (1.3) when f satisfies

(4) there exist $q \in (p, p_s^*)$ and $h(x) \in C(\mathbb{R}^N, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} h(x) = 0$ such that
$$|f(x,t)| \leq h(x)|t|^{q-1} \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Recently, when $f(x,u) + g(x) = \lambda\omega(x)|u|^{r-2}u - h(x)|u|^{q-2}u$, where $h(x)$ is a nonnegative function satisfying some ratio of integration with $\omega(x)$, $1 < r < q < \infty$, Pucci et al. showed the existence and multiplicity of entire solutions for problem (1.3) in [20]. Later, for the critical case of $f(x,u)+g(x) = \alpha|u|^{p_s^*-2}u+\beta\kappa(x)|u|^{q-2}u$, Wang and Zhang [28] showed the existence of infinitely many solutions which tend to zero under the appropriate positive parameters α and β. For more results on different forms of problem (1.3), readers can see [9, 17, 25, 32, 35] and the references therein. However, there seem to be few articles studying fractional p-Laplacian Kirchhoff type problems with steep potential well and concave-convex nonlinearities. We note that, when the concave-convex term presents as $f(x)|u|^{q-2}u + g(x)|u|^{r-2}u$ with $1 < r < p < q < p_s^*$, Xiong et al. [33] explored the existence and multiplicity of nontrivial solutions to problem (1.1). Specifically, if $p = 2$, $g(x,u) = \mu g(x)u^q$ for all $\lambda > 0$ and $0 < q < 1$, Shao and Chen [21] also obtained the existence and multiplicity of two nontrivial solutions for problem (1.1).

Motivated by the works mentioned above, especially by [16, 19, 21, 33], the purpose of this paper is to study the fractional p-Laplacian Kirchhoff type problem (1.1) involving steep potential well and more general concave-convex nonlinearities. More precisely, we follow the variational methods and use an interesting truncation technique as e.g. in [16] to prove the existence and asymptotic behavior of the nontrivial solutions for problem (1.1). Furthermore, we also obtain the nonexistence result of nontrivial solutions to this problem. It is significantly different from the previous works. When dealing with problem (1.1), we may encounter several difficulties as follows:

(1) We cannot directly generalize minimum and sufficient for research problem (1.1) with fractional p-Laplacian Kirchhoff type due to the set $\Omega \times \Omega$ is strictly contained in $\mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c)$, so we need to introduction a new fractional space as [9, 22, 31] to solve this difficulty;
(2) Note that V, f and g are not required to be radial and $\frac{f(x,t)}{|t|^{p-1}}$ may not be increasing in $(-\infty, 0)$ and $(0, +\infty)$ hinders us from borrowing Nehari manifold and fibering methods such as following [9, 24, 27], we will use the truncation technique to overcome it;
(3) Another difficulty is the presence of nonlocal term $(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy)(-\Delta)^s_p u$ and the lack of compactness, which means that using the variational method directly in a standard way is not feasible. Then we can restore compactness by constructing some effective inequalities for $b \to 0$ and $\lambda \to \infty$.

To state our main results, we assume that V satisfies:

(A1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \geq 0$ on $\mathbb{R}^N$;
(A2) there exists $V_0 > 0$ such that the set $\{V < V_0\} := \{x \in \mathbb{R}^N : V(x) < V_0\}$ is nonempty and has finite Lebesgue measure;
(A3) $\Omega = \operatorname{int} V^{-1}(0)$ is a nonempty open set and has smooth boundary with $\overline{\Omega} = V^{-1}(0)$.

Considering that $F(x,t) = \int_0^t f(x,\tau)d\tau$ and $G(x,t) = \int_0^t g(x,\tau)d\tau$ are the primitives of f and g respectively, we assume the following conditions:

(A4) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $f(x,t) = o(|t|^{p-2}t)$ as $|t| \to 0$ uniformly for $x \in \mathbb{R}^N$;
(A5) there exist constants $C > 0$ and $q \in (p, p^*_s)$ such that $|f(x,t)| \leq C(1 + |t|^{q-1})$, where $p^*_s = \frac{Np}{N-sp}$ is the fractional critical exponent;
(A6) there exists $\mu > 0$ such that $f(x,t)t \geq \mu F(x,t) > 0$ for all $p < \mu < p^*_s$ and $t \in \mathbb{R} \setminus \{0\}$;
(A7) $g \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exists $h(x) \in L^{\frac{p}{p-r}}(\mathbb{R}^N, \mathbb{R}^+)$ such that $|g(x,t)| \leq h(x)|t|^{r-1}$ for all $r \in (1, p)$;
(A8) there exists $\nu \in (1, p)$ such that $\nu G(x,t) \geq tg(x,t) > 0$ for all $t \in \mathbb{R} \setminus \{0\}$.

Now, on the existence and asymptotic behavior of nontrivial solutions for problem (1.1) we provide the following theorems.

Theorem 1.1. *Suppose that (A1)–(A8) hold. Then for each fixed $a > 0$, there exist $\widehat{b}, \Lambda, \Pi > 0$ such that $b \in (0, \widehat{b})$, $\lambda > \Lambda$ and $0 < |h|_{\frac{p}{p-r}} < \Pi$, problem (1.1) possesses at least two different nontrivial solutions $u^{(i)}_{b,\lambda}$ $(i = 1, 2)$.*

Theorem 1.2. *Let $u^{(i)}_{b,\lambda}$ $(i = 1, 2)$ be two solutions of problem (1.1) given by Theorem 1.1. Then for each $b \in (0, \widehat{b})$ and $u^{(i)}_{b,\lambda_n} \to u^{(i)}_b$ in $W^{s,p}(\mathbb{R}^N)$ as $\lambda_n \to \infty$ up to a subsequence, where $u^{(1)}_b \neq u^{(2)}_b$ are two nontrivial solutions of*

$$\begin{gathered}\Big(a + b\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)(-\Delta)^s_p u = f(x,u) + g(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.\end{gathered} \tag{1.4}$$

Theorem 1.3. *Let $a = 1$ and $u^{(i)}_{b,\lambda} (i = 1, 2)$ be two solutions of problem (1.1) given by Theorem 1.1. Then for each fixed $\lambda > \Lambda$ and $u^{(i)}_{b_n,\lambda} \to u^{(i)}_\lambda$ in $W^{s,p}(\mathbb{R}^N)$ as $b_n \to 0$ up to a subsequence, where $u^{(1)}_\lambda \neq u^{(2)}_\lambda$ are two nontrivial solutions of*

$$\begin{gathered}(-\Delta)^s_p u + \lambda V(x)|u|^{p-2}u = f(x,u) + g(x,u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N).\end{gathered} \tag{1.5}$$

Theorem 1.4. *Let $a = 1$ and $u^{(i)}_{b,\lambda} (i = 1, 2)$ be two solutions of problem (1.1) given by Theorem 1.1. Then for $u^{(i)}_\lambda \to \widetilde{u}^{(i)}$ in $W^{s,p}(\mathbb{R}^N)$ as $\lambda \to \infty$ and $b \to 0$ up to a subsequence, where $\widetilde{u}^{(1)} \neq \widetilde{u}^{(2)}$ are two nontrivial solutions of*

$$(-\Delta)^s_p u = f(x,u) + g(x,u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Next, we investigate the nonexistence of nontrivial solutions.

Theorem 1.5. *Suppose that* (A1)–(A8) *hold. If* $a > a_1 := \mathfrak{C}|\overline{\Omega}|^{\frac{p_s^*-p}{p_s^*}}$ *and for each fixed* $\lambda > 0$, *then there exists a constant* $a_2 > 0$ *such that for all* $b > 0$ *and* $a > \max\{a_1, a_2\}$, *problem* (1.1) *has no nontrivial solution.*

Remark 1.6. Analysis of conditions and regularity of solutions:

(i) $(A1)$–$(A3)$ were first proposed by Bartsch and Wang in [2], where $\lambda V(x)$ represents a steep potential well with a bottom of $V^{-1}(0)$, and its depth is controlled by λ. (A4) and (A5) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon |t|^{p-1} + C_\varepsilon |t|^{q-1}. \tag{1.6}$$

(ii) Combined with the regularity theory for the fractional p-Laplacian equations established by Lannizzotto et al. [11, 12], the kernel function of the operator $(-\Delta)_p^s$ satisfies standard singularity and meets the structural requirements of the regularity theory. Under assumptions (A1)–(A3), the potential V ensures the local boundedness of the coefficients in the equation. Furthermore, (A4)–(A8) guarantee that f and g satisfy the requirements of local boundedness and controlled growth for nonlinear terms in the theory. Consequently, all nontrivial solutions (denoted by u) obtained in this paper are globally Hölder continuous for $i = 1, 2$:

- the solutions $u_{b,\lambda}^{(i)}$ in Theorem 1.1 and $u_\lambda^{(i)}$ in Theorem 1.3 satisfy $u \in C^\alpha_{\rm loc}(\mathbb{R}^N)$ with some $\alpha \in (0, s]$;
- the solutions $u_b^{(i)}$ in Theorem 1.2 and $\widetilde{u}^{(i)}$ in Theorem 1.4 satisfy $u \in C^\alpha(\overline{\Omega})$ with some $\alpha \in (0, s]$, and $\widetilde{u}^{(i)}$ attain the optimal exponent $\alpha = s$;
- trivial solutions (the nonexistence case in Theorem 1.5) naturally enjoy $C^\infty(\mathbb{R}^N)$ regularity.

(iii) In the present paper, our main results complement and develop the relevant results on fractional p-Laplacian Kirchhoff type problems in [19, 26, 31, 33] by assigning more general conditions to the nonlinear terms f and g.

The rest of this article is arranged as follows. In the next section, we present some definitions and useful results. In Sections 3 and 4, we provide proofs for Theorems 1.1–1.4 and finally in Section 5 we prove Theorem 1.5.

2. Preliminaries

In this section, we will construct a variational framework and provide preliminary propositions. Before that, we need to some notations. $\mathcal{C}, C, C_1, C_2, \cdots$ denote positive constants (possibly different), $|\cdot|_\nu$ is the usual norm of the space $L^\nu(\mathbb{R}^N)$ for all $1 \le \nu \le \infty$. If the subsequence of sequence $\{u_n\}$ is taken, it will be denoted again as $\{u_n\}$. $o(1)$ denotes any quantity which tends to zero when $n \to \infty$. Moreover, define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

endowed with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = ([u]_{s,p}^p + |u|_p^p)^{1/p}$$

for

$$[u]_{s,p} := \Big(\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\, dx\, dy \Big)^{1/p}.$$

Now we recall the embedding relationships of the fractional Sobolev spaces into Lebesgue spaces.

