Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 16, pp. 1–23.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.16

EXISTENCE AND STABILITY OF SOLUTIONS FOR A VISCOELASTIC COUPLED SYSTEM OF TWO WAVE EQUATIONS

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Abstract. In this work, we consider the coupled viscoelastic wave system

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u\,ds + \alpha v = 0, \quad \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + \alpha u = 0, \quad \text{in } \Omega \times (0,T), \\ u = v = 0, \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad u_t(0) = u_1, \quad v(0) = v_0, \quad v_t(0) = v_1, \quad \text{in } \Omega, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\alpha > 0$, and the initial data belong to suitable spaces. The relaxation function g satisfies $g'(t) \le -\xi(t)G(g(t))$, where G is an increasing and convex function near the origin and ξ is a non-increasing function. We first prove the well-posedness of the problem, and then we establish a general decay rate of the system energy, highlighting the influence of the relaxation function on the stability of the solutions. Numerical tests were also conducted to validate our theoretical findings.

1. Introduction

Wave equations with memory effects have been a central topic in the analysis of partial differential equations, owing to their applications in physics, engineering, and material science. The focus has been on the existence, stability, and energy decay of solutions, with significant progress made for both single and coupled wave equations. To motivate our work, data collected and results are reviewed to provide a thorough study of both the single and coupled viscoelastic wave equation. We start with the pioneer works of Dafermos [10, 11], where the author showed that solutions decay to zero for smooth, monotone relaxation functions g, but the decay rate remained unspecified. Cavalcanti et al. [7] studied

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds - \gamma\Delta u_t = 0, \quad \text{in } \Omega \times \mathbb{R}_+^*$$

proving global existence for $\gamma \ge 0$ and exponential energy decay for $\gamma > 0$. Messaoudi and Tatar [25, 26] extended these results to include source terms, showing exponential and polynomial decay depending on the rate of decay of g(t). Cavalcanti et al. [8] considered a viscoelastic wave equation with a local frictional damping, where the kernel g satisfies, for two positive constants ξ_1 and ξ_2,

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t), \quad \forall t \in \mathbb{R}_+.$$

Under the above assumptions and subject to certain restrictions on the control zone, they established an exponential decay result. Berrimi and Messaoudi [5] later improved this result by proving that the viscoelastic dissipation alone is sufficient to stabilize the system, and subsequently extended it to the case where a source term competes with the dissipation [6]. Further contributions including the analysis of nonlinear damping with polynomially decaying kernels [9] and global existence with uniform decay for source-type problems [17] appeared later. Without restricting $g(t)$

2020 *Mathematics Subject Classification.* 35L51, 35B35, 35B40, 49K40, 93D15.
Key words and phrases. Viscoelasticity; energy method; wave equation; general stability.

Submitted September 24, 2025. Published February 19, 2026.

to exponential or polynomial decay, Messaoudi [20, 21] generalized all the existing decay results by considering the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\, d\tau = b|u|^\gamma u, \quad b \geq 0,$$

where $g'(t) \leq -\xi(t)g(t)$, for a non-increasing differentiable $\xi(t)$. These latter results inspired other authors and more results have been established, see [33, 14, 32, 34, 23].

Further improvement was establish by Messaoudi and Al-Khulaifi [24], who addressed the problem in [7], for $\gamma = 0$ and for a kernel satisfying, for $1 \leq p \leq \frac{3}{2}$, the condition

$$g'(t) \leq -\xi(t)g^p(t), \quad \forall t \in \mathbb{R}_+.$$

They proved that the energy of solutions converges to zero at the same rate as $g(t)$, provided $g(t)$ decays faster than s^{-2} at infinity. Later, Mustafa [27] extended his earlier joint work with Messaoudi [28], which dealt with kernels satisfying $g'(t) \leq -G(g(t))$, to the more general condition

$$g'(t) \leq -\xi(t)G(g(t)), \quad \forall t \in \mathbb{R}_+,$$

where G is strictly increasing and convex function. This extension provided explicit energy decay estimates and encompassed a wider range of decay behaviors.

Coupled wave systems also have consistently drawn attention due to their fundamental role in understanding stability and long-term dynamics. In a recent contribution, Guesmia, Messaoudi, and Zahri [13] analyzed an abstract coupled system involving a viscoelastic component interacting with a purely elastic one. Precisely, they looked into the problem

$$\begin{gathered}
u_{tt}(t) + Au(t) - \int_0^t g(s)A^\theta u(t-s)\, ds + \alpha u + \beta Bv(t) = f(u), \quad t > 0, \\
v_{tt}(t) + Av(t) + \beta Bu(t) = 0, \quad t > 0, \\
u(0) = u_0, \quad u_t(0) = u_1, \quad v(0) = v_0, \quad v_t(0) = v_1,
\end{gathered}$$

where A is a self-adjoint positive operator on a Hilbert space H, $\theta \in [0,1]$, $\alpha > 0$, and g is a positive nonincreasing relaxation function. Their analysis established general decay rates for the system energy, showing that the asymptotic behavior depends strongly on the relaxation function and the coupling parameter. Numerical experiments were also provided, highlighting the impact of memory effects on the qualitative behavior of solutions. Earlier, Feng and Li [12] considered a viscoelastic Lamé system with strong damping terms. For two distinct relaxation kernels g_1 and g_2, they assumed conditions of the form $g_i'(t) \leq -\xi_i(t)H_i(g_i(t))$ $(i = 1, 2)$, which led to explicit and general decay results. Their work improved earlier results of Beniani, Taouaf, and Benaissa [4], who had previously established a general decay property for similar coupled viscoelastic structures. Further investigations were carried out by Jin et al. [15], who studied a coupled system of two abstract evolution equations with finite memory, given by

$$\begin{gathered}
u_{tt}(t) + Au(t) - \int_0^t g(s)Au(t-s)\, ds + \alpha u + \beta Bv(t) = f(u), \quad t > 0, \\
v_{tt}(t) + Av(t) + \beta Bu(t) = 0, \quad t > 0,
\end{gathered}$$

where A and B are operators, $\alpha \geq 0$, $\beta > 0$, and f is a nonlinear source. Under appropriate assumptions on g and structural conditions on A, B, and f, they proved polynomial stability with decay rates of order t^{-1}. Several extensions and related contributions in this direction may be found in [31, 3]. In addition to viscoelastic systems, purely elastic coupled wave models have also been studied. For instance, a weakly coupled system

$$\begin{gathered}
u_{tt} - \Delta u + u_t + \alpha v = 0, \quad \text{in } \Omega \times (0, \infty), \\
v_{tt} - \Delta v + \alpha u = 0, \quad \text{in } \Omega \times (0, \infty),
\end{gathered}$$

was analyzed in [19], where $\Omega \subset \mathbb{R}^n$ is a bounded domain and α a positive constant. The authors showed that the energy decays polynomially, with the rate being optimal, and provided numerical validation in the one-dimensional case. Similar results on stability and decay rates had earlier been obtained for systems with boundary damping, notably by Komornik et al. [16] and Aassila [1, 2], where exponential stability was achieved in the case of linear damping and polynomial stability in the case of nonlinear damping mechanisms.

Motivated by the findings in [13], the present work investigates the coupled viscoelastic wave system

$$\begin{gathered} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u\,ds + \alpha v = 0, \quad \text{in } \Omega\times(0,T), \\ v_{tt} - \Delta v + \alpha u = 0, \quad \text{in } \Omega\times(0,T), \\ u = v = 0, \quad \text{on } \partial\Omega\times(0,T), \\ u(0)=u_0,\ u_t(0)=u_1,\ v(0)=v_0,\ v_t(0)=v_1, \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\alpha > 0$, and the initial data belong to suitable spaces. The relaxation function g satisfies $g'(t) \le -\xi(t)G(g(t))$, where G is an increasing and convex function near the origin and ξ is a nonincreasing function. We establish, in details, an existence and uniqueness result, then we prove a general and explicit result for energy decay and provide some numerical illustration to validate our theoretical findings.

The structure of this paper is as follows: the next section presents some assumptions, remarks and functionals needed in our results. In Section 3, we establish the existence of solutions by combining energy estimates with the Faedo-Galerkin method. Section 4 presents several technical lemmas and contains the statement and proof of the decay results. In section 5, two numerical tests are conducted aiming to verify our decay result.

2. Preliminaries

This section presents the foundation for our main result. We set

$$\begin{gathered} (h\star\varphi)(t) = \int_0^t h(t-s)\varphi(s)ds,\ (h\diamond\varphi)(t) = \int_0^t h(t-s)\|\varphi(t)-\varphi(s)\|ds, \\ (h\circ\varphi)(t) = \int_0^t h(t-s)\int_\Omega \|\varphi(t)-\varphi(s)\|^2 dx\,ds. \end{gathered}$$

For the relaxation function g, we assume:

(A1) $g:\mathbb{R}^+\to\mathbb{R}^+$ is a C^1 function satisfying

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)\,ds = l > 0.$$

(A2) There exists a C^1 function $G:\mathbb{R}^+\to\mathbb{R}^+$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0,r]$, $r \le g(0)$, with $G(0) = G'(0) = 0$, such that

$$g'(t) \le -\xi(t)G(g(t)), \quad \forall t \ge 0, \tag{2.1}$$

where ξ is a positive non-increasing differentiable function.

(A3) The constant α is such that

$$0 < \alpha < \frac{\sqrt{l}}{C_p}, \tag{2.2}$$

where C_p is the Poincaré constant.