Lemma 2.1 (see [5])**.** *Define* $\mathcal{D}^{s,p}(\mathbb{R}^N)$ *as the closure of* $C_0^\infty(\mathbb{R}^N)$ *with respect to* $[u]_{s,p}$. *Let* $s \in (0,1)$ *and* $p \in [1, \infty)$. *Then there exists the optimal Sobolev embedding constant* $\mathfrak{C} := \mathfrak{C}(N, s, p) > 0$ *such that*

$$|u|_{p_s^*}^p \le \mathfrak{C}[u]_{s,p}^p$$

for every $u \in \mathcal{D}^{s,p}(\mathbb{R}^N)$. *Moreover,* $W^{s,p}(\mathbb{R}^N)$ *is continuously embedded in* $L^q(\mathbb{R}^N)$ *for any* $q \in [p, p_s^*]$ *and compactly in* $L^q_{\rm loc}(\mathbb{R}^N)$ *whenever* $q \in [1, p_s^*)$.

Set

$$X=\Big\{u\in W^{s,p}(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)|u|^p dx<\infty\Big\}$$

with the norm

$$\|u\|=\Big(a[u]_{s,p}^p+\int_{\mathbb{R}^N}V(x)|u|^p dx\Big)^{1/p}.$$

For $\lambda>0$, we also need to provide the following norm

$$\|u\|_\lambda=\Big(a[u]_{s,p}^p+\int_{\mathbb{R}^N}\lambda V(x)|u|^p dx\Big)^{1/p}.$$

If $\lambda\ge 1$, then $\|u\|\le\|u\|_\lambda$. Define $X_\lambda=(X,\|u\|_\lambda)$, it follows from (A1) and (A2) that

$$\|u\|_{W^{s,p}(\mathbb{R}^N)}^p\le\max\Big\{\frac{\mathfrak{C}|\{V<V_0\}|^{\frac{p_s^*-p}{p_s^*}}+1}{a},\frac{1}{V_0}\Big\}\|u\|^p,$$

then X is continuously embedded in $W^{s,p}(\mathbb{R}^N)$. Let $\Lambda_1=\frac{a}{\mathfrak{C}V_0|\{V<V_0\}|^{\frac{p_s^*-p}{p_s^*}}}$. As in the idea of [23], for any $\kappa\in[p,p_s^*]$, we also have

$$\int_{\mathbb{R}^N}|u|^\kappa dx\le\Big(\frac{\mathfrak{C}}{a}\Big)^{\kappa/p}|\{V<V_0\}|^{\frac{p_s^*-\kappa}{p_s^*}}\|u\|_\lambda^\kappa\quad\text{for all }\lambda\ge\Lambda_1.\tag{2.1}$$

Furthermore, we give additional definitions about fractional space, for more details see [5, 9, 22, 31]. Set $\Omega\subset\mathbb{R}^N$ be an open bounded set with smooth boundary, $\Omega^c=\mathbb{R}^N\setminus\Omega$, $\mathcal{O}=(\Omega^c\times\Omega^c)\subset\mathbb{R}^{2N}$ and $\mathcal{Q}=\mathbb{R}^{2N}\setminus\mathcal{O}$. It is defined in the usual fractional space $W^{s,p}(\Omega)$ endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)}=\Big(\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy+\int_\Omega|u|^p dx\Big)^{1/p}.$$

The fractional space E is defined by

$$E=\Big\{u\in L^p(\Omega):\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\in L^p(\mathcal{Q})\Big\},$$

and endowed with the norm

$$\|u\|_E=\Big(\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy+\int_\Omega|u|^p dx\Big)^{1/p}.$$

Moreover, let $E_0=\{u\in E:u=0\text{ a.e. in }\Omega^c\}$. Then we have the following results.

Lemma 2.2 (see [9, 31]). *The following claims are established:*

- (i) *if $v\in E$, then $v\in W^{s,p}(\Omega)$ and $\|v\|_{W^{s,p}(\Omega)}\le\|v\|_E$;*
- (ii) *if $v\in E_0$, then $v\in W^{s,p}(\mathbb{R}^N)$ and $\|v\|_{W^{s,p}(\Omega)}\le\|v\|_{W^{s,p}(\mathbb{R}^N)}\le\|v\|_E$;*
- (iii) *there exists a constant $\mathfrak{C}:=\mathfrak{C}(N,s,p)>0$ such that for any $v\in E_0$,*
$$|v|_{L^{p_s^*}(\Omega)}^p\le\mathfrak{C}[v]_{s,p}^p;$$
- (iv) *there exists a constant $\widetilde{\mathfrak{C}}:=\widetilde{\mathfrak{C}}(N,s,p,\Omega)>1$, such that for any $u\in E_0$,*
$$\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\le\|u\|_E^p\le\widetilde{\mathfrak{C}}\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy,$$
i.e.
$$\|u\|_{E_0}=\Big(\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{1/p}$$
is a norm on E_0 and is equivalent to the norm on E;
- (v) *$(E_0,\|\cdot\|_{E_0})$ is a reflexive Banach space;*
- (vi) *let $\{u_n\}\subset E_0$ be a bounded sequence. Then, there exists $u\in L^q(\mathbb{R}^N)$ such that up to a subsequence, $u_n\to u$ in $L^q(\mathbb{R}^N)$ as $n\to\infty$ for every $q\in[1,p_s^*)$.*

Proceeding as in [30, Lemma 1] and Lemma 2.2 in [34], it is easy to verify the validity of the following lemma.

Lemma 2.3. *Suppose that* (A1)–(A5), (A7) *are satisfied. Then*

(i) $\langle \mathcal{F}'(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v dx$ *and* $\mathcal{F}' : X_\lambda \to X_\lambda'$ *is weakly continuous;*

(ii) $\langle \mathcal{G}'(u), v\rangle = \int_{\mathbb{R}^N} g(x,u)v dx$ *and* $\mathcal{G}' : X_\lambda \to X_\lambda'$ *is weakly continuous, where* $\mathcal{F}(u) := \int_{\mathbb{R}^N} f(x,u)v dx$ *and* $\mathcal{G}(u) := \int_{\mathbb{R}^N} g(x,u)v dx$. *In particular, if* $u_n \rightharpoonup u$ *in* X_λ, *we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |g(x,u_n) - g(x,u)|^{\frac{p}{p-1}} dx = 0.$$

As a consequence, related to problem (1.1) we have the energy functional $\mathcal{J}_{b,\lambda} : X_\lambda \to \mathbb{R}$ given by

$$\mathcal{J}_{b,\lambda}(u) = \frac{1}{p}\|u\|_\lambda^p + \frac{b}{2p}[u]_{s,p}^{2p} - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} G(x,u)dx.$$

For all $u \in X_\lambda$ and $v \in C_0^\infty(\mathbb{R}^N)$, it is easy to see that $\mathcal{J}_{b,\lambda} \in C^1(X_\lambda, \mathbb{R})$ and

$$\begin{aligned}\langle \mathcal{J}_{b,\lambda}'(u), v\rangle &= (a + b[u]_{s,p}^p) \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+sp}}\,dx\,dy \\ &\quad + \int_{\mathbb{R}^N} \lambda V(x)|u|^{p-2}uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx - \int_{\mathbb{R}^N} g(x,u)v\,dx.\end{aligned}$$

Clearly, if $u \in X_\lambda$ is a critical point of $\mathcal{J}_{b,\lambda}$, then u is a solution of problem (1.1).

Next, we introduce a variant of the Mountain Pass Theorem [7] which is considered the Cerami condition. Let $\mathcal{B}$ be a Banach space and $\mathcal{J} \in C^1(\mathcal{B}, \mathbb{R})$. If a sequence $\{u_n\} \subset \mathcal{B}$ is said to be a Cerami sequence (in short $(Ce)_c$ sequence) at the level $c \in \mathbb{R}$ when $\mathcal{J}(u_n) \to c$ and $(1 + \|u_n\|_{\mathcal{B}})\|\mathcal{J}'(u_n)\|_{\mathcal{B}^*} \to 0$, where $\mathcal{B}^*$ denotes the dual space of $\mathcal{B}$.

Theorem 2.4 (see [7])**.** *Let* $\mathcal{B}$ *be a real Banach space. Suppose that* $\mathcal{J} \in C^1(\mathcal{B}, \mathbb{R})$ *satisfies*

$$\max\{\mathcal{J}(0), \mathcal{J}(e)\} \le \mu < \eta \le \inf_{\|u\|_{\mathcal{B}}=\rho} \mathcal{J}(u)$$

for some $\eta > \mu, \rho > 0$ *and* $e \in \mathcal{B}$ *with* $\|e\|_{\mathcal{B}} > \rho$. *Let* $c \ge \eta$ *be characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \mathcal{J}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], \mathcal{B}) : \gamma(0) = 0, \gamma(1) = e\}$ *is the set of continuous paths joining* 0 *and* e. *Then there exists a sequence* $\{u_n\} \subset \mathcal{B}$ *such that*

$$\mathcal{J}(u_n) \to c \ge \eta \text{ and } (1 + \|u_n\|_{\mathcal{B}})\|\mathcal{J}'(u_n)\|_{\mathcal{B}^*} \to 0.$$

3. Existence of nontrivial solutions

In this section, we study the existence of two nontrivial solutions for problem (1.1) and provide a proof of Theorem 1.1. For this, we borrow the truncation technique. As in [16], define the cut-off function $\varphi \in C^1([0,\infty), \mathbb{R})$ satisfying $\varphi(t) = 1$ if $t \in [0,1]$, $0 \le \varphi(t) \le 1$ if $t \in (1,2)$, $\varphi(t) = 0$ if $t \in [2,\infty)$, $\max_{t>0} |\varphi'(t)| \le 2$ and $\varphi'(t) \le 0$ for any $t \in (0,\infty)$. Furthermore, set

$$\widehat{\Theta} = \frac{(\mu + r)|h|_{\frac{p}{p-r}}}{\mu r(\Lambda_1 V_0)^{r/p}} \quad \text{for each } \beta > \Big(\frac{r\widehat{\Theta}}{2^{\frac{p-r}{p}} p\Theta}\Big)^{\frac{1}{p-r}}$$

with $\Theta = \frac{\mu-p}{\mu p}$. We move to consider the truncated functional $\mathcal{J}_{b,\lambda}^\beta : X_\lambda \to \mathbb{R}$ defined by

$$\mathcal{J}_{b,\lambda}^\beta(u) = \frac{1}{p}\|u\|_\lambda^p + \frac{b}{2p}\varphi\Big(\frac{\|u\|_\lambda^p}{\beta^p}\Big)[u]_{s,p}^{2p} - \int_{\mathbb{R}^N} F(x,u)dx - \int_{\mathbb{R}^N} G(x,u)dx,$$

where φ is a smooth cut-off function such that

$$\varphi\Big(\frac{\|u\|_\lambda^p}{\beta^p}\Big) = \begin{cases} 1, & \|u\|_\lambda \le \beta, \\ 0, & \|u\|_\lambda \ge 2^{1/p}\beta. \end{cases}$$

It is easy to see that $\mathcal{J}_{b,\lambda}^\beta$ is of class C^1. In addition, for all $u, v \in X_\lambda$ we find that

$$\langle (\mathcal{J}_{b,\lambda}^\beta)'(u), v\rangle = \Big(a \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+sp}}\,dx\,dy$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\lambda V(x)|u|^{p-2}uv\,dx\Big)\Big[1+\frac{b}{2\beta^p}\varphi'\Big(\frac{\|u\|_\lambda^p}{\beta^p}\Big)[u]_{s,p}^{2p}\Big]\\
&+b\varphi\Big(\frac{\|u\|_\lambda^p}{\beta^p}\Big)[u]_{s,p}^p\int\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+sp}}\,dx\,dy\\
&-\int_{\mathbb{R}^N}f(x,u)v\,dx-\int_{\mathbb{R}^N}g(x,u)v\,dx.
\end{aligned}
$$

Now we check that the truncated functional $\mathcal{J}_{b,\lambda}^\beta$ satisfies the mountain pass geometry.

Lemma 3.1. *Suppose that* (A1)–(A7) *are satisfied. Then for all* $\lambda\geq\Lambda_1$,

(i) *there exist* $\Pi_1,\rho,\eta>0$ *(independent of* β,b *and* λ*) such that for all* $0<|h|_{\frac{p}{p-r}}<\Pi_1$,

$$\inf\{\mathcal{J}_{b,\lambda}^\beta(u):u\in X_\lambda\quad\text{with}\quad\|u\|_\lambda=\rho\}\geq\eta;$$

(ii) *there exist* $\widehat{b}>0$ *(independent of* β *and* λ*) and* $e\in C_0^\infty(\Omega)$ *with* $\|e\|_\lambda>\rho$ *such that* $\mathcal{J}_{b,\lambda}^\beta(e)<0$ *for all* $b\in(0,\widehat{b})$.

Proof. Set $\varepsilon=\frac{\Lambda_1V_0}{2}$. We can use (1.6), (2.1), (A7) and Hölder inequality to obtain

$$
\begin{aligned}
\mathcal{J}_{b,\lambda}^\beta(u)&\geq\frac{1}{p}\|u\|_\lambda^p-\int_{\mathbb{R}^N}F(x,u)dx-\int_{\mathbb{R}^N}G(x,u)dx\\
&\geq\frac{1}{p}\|u\|_\lambda^p-\frac{\varepsilon}{p}\int_{\mathbb{R}^N}|u|^pdx-\frac{C_\varepsilon}{q}\int_{\mathbb{R}^N}|u|^qdx-\frac{1}{r}\int_{\mathbb{R}^N}h(x)|u|^rdx\\
&\geq\frac{1}{p}\Big(1-\frac{\varepsilon\mathfrak{C}}{a}|\{V<V_0\}|^{\frac{p_s^*-p}{p_s^*}}\Big)\|u\|_\lambda^p-\frac{C_\varepsilon}{q}\Big(\frac{\mathfrak{C}}{a}\Big)^{q/p}|\{V<V_0\}|^{\frac{p_s^*-q}{p_s^*}}\|u\|_\lambda^q\\
&\quad-\frac{|h|_{\frac{p}{p-r}}}{r}\Big(\frac{\mathfrak{C}}{a}\Big)^{r/p}|\{V<V_0\}|^{\frac{r(p_s^*-p)}{pp_s^*}}\|u\|_\lambda^r\\
&=\frac{1}{2p}\|u\|_\lambda^p-\frac{C_{\frac{\Lambda_1V_0}{2}}}{q}\Big(\frac{\mathfrak{C}}{a}\Big)^{q/p}\Big(\frac{a}{\mathfrak{C}\Lambda_1V_0}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\|u\|_\lambda^q-\frac{|h|_{\frac{p}{p-r}}}{r}\Big(\frac{1}{\Lambda_1V_0}\Big)^{r/p}\|u\|_\lambda^r\\
&:=\|u\|_\lambda^r\Big(\frac{1}{2p}\|u\|_\lambda^{p-r}-A\|u\|_\lambda^{q-r}-B|h|_{\frac{p}{p-r}}\Big),
\end{aligned}
$$

where

$$A=\frac{C_{\frac{\Lambda_1V_0}{2}}}{q}\Big(\frac{\mathfrak{C}}{a}\Big)^{q/p}\Big(\frac{a}{\mathfrak{C}\Lambda_1V_0}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\quad\text{and}\quad B=\frac{1}{r}\Big(\frac{1}{\Lambda_1V_0}\Big)^{r/p}.$$

By simple calculation, for each $t\geq0$, we can obtain that $\Psi(t):=\frac{1}{2p}t^{p-r}-At^{q-r}$ admits a unique maximum point at $t_0=\Big(\frac{p-r}{2Ap(q-r)}\Big)^{\frac{1}{q-p}}$ and its maximum is

$$\max_{t\geq0}\Psi(t)=\Psi(t_0)=\frac{q-p}{2p(q-r)}\Big(\frac{p-r}{2Ap(q-r)}\Big)^{\frac{p-r}{q-p}}>0.$$

Therefore, it is easy to deduce that there is $t_0:=\rho=\|u\|_\lambda>0$ such that

$$\mathcal{J}_{b,\lambda}^\beta(u)\geq\eta:=t_0^r\Big(\Psi(t_0)-B|h|_{\frac{p}{p-r}}\Big)>0,$$

provided that

$$|h|_{\frac{p}{p-r}}<\Pi_1:=r(q-p)\Big(\frac{a}{2\mathfrak{C}p(q-r)}\Big)^{\frac{q-r}{q-p}}\Big(\frac{q(p-r)}{C_{\frac{\Lambda_1V_0}{2}}}\Big)^{\frac{p-r}{q-p}}\Big(\frac{\mathfrak{C}\Lambda_1V_0}{a}\Big)^{\frac{(p_s^*-q)(p-r)}{(p_s^*-p)(q-p)}+\frac{r}{p}},$$

which completes the proof of (i).

To prove (ii), we define the functional $\mathcal{I}_\lambda:X_\lambda\to\mathbb{R}$ by

$$\mathcal{I}_\lambda(u)=\frac{1}{p}\|u\|_\lambda^p-\int_{\mathbb{R}^N}F(x,u)dx-\int_{\mathbb{R}^N}G(x,u)dx.$$

By (A4)–(A6) and (A8), there exist positive constants $C_i (i = 1, 2, \cdots, 4)$ such that

$$F(x,t) \geq C_1|t|^{\mu} - C_2|t|^p \quad \text{and} \quad G(x,t) \geq C_3|t|^{\nu} - C_4 \tag{3.1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Let $\phi \in C_0^{\infty}(\Omega)$ be a fixed smooth positive function. Using (3.1), then one sees that

$$\begin{aligned}
\mathcal{I}_{\lambda}(l\phi) &= \frac{al^p}{p}\int_{\mathcal{Q}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - \int_{\Omega} F(x,l\phi)dx - \int_{\Omega} G(x,l\phi)\,dx \\
&\leq \frac{al^p}{p}\int_{\mathcal{Q}} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy - C_1 l^{\mu}\int_{\Omega}|\phi|^{\mu}dx + C_2 l^p \int_{\Omega}|\phi|^p dx - C_3 l^{\nu}\int_{\Omega}|\phi|^{\nu}dx + C_4|\Omega| \\
&\to -\infty \quad \text{as } l \to +\infty.
\end{aligned}$$

Then, there exists $e \in C_0^{\infty}(\Omega)$ with $\|e\|_{\lambda} > \rho$ such that $\mathcal{I}_{\lambda}(e) \leq -1$. Then there exists $\widehat{b} > 0$ such that

$$\begin{aligned}
\mathcal{J}_{b,\lambda}^{\beta}(e) &= \mathcal{I}_{\lambda}(e) + \frac{b}{2p}\varphi\Big(\frac{\|e\|_{\lambda}^p}{\beta^p}\Big)\Big(\int_{\mathcal{Q}} \frac{|e(x)-e(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2 \\
&\leq -1 + \frac{b}{2p}\Big(\int_{\mathcal{Q}} \frac{|e(x)-e(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2 < 0
\end{aligned}$$

for all $b \in (0, \widehat{b})$. This completes the proof. □

Now we define the mountain pass value

$$c_{b,\lambda}^{\beta} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}_{b,\lambda}^{\beta}(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X_{\lambda}) : \gamma(0) = 0, \gamma(1) = e\}$. Then by Lemma 3.1 and Theorem 2.4, we deduce that for all $\lambda \geq \Lambda_1$ and $b \in (0, \widehat{b})$, there exists a $(Ce)_{c_{b,\lambda}^{\beta}}$ sequence $\{u_n\} \subset X_{\lambda}$ such that

$$\mathcal{J}_{b,\lambda}^{\beta}(u_n) \to c_{b,\lambda}^{\beta} \geq \eta > 0 \quad \text{and} \quad (1 + \|u_n\|_{\lambda})\|(\mathcal{J}_{b,\lambda}^{\beta})'(u_n)\|_{X_{\lambda}^*} \to 0. \tag{3.2}$$

In the following lemma, we provide an estimate on the upper bound of $c_{b,\lambda}^{\beta}$ which plays a key role in the use of truncation technique.

Lemma 3.2. *Suppose that* (A1)–(A6), (A8) *are satisfied. Then for all $\lambda \geq \Lambda_1$ and $b \in (0, \widehat{b})$, there exists $M > 0$ (independent of β, b and λ) such that $c_{b,\lambda}^{\beta} \leq M$.*

Proof. For each $0 < e \in C_0^{\infty}(\Omega)$, we have

$$\begin{aligned}
\mathcal{J}_{b,\lambda}^{\beta}(te) &= \frac{t^p}{p}\|e\|_{\lambda}^p + \frac{bt^{2p}}{2p}\varphi\Big(\frac{t^p\|e\|_{\lambda}^p}{\beta^p}\Big)[e]_{s,p}^{2p} - \int_{\mathbb{R}^N} F(x,te)dx - \int_{\mathbb{R}^N} G(x,te)dx \\
&\leq \frac{at^p}{p}\int_{\mathcal{Q}} \frac{|e(x)-e(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \frac{\widehat{b}t^{2p}}{2p}\Big(\int_{\mathcal{Q}} \frac{|e(x)-e(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2 \\
&\quad - C_1 t^{\mu}\int_{\Omega}|e|^{\mu}dx + C_2 t^p\int_{\Omega}|e|^p dx - C_3 t^{\nu}\int_{\Omega}|e|^{\nu}dx + C_4|\Omega|.
\end{aligned}$$

Consequently, there exists a constant $M > 0$ such that

$$c_{b,\lambda}^{\beta} \leq \max_{t\in[0,1]} \mathcal{J}_{b,\lambda}^{\beta}(te) \leq M.$$

This completes the proof. □

Next, up to a subsequence, we show that the sequence $\{u_n\}$ given by (3.2) satisfies $\|u_n\|_{\lambda} \leq \beta$ for the selected appropriate $\beta > 0$ which is defined at the beginning of this section.