Lemma 2.1 ([30]). *For any function $h \in C^1(\mathbb{R})$ and any $k \in H^1(0,T)$, we have*

$$(h\star k)(t)k_t = -\frac{1}{2}h(t)\|k\|^2 + \frac{1}{2}(h'\circ k)(t) - \frac{1}{2}\frac{d}{dt}\Big\{(h\circ k)(t) - \Big(\int_0^t h(\tau)d\tau\Big)\|k\|^2\Big\}.$$

Lemma 2.2 ([20]). *For $u \in H_0^1(\Omega)$, we have*

$$\int_\Omega \Big(\int_0^t g(t-s)(u(t)-u(s))\,ds\Big)^2 dx \le (1-l)C_p^2(g\circ\nabla u)(t), \tag{2.3}$$

where l is given in (A1).

3. Existence of weak solutions

Now, we examine the well-posedness of problem (1.1). To establish this, we will combine some energy estimates and the Faedo-Galerkin method.

Theorem 3.1. *Assume that* (A1)–(A3) *hold. Then, for each* $(u_0, v_0) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^2$ *and* $(u_1, v_1) \in \left(H_0^1(\Omega)\right)^2$, *problem* (1.1) *has a unique strong solution* (u, v) *satisfying, for every* $T > 0$,

$$
\begin{gathered}
u, v \in L^\infty\left([0,T), H^2(\Omega) \cap H_0^1(\Omega)\right), \\
u_t, v_t \in L^2\left((0,T), H_0^1(\Omega)\right) \cap H_0^1\left((0,T) \times \Omega\right), \\
u_{tt}, v_{tt} \in L^\infty\left((0,T), L^2(\Omega)\right).
\end{gathered}
$$

Proof.
Step 1. Faedo-Galerkin approximation. We take $\{\phi_j\}_{j=1}^\infty$ to be the eigenfunctions of the Laplacian operator subject to Dirichlet boundary conditions, satisfying $-\Delta\phi_i = \lambda_i\phi_i$. Then $\{\phi_j\}_{j=1}^\infty$ forms an orthogonal basis of $H_0^1(\Omega)$, $H^2(\Omega) \cap H_0^1(\Omega)$, and $L^2(\Omega)$. Let V_N be spanned by the N first functions $\phi_1, \phi_2, \dots, \phi_N$. We seek an approximate solution (u, v) of the form

$$
u^N(x,t) = \sum_{j=1}^N a_j(t)\phi_j(x), \quad v^N(x,t) = \sum_{j=1}^N b_j(t)\phi_j(x), \quad t \in (0,T),
$$

which satisfies

$$
\begin{gathered}
\int_\Omega u_{tt}^N \phi_j dx + \int_\Omega \nabla u^N.\nabla\phi_j dx - \int_\Omega \int_0^t g(t-s)\nabla u^N(s).\nabla\phi_j \,ds\,dx + \alpha \int_\Omega v^N \phi_j \,dx = 0, \\
\int_\Omega v_{tt}^N \phi_j dx + \int_\Omega \nabla v^N.\nabla\phi_j dx + \alpha \int_\Omega u^N \phi_j \,dx = 0, \quad 1 \le j \le N
\end{gathered}
$$

and

$$
\begin{gathered}
u^N(0) = u_0^N = \sum_{i=1}^N \langle u_0, \phi_i\rangle\phi_i \to u_0, \\
v^N(0) = v_0^N = \sum_{i=1}^N \langle v_0, \phi_i\rangle\phi_i \to v_0 \quad \text{in } H^2(\Omega) \cap H_0^1(\Omega), \\
u_t^N(0) = u_1^N = \sum_{i=1}^N \langle u_1, \phi_i\rangle\phi_i \to u_1, \\
v_t^N(0) = v_1^N = \sum_{i=1}^N \langle v_1, \phi_i\rangle\phi_i \to v_1 \quad \text{in } H_0^1(\Omega).
\end{gathered} \tag{3.1}
$$

Standard results of ordinary differential equations guarantee the existence of a unique local solution (u^N, v^N) of (3)-(3.1) defined on $[0, t_N), 0 < t_N \le T$. Our next step is to obtain a priori estimates that enable us to extend the solution to the entire interval.

Step 2. A priori estimate 1. By multiplying the first equation in (3) by $a_j'(t)$ and summing the results over j from 1 to N, we obtain, for all $0 < t \le t_N$,

$$
\frac{d}{dt}\Big(\frac{1}{2}\|u_t^N(t)\|^2 + \frac{1}{2}\|\nabla u^N(t)\|^2\Big) + \alpha \int_\Omega v^N(t)\,u_t^N(t)\,dx \quad - \int_0^t g(t-s)\int_\Omega \nabla u^N(s)\nabla u_t^N(t)\,dxds = 0. \tag{3.2}
$$

Using Lemma 2.1 for the last term of the left-hand side of (3.2) we find that

$$
\begin{aligned}
&\frac{d}{2dt}\Big(\|u_t^N(t)\|^2 + \Big[1 - \int_0^t g(s)ds\Big]\|\nabla u^N(t)\|^2 + (g \circ \nabla u^N)(t)\Big) + \alpha \int_\Omega v^N(t)\,u_t^N(t)dx \\
&+ \frac{1}{2}g(t) \parallel \nabla u^N(t) \parallel^2 - \frac{1}{2}(g' \circ \nabla u^N)(t) = 0.
\end{aligned} \tag{3.3}
$$

Multiplying the second equation in (3) by $b_j'(t)$ and summing with respect to j, we obtain

$$\frac{d}{2dt}\Big(\|v_t^N(t)\|^2+\|\nabla v^N(t)\|^2\Big)+\alpha\int_\Omega u^N(t)\,v_t^N(t)\,dx=0,\quad 0<t\le t_N. \tag{3.4}$$

Then, combining (3.3) and (3.4) yields

$$\begin{aligned}&\frac{d}{2dt}\Big(\|u_t^N(t)\|^2+\Big[1-\int_0^t g(s)ds\Big]\|\nabla u^N(t)\|^2+(g\circ\nabla u^N)(t)+\|v_t^N(t)\|^2+\|\nabla v^N(t)\|^2\\&+2\alpha\int_\Omega v^N(t)u^N(t)dx\Big)\\&=-\frac{1}{2}g(t)\parallel\nabla u^N(t)\parallel^2+\frac{1}{2}(g'\circ\nabla u^N)(t).\end{aligned} \tag{3.5}$$

By assumptions (A1)–(A3) we obtain

$$\begin{aligned}&\frac{d}{2dt}\Big(\|u_t^N(t)\|^2+\Big[1-\int_0^t g(s)ds\Big]\|\nabla u^N(t)\|^2+(g\circ\nabla u^N)(t)+\|v_t^N(t)\|^2+\|\nabla v^N(t)\|^2\\&+2\alpha\int_\Omega v^N(t)u^N(t)dx\Big)\le 0.\end{aligned} \tag{3.6}$$

Integrating (3.6) over $(0,t)$ yields

$$E_N(t)\le E_N(0), \tag{3.7}$$

where

$$\begin{aligned}E_N(t)&=\|u_t^N(t)\|^2+\Big[1-\int_0^t g(s)ds\Big]\|\nabla u^N(t)\|^2+(g\circ\nabla u^N)(t)+\|v_t^N(t)\|^2+\|\nabla v^N(t)\|^2\\&\quad+2\alpha\int_\Omega v^N(t)u^N(t)dx,\\E_N(0)&=\|u_1^N\|^2+\|v_1^N\|^2+\|\nabla u_0^N\|^2+\|\nabla v_0^N\|^2+2\alpha\int_\Omega v_0^N u_0^N\,dx.\end{aligned}$$

Using Young's and Poincaré's inequalities, for $\varepsilon_1>0$, we have

$$2\int_\Omega v^N(t)u^N(t)dx\ge-\varepsilon_1 C_p\|\nabla u^N(t)\|^2-\frac{C_p}{\varepsilon_1}\|\nabla v^N(t)\|^2 \tag{3.8}$$

and

$$2\alpha\int_\Omega v_0^N u_0^N\,dx\le\alpha C_p\|\nabla u_0^N\|^2+\alpha C_p\|\nabla v_0^N\|^2. \tag{3.9}$$

Thus

$$E_N(0)\le\|u_1^N\|^2+\|v_1^N\|^2+(1+\alpha C_p)\|\nabla u_0^N\|^2+(1+\alpha C_p)\|\nabla v_0^N\|^2. \tag{3.10}$$

Since the sequences $(u_0^N)_{N\in\mathbb{N}}$, $(u_1^N)_{N\in\mathbb{N}}$, $(v_0^N)_{N\in\mathbb{N}}$, $(v_1^N)_{N\in\mathbb{N}}$ converge, there exists a positive constant L_1, independent of N, such that

$$E_N(0)\le L_1. \tag{3.11}$$

On the other hand,

$$\begin{aligned}E_N(t)&\ge\|u_t^N(t)\|^2+\Big[1-\int_0^t g(s)ds-\alpha\varepsilon_1 C_p\Big]\|\nabla u^N(t)\|^2+(g\circ\nabla u^N)(t)+\|v_t^N(t)\|^2\\&\quad+\Big(1-\frac{\alpha C_p}{\varepsilon_1}\Big)\|\nabla v^N(t)\|^2.\end{aligned} \tag{3.12}$$

Let us select ε_1 such that it satisfies $\alpha C_p\le\varepsilon_1\le\frac{l}{\alpha C_p}$. Then, there exists $C>0$, such that

$$E_N(t)\ge C\left(\|u_t^N(t)\|^2+\|\nabla u^N(t)\|^2+(g\circ\nabla u^N)(t)+\|v_t^N(t)\|^2+\|\nabla v^N(t)\|^2\right). \tag{3.13}$$

From (3.7) and (3.11), we deduce

$$\|u_t^N(t)\|^2+\|v_t^N(t)\|^2+\|\nabla u^N(t)\|^2+\|\nabla v^N(t)\|^2+(g\circ\nabla u^N)(t)\le\frac{L_1}{C},\quad\forall t\le t_N\le T, \tag{3.14}$$

where $\frac{L_1}{C}$ is a constant independent of t and N.

A priori estimate 2. In (3), we substitute ϕ_j by $-\Delta\phi_j$ then multiply the equation $(3)_1$ by $a_j'(t)$ and the second equation $(3)_2$ by $b_j'(t)$, sum the resulting equations, and use (A1)–(A3), we arrive at

$$\begin{aligned}&\frac{d}{2dt}\Big[\|\nabla u_t^N(t)\|^2+\Big(1-\int_0^t g(s)\,ds\Big)\|\Delta u^N(t)\|^2+(g\circ\Delta u^N)(t)+\|\nabla v_t^N(t)\|^2+\|\Delta v^N(t)\|^2\\&+2\alpha\int_\Omega \nabla v^N(t)\nabla u^N(t)dx\Big]-\frac12\left(g'\circ\Delta u^N\right)(t)+\frac12 g(t)\|\Delta u^N(t)\|^2\le 0.\end{aligned}$$

Applying Young's and Poincaré's inequalities, we have for $\varepsilon_3>0$,

$$2\int_\Omega \nabla v^N(t)\nabla u^N(t)dx\ge -\varepsilon_3 C_p\|\Delta u^N(t)\|^2-\frac{C_p}{\varepsilon_3}\|\Delta v^N(t)\|^2,$$

and

$$2\int_\Omega \nabla v_0^N\nabla u_0^N dx\le C_p\|\Delta u_0^N\|^2+C_p\|\Delta v_0^N\|^2.$$

Then

$$\begin{aligned}&\|\nabla u_1^N\|^2+\|\nabla v_1^N\|^2+\|\Delta u_0^N\|^2+\|\Delta v_0^N\|^2+2\alpha\int_\Omega \nabla v_0^N\nabla u_0^N dx\\&\le \|\nabla u_1^N\|^2+\|\nabla v_1^N\|^2+(1+\alpha C_p)\|\Delta u_0^N\|^2+(1+\alpha C_p)\|\Delta v_0^N\|^2.\end{aligned}\tag{3.15}$$

Since the sequences $(u_0^N)_{N\in\mathbb N}$, $(u_1^N)_{N\in\mathbb N}, (v_0^N)_{N\in\mathbb N}, (v_1^N)_{N\in\mathbb N}$ converge, there exists a positive constant L_2, independent of N, such that

$$\|\nabla u_1^N\|^2+\|\nabla v_1^N\|^2+\|\Delta u_0^N\|^2+\|\Delta v_0^N\|^2+2\alpha\int_\Omega \nabla v_0^N\nabla u_0^N dx\le L_2.\tag{3.16}$$

Furthermore,

$$\begin{aligned}&\|\nabla u_t^N(t)\|^2+\Big(1-\int_0^t g(s)\,ds\Big)\|\Delta u^N(t)\|^2+(g\circ\Delta u^N)(t)+\|\nabla v_t^N(t)\|^2+\|\Delta v^N(t)\|^2\\&+2\alpha\int_\Omega \nabla v^N(t)\nabla u^N(t)dx\\&\ge \|\nabla u_t^N(t)\|^2+\Big(1-\int_0^t g(s)\,ds-\varepsilon_3 C_p\Big)\|\Delta u^N(t)\|^2\\&\quad+(g\circ\Delta u^N)(t)+\|\nabla v_t^N(t)\|^2+\Big(1-\frac{\alpha C_p}{\varepsilon_3}\Big)\|\Delta v^N(t)\|^2.\end{aligned}$$

We choose ε_3 such that it satisfies $\alpha C_p\le\varepsilon_3\le\frac{l}{\alpha C_p}$. Then there exists $C>0$, such that

$$\begin{aligned}&\|\nabla u_t^N(t)\|^2+\Big(1-\int_0^t g(s)\,ds\Big)\|\Delta u^N(t)\|^2+(g\circ\Delta u^N)(t)+\|\nabla v_t^N(t)\|^2+\|\Delta v^N(t)\|^2\\&+2\alpha\int_\Omega \nabla v^N(t)\nabla u^N(t)dx\\&\ge C\Big(\|\nabla u_t^N(t)\|^2+\|\Delta u^N(t)\|^2+(g\circ\Delta u^N)(t)+\|\nabla v_t^N(t)\|^2+\|\Delta v^N(t)\|^2\Big).\end{aligned}\tag{3.17}$$

From (3.15), (3.16) and (3.17), it follows that

$$\|\nabla u_t^N(t)\|^2+\|\Delta u^N(t)\|^2+(g\circ\Delta u^N)(t)+\|\nabla v_t^N(t)\|^2+\|\Delta v^N(t)\|^2\le\frac{L_2}{C},\ \forall\, t\le t_N\le T,\tag{3.18}$$

where $\frac{L_2}{C}$ is a constant independent of t and N.

A priori estimate 3. By multiplying the first equation in $(3)_1$ by $a_j''(t)$ and summing the results over j from 1 to k, we obtain

$$\begin{aligned}&\int_\Omega |u_{tt}^N(t)|^2dx\\&=\int_\Omega \Delta u^N(t)u_{tt}^N(t)\,dx-\int_\Omega\int_0^t g(t-s)\Delta u^N(s)u_{tt}^N(t)\,ds\,dx-\alpha\int_\Omega v^N(t)u_{tt}^N(t)\,dx\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Delta u^N(t) u^N_{tt}(t)\,dx + \int_\Omega u^N_{tt}(t) \int_0^t g(t-s)\Big(\Delta u^N(t) - \Delta u^N(s)\Big)\,ds\,dx \\
&\quad - \int_0^t g(s)ds \int_\Omega u^N_{tt}(t)\Delta u^N(t)dx - \alpha \int_\Omega v^N(t) u^N_{tt}(t)\,dx \\
&\le \varepsilon \int_\Omega |u^N_{tt}(t)|^2 dx + \frac{1}{4\varepsilon}\int_\Omega |\Delta u^N(t)|^2\,dx + \varepsilon \int_\Omega |u^N_{tt}(t)|^2 dx + \frac{1-l}{4\varepsilon}(g\circ \Delta u^N)(t) \\
&\quad + \varepsilon(1-l)\int_\Omega |u^N_{tt}(t)|^2 dx + \frac{1-l}{4\varepsilon}\int_\Omega |\Delta u^N(t)|^2 dx + \varepsilon\int_\Omega |u^N_{tt}(t)|^2 dx + \frac{\alpha^2}{4\varepsilon}\int_\Omega |v^N(t)|^2 dx \\
&\le \varepsilon(4-l)\int_\Omega |u^N_{tt}(t)|^2 dx + \frac{2-l}{4\varepsilon}\int_\Omega |\Delta u^N(t)|^2 dx + \frac{1-l}{4\varepsilon}(g\circ\Delta u^N)(t) + \frac{\alpha^2}{4\varepsilon}\int_\Omega |v^N(t)|^2 dx.
\end{aligned}
$$

Hence, for some $c>0$, we have

$$
(1-\varepsilon(4-l))\int_\Omega |u^N_{tt}(t)|^2 dx \le \frac{c}{4\varepsilon}(L_1+L_2). \tag{3.19}
$$

We take $\varepsilon>0$ small enough, so

$$
\|u^N_{tt}(t)\|^2 \le \tilde c(L_1+L_2) = L_3. \tag{3.20}
$$

Similarly, we obtain $\|v^N_{tt}(t)\|^2 \le L_4$, where L_3, L_4 are constants independent of t and N.

From (3.14), (3.18) and (3.20), we conclude that

$$
\begin{gathered}
(u^N)_N \text{ and } (v^N)_N \text{ are bounded in } L^\infty((0,T), H^2(\Omega)\cap H^1_0(\Omega)), \\
(u^N_t)_N \text{ and } (v^N_t)_N \text{ are bounded in } L^\infty((0,T), H^1_0(\Omega)), \\
(u^N_{tt})_N \text{ and } (v^N_{tt})_N \text{ are bounded in } L^\infty((0,T), L^2(\Omega)).
\end{gathered} \tag{3.21}
$$

Step 3. From (3.21), there exist subsequences of $(u^N)_N$ and $(v^N)_N$ (still denoted by $(u^N)_N$ and $(v^N)_N$), and two functions $u,v:\Omega\times[0,T)\to\mathbb{R}$, such that

$$
\begin{gathered}
u^N \overset{\star}{\rightharpoonup} u \text{ and } v^N \overset{\star}{\rightharpoonup} v \quad \text{in } L^\infty((0,T), H^2(\Omega)\cap H^1_0(\Omega)), \\
u^N_t \overset{\star}{\rightharpoonup} u_t \text{ and } v^N_t \overset{\star}{\rightharpoonup} v_t \text{ in } L^\infty((0,T), H^1_0(\Omega)), \\
u^N_{tt} \overset{\star}{\rightharpoonup} u_{tt} \text{ and } v^N_{tt} \overset{\star}{\rightharpoonup} v_{tt} \text{ in } L^\infty((0,T), L^2(\Omega)).
\end{gathered} \tag{3.22}
$$

Thanks to convergences (3.1) and (3.22), taking $N\to+\infty$ in (3), we obtain

$$
\begin{gathered}
\int_\Omega u_{tt}\phi_j dx + \int_\Omega \nabla u.\nabla\phi_j dx - \int_\Omega\int_0^t g(t-s)\nabla u(s).\nabla\phi_j\,ds\,dx + \alpha\int_\Omega v\phi_j\,dx = 0, \\
\int_\Omega v_{tt}\phi_j dx + \int_\Omega \nabla v.\nabla\phi_j dx + \alpha\int_\Omega u\phi_j\,dx = 0, \quad 1\le j\le N.
\end{gathered} \tag{3.23}
$$

Consequently, for every $\phi\in H^1_0(\Omega)\cap H^2(\Omega)$,

$$
\begin{gathered}
\int_\Omega \Big[u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\,ds\,dx + \alpha v\Big]\phi\,dx = 0, \\
\int_\Omega [v_{tt} - \Delta v + \alpha u]\phi\,dx = 0.
\end{gathered}
$$

Therefore, (u,v) solves the equations in (1.1).