Lemma 3.3. *Suppose that* (A1)–(A3), (A6)–(A8) *are satisfied, and let $\beta = \big(\frac{M+1}{\Theta}\big)^{1/p}$. If $\{u_n\} \subset X_{\lambda}$ is a sequence satisfying* (3.2)*, up to a subsequence, then there exists $\Pi_2 > 0$ such that for all $b \in (0, \widehat{b})$ and $0 < |h|_{\frac{p}{p-r}} < \Pi_2$, we have $\|u_n\|_{\lambda} \leq \beta$ for all $\lambda \geq \Lambda_1$.*

Proof. We first prove that $\|u_n\|_\lambda \le 2^{1/p}\beta$ as $n \to \infty$. Suppose by contradiction that, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n\|_\lambda > 2^{1/p}\beta$. We introduce the function $\Phi : [0, +\infty) \to \mathbb{R}$ defined by $\Phi(t) = \Theta t^p - \widehat{\Theta} t^r$.

A direct calculation shows that there exists a unique number $t_0 = \big(\frac{r\widehat{\Theta}}{p\Theta}\big)^{\frac{1}{p-r}} > 0$ such that $\min_{t\ge 0} \Phi(t) = \Phi(t_0)$. Thus, using (A6)–(A8), (2.1) and Hölder inequality lead to

$$
\begin{aligned}
c^\beta_{b,\lambda} &= \lim_{n\to\infty} \Big[\mathcal{J}^\beta_{b,\lambda}(u_n) - \frac{1}{\mu}\langle (\mathcal{J}^\beta_{b,\lambda})'(u_n), u_n\rangle\Big] \\
&= \lim_{n\to\infty} \Big[\Big(\frac{1}{p} - \frac{1}{\mu} - \frac{b}{2\mu\beta^p}\varphi'\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p}\Big)\|u_n\|^p_\lambda + \Big(\frac{b}{2p} - \frac{b}{\mu}\Big)\varphi\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p} \\
&\quad - \int_{\mathbb{R}^N} \Big(F(x, u_n) - \frac{1}{\mu} f(x, u_n)u_n\Big)dx - \int_{\mathbb{R}^N} \Big(G(x, u_n) - \frac{1}{\mu} g(x, u_n)u_n\Big)dx\Big] \\
&\ge \lim_{n\to\infty} \Big[\Big(\frac{\mu - p}{\mu p} - \frac{b}{2\mu\beta^p}\varphi'\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p}\Big)\|u_n\|^p_\lambda - \frac{b(2p-\mu)}{2\mu p}\varphi\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p} \\
&\quad - \int_{\mathbb{R}^N} \Big(|G(x, u_n)| + \frac{1}{\mu}|g(x, u_n)u_n|\Big)dx\Big] \\
&\ge \lim_{n\to\infty} \Big[\Big(\frac{\mu - p}{\mu p} - \frac{b}{2\mu\beta^p}\varphi'\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p}\Big)\|u_n\|^p_\lambda - \frac{b(2p-\mu)}{2\mu p}\varphi\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p} \\
&\quad - \frac{(\mu + r)|h|_{\frac{p}{p-r}}}{\mu r}\Big(\frac{1}{\Lambda_1 V_0}\Big)^{r/p}\|u_n\|^r_\lambda\Big] \\
&= \lim_{n\to\infty} \Big[\frac{\mu - p}{\mu p}\|u_n\|^p_\lambda - \frac{(\mu + r)|h|_{\frac{p}{p-r}}}{\mu r(\Lambda_1 V_0)^{r/p}}\|u_n\|^r_\lambda - \frac{b}{2\mu\beta^p}\varphi'\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p}\|u_n\|^p_\lambda \\
&\quad - \frac{b(2p-\mu)}{2\mu p}\varphi\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p}\Big] \\
&\ge 2M + \frac{3}{2} + \Big[\frac{1}{2} - \widehat{\Theta}\Big(\frac{2(M+1)}{\Theta}\Big)^{r/p}\Big] \\
&> 2M + \frac{3}{2},
\end{aligned}
$$

provided that

$$
|h|_{\frac{p}{p-r}} < \Pi_2 := \min\Big\{\frac{\mu r}{\mu + r}\Big[\frac{\Lambda_1 V_0(\mu - p)}{2^{\frac{p+r}{r}}\mu p(M+1)}\Big]^{r/p}, \frac{2\mu p(M+1)}{\mu + r}\Big[\frac{\Lambda_1 V_0(\mu - p)}{2\mu p(M+1)}\Big]^{r/p}\Big\},
$$

which is a contradiction by Lemma 3.2.

To finish the proof of the lemma, let us argue by contradiction. Assuming that there is no subsequence of $\{u_n\}$ which is uniformly bounded by β. Then we infer that $\beta < \|u_n\|_\lambda \le 2^{1/p}\beta$ as $n \to \infty$. Using a calculation method similar to the above and borrowing the fact that φ is non-increasing, we know that

$$
\begin{aligned}
c^\beta_{b,\lambda} &= \lim_{n\to\infty} \Big[\mathcal{J}^\beta_{b,\lambda}(u_n) - \frac{1}{\mu}\langle (\mathcal{J}^\beta_{b,\lambda})'(u_n), u_n\rangle\Big] \\
&\ge \lim_{n\to\infty} \Big[\frac{\mu - p}{\mu p}\|u_n\|^p_\lambda - \frac{(\mu + r)|h|_{\frac{p}{p-r}}}{\mu r(\Lambda_1 V_0)^{r/p}}\|u_n\|^r_\lambda \\
&\quad - \frac{b}{2\mu\beta^p}\varphi'\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p}\|u_n\|^p_\lambda - \frac{b(2p-\mu)}{2\mu p}\varphi\Big(\frac{\|u_n\|^p_\lambda}{\beta^p}\Big)[u_n]^{2p}_{s,p}\Big] \\
&\ge \liminf_{n\to\infty} \Big[\frac{\mu - p}{\mu p}\|u_n\|^p_\lambda - \frac{(\mu + r)|h|_{\frac{p}{p-r}}}{\mu r(\Lambda_1 V_0)^{r/p}}\|u_n\|^r_\lambda - \frac{b|2p-\mu|}{2a^2\mu p}\|u_n\|^{2p}_\lambda\Big] \\
&\ge \Theta\beta^p - \widehat{\Theta}\Big(2^{1/p}\beta\Big)^r - \frac{b|2p-\mu|}{2a^2\mu p}\Big(2^{1/p}\beta\Big)^{2p} \\
&= M + \frac{1}{2} + \Big[\frac{1}{2} - \widehat{\Theta}\Big(\frac{2(M+1)}{\Theta}\Big)^{r/p}\Big] - \frac{2b|2p-\mu|}{\mu p}\Big(\frac{M+1}{a\Theta}\Big)^2,
\end{aligned}
$$

provided that $0 < |h|_{\frac{p}{p-r}} < \Pi_2$, there exists a constant $\widehat{b} > 0$ such that for each fixed $a > 0$ and $b \in (0, \widehat{b})$, this is a contradiction by choosing $\widehat{b}$ as small enough. This completes the proof. □

Remark 3.4. If the sequence $\{u_n\} \subset X_\lambda$ is obtained in Lemma 3.3, then $\{u_n\}$ is also a bounded (Ce) sequence of $\mathcal{J}_{b,\lambda}$ satisfying $\|u_n\|_\lambda \le \beta$. By the definition of the truncation function φ, one can see that

$$\mathcal{J}_{b,\lambda}(u_n) \to c^{\beta}_{b,\lambda} \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|\mathcal{J}'_{b,\lambda}(u_n)\|_{X^*_\lambda} \to 0.$$

Lemma 3.5. *Suppose that* (A1)–(A3), (A6)–(A8) *are satisfied. Let* $\kappa \in [p, p^*_s)$ *and* $\{u_n\}$ *be a bounded* $(Ce)_{c^{\beta}_{b,\lambda}}$ *sequence. Then there is a subsequence again denoted as* $\{u_n\}$ *such that for each* $\varepsilon > 0$, *there exists* $R_\varepsilon > 0$ *such that for all* $R \ge R_\varepsilon$,

$$\limsup_{n\to\infty} \int_{B_n \setminus B_R} |u_n|^\kappa dx \le \varepsilon,$$

where $B_k = \{x \in \mathbb{R}^N : |x| \le k\}$.

The proof of the above lemma can be found in [15], so we omit it here.

Lemma 3.6. *Suppose that* (A4) *and* (A5) *are satisfied. If* $u_n \rightharpoonup u$ *in* $W^{s,p}(\mathbb{R}^N)$ *and* $v_n := u_n - u$, *then*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |[f(x, u_n) - f(x, v_n) - f(x, u)]\psi| dx = 0$$

uniformly in $\psi \in W^{s,p}(\mathbb{R}^N)$ *with* $\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \le 1$.