Step 4. Initial conditions. First, by Lions' lemma [18] and since

$$
\begin{gathered}
u^N \overset{\star}{\rightharpoonup} u \quad \text{in } L^\infty((0,T), H^2(\Omega)\cap H^1_0(\Omega)), \\
u^N_t \overset{\star}{\rightharpoonup} u_t \quad \text{in } L^\infty((0,T), H^1_0(\Omega)),
\end{gathered}
$$

we deduce, up to a subsequence, that

$$
u^N \to u \quad \text{in } C([0,T], H^1_0(\Omega)). \tag{3.24}
$$

Therefore, $u^N(\cdot,0)$ is defined and

$$
u^N(\cdot,0)\to u(\cdot,0) \quad \text{in } H^1_0(\Omega).
$$

But,
$$u^N(\cdot,0)=u_0^N\to u_0 \text{ in } H^2(\Omega)\cap H_0^1(\Omega).$$
Then $u(\cdot,0)=u_0$. Similarly, we obtain $v(\cdot,0)=v_0$.

Noting that
$$u_{tt}^N \overset{\star}{\rightharpoonup} u_{tt} \quad\text{and}\quad v_{tt}^N \overset{\star}{\rightharpoonup} v_{tt} \quad\text{in } L^\infty((0,T),L^2(\Omega)),$$
we obtain $u_t,v_t\in C([0,T),L^2(\Omega))$, as in lions [18], and thus $u_t(\cdot,0)$ has a meaning. Furthermore
$$u_t^N(\cdot,0)\to u_t(\cdot,0) \quad\text{and}\quad v_t^N(\cdot,0)\to v_t(\cdot,0) \quad\text{in } L^2(\Omega).$$
But,
$$u_t^N(\cdot,0)=u_1^N\to u_1 \quad\text{and}\quad v_t^N(\cdot,0)=v_1^N\to v_1 \quad\text{in } H_0^1(\Omega).$$
Hence, $u_t(x,0)=u_1(x)$ and $v_t(x,0)=v_1(x)$.

Uniqueness. Assume that problem (1.1) has two weak solutions, (u,v) and $(\overline{u},\overline{v})$. Then, $(\mathbf{u},\mathbf{v})=(u-\overline{u},v-\overline{v})$ satisfies
$$\mathbf{u}_{tt}-\Delta\mathbf{u}+\int_0^t g(t-s)\Delta\mathbf{u}\,ds+\alpha\mathbf{v}=0, \quad \text{in } \Omega\times(0,T), \tag{3.25}$$
$$\mathbf{v}_{tt}-\Delta\mathbf{v}+\alpha\mathbf{u}=0, \quad \text{in } \Omega\times(0,T), \tag{3.26}$$
$$\mathbf{u}(x,t)=\mathbf{v}(x,t)=0, \quad \text{on } \partial\Omega\times(0,T), \tag{3.27}$$
$$\mathbf{u}(x,0)=\mathbf{u}_t(x,0)=0,\ \mathbf{v}(x,0)=\mathbf{v}_t(x,0)=0, \quad \text{in } \Omega. \tag{3.28}$$
Multiplying (3.25) by $\mathbf{u}_t$ and (3.26) by $\mathbf{v}_t$, integrating over Ω, and adding the two resulting equations, we find
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}_t(t)\|^2+\Big[1-\int_0^t g(s)ds\Big]\|\nabla\mathbf{u}(t)\|^2+(g\circ\nabla\mathbf{u})(t)\\
&+2\alpha\int_\Omega \mathbf{v}\ \mathbf{u}\,dx+\|\mathbf{v}_t(t)\|^2+\|\nabla\mathbf{v}(t)\|^2\Big)\\
&=-\frac{1}{2}g(t)\parallel\nabla\mathbf{u}(t)\parallel^2+\frac{1}{2}(g'\circ\nabla\mathbf{u})(t).
\end{aligned}\tag{3.29}$$
Using assumptions (A1)–(A3), and integrating over $(0,t)$, $t\le T$, we obtain
$$\|\mathbf{u}_t(t)\|^2+\Big[1-\int_0^t g(s)ds\Big]\|\nabla\mathbf{u}(t)\|^2+(g\circ\nabla\mathbf{u})(t)+2\alpha\int_\Omega \mathbf{v}\ \mathbf{u}dx+\|\mathbf{v}_t(t)\|^2+\|\nabla\mathbf{v}(t)\|^2\le 0.$$
From Young and Poincaré's inequalities, we obtain
$$\begin{aligned}
&\|\mathbf{u}_t(t)\|^2+\Big[1-\int_0^t g(s)ds-\alpha\varepsilon_1 C_p\Big]\|\nabla\mathbf{u}(t)\|^2+(g\circ\nabla\mathbf{u})(t)\\
&+\|\mathbf{v}_t(t)\|^2+\Big(1-\frac{\alpha C_p}{\varepsilon_1}\Big)\|\nabla\mathbf{v}(t)\|^2\le 0.
\end{aligned}$$
We select ε_1 such that it satisfies $\alpha C_p\le\varepsilon_1\le\frac{l}{\alpha C_p}$. Then
$$\|\mathbf{u}_t\|^2+\|\nabla\mathbf{u}\|^2+(g\circ\nabla\mathbf{u})(t)+\|\mathbf{v}_t\|^2+\|\nabla\mathbf{v}\|^2=0.$$
Hence, $\mathbf{u}_t(\cdot,t)=\mathbf{v}_t(\cdot,t)=0$ and $\nabla\mathbf{u}(\cdot,t)=\nabla\mathbf{v}(\cdot,t)=0$, for all $t\in(0,T)$. This implies that $\mathbf{u}=\ \mathbf{v}=0$ on $\Omega\times(0,T)$, given that $\mathbf{u}=\mathbf{v}=0$ on $\partial\Omega\times(0,T)$. This establishes the uniqueness. □

4. Asymptotic behavior

Now, we introduce the first and the second order energy functionals corresponding to system (1.1),
$$\begin{aligned}
E(t)=&\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\|v_t(t)\|^2+\frac{1}{2}\Big(1-\int_0^t g(s)\,ds\Big)\|\nabla u(t)\|^2+\frac{1}{2}\|\nabla v(t)\|^2\\
&+\frac{1}{2}(g\circ\nabla u)(t)+\alpha\int_\Omega uv\,dx
\end{aligned}\tag{4.1}$$

and

$$\begin{aligned}\mathsf{e}(t) &= \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\|\nabla v_t(t)\|^2 + \frac{1}{2}\|\Delta v(t)\|^2 + \frac{1}{2}\Big(1 - \int_0^t g(s)\,ds\Big)\|\Delta u(t)\|^2 \\ &\quad + \frac{1}{2}(g \circ \Delta u)(t) + \alpha \int_\Omega \nabla v \nabla u dx.\end{aligned} \tag{4.2}$$

Lemma 4.1. *Under the conditions of Theorem 3, there exists a constant $m > 0$ such that, for all $t \geq 0$,*

$$E(t) \geq m\left(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + (g \circ \nabla u)(t)\right), \tag{4.3}$$

$$\mathsf{e}(t) \geq m\left(\|\nabla u_t(t)\|^2 + \|\nabla v_t(t)\|^2 + \|\Delta v(t)\|^2 + \|\Delta u(t)\|^2 + (g \circ \Delta u)(t)\right). \tag{4.4}$$

Proof. Applying Cauchy-Schwarz and Young's inequalities, for $\varepsilon_1 > 0$, we obtain

$$E(t) \geq \frac{1}{2}\Big(\|u_t(t)\|^2 + \|v_t(t)\|^2 + [l - \alpha\varepsilon_1 C_p]\,\|\nabla u(t)\|^2 + \big(1 - \frac{\alpha C_p}{\varepsilon_1}\big)\|\nabla v(t)\|^2 + (g \circ \nabla u)(t)\Big),$$

for all $t \geq 0$. Keeping in mind (2.2), we can select ε_1 such that $\alpha C_p \leq \varepsilon_1 \leq \frac{l}{\alpha C_p}$. Consequently, we derive (4.3) with $m = \frac{1}{2}\min\{l - \alpha\varepsilon_1 C_p, 1 - \frac{\alpha C_p}{\varepsilon_1}\}$. Using similar reasoning, (4.4) can also be established with the same constant. □

Lemma 4.2. *Under the conditions of Theorem 3.1, the energy functional satisfies*

$$E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 \leq 0, \tag{4.5}$$

$$\mathsf{e}'(t) = \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u(t)\|^2 \leq 0. \tag{4.6}$$

Proof. To obtain (4.5), we multiply the first equation of system (1.1) by u_t, the second one by v_t, we integrate each one over Ω, utilizing the boundary conditions, it follows that

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^2 + \alpha\int_\Omega u\,v\Big)dx \\ &- \int_0^t g(t-s)\int_\Omega \nabla u(s)\nabla u_t\,dxds = 0.\end{aligned}$$

Applying Lemma 2.1, we arrive at

$$\begin{aligned}&\frac{d}{2dt}\Big(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \Big[1 - \int_0^t g(s)ds\Big]\|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla v(t)\|^2 + (g \circ \nabla u)(t) + \alpha\int_\Omega v\,udx\Big) \\ &+ \frac{1}{2}g(t)\parallel \nabla u(t) \parallel^2 - \frac{1}{2}(g' \circ \nabla u)(t) = 0.\end{aligned}$$

To obtain (4.6), we multiply the equation $(1.1)_1$ and $(1.1)_2$ by $-\Delta u_t$ and $-\Delta v_t$, respectively, and sum, to obtain

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|\nabla v_t(t)\|^2 + \frac{1}{2}\|\Delta v(t)\|^2 + \alpha\int_\Omega \nabla v\nabla u\Big)dx \\ &- \int_0^t g(t-s)\int_\Omega \Delta u(s)\Delta u_t(t)\,dxds = 0.\end{aligned}$$

Applying Lemma 1, we arrive at

$$\begin{aligned}&\frac{d}{2dt}\Big(\|\nabla u_t(t)\|^2 + \|\nabla v_t(t)\|^2 + \|\Delta v(t)\|^2 + \Big[1 - \int_0^t g(s)ds\Big]\|\Delta u(t)\|^2 \\ &+ (g \circ \Delta u)(t) + 2\alpha\int_\Omega \nabla v\nabla udx\Big) - \frac{1}{2}(g' \circ \Delta u)(t) + \frac{1}{2}g(t)\|\Delta u(t)\|^2 = 0.\end{aligned}$$

Then our conclusion follows. □

Now we construct a Lyapunov functional L equivalent to $E + e$, with which we can show the desired result. Similarly to [15], we define

$$C_\mu = \int_0^\infty \frac{g^2(s)}{\mu g(s) - g'(s)}\, ds, \quad h(t) = \mu g(t) - g'(t) \quad \text{for } 0 < \mu < 1.$$

Remark 4.3 ([29])**.** (1) The well-known Jensen's inequality will be of essential use in establishing our main result. For the sake of completeness, let us present it here. If F is a convex function on $[a, b]$, $f : \Omega \to [a, b]$, and h are integrable functions on Ω with $h(x) \geq 0$ and $\int_\Omega h(x)\, dx = \kappa > 0$, then Jensen's inequality states that

$$F\Big(\frac{1}{\kappa}\int_\Omega f(x)h(x)\, dx\Big) \leq \frac{1}{\kappa}\int_\Omega F(f(x))h(x)\, dx.$$

(2) From condition (A1), it follows that $\lim_{t\to+\infty} g(t) = 0$. Thus, there exists $t_0 \geq 0$ large enough such that

$$g(t_0) = r \quad \text{and} \quad g(t) \leq r, \forall t \geq t_0.$$

Since g and ζ are positive, non-increasing and continuous functions, and G is a positive and continuous function, we have for all $t \in [0, t_0]$,

$$0 < g(t_0) \leq g(t) \leq g(0), \quad 0 < \zeta(t_0) \leq \zeta(t) \leq \zeta(0).$$

This implies the existence of two positive constants a and b such that

$$a \leq \zeta(t)G(g(t)) \leq b.$$

As a result, we have, for all $t \in [0, t_0]$,

$$g'(t) \leq -\zeta(t)G(g(t)) \leq -a,$$

consequently,

$$g'(t) \leq -\frac{a}{g(0)}g(0) \leq -\frac{a}{g(0)}g(t).$$

(3) Given that

$$\frac{\mu g^2(s)}{\mu g(s) - g'(s)} < g(s),$$

and applying the Lebesgue dominated convergence theorem, we easily check that

$$\mu C_\mu = \int_0^\infty \frac{\mu g^2(s)}{\mu g(s) - g'(s)}\, ds \to 0 \quad \text{as } \mu \to 0.$$

(4) If G is a strictly increasing and strictly convex C^2 function on $(0, r]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$ which is strictly increasing and strictly convex C^2 function on $(0, \infty)$. See [29].

Lemma 4.4. *Let (u, v) be the solution of* (1.1). *Consider the functional*

$$\psi(t) = \int_\Omega u u_t\, dx + \int_\Omega v v_t\, dx.$$

Then, for any $\epsilon > 0$,

$$\psi'(t) \leq \|u_t(t)\|^2 + \|v_t(t)\|^2 - (l - \epsilon)\|\nabla u(t)\|^2 - \|\nabla v(t)\|^2 - 2\alpha \int_\Omega u\, v dx + \frac{C_\mu}{4\epsilon}(h \circ \nabla u)(t). \quad (4.7)$$

Proof. Using system (1.1), a direct computation leads to

$$\begin{aligned} \psi'(t) &= \|u_t(t)\|^2 + \|v_t(t)\|^2 - \|\nabla u(t)\|^2 - \|\nabla v(t)\|^2 + \int_\Omega \nabla u(t)\Big(\int_0^t g(t-s)\nabla u(s) ds\Big) dx \\ &\quad - 2\alpha \int_\Omega u\, v dx. \end{aligned} \quad (4.8)$$

Using Young's inequality and the fact that $\int_0^t g(s)\,ds \le \int_0^\infty g(s)\,ds = 1-l$, we estimate the fifth term in (4.8) to obtain

$$\begin{aligned} \psi'(t) &\le \|u_t(t)\|^2 + \|v_t(t)\|^2 - l\|\nabla u(t)\|^2 - \|\nabla v(t)\|^2 + \epsilon\|\nabla u(t)\|^2 \\ &\quad + \frac{1}{4\epsilon}\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2 dx - 2\alpha\int_\Omega u\,v dx. \end{aligned} \tag{4.9}$$

Now, the Cauchy-Schwarz inequality gives

$$\begin{aligned} &\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|\,ds\Big)^2 dx \\ &= \int_\Omega \Big(\int_0^t \frac{g(t-s)}{\sqrt{\mu g(t-s) - g'(t-s)}}\sqrt{\mu g(t-s) - g'(t-s)}|\nabla u(s) - \nabla u(t)|\,ds\Big)^2 dx \\ &\le \Big(\int_0^t \frac{g^2(s)}{\mu g(s) - g'(s)}\,ds\Big)\int_\Omega \Big(\int_0^t [\mu g(t-s) - g'(t-s)]|\nabla u(s) - \nabla u(t)|^2\,ds\Big)\,dx \\ &\le C_\mu (h\circ\nabla u)(t). \end{aligned} \tag{4.10}$$

By substituting (4.10) into (4.9), we obtain (4.7). □

Lemma 4.5. *Let (u,v) be the solution of (1.1). Then, the functional*

$$\chi(t) = -\int_\Omega u_t \int_0^t g(t-s)(u(t) - u(s))\,ds\,dx,$$

satisfies, for any $\delta_1 > 0$, the estimate

$$\begin{aligned} \chi'(t) &\le \Big(\delta_1 - \int_0^t g(s)\,ds\Big)\|u_t(t)\|^2 + \delta_1\|\nabla u(t)\|^2 + \alpha^2 C_p\frac{\delta_1}{2}\|\nabla v(t)\|^2 \\ &\quad + \Big(C + C_\mu\Big(1 + \frac{C}{\delta_1}\Big)\Big)(h\circ\nabla u)(t), \end{aligned} \tag{4.11}$$

where $C = \max\{c_1, c_2\}$, with $c_1 = \frac{C_p\big(\alpha(1-l)+g(0)\big)}{2}$, $c_2 = c + \frac{C_p(1+\mu^2)}{2}$.