Proof. The proof is similar to the proof in [6]. For the convenience of readers, here we provide the process of proof. Let $\xi : [0, +\infty) \to [0, 1]$ be a smooth cut-off function satisfying $\xi(t) = 1$ if $|t| \le 1$ and $\xi(t) = 0$ if $|t| \ge p$. We define $\bar{u}_n(x) = \xi(\frac{p|x|}{n})u(x)$. Obviously,

$$\|\bar{u}_n - u\|_\lambda \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Going if necessary to a subsequence, define $w_n = u - \bar{u}_n$. It follows from (3.3) and any $\psi \in W^{s,p}(\mathbb{R}^N)$ that

$$\lim_{n\to\infty} \sup_{\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} [f(x, \bar{u}_n) - f(x, u)]\psi dx = 0. \tag{3.4}$$

Moreover, by (A4), (A5), Lemma 3.5 and [15, Lemma 3.3], a standard argument shows that

$$\lim_{n\to\infty} \sup_{\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} [f(x, u_n) - f(x, u_n - \bar{u}_n) - f(x, \bar{u}_n)]\psi dx = 0. \tag{3.5}$$

Next, we prove that

$$\lim_{n\to\infty} \sup_{\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} [f(x, v_n + w_n) - f(x, v_n)]\psi dx = 0. \tag{3.6}$$

We define $\bar{F}(x, 0) = 0$ and $\bar{F}(x, t) = \frac{f(x,t)}{|t|^{p-2}t}$ if $t \ne 0$. By (A4) and (A5), $\bar{F}$ is continuous at $t = 0$ and $|\bar{F}(x, t)| \le C_1(1+|t|^{q-p})$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. For any $\sigma > 0$, we set $A^\sigma_n = \{x \in \mathbb{R}^N : |v_n(x)| \le \sigma\}$ and $B^\sigma_n = \mathbb{R}^N \setminus A^\sigma_n$. Then the Lebesgue measure

$$|B^\sigma_n| \le \frac{1}{\sigma^q} \int_{B^\sigma_n} |v_n|^q dx \le \frac{C_2}{\sigma^q} \to 0 \ \text{ as } \sigma \to \infty.$$

Hence, for some $\bar{\sigma} > 0$ and any $\varepsilon > 0$, by (A4), (A5), Hölder inequality and the boundedness of $\{w_n\}$, we have

$$\Big| \int_{B^{\bar{\sigma}}_n} [f(x, v_n + w_n) - f(x, v_n)]\psi dx \Big| \le \varepsilon \tag{3.7}$$

uniformly in $\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \le 1$. By the uniformly continuity of $\bar{F}$ on $\mathbb{R}^N \times [-\bar{\sigma}, \bar{\sigma}]$, there exists $\delta := \delta(\varepsilon, \bar{\sigma}) > 0$ such that $|w| < \delta$ and

$$|\bar{F}(x, t + w) - \bar{F}(x, t)| \le \varepsilon \quad \text{for all } (x, t) \in \mathbb{R}^N \times [-\bar{\sigma}, \bar{\sigma}]. \tag{3.8}$$

Similarly, we set $C_n^\delta = \{x \in \mathbb{R}^N : |w_n(x)| \leq \delta\}$ and $D_n^\delta = \mathbb{R}^N \setminus C_n^\delta$. Since $|A_n^{\bar\sigma} \bigcap D_n^\delta| \to 0$ as $n \to \infty$, it is easy to know that there exists $N_1 \in \mathbb{N}^+$ such that $n \geq N_1$ and

$$\Big| \int_{A_n^{\bar\sigma} \bigcap D_n^\delta} [f(x, v_n + w_n) - f(x, v_n)]\psi dx\Big| \leq \varepsilon \tag{3.9}$$

uniformly in $\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \leq 1$. By using (3.3), there exists $N_2 \in \mathbb{N}^+$ ensures that both $|w_n|_p < \varepsilon$ and $|w_n|_q < \varepsilon$ hold for any $n \geq N_2 \geq N_1$. Thus, from (3.8) we infer that

$$\Big| \int_{A_n^{\bar\sigma} \bigcap C_n^\delta} [\bar F(x, v_n + w_n) - \bar F(x, v_n)]|v_n|^{p-2} v_n \psi dx\Big| \leq C_3\varepsilon \tag{3.10}$$

uniformly in $\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \leq 1$. Based on (3.8)–(3.10), the boundedness of $\{v_n\}$ and $\{w_n\}$, Hölder inequality and the inequalities

$$\begin{gathered} |a+b|^\alpha \leq C_\alpha(|a|^\alpha + |b|^\alpha), \quad \alpha \in (0, +\infty), \\ \| a|^{\alpha-2}a - |b|^{\alpha-2}b| \leq C_\alpha(|a| + |b|)^{\alpha-2}|a-b|, \quad \alpha \in (2, +\infty), \end{gathered} \tag{3.11}$$

one has

$$\begin{aligned} &\Big| \int_{A_n^{\bar\sigma}} [f(x, v_n + w_n) - f(x, v_n)]\psi dx\Big| \\ &\leq \Big| \int_{A_n^{\bar\sigma} \bigcap C_n^\delta} [f(x, v_n + w_n) - f(x, v_n)]\psi dx\Big| \\ &\quad + \Big| \int_{A_n^{\bar\sigma} \bigcap D_n^\delta} [f(x, v_n + w_n) - f(x, v_n)]\psi dx\Big| \\ &\leq \Big| \int_{A_n^{\bar\sigma} \bigcap D_n^\delta} [f(x, v_n + w_n) - f(x, v_n)]\psi dx\Big| \\ &\quad + \Big| \int_{A_n^{\bar\sigma} \bigcap C_n^\delta} [\bar F(x, v_n + w_n) - \bar F(x, v_n)]|v_n|^{p-2} v_n \psi dx\Big| \\ &\quad + \Big| \int_{A_n^{\bar\sigma} \bigcap C_n^\delta} \bar F(x, v_n + w_n)[|v_n + w_n|^{p-2}(v_n + w_n) - |v_n|^{p-2}v_n]\psi dx\Big| \\ &\leq \Big| \int_{A_n^{\bar\sigma} \bigcap D_n^\delta} [f(x, v_n + w_n) - f(x, v_n)]\psi dx\Big| \\ &\quad + \Big| \int_{A_n^{\bar\sigma} \bigcap C_n^\delta} [\bar F(x, v_n + w_n) - \bar F(x, v_n)]|v_n|^{p-2} v_n \psi dx\Big| \\ &\quad + C_4 \int_{\mathbb{R}^N} (1 + |v_n + w_n|^{q-p})(|v_n|^{p-2}|w_n| + |w_n|^{p-1})|\psi| dx \\ &\leq \varepsilon + C_3\varepsilon + C_5(\varepsilon + \varepsilon^{q-1} + \varepsilon^{p-1} + \varepsilon^{q-p+1}) \end{aligned}$$

uniformly in $\|\psi\|_{W^{s,p}(\mathbb{R}^N)} \leq 1$. This, together with (3.7), implies that (3.6). Hence, it follows from (3.4)–(3.6) that Lemma holds. □

We are now preparing to prove the following compactness conditions of $\mathcal{J}_{b,\lambda}$.

Lemma 3.7. *Suppose that* (A1)–(A5), (A7) *are satisfied, and let* $\beta = \big(\frac{M+1}{\Theta}\big)^{1/p}$. *Then there exists* $\Lambda > 0$ *such that, for each* $b \in (0, \widehat{b})$ *and* $\lambda > \Lambda$, *if* $\{u_n\} \subset X_\lambda$ *is a sequence satisfying* (3.2), *then* $\{u_n\}$ *has a convergent subsequence in* X_λ.

Proof. By Lemma 3.3 and Remark 3.4, we see that the bounded $(Ce)_{c_{b,\lambda}^\beta}$ sequence $\{u_n\}$ satisfying $\|u_n\|_\lambda \leq \beta$. Then there exist $u \in X_\lambda$ and $L > 0$ such that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } X_\lambda, \\ u_n \to u \quad \text{strongly in } L^\kappa_{\rm loc}(\mathbb{R}^N) \quad \text{for all } \kappa \in [p, p_s^*), \\ u_n \to u \quad \text{a.e. in } \mathbb{R}^N, \\ [u]^p_{s,p} \leq \lim_{n\to\infty} [u_n]^p_{s,p} = L^p. \end{gathered} \tag{3.12}$$

Since X_λ is continuously embedded in $W^{s,p}(\mathbb{R}^N)$, there exists a constant $C > 0$ such that $\|u_n\|_{W^{s,p}(\mathbb{R}^N)} \le C\beta$. Then, $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$ implies that

$$\begin{aligned}
&\lim_{n\to\infty} \int\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}[u_n(x)-u_n(y)][v(x)-v(y)]}{|x-y|^{N+sp}}\,dx\,dy \\
&= \int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+sp}}\,dx\,dy..
\end{aligned} \tag{3.13}$$

Combining Lemma 2.3 and Hölder inequality, for all $v \in X_\lambda$, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(x,u_n)v\,dx = \int_{\mathbb{R}^N} f(x,u)v\,dx, \tag{3.14}$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x,u_n)v\,dx = \int_{\mathbb{R}^N} g(x,u)v\,dx. \tag{3.15}$$

Then, by (3.12)–(3.15), for any $v \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
o(1) &= \langle \mathcal{J}'_{b,\lambda}(u_n), v\rangle \\
&\to (a+bL^p)\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+sp}}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N} \lambda V(x)|u|^{p-2}uv\,dx - \int_{\mathbb{R}^N} f(x,u)v\,dx - \int_{\mathbb{R}^N} g(x,u)v\,dx \quad \text{as } n\to\infty,
\end{aligned}$$

which yields

$$\|u\|_\lambda^p + bL^p[u]_{s,p}^p - \int_{\mathbb{R}^N} f(x,u)u\,dx - \int_{\mathbb{R}^N} g(x,u)u\,dx = 0. \tag{3.16}$$

Set $v_n = u_n - u$. By $\|u_n\|_\lambda \le \beta$ one has

$$\|u\|_\lambda \le \liminf_{n\to\infty} \|u_n\|_\lambda \le \beta,$$

leading to

$$\|v_n\|_\lambda = \|u_n - u\|_\lambda \le 2\beta. \tag{3.17}$$

From the Brézis-Lieb Lemma [3] we know that

$$\|v_n\|_\lambda^p = \|u_n\|_\lambda^p - \|u\|_\lambda^p + o(1). \tag{3.18}$$

Moreover, in view of (A2), (3.12) and Hölder inequality, for any $\kappa \in [p, p_s^*)$, one has

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^p dx &= \int_{\{V\ge V_0\}} |v_n|^p dx + \int_{\{V<V_0\}} |v_n|^p dx \\
&\le \frac{1}{\lambda V_0}\int_{\mathbb{R}^N} \lambda V|v_n|^p dx + |\{V<V_0\}|^{\frac{\kappa-p}{\kappa}} \Big(\int_{\{V<V_0\}} |v_n|^\kappa dx\Big)^{\frac{p}{\kappa}} \\
&\le \frac{1}{\lambda V_0}\|v_n\|_\lambda^p + o(1).
\end{aligned}$$

This implies

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^q dx &= \int_{\mathbb{R}^N} \Big(|v_n|^{\frac{p(p_s^*-q)}{p_s^*-p}} |v_n|^{\frac{p_s^*(q-p)}{p_s^*-p}}\Big) dx \\
&\le \Big(\int_{\mathbb{R}^N} |v_n|^p dx\Big)^{\frac{p_s^*-q}{p_s^*-p}} \Big(\int_{\mathbb{R}^N} |v_n|^{p_s^*} dx\Big)^{\frac{q-p}{p_s^*-p}} \\
&\le \Big(\frac{1}{\lambda V_0}\|v_n\|_\lambda^p\Big)^{\frac{p_s^*-q}{p_s^*-p}} \Big(\mathfrak{C}\int\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{\frac{p_s^*(q-p)}{p(p_s^*-p)}} + o(1) \\
&\le \Big(\frac{1}{\lambda V_0}\Big)^{\frac{p_s^*-q}{p_s^*-p}} \Big(\frac{\mathfrak{C}}{a}\Big)^{\frac{p_s^*(q-p)}{p(p_s^*-p)}} \|v_n\|_\lambda^q + o(1).
\end{aligned} \tag{3.19}$$