Proof. From (1.1) and integration by parts, we obtain

$$\begin{aligned} \chi'(t) &= \Big(1 - \int_0^t g(s)\,ds\Big)\int_\Omega \nabla u(t)\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &\quad + \int_\Omega \Big(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,ds\Big)^2 dx - \int_\Omega u_t\int_0^t g'(t-s)(u(t) - u(s))\,ds\,dx \\ &\quad - \Big(\int_0^t g(s)\,ds\Big)\|u_t(t)\|^2 + \alpha\int_\Omega v\int_0^t g(t-s)(u(t) - u(s))\,ds\,dx. \end{aligned} \tag{4.12}$$

By Cauchy-Schwartz inequality, the second term of (4.12) satisfies

$$\int_\Omega \Big(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))\,ds\Big)^2 dx \le C_\mu (h\circ\nabla u)(t). \tag{4.13}$$

Similarly, for any $\delta_1 > 0$, we have

$$\begin{aligned} &\Big(1 - \int_0^t g(s)\,ds\Big)\int_\Omega \nabla u\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))\,ds\,dx \\ &\le \delta_1\|\nabla u(t)\|^2 + \frac{(1-g_0)^2\,C_\mu}{4\delta_1}\,(h\circ\nabla u)(t) \\ &\le \delta_1\|\nabla u(t)\|^2 + \frac{cC_\mu}{\delta_1}(h\circ\nabla u)(t), \end{aligned} \tag{4.14}$$

$$
\begin{aligned}
&-\int_\Omega u_t \int_0^t g'(t-s)(u(t)-u(s))\,ds\,dx\\
&=\int_\Omega u_t \int_0^t h(t-s)(u(t)-u(s))\,ds\,dx-\int_\Omega u_t \int_0^t \mu g(t-s)(u(t)-u(s))\,ds\,dx\\
&\le \delta_1\|u_t(t)\|^2+\frac{1}{2\delta_1}\int_\Omega\Big(\int_0^t\sqrt{h(t-s)}\sqrt{h(t-s)}|u(s)-u(t)|\,ds\Big)^2dx\\
&\quad+\frac{\mu^2}{2\delta_1}\int_\Omega\Big(\int_0^t g(t-s)|u(s)-u(t)|\,ds\Big)^2dx\\
&\le \delta_1\|u_t(t)\|^2+\frac{\int_0^t h(s)\,ds}{2\delta_1}(h\circ u)(t)+\frac{\mu^2C_\mu}{2\delta_1}(h\circ u)(t)\\
&\le \delta_1\|u_t(t)\|^2+\frac{C_p\big(\alpha(1-l)+g(0)\big)}{2\delta_1}(h\circ\nabla u)(t)+\frac{C_p\mu^2C_\mu}{2\delta_1}(h\circ\nabla u)(t)
\end{aligned}
\tag{4.15}
$$

and

$$
\alpha\int_\Omega v\int_0^t g(t-s)(u(t)-u(s))\,ds\,dx\le \alpha^2C_p\,\frac{\delta_1}{2}\|\nabla v(t)\|^2+C_p\frac{C_\mu}{2\delta_1}(h\circ\nabla u)(t). \tag{4.16}
$$

Collecting (4.13)–(4.16), we obtain the required estimate. □

Lemma 4.6. *Assume that* (A1)–(A3) *hold. Then, the functional*

$$
I(t)=\int_\Omega u_{tt}\,v_t dx-\int_\Omega v_{tt}\,u_t dx
$$

satisfies along the solution of (1.1) *and for any* $\delta_2>0$,

$$
I'(t)\le-(\alpha-\delta_2)\|v_t(t)\|^2+\alpha\|u_t(t)\|^2+\frac{Cg^2(t)}{\delta_2}\|\Delta u(t)\|^2-\frac{C}{\delta_2}(g'\circ\Delta u)(t). \tag{4.17}
$$

Proof. By using equations (1.1) and integrating by parts, we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u_{tt}\,v_t dx&=\int_\Omega u_{tt}\,v_{tt}dx-\int_\Omega \nabla u_t\nabla v_t dx-\alpha\|v_t(t)\|^2\\
&\quad+\int_\Omega v_t\int_0^t g'(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx-g(t)\int_\Omega \Delta u v_t dx,
\end{aligned}
$$

and

$$
-\frac{d}{dt}\int_\Omega v_{tt}\,u_t dx=-\int_\Omega v_{tt}\,u_{tt}dx+\int_\Omega \nabla u_t\nabla v_t dx+\alpha\|u_t(t)\|^2.
$$

Addition of the last two identities leads to

$$
I'(t)=\int_\Omega v_t\int_0^t g'(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx-g(t)\int_\Omega \Delta u v_t dx-\alpha\|v_t(t)\|^2+\alpha\|u_t(t)\|^2
$$

and Young's inequality yields

$$
\begin{gathered}
\int_\Omega v_t\int_0^t g'(t-s)(\Delta u(t)-\Delta u(s))\,ds\,dx\le\frac{\delta_2}{2}\|v_t(t)\|^2-\frac{C}{\delta_2}(g'\circ\Delta u)(t),\\
g(t)\int_\Omega \Delta u v_t dx\le\frac{\delta_2}{2}\|v_t(t)\|^2+\frac{Cg^2(t)}{\delta_2}\|\Delta u(t)\|^2.
\end{gathered}
$$

We finally get the estimate (4.17). □

Lemma 4.7 ([29])**.** *Assume that* (A1)–(A3) *hold. Then*

$$
K(t)=\int_\Omega\int_0^t f(t-s)|\nabla u(s)|^2\,ds\,dx,
$$

where $f(t)=\int_t^\infty g(s)\,ds$, *satisfies along the solution of* (1.1), *the estimate*

$$
K'(t)\le-\frac{1}{2}(g\circ\nabla u)(t)+3(1-l)\int_\Omega|\nabla u(t)|^2\,dx. \tag{4.18}
$$

At this stage, we introduce the Lyapunov functional

$$L(t) = \eta(E(t) + e(t)) + \eta_1\psi(t) + \eta_2\chi(t) + I(t), \tag{4.19}$$

where $\eta, \eta_1, \eta_2 > 0$, are constants to be suitably specified later.

Proposition 4.8. *There exist $\rho_i > 0$, $(i = 1, 2)$ such that*

$$\rho_1(E(t) + e(t)) \le L(t) \le \rho_2(E(t) + e(t)),\ t \ge 0. \tag{4.20}$$

Proof. Using Young's and Cauchy-Schwarz inequalities, we see that

$$\begin{aligned}\psi(t) &\le \frac{1}{2}\int_\Omega |u|^2\,dx + \frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{1}{2}\int_\Omega |v|^2\,dx + \frac{1}{2}\int_\Omega |v_t|^2\,dx\\ &\le \frac{C_p}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2 + \frac{C_p}{2}\|\nabla v(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2,\end{aligned}$$

and

$$\begin{aligned}\chi(t) &\le \frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{1}{2}\int_\Omega \Big(\int_0^t g(t-s)|u(t) - u(s)|\,ds\Big)^2\,dx\\ &\le \frac{1}{2}\|u_t(t)\|^2 + \frac{C_p^2}{2}(1-l)(g\circ\nabla u)(t).\end{aligned}$$

Similarly, using (1.1), we obtain

$$\begin{aligned}I(t) &\le \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2 + \int_\Omega |v_t|\int_0^t g(t-s)|\Delta u(s) - \Delta u(t)|\,ds\,dx + \frac{(1-l)}{2}\|v_t(t)\|^2\\ &\quad + \frac{(1-l)}{2}\|\Delta u(t)\|^2 + \frac{\alpha}{2}C_p\|\nabla v(t)\|^2 + \frac{\alpha}{2}\|v_t(t)\|^2 + \frac{1}{2}\|\Delta v(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2\\ &\quad + \frac{\alpha}{2}C_p\|\nabla u(t)\|^2 + \frac{\alpha}{2}\|u_t(t)\|^2\\ &\le \Big(\frac{1}{2} + \frac{(1-l)}{2}\Big)\|\Delta u(t)\|^2 + \Big(1 + \frac{(1-l)}{2} + \frac{\alpha}{2}\Big)\|v_t(t)\|^2 + \frac{(1-l)}{2}(g\circ\Delta u)(t)\\ &\quad + \frac{\alpha}{2}C_p\|\nabla v(t)\|^2 + \frac{\alpha}{2}C_p\|\nabla u(t)\|^2 + \frac{1}{2}\|\Delta v(t)\|^2 + \Big(\frac{1}{2} + \frac{\alpha}{2}\Big)\|u_t(t)\|^2.\end{aligned}$$

Then

$$\begin{aligned}|L(t) - \eta(E(t) + e(t))| &\le \frac{C_p}{2}(\eta_1 + \alpha)\|\nabla u(t)\|^2 + \Big(\frac{\eta_1}{2} + \frac{\eta_2}{2} + \frac{\alpha}{2} + \frac{1}{2}\Big)\|u_t(t)\|^2 + \frac{C_p\eta_1}{2}\|\nabla v(t)\|^2\\ &\quad + \Big(\frac{\eta_1}{2} + 1 + \frac{(1-l)}{2} + \frac{\alpha}{2}\Big)\|v_t(t)\|^2 + \frac{C_p^2\eta_2}{2}(1-l)(g\circ\nabla u)(t)\\ &\quad + \Big(\frac{1}{2} + \frac{(1-l)}{2}\Big)\|\Delta u(t)\|^2 + \frac{1}{2}\|\Delta v(t)\|^2 + \frac{(1-l)}{2}(g\circ\Delta u)(t)\\ &\le C\,(\eta_1 + \eta_2)\Big(\|\nabla u(t)\|^2 + \|u_t(t)\|^2 + \|\nabla v(t)\|^2 + \|v_t(t)\|^2 + (g\circ\nabla u)(t)\\ &\quad + \|\Delta u(t)\|^2 + \|\Delta v(t)\|^2 + (g\circ\Delta u)(t)\Big).\end{aligned}$$

Therefore,

$$|L(t) - \eta(E(t) + e(t))| \le C\,(\eta_1 + \eta_2)\,(E(t) + e(t)),$$

which yields

$$(\eta - C\,(\eta_1 + \eta_2))(E(t) + e(t)) \le L(t) \le (\eta + C\,(\eta_1 + \eta_2))(E(t) + e(t)).$$

So, by choosing $\eta > C\,(\eta_1 + \eta_2)$, we obtain (4.20). □

Lemma 4.9. *Assume that conditions* (A1)–(A3) *hold. Then, for any $M, N > 0$ and for suitable choice of $\eta, \eta_1, \eta_2 > 0$, we have*

$$\begin{aligned}L'(t) &\le -\frac{l\alpha}{8}\|v_t(t)\|^2 - \Big(\frac{3\alpha}{2} - \frac{3\alpha l}{N}\Big)\|u_t(t)\|^2 - \alpha^2\int_\Omega u\,v dx - \Big(\frac{\alpha}{2}(l-\epsilon) - \frac{3\alpha l}{N}\Big)\|\nabla u(t)\|^2\\ &\quad - \Big(\frac{\alpha}{2} - 3C_p\alpha^3\frac{l}{2N}\Big)\|\nabla v(t)\|^2 + \frac{1}{2M}(g\circ\nabla u)(t), \quad \text{for all } t \ge t_0,\end{aligned} \tag{4.21}$$

where t_0 is introduced in remark 4.3.