By Lemma 3.6, $v_n \rightharpoonup 0$ in $W^{s,p}(\mathbb{R}^N)$ and $v_n \to 0$ in $L^{\kappa}_{\rm loc}(\mathbb{R}^N)$ for all $\kappa \in [p, p_s^*)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N}[f(x,u_n)u_n - f(x,u)u]dx &= \int_{\mathbb{R}^N}[f(x,u_n) - f(x,v_n) - f(x,u)u_n]dx \\ &\quad + \int_{\mathbb{R}^N}[f(x,v_n)v_n + f(x,v_n)u + f(x,u)v_n]dx \\ &= \int_{\mathbb{R}^N} f(x,v_n)v_n\,dx + o(1).\end{aligned} \tag{3.20}$$

Based on (1.6), (2.1), (3.17), (3.19), Hölder inequality and let $\varepsilon = \frac{\Lambda_1 V_0}{2}$, one has

$$\begin{aligned}&\int_{\mathbb{R}^N} f(x,v_n)v_n\,dx \\ &\le \frac{\Lambda_1 V_0}{2}\int_{\mathbb{R}^N}|v_n|^p dx + C_{\frac{\Lambda_1 V_0}{2}}\int_{\mathbb{R}^N}|v_n|^q dx \\ &\le \frac{\Lambda_1 V_0}{2}\int_{\mathbb{R}^N}|v_n|^p dx + C_{\frac{\Lambda_1 V_0}{2}}\Big(\int_{\mathbb{R}^N}|v_n|^q dx\Big)^{\frac{q-p}{q}}\Big(\int_{\mathbb{R}^N}|v_n|^q dx\Big)^{\frac{p}{q}} \\ &\le \Big[\frac12 + C_{\frac{\Lambda_1 V_0}{2}}\Big((2\beta)^q\Big(\frac{1}{\lambda V_0}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\Big(\frac{\mathfrak{C}}{a}\Big)^{\frac{p_s^*(q-p)}{p(p_s^*-p)}}\Big)^{\frac{q-p}{q}}\frac{\mathfrak{C}}{a}\Big(\frac{a}{\mathfrak{C}\Lambda_1 V_0}\Big)^{\frac{p(p_s^*-q)}{q(p_s^*-p)}}\Big]\|v_n\|_\lambda^p + o(1).\end{aligned} \tag{3.21}$$

Furthermore, by applying Lemma 2.3, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}[g(x,u_n)u_n - g(x,u)u]dx \\ &= \int_{\mathbb{R}^N} g(x,u)(u_n-u)dx + \int_{\mathbb{R}^N}[g(x,u_n) - g(x,u)u_n]dx \\ &\le \int_{\mathbb{R}^N} g(x,u)v_n\,dx + \Big(\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^{\frac{p}{p-1}}dx\Big)^{\frac{p-1}{p}}|u_n|_p \\ &\to 0 \quad \text{as } n\to\infty.\end{aligned} \tag{3.22}$$

Observe that $\langle \mathcal{J}'_{b,\lambda}(u_n), u_n\rangle = o(1)$, then from (3.12), (3.16), (3.18) and (3.20)–(3.22), we have

$$\begin{aligned}o(1) &= \|v_n\|_\lambda^p + bL^{2p} - bL^p[u]_{s,p}^p - \int_{\mathbb{R}^N}[f(x,u_n)u_n - f(x,u)u]dx \\ &\quad - \int_{\mathbb{R}^N}[g(x,u_n)u_n - g(x,u)u]dx + o(1) \\ &\ge \|v_n\|_\lambda^p - \int_{\mathbb{R}^N}[f(x,u_n)u_n - f(x,u)u]dx + o(1) \\ &\ge \Big[\frac12 - C_{\frac{\Lambda_1 V_0}{2}}\Big((2\beta)^q\Big(\frac{1}{\lambda V_0}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\Big(\frac{\mathfrak{C}}{a}\Big)^{\frac{p_s^*(q-p)}{p(p_s^*-p)}}\Big)^{\frac{q-p}{q}}\frac{\mathfrak{C}}{a}\Big(\frac{a}{\mathfrak{C}\Lambda_1 V_0}\Big)^{\frac{p(p_s^*-q)}{q(p_s^*-p)}}\Big]\|v_n\|_\lambda^p + o(1),\end{aligned}$$

which yields that there exists

$$\Lambda_2 = \frac{(2\beta)^{\frac{q(p_s^*-p)}{p_s^*-q}}}{V_0}|\{V<V_0\}|^{\frac{p(p_s^*-p)}{p_s^*(q-p)}}\Big(2C_{\frac{\Lambda_1 V_0}{2}}\Big)^{\frac{q(p_s^*-p)}{(q-p)(p_s^*-q)}}\Big(\frac{\mathfrak{C}}{a}\Big)^{\frac{pq(p_s^*-p)+p_s^*(q-p)^2}{p(q-p)(p_s^*-q)}}$$

such that $v_n \to 0$ strongly in X_λ for all $\lambda > \Lambda := \max\{1, \Lambda_1, \Lambda_2\}$. This completes the proof. □

Let $\mathcal{J}_{b,\lambda}(u)|_{E_0}$ be a restriction of $\mathcal{J}_{b,\lambda}$ on E_0, that is

$$\begin{aligned}\mathcal{J}_{b,\lambda}(u)|_{E_0} &= \frac{a}{p}\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \frac{b}{2p}\Big(\int_{\mathcal{Q}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2 \\ &\quad - \int_\Omega F(x,u)dx - \int_\Omega G(x,u)dx.\end{aligned} \tag{3.23}$$

We define

$$\widehat{c} = \inf_{\gamma\in\widehat{\Gamma}}\max_{t\in[0,1]}\mathcal{J}_{b,\lambda}|_{E_0}(\gamma(t)),$$

where

$$\widehat{\Gamma} = \{\gamma \in C([0,1], E_0) : \gamma(0) = 0, \gamma(1) = e\}.$$

Indeed, it is easily seen that $\widehat{c}$ is independent of λ. Furthermore, if (A4)–(A8) hold, then similarly to the discussion of Lemmas 3.1–3.3 and Remark 3.4, there exist constants $\widehat{b}, \Pi_3 > 0$ such that for all $b \in (0, \widehat{b})$ and $0 < |h|_{\frac{p}{p-r}} < \Pi_3$, using Theorem 2.4 and $E_0 \subset X_\lambda$ for all $\lambda > 0$, we can conclude that $0 < \eta \le c^\beta_{b,\lambda} \le \widehat{c}$ for all $\lambda \ge \Lambda_1$. Now, we can choose $\widehat{M} > \widehat{c}$. Thus

$$0 < \eta \le c^\beta_{b,\lambda} \le \widehat{c} < \widehat{M} \quad \text{for all } \lambda \ge \Lambda_1. \tag{3.24}$$

Proof of Theorem 1.1. Let β be defined as in Lemma 3.3. By Lemmas 3.1–3.3 and Remark 3.4, there exist constants $\Pi := \min\{\Pi_1, \Pi_2, \Pi_3\}, \widehat{b} > 0$ such that for all $b \in (0, \widehat{b})$ and $0 < |h|_{\frac{p}{p-r}} < \Pi$, $\mathcal{J}_{b,\lambda}$ possesses a (Ce) sequence $\{u_n\} \subset X_\lambda$ at the mountain pass level $c^\beta_{b,\lambda}$ for all $\lambda \ge \Lambda_1$, after passing to a subsequence, $\{u_n\}$ satisfying

$$\sup_{n\in\mathbb{N}^+} \|u_n\|_\lambda \le \beta, \quad \mathcal{J}_{b,\lambda}(u_n) \to c^\beta_{b,\lambda} \quad \text{and} \quad (1 + \|u_n\|_\lambda)\|\mathcal{J}'_{b,\lambda}(u_n)\|_{X^*_\lambda} \to 0.$$

So by Lemma 3.7, we can see that there exists $\Lambda > 0$ such that $\{u_n\}$ has a convergent subsequence in X_λ for all $\lambda > \Lambda$ and $b \in (0, \widehat{b})$. We may assume that $u_n \to u^{(1)}_{b,\lambda}$ as $n \to \infty$, and thus

$$0 < \|u^{(1)}_{b,\lambda}\|_\lambda \le \beta, \quad \mathcal{J}_{b,\lambda}(u^{(1)}_{b,\lambda}) = c^\beta_{b,\lambda} \quad \text{and} \quad \mathcal{J}'_{b,\lambda}(u^{(1)}_{b,\lambda}) = 0.$$

Consequently, $u^{(1)}_{b,\lambda}$ is a nontrivial solution for problem (1.1).

The second nontrivial solution of problem (1.1) will be constructed by the local minimization. Now we show that there exists $\psi \in X_\lambda$ such that $\mathcal{J}_{b,\lambda}(l\psi) < 0$ for all $l > 0$ small enough. By (3.1) and (3.23), take $\psi \in E_0$ with $\int_\Omega |\psi|^\nu dx > 0$, we obtain

$$\begin{aligned}
\mathcal{J}_{b,\lambda}(l\psi) &= \frac{al^p}{p}\int_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \frac{bl^{2p}}{2p}\Big(\int_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2 \\
&\quad - \int_\Omega F(x, l\psi)dx - \int_\Omega G(x, l\psi)dx \\
&\le \frac{al^p}{p}\int_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + \frac{bl^{2p}}{2p}\Big(\int_{\mathcal{Q}} \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2 \\
&\quad - C_1 l^\mu \int_\Omega |\psi|^\mu dx + C_2 l^p \int_\Omega |\psi|^p dx - C_3 l^\nu \int_\Omega |\psi|^\nu dx + C_4|\Omega| \\
&< 0 \quad \text{as } l \to 0.
\end{aligned} \tag{3.25}$$

Then we can deduce that the minimum of the (weakly lower semi-continuous) functional $\mathcal{J}_{b,\lambda}$ on any closed ball in X_λ with center 0 and radius $\varrho < \rho$ satisfying $\mathcal{J}_{b,\lambda}(u) \ge 0$ for any $u \in X_\lambda$ with $\|u\|_\lambda = \varrho$ is achieved in the corresponding open ball and thus draw out a nontrivial solution $u^{(2)}_{b,\lambda}$ of problem (1.1) satisfying $\mathcal{J}_{b,\lambda}(u^{(2)}_{b,\lambda}) < 0$ and $\|u^{(2)}_{b,\lambda}\| < \varrho$. In addition, (3.25) means that there exist $l_0 > 0$ and $\alpha < 0$ (independent of λ) such that $\mathcal{J}_{b,\lambda}(l_0\psi) = \alpha$ and $\|l_0\psi\| < \varrho$. Then it is clear that

$$\mathcal{J}_{b,\lambda}(u^{(2)}_{b,\lambda}) \le \alpha < 0 < \eta \le c^\beta_{b,\lambda} = \mathcal{J}_{b,\lambda}(u^{(1)}_{b,\lambda}) \tag{3.26}$$

for all $b \in (0, \widehat{b})$, $\lambda > \Lambda$ and $0 < |h|_{\frac{p}{p-r}} < \Pi$. The proof is complete. □

4. Asymptotic behavior of nontrivial solutions

In this section, we discuss the asymptotic behavior of the desired solutions in Theorem 1.1 and provide the proofs of Theorems 1.2–1.4.