Proof. By combining (4.5), (4.6), (4.7), (4.11) and (4.17), we obtain

$$\begin{aligned}
L'(t) \le &\frac{\eta}{2}(g'\circ\nabla u)(t) - \frac{\eta}{2}g(t)\|\nabla u(t)\|^2 + \frac{\eta}{2}(g'\circ\Delta u)(t) - \frac{\eta}{2}g(t)\|\Delta u(t)\|^2 + \eta_1\|u_t(t)\|^2 + \eta_1\|v_t(t)\|^2\\
&- \eta_1\|\nabla v(t)\|^2 - \eta_1(l-\epsilon)\|\nabla u(t)\|^2 - 2\alpha\eta_1\int_\Omega uv\,dx + \frac{C_\mu\eta_1}{4\epsilon}(h\circ\nabla u)(t) + \eta_2\delta_1\|\nabla u(t)\|^2\\
&+ \eta_2\Big(\delta_1 - \int_0^t g(s)\,ds\Big)\|u_t(t)\|^2 + \eta_2\alpha^2C_p\frac{\delta_1}{2}\|\nabla v(t)\|^2 + \eta_2\Big(C + C_\mu\Big(1+\frac{C}{\delta_1}\Big)\Big)(h\circ\nabla u)(t)\\
&- (\alpha-\delta_2)\|v_t(t)\|^2 + \alpha\|u_t(t)\|_2^2 + \frac{Cg^2(t)}{\delta_2}\|\Delta u(t)\|_2^2 - \frac{C}{\delta_2}(g'\circ\Delta u)(t).
\end{aligned}$$

Letting $g_0 = \int_0^{t_0} g(s)\,ds$ and recalling that $g' = (\mu g - h)$, we obtain

$$\begin{aligned}
L'(t) \le &-(\alpha-\delta_2-\eta_1)\|v_t(t)\|^2 - (\eta_2(g_0-\delta_1) - \eta_1 - \alpha)\|u_t(t)\|^2 - 2\alpha\eta_1\int_\Omega u\,v dx\\
&- (\eta_1(l-\epsilon) - \eta_2\delta_1)\|\nabla u(t)\|^2 - \Big(\eta_1 - \eta_2C_p\alpha^2\frac{\delta_1}{2}\Big)\|\nabla v(t)\|^2 + \Big(\frac{\eta}{2} - \frac{C}{\delta_2}\Big)(g'\circ\Delta u)(t)\\
&- g(t)\Big(\frac{\eta}{2} - \frac{Cg(t)}{\delta_2}\Big)\|\Delta u(t)\|^2 - \Big(\frac{\eta}{2} - C\eta_2 - C_\mu\Big(\frac{\eta_1}{4\epsilon} + \eta_2\Big(1+\frac{C}{\delta_1}\Big)\Big)\Big)(h\circ\nabla u)(t)\\
&- \frac{\eta}{2}g(t)\|\nabla u(t)\|^2 + \frac{\eta}{2}\mu(g\circ\nabla u)(t), \quad \forall t\ge t_0.
\end{aligned}$$

Taking $\delta_2 \le \frac{\alpha l}{4}$, $\delta_1 = \frac{lg_0}{N}$ ($N\ge 12$, to be chosen later) and $\eta_2 = \frac{3\alpha}{g_0}$, we find that for all $t \ge t_0$,

$$\begin{aligned}
L'(t) \le &-(\alpha-\delta_2-\eta_1)\|v_t(t)\|^2 - \Big(2\alpha - \frac{3\alpha l}{N} - \eta_1\Big)\|u_t(t)\|^2 - 2\alpha\eta_1\int_\Omega u\,v dx\\
&- \Big(\eta_1(l-\epsilon) - \frac{3\alpha l}{N}\Big)\|\nabla u(t)\|^2 - \Big(\eta_1 - 3C_p\alpha^3\frac{l}{2N}\Big)\|\nabla v(t)\|^2 + \Big(\frac{\eta}{2} - \frac{C}{\delta_2}\Big)(g'\circ\Delta u)(t)\\
&- g(t)\Big(\frac{\eta}{2} - \frac{Cg(t)}{\delta_2}\Big)\|\Delta u(t)\|^2 - \Big(\frac{\eta}{2} - C\frac{3\alpha}{g_0} - C_\mu\Big(\frac{\eta_1}{4\epsilon} + \frac{3\alpha}{g_0}\Big(1+\frac{NC}{lg_0}\Big)\Big)\Big)(h\circ\nabla u)(t)\\
&+ \frac{\eta\mu}{2}(g\circ\nabla u)(t) - \frac{\eta}{2}g(t)\|\nabla u(t)\|^2.
\end{aligned}$$

For $0 < \epsilon < \frac{l^2}{2}$, we choose η_1 such that

$$\begin{gathered}
(l-\epsilon)\eta_1 - \frac{3\alpha l}{N} > \frac{l\alpha}{4}(1-l), \quad \alpha - \delta_2 - \eta_1 > \frac{l\alpha}{8},\\
2\alpha - \frac{3\alpha l}{N} - \eta_1 > \frac{\alpha(1+l)}{4}, \quad \eta_1 - 3C_p\alpha^3\frac{l}{2N} > \frac{\alpha}{4}.
\end{gathered} \tag{4.22}$$

In particular, we take $\eta_1 = \frac{\alpha}{2}$ and N large so that

$$\frac{\alpha}{2} - 3C_p\alpha^3\frac{l}{2N} > \frac{\alpha}{4},$$

and the inequalities (4.22) are satisfied.
By applying Remark 4.3, there exists $0 < \mu_0 < 1$, such that if $\mu < \mu_0$, then

$$\mu C_\mu < \frac{1}{8\big(\frac{\alpha}{16\epsilon} + \frac{3\alpha}{g_0}\big(1+\frac{NC}{lg_0}\big)\big)}.$$

Now, we choose η large enough and μ so that

$$\frac{\eta}{2} - \frac{C}{\delta_2} > 0, \quad \frac{\eta}{2} - \frac{3\alpha C}{g_0} > 0 \quad \text{and} \quad \mu = \frac{1}{M\eta} < \mu_0,$$

which gives

$$\frac{\eta}{2} - C\frac{3\alpha}{g_0} - C_\mu\Big(\frac{\alpha}{16\epsilon} + \frac{3\alpha}{g_0}\Big(1+\frac{NC}{lg_0}\Big)\Big) > 0.$$

Hence,

$$\begin{aligned}L'(t) &\le -\frac{l\alpha}{8}\|v_t(t)\|^2 - \Big(\frac{3\alpha}{2} - \frac{3\alpha l}{N}\Big)\|u_t(t)\|^2 - \alpha^2 \int_\Omega uv\,dx - \Big(\frac{\alpha}{2}(l-\epsilon) - \frac{3\alpha l}{N}\Big)\|\nabla u(t)\|^2 \\ &\quad - \Big(\frac{\alpha}{2} - 3C_p\alpha^3\frac{l}{2N}\Big)\|\nabla v(t)\|^2 + \frac{1}{2M}(g\circ\nabla u)(t).\end{aligned}$$

We further choose ϵ even smaller, N and η larger, such that (4.20) remains valid, we obtain (4.21). □

Theorem 4.10. *Assume that* (A1)–(A3) *hold and* $u_0, v_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $u_1, v_1 \in H_0^1(\Omega)$. *Then, there exist positive constants* k_0, C *and* γ *such that the energy functional* E *satisfies*

$$E(t) \le k_0 \frac{(E+e)(t_0)}{\int_{t_0}^t \xi(s)ds}, \quad \forall t \ge t_0, \quad \text{if } G \text{ is linear.} \tag{4.23}$$

$$E(t) \le CG_2^{-1}\Big(\frac{\gamma}{\int_{t_0}^t \xi(s)ds}\Big), \quad \forall t > t_0, \quad \text{if } G \text{ is non linear,} \tag{4.24}$$

where $G_2(\tau) = \tau G'(\tau)$.