Proof of Theorem 1.2. If $u^{(i)}_{b,\lambda}(i=1,2)$ are two different nontrivial solutions of problem (1.1) given by Theorem 1.1. For each fixed $b\in(0,\widehat{b})$, then for any $0<|h|_{\frac{p}{p-r}}<\Pi$ and sequence $\lambda_n\to\infty$, up to a subsequence, let $u^{(i)}_n=u^{(i)}_{b,\lambda_n}$ be the critical points of $\mathcal{J}_{b,\lambda_n}$, by (3.24) and (3.26), we obtain

$$\mathcal{J}_{b,\lambda_n}(u^{(2)}_n)\le\alpha<0<\eta\le c^{\beta}_{b,\lambda_n}=\mathcal{J}_{b,\lambda_n}(u^{(1)}_n)<\widehat{M}. \tag{4.1}$$

By using Theorem 1.1, there exists a constant $\mathcal{C}>0$ independent of λ_n such that

$$0<\|u^{(i)}_n\|_{\lambda_n}\le\mathcal{C}\quad\text{for all } n. \tag{4.2}$$

Hence, passing to a subsequence, we assume that

$$\begin{gathered}u^{(i)}_n\rightharpoonup u^{(i)}_b\quad\text{weakly in } X,\\ u^{(i)}_n\to u^{(i)}_b\quad\text{strongly in } L^{\kappa}_{\rm loc}(\mathbb{R}^N),\ \text{for all }\kappa\in[p,p^*_s),\\ u^{(i)}_n\to u^{(i)}_b\quad\text{a.e. in }\mathbb{R}^N.\end{gathered} \tag{4.3}$$

By using Fatou's lemma and (4.2), we reach

$$\int_{\mathbb{R}^N}V(x)|u^{(i)}_b|^pdx\le\liminf_{n\to\infty}\int_{\mathbb{R}^N}V(x)|u^{(i)}_n|^pdx\le\liminf_{n\to\infty}\frac{\|u^{(i)}_n\|^p_{\lambda_n}}{\lambda_n}=0,$$

which yields $u^{(i)}_b=0$ a.e. in $\mathbb{R}^N\setminus\overline{V^{-1}(0)}$, and $u^{(i)}_b\in E_0$ by (A3).

Below we show that $u^{(i)}_n\to u^{(i)}_b$ strongly in $L^{\kappa}(\mathbb{R}^N)$ for any $p\le\kappa<p^*_s$. Otherwise, by Lions' vanishing lemma (see [29]) there exist $\delta>0,R_0>0$ and $x_n\in\mathbb{R}^N$ such that

$$\int_{B_{R_0}(x_n)}|u^{(i)}_n-u^{(i)}_b|^pdx\ge\delta.$$

Let $x_n\to\infty$, then $|B_{R_0}(x_n)\cap\{V<V_0\}|\to0$. By (4.3) and Hölder inequality, one has

$$\lim_{n\to\infty}\int_{B_{R_0}(x_n)\cap\{V<V_0\}}|u^{(i)}_n-u^{(i)}_b|^pdx=0.$$

Consequently, we can deduce that

$$\begin{aligned}\|u^{(i)}_n\|^p_{\lambda_n}&\ge\lambda_nV_0\int_{B_{R_0}(x_n)\cap\{V\ge V_0\}}|u^{(i)}_n|^pdx\\&=\lambda_nV_0\Big(\int_{B_{R_0}(x_n)\cap\{V\ge V_0\}}|u^{(i)}_n-u^{(i)}_b|^pdx+\int_{B_{R_0}(x_n)\cap\{V\ge V_0\}}|u^{(i)}_b|^pdx\Big)+o(1)\\&\ge\lambda_nV_0\Big[\int_{B_{R_0}(x_n)}|u^{(i)}_n-u^{(i)}_b|^pdx-\int_{B_{R_0}(x_n)\cap\{V<V_0\}}|u^{(i)}_n-u^{(i)}_b|^pdx\Big]+o(1)\\&\to\infty\quad\text{as } n\to\infty,\end{aligned}$$

which contradicts (4.2). Furthermore, applying (3.20), (3.22) and $u^{(i)}_n\to u^{(i)}_b$ in $L^{\kappa}(\mathbb{R}^N)$ for any $[p,p^*_s)$, we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u^{(i)}_n)u^{(i)}_n\,dx=\int_{\mathbb{R}^N}f(x,u^{(i)}_b)u^{(i)}_bdx, \tag{4.4}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}g(x,u^{(i)}_n)u^{(i)}_n\,dx=\int_{\mathbb{R}^N}g(x,u^{(i)}_b)u^{(i)}_bdx. \tag{4.5}$$

We next prove that $u^{(i)}_n\to u^{(i)}_b$ in $W^{s,p}(\mathbb{R}^N)$. Since $\langle\mathcal{J}'_{b,\lambda_n}(u^{(i)}_n),u^{(i)}_n\rangle=\langle\mathcal{J}'_{b,\lambda_n}(u^{(i)}_n),u^{(i)}_b\rangle=0$, then we have

$$\|u^{(i)}_n\|^p_{\lambda_n}+b[u^{(i)}_n]^{2p}_{s,p}=\int_{\mathbb{R}^N}f(x,u^{(i)}_n)u^{(i)}_n\,dx+\int_{\mathbb{R}^N}g(x,u^{(i)}_n)u^{(i)}_n\,dx \tag{4.6}$$

and

$$a\int\int_{\mathbb{R}^{2N}}\frac{|u^{(i)}_n(x)-u^{(i)}_n(y)|^{p-2}[u^{(i)}_n(x)-u^{(i)}_n(y)][u^{(i)}_b(x)-u^{(i)}_b(y)]}{|x-y|^{N+sp}}\,dx\,dy$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\lambda_n V(x)|u_n^{(i)}|^{p-2}u_n^{(i)}u_b^{(i)}dx\\
&=-b[u_n^{(i)}]_{s,p}^p\int\int_{\mathbb{R}^{2N}}\frac{|u_n^{(i)}(x)-u_n^{(i)}(y)|^{p-2}[u_n^{(i)}(x)-u_n^{(i)}(y)][u_b^{(i)}(x)-u_b^{(i)}(y)]}{|x-y|^{N+sp}}\,dx\,dy\\
&\quad+\int_{\mathbb{R}^N}f(x,u_n^{(i)})u_b^{(i)}dx+\int_{\mathbb{R}^N}g(x,u_n^{(i)})u_b^{(i)}dx.
\end{aligned}
$$

Combining this with (4.4)–(4.6) and $u_b^{(i)}=0$ a.e. in $\mathbb{R}^N\setminus\overline{V^{-1}(0)}$, it is easy to check that

$$
\|u_n^{(i)}\|_{\lambda_n}^p+b[u_n^{(i)}]_{s,p}^{2p}=\int_{\mathbb{R}^N}f(x,u_b^{(i)})u_b^{(i)}dx+\int_{\mathbb{R}^N}g(x,u_b^{(i)})u_b^{(i)}dx+o(1) \tag{4.7}
$$

and

$$
\|u_b^{(i)}\|^p+b[u_n^{(i)}]_{s,p}^p[u_b^{(i)}]_{s,p}^p=\int_{\mathbb{R}^N}f(x,u_b^{(i)})u_b^{(i)}dx+\int_{\mathbb{R}^N}g(x,u_b^{(i)})u_b^{(i)}dx+o(1). \tag{4.8}
$$

Passing to subsequence if necessary, there exist constants $L_1,L_2>0$ such that

$$
\|u_n^{(i)}\|_{\lambda_n}^p\to L_1,\ [u_n^{(i)}]_{s,p}^p\to L_2, \tag{4.9}
$$

$$
[u_b^{(i)}]_{s,p}^p\le\liminf_{n\to\infty}[u_n^{(i)}]_{s,p}^p=L_2. \tag{4.10}
$$

Thus, by (4.7)–(4.10), we infer that

$$
L_1+bL_2^2=\|u_b^{(i)}\|^p+bL_2[u_b^{(i)}]_{s,p}^p\le\|u_b^{(i)}\|^p+bL_2^2.
$$

Then $L_1\le\|u_b^{(i)}\|^p$. In addition, weakly lower semi-continuity of norm implies that

$$
\|u_b^{(i)}\|^p\le\liminf_{n\to\infty}\|u_n^{(i)}\|^p\le\limsup_{n\to\infty}\|u_n^{(i)}\|^p\le\lim_{n\to\infty}\|u_n^{(i)}\|_{\lambda_n}^p=L_1. \tag{4.11}
$$

Therefore, $u_n^{(i)}\to u_b^{(i)}$ in X, and so $u_n^{(i)}\to u_b^{(i)}$ in $W^{s,p}(\mathbb{R}^N)$. Since $\langle\mathcal{J}'_{b,\lambda_n}(u_n^{(i)}),\phi\rangle=0$, it is easy to check that

$$
\begin{aligned}
&\Big(a+b\int_{\mathcal{Q}}\frac{|u_b^{(i)}(x)-u_b^{(i)}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)\int_{\mathcal{Q}}\frac{|u_b^{(i)}(x)-u_b^{(i)}(y)|^{p-2}[u_b^{(i)}(x)-u_b^{(i)}(y)][\phi(x)-\phi(y)]}{|x-y|^{N+sp}}\,dx\,dy\\
&=\int_\Omega f(x,u_b^{(i)})\phi dx+\int_\Omega g(x,u_b^{(i)})\phi dx
\end{aligned}
$$

for any $\phi\in C_0^\infty(\Omega)$, that is, $u_b^{(i)}$ are two solutions of problem (1.4) by the density of $C_0^\infty(\Omega)$ in X_0. Moreover, from (4.1) and α,η are independent of λ, we deduce that

$$
\begin{aligned}
&\frac{a}{p}\int_{\mathcal{Q}}\frac{|u_b^{(1)}(x)-u_b^{(1)}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy+\frac{b}{2p}\Big(\int_{\mathcal{Q}}\frac{|u_b^{(1)}(x)-u_b^{(1)}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2\\
&-\int_\Omega F(x,u_b^{(1)})dx-\int_\Omega G(x,u_b^{(1)})dx\ge\eta>0
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{a}{p}\int_{\mathcal{Q}}\frac{|u_b^{(2)}(x)-u_b^{(2)}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy+\frac{b}{2p}\Big(\int_{\mathcal{Q}}\frac{|u_b^{(2)}(x)-u_b^{(2)}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^2\\
&-\int_\Omega F(x,u_b^{(2)})dx-\int_\Omega G(x,u_b^{(2)})dx\le\alpha<0,
\end{aligned}
$$

which implies that $u_b^{(i)}\ne0$ and $u_b^{(1)}\ne u_b^{(2)}$. This completes the proof. □

Proof of Theorem 1.3. Let $a=1$ and for each fixed $\lambda>\Lambda$. Then for all $b_n\in(0,\hat b)$ small enough, let $u_n^{(i)}=u_{b_n,\lambda}^{(i)}(i=1,2)$ be the critical points of $\mathcal{J}_{b_n,\lambda}$ given by Theorem 1.1, we have

$$
0<\|u_n^{(i)}\|_\lambda\le\mathcal{C}\quad\text{for all } n.
$$

Going if necessary to a subsequence, we may assume that $u_n^{(i)} \rightharpoonup u_\lambda^{(i)}$ in X_λ. Since $\mathcal{J}'_{b_n,\lambda}(u_n^{(i)}) = 0$, we can obtain that $u_n^{(i)} \to u_\lambda^{(i)}$ in X_λ similar to prove Lemma 3.7. Moreover, it follows from (3.12)–(3.15) and $\langle \mathcal{J}'_{b_n,\lambda}(u_n^{(i)}), v\rangle = 0$ that

$$\begin{aligned}\lim_{n\to\infty} \langle \mathcal{J}'_{b_n,\lambda}(u_n^{(i)}), v\rangle &= a\int\int_{\mathbb{R}^{2N}} \frac{|u_\lambda^{(i)}(x)-u_\lambda^{(i)}(y)|^{p-2}[u_\lambda^{(i)}(x)-u_\lambda^{(i)}(y)][v(x)-v(y)]}{|x-y|^{N+sp}}\,dx\,dy\\ &\quad + \int_{\mathbb{R}^N} \lambda V(x)|u_\lambda^{(i)}|^{p-2}u_\lambda^{(i)} v\,dx - \int_{\mathbb{R}^N} f(x,u_\lambda^{(i)})v\,dx - \int_{\mathbb{R}^N} g(x,u_\lambda^{(i)})v\,dx = 0,\end{aligned}$$

which, together with the proof idea of Theorem 1.2, implies that $u_\lambda^{(1)} \neq u_\lambda^{(2)}$ are two nontrivial solutions of problem (1.5). This completes the proof. □

The proof Theorem 1.4 is similar to that of Theorem 1.2, so we omit it here.

5. Nonexistence of nontrivial solutions

In this section, we argue by contradiction to prove the nonexistence result of nontrivial solutions for problem (1.1) as a is large enough.

Proof of Theorem 1.5. We define $V_n = \{x \in \mathbb{R}^N : V(x) < \frac{1}{n+k}\}$ with $\frac{1}{k} < V_0$. Then $V_{n+1} \subset V_n$ for any $n \in \mathbb{N}^+$. Observing that condition (A2) yields $|V_1| = |\{x \in \mathbb{R}^N : V(x) < \frac{1}{1+k} < V_0\}| < \infty$. Then it is easy to see that $\lim_{n\to\infty} |V_n| = |\lim_{n\to\infty} V_n|$. Using (A3), we have $\lim_{n\to\infty} V_n = \overline{\Omega}$. Therefore, $\lim_{n\to\infty} |V_n| = |\overline{\Omega}|$. Thus, by virtue of $a > a_1$, we can choose $\varepsilon = (\frac{a}{\mathfrak{C}})^{\frac{p_s^*}{p_s^*-p}} - |\overline{\Omega}|$, there exists $n_0 \in \mathbb{N}^+$ such that

$$|V_n| - |\overline{\Omega}| < \varepsilon \quad \text{for all } n > n_0,$$

which means that $|V_{n_0+1}| < (\frac{a}{\mathfrak{C}})^{\frac{p_s^*}{p_s^*-p}}$, so one can infer that

$$a > \mathfrak{C}|\{x \in \mathbb{R}^N : V(x) < K_0\}|^{\frac{p_s^*}{p_s^*-p}} \quad \text{with } K_0 := \frac{1}{n_0+1+k}. \tag{5.1}$$

As shown in [23], for any $\lambda > 0$ and $\kappa \in [p, p_s^*]$, it follows from (5.1) that

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^\kappa dx &\le \Big(\max\Big\{\frac{a}{\lambda\mathfrak{C}K_0}, \frac{a}{\mathfrak{C}}\Big\}\Big)^{\frac{p_s^*-\kappa}{p_s^*-p}} \Big(\frac{\mathfrak{C}}{a}\Big)^{\kappa/p} \|u\|_\lambda^\kappa\\ &:= \Big(\frac{\widetilde{\Lambda} a}{\mathfrak{C}}\Big)^{\frac{p_s^*-\kappa}{p_s^*-p}} \Big(\frac{\mathfrak{C}}{a}\Big)^{\kappa/p} \|u\|_\lambda^\kappa,\end{aligned} \tag{5.2}$$

where $\widetilde{\Lambda} := \max\{\frac{1}{\lambda K_0}, 1\}$. Let u be a nontrivial solution of problem (1.1) and $\varepsilon = \frac{1}{2\widetilde{\Lambda}}$. Combining (A4)–(A8), (1.6), (5.2) and Hölder inequality, we reach

$$\begin{aligned}0 &= \langle \mathcal{J}'_{b,\lambda}(u), u\rangle\\ &= \|u\|_\lambda^p + b[u]_{s,p}^{2p} - \int_{\mathbb{R}^N} f(x,u)u\,dx - \int_{\mathbb{R}^N} g(x,u)u\,dx\\ &\ge \|u\|_\lambda^p - \frac{1}{2\widetilde{\Lambda}}\int_{\mathbb{R}^N} |u|^p dx - C_{\frac{1}{2\widetilde{\Lambda}}}\int_{\mathbb{R}^N} |u|^q dx - \int_{\mathbb{R}^N} h(x)|u|^r dx\\ &\ge \frac{1}{2}\|u\|_\lambda^p - C_{\frac{1}{2\widetilde{\Lambda}}}\int_{\mathbb{R}^N} |u|^q dx - |h|_{\frac{p}{p-r}}\Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{r/p}\\ &\ge \frac{1}{2}\|u\|_\lambda^p - C_{\frac{1}{2\widetilde{\Lambda}}}\Big(\frac{\widetilde{\Lambda} a}{\mathfrak{C}}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\Big(\frac{\mathfrak{C}}{a}\Big)^{q/p}\|u\|_\lambda^q - \widetilde{\Lambda}^{r/p}|h|_{\frac{p}{p-r}}\|u\|_\lambda^r.\end{aligned} \tag{5.3}$$

On the other hand, from (3.11) and (5.3), we obtain positive constants $C_i (i = 1, 2, ..., 5)$ such that

$$\begin{aligned}&\frac{1}{2}\|u\|_\lambda^p - C_{\frac{1}{2\widetilde{\Lambda}}}\Big(\frac{\widetilde{\Lambda} a}{\mathfrak{C}}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\Big(\frac{\mathfrak{C}}{a}\Big)^{q/p}\|u\|_\lambda^q - \widetilde{\Lambda}^{r/p}|h|_{\frac{p}{p-r}}\|u\|_\lambda^r\\ &= \frac{1}{2}\Big(a[u]_{s,p}^p + \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big) - C_{\frac{1}{2\widetilde{\Lambda}}}\Big(\frac{\widetilde{\Lambda} a}{\mathfrak{C}}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\Big(\frac{\mathfrak{C}}{a}\Big)^{q/p}\Big(a[u]_{s,p}^p + \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big)^{q/p}\end{aligned}$$

$$
\begin{aligned}
&\quad - \widetilde{\Lambda}^{r/p}|h|_{\frac{p}{p-r}}\Big(a[u]_{s,p}^p + \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big)^{r/p} \\
&\geq \frac{1}{2}\Big(a[u]_{s,p}^p + \int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big) - \widetilde{\Lambda}^{r/p}|h|_{\frac{p}{p-r}} C_{\frac{r}{p}}\Big[(a[u]_{s,p}^p)^{r/p} + \Big(\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big)^{r/p}\Big] \\
&\quad - C_{\frac{1}{2\widetilde{\Lambda}}}\Big(\frac{\widetilde{\Lambda}a}{\mathfrak{C}}\Big)^{\frac{p_s^*-q}{p_s^*-p}}\Big(\frac{\mathfrak{C}}{a}\Big)^{q/p} C_{\frac{q}{p}}\Big[(a[u]_{s,p}^p)^{q/p} + \Big(\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big)^{q/p}\Big] \\
&= \frac{a}{2}[u]_{s,p}^p + \frac{1}{2}\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx - C_{\frac{1}{2\widetilde{\Lambda}}}\Big(\frac{\widetilde{\Lambda}a}{\mathfrak{C}}\Big)^{\frac{p_s^*-q}{p_s^*-p}} \mathfrak{C}^{q/p} C_{\frac{q}{p}}[u]_{s,p}^q \\
&\quad - C_{\frac{1}{2\widetilde{\Lambda}}}\Big(\frac{\widetilde{\Lambda}a}{\mathfrak{C}}\Big)^{\frac{p_s^*-q}{p_s^*-p}} C_{\frac{q}{p}}\Big(\frac{\mathfrak{C}}{a}\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big)^{q/p} - \widetilde{\Lambda}^{r/p}|h|_{\frac{p}{p-r}} C_{\frac{r}{p}} a^{r/p}[u]_{s,p}^r \\
&\quad - \widetilde{\Lambda}^{r/p}|h|_{\frac{p}{p-r}} C_{\frac{r}{p}}\Big(\int_{\mathbb{R}^N} \lambda V(x)|u|^p dx\Big)^{r/p} \\
&:= \frac{a}{2} + C_1 - C_2 a^{\frac{p_s^*-q}{p_s^*-p}} - C_3 a^{\frac{p_s^*-q}{p_s^*-p} - \frac{q}{p}} - C_4 a^{r/p} - C_5 > 0,
\end{aligned}
$$

provided that $a > a_2$ by choosing $a_2 > 0$ sufficiently large. This contradicts (5.3) and completes the proof. □

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Shuwen He (corresponding author)
School of Mathematics, Sichuan University, Chengdu 610064, China
Email address: shuwenxueyi@163.com

Shiqing Zhang
School of Mathematics, Sichuan University, Chengdu 610064, China
Email address: zhangshiqing@scu.edu.cn