Proof. Recalling (4.21), then for some constant $k > 0$ and for all $t \ge t_0$, we obtain

$$\begin{aligned}L'(t) &\le -\frac{l\alpha}{8}\|v_t(t)\|^2 - \Big(\frac{3\alpha}{2} - \frac{3\alpha l}{N}\Big)\|u_t(t)\|^2 - \alpha^2 \int_\Omega u\,vdx - \Big(\frac{\alpha}{2}(l-\epsilon) - \frac{3\alpha l}{N}\Big)\|\nabla u(t)\|^2 \\ &\quad - \Big(\frac{\alpha}{2} - 3C_p\alpha^3\frac{l}{2N}\Big)\|\nabla v(t)\|^2 + \frac{1}{2M}(g\circ\nabla u)(t) \\ &\le -kE(t) + c(g\circ\nabla u)(t).\end{aligned} \tag{4.25}$$

Case (I): $G(t)$ **is linear.** Multiplying (4.25) by $\xi(t)$ and using (A1) and (4.5), we find

$$\begin{aligned}\xi(t)L'(t) &\le -k\xi(t)E(t) + c\xi(t)\int_0^t g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\ &\le -k\xi(t)E(t) - cE'(t), \quad \forall t \ge t_0.\end{aligned}$$

As ξ is non-increasing and $L \ge 0$, we reach

$$(\xi(t)L(t))' \le \xi(t)L'(t) \le -k\xi(t)E(t) - cE'(t), \quad \forall t \ge t_0;$$

which yields

$$\xi(t)E(t) \le -\frac{1}{k}(\xi L + cE)'(t), \quad \forall t \ge t_0.$$

Simple integration over (t_0, t) leads to

$$\begin{aligned}E(t)\int_{t_0}^t \xi(s)ds &\le \int_{t_0}^t \xi(s)E(s)ds \\ &\le -\int_{t_0}^t \frac{1}{k}(\xi L + cE)'(s)ds \\ &\le \frac{1}{k}(\xi L + cE)(t_0), \quad \forall t \ge t_0.\end{aligned}$$

Using $\xi L + cE \sim E + e$, estimate (4.23) is established.

Case (II): $G(t)$ **nonlinear.** As in [27], we use (4.5) and (4.3) to deduce that, for any $t \ge t_0$,

$$\begin{aligned}&\int_0^{t_0} g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\ &\le -\frac{g(0)}{a}\int_0^{t_0} g'(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2\,dx\,ds \\ &\le -cE'(t),\end{aligned}$$

which can be utilized in (4.25), to obtain, for all $t \geq t_0$,

$$\begin{aligned} L'(t) &\leq -kE(t) + c(g \circ \nabla u)(t) \\ &\leq -kE(t) - cE'(t) + c\int_{t_0}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \,dx\,ds. \end{aligned}$$

By taking $F(t) = L(t) + cE(t)$, which is equivalent to $E(t) + e(t)$, we infer that

$$F'(t) \leq -kE(t) + c\int_{t_0}^{t} g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \,dx\,ds. \tag{4.26}$$

We use Lemmas 4.7 and 4.9 to conclude that, for $\beta > 0$,

$$\mathcal{L}(t) = L(t) + \beta K(t)$$

is nonnegative and for all $t \geq t_0$ satisfies

$$\begin{aligned} \mathcal{L}'(t) &\leq -\frac{l\alpha}{8}\|v_t(t)\|^2 - \Big(\frac{3\alpha}{2} - \frac{3\alpha l}{N}\Big)\|u_t(t)\|^2 - \alpha^2 \int_\Omega u\,v dx - \Big(\frac{\alpha}{2}(l-\epsilon) - \frac{3\alpha l}{N}\Big)\|\nabla u(t)\|^2 \\ &\quad - \Big(\frac{\alpha}{2} - 3C_p\alpha^3 \frac{l}{2N}\Big)\|\nabla v(t)\|^2 + \frac{1}{2M}(g \circ \nabla u)(t) - \frac{\beta}{2}(g \circ \nabla u)(t) + 3\beta(1-l)\|\nabla u(t)\|^2 \\ &\leq -\frac{l\alpha}{8}\|v_t(t)\|^2 - \Big(\frac{3\alpha}{2} - \frac{3\alpha l}{N}\Big)\|u_t(t)\|^2 - \alpha^2 \int_\Omega u\,v dx \\ &\quad - \Big(\frac{\alpha}{2}(l-\epsilon) - \frac{3\alpha l}{N} - 3\beta(1-l)\Big)\|\nabla u(t)\|^2 \\ &\quad - \Big(\frac{\alpha}{2} - 3C_p\alpha^3 \frac{l}{2N}\Big)\|\nabla v(t)\|^2 - \Big(\frac{\beta}{2} - \frac{1}{2M}\Big)(g \circ \nabla u)(t). \end{aligned}$$

By selecting β sufficiently small and then M sufficiently large such that

$$\begin{gathered} \frac{\alpha}{2}(l-\epsilon) - \frac{3\alpha l}{N} - 3\beta(1-l) > 0, \\ \frac{\beta}{2} - \frac{1}{2M} > 0, \end{gathered}$$

we easily obtain $\mathcal{L}'(t) \leq -dE(t)$, where d is some positive constant. So,

$$d\int_{t_0}^{t} E(s)\,ds \leq \mathcal{L}(t_0) - \mathcal{L}(t) \leq \mathcal{L}(t_0),$$

which implies that

$$\int_0^\infty E(s)\,ds < \infty. \tag{4.27}$$

As in [27], we take

$$J(t) := q\int_{t_0}^{t} \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \,dx\,ds,$$

where (4.27) allows us to choose a constant $0 < q < 1$ so that

$$J(t) < 1, \quad \forall t \geq t_0. \tag{4.28}$$

Then, we take

$$\lambda(t) := -\int_{t_0}^{t} g'(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 \,dx\,ds$$

and observe that $\lambda(t) \leq -cE'(t)$. Since G is strictly convex on $(0, r]$ and $G(0) = 0$, it follows that

$$G(\theta z) \leq \theta G(z), \quad \text{for } 0 \leq \theta \leq 1 \text{ and } z \in (0, r].$$

By employing the above estimate, hypothesis (A1), (4.28) and Jensen's inequality, we obtain

$$\lambda(t) = \frac{1}{J(t)}\int_{t_0}^{t} J(t)(-g'(s)) \int_\Omega \big(|\nabla u(t) - \nabla u(t-s)|^2\big)\,dx\,ds$$

$$
\begin{aligned}
&\geq \frac{1}{J(t)}\int_{t_0}^{t} J(t)\xi(s)G(g(s))\int_\Omega \left(|\nabla u(t)-\nabla u(t-s)|^2\right)\,dx\,ds\\
&\geq \frac{\xi(t)}{J(t)}\int_{t_0}^{t} G\left(J(t)g(s)\right)\int_\Omega \left(|\nabla u(t)-\nabla u(t-s)|^2\right)\,dx\,ds\\
&\geq \frac{\xi(t)}{q}G\Big(\frac{1}{J(t)}\int_{t_0}^{t} J(t)g(s)q\int_\Omega \left(|\nabla u(t)-\nabla u(t-s)|^2\right)\,dx\,ds\Big)\\
&= \frac{\xi(t)}{q}G\Big(q\int_{t_0}^{t} g(s)\int_\Omega \left(|\nabla u(t)-\nabla u(t-s)|^2\right)\,dx\,ds\Big)\\
&= \frac{\xi(t)}{q}\overline{G}\Big(q\int_{t_0}^{t} g(s)\int_\Omega \left(|\nabla u(t)-\nabla u(t-s)|^2\right)\,dx\,ds\Big),
\end{aligned}
$$

where $\overline{G}$ is an extension of G such that $\overline{G}$ is strictly increasing and strictly convex C^2 function on $(0,\infty)$. This indicates that

$$
\int_{t_0}^{t} g(s)\int_\Omega \left(|\nabla u(t)-\nabla u(t-s)|^2\right)\,dx\,ds \leq \frac{1}{q}\overline{G}^{-1}\Big(\frac{q\lambda(t)}{\xi(t)}\Big),
$$

and (4.26) becomes

$$
F'(t) \leq -kE(t)+\frac{c}{q}\overline{G}^{-1}\Big(\frac{q\lambda(t)}{\xi(t)}\Big),\quad \forall t\geq t_0. \tag{4.29}
$$

For $\varepsilon_0<r$, using (4.4), and the fact that $E'\leq 0$, $\overline{G}'>0$, $\overline{G}''>0$, we find that the functional F_1, defined by

$$
F_1(t):=\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)F(t),
$$

satisfies

$$
\begin{aligned}
F_1'(t) &= \frac{\varepsilon_0 E'(t)}{E(0)}\overline{G}''\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)F(t)+\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)F'(t)\\
&\leq -kE(t)\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\frac{c}{q}\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)\overline{G}^{-1}\Big(\frac{q\lambda(t)}{\xi(t)}\Big).
\end{aligned} \tag{4.30}
$$

Consider $\overline{G}^*$ the convex conjugate of $\overline{G}$ in the sense of Young [3, pp. 61-64], given by

$$
\overline{G}^* = s(\overline{G}')^{-1}(s)-\overline{G}[(\overline{G}')^{-1}(s)], \tag{4.31}
$$

and which satisfies the generalized Young inequality

$$
AB\leq \overline{G}^*(A)+\overline{G}(B). \tag{4.32}
$$

Then, with $A=\overline{G}'(\frac{\varepsilon_0 E(t)}{E(0)})$ and $B=\overline{G}^{-1}(\frac{q\lambda(t)}{\xi(t)})$, and using (3) and (4.5)–(4.3), we attain

$$
\begin{aligned}
F_1'(t) &\leq -kE(t)\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\frac{c}{q}\overline{G}^*\Big(\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)\Big)+\frac{c}{q}\frac{q\lambda(t)}{\xi(t)}\\
&\leq -kE(t)\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\frac{c}{q}\varepsilon_0\frac{E(t)}{E(0)}\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\frac{c\lambda(t)}{\xi(t)}\\
&\leq -kE(t)\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\widehat{c}\varepsilon_0\frac{E(t)}{E(0)}\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\frac{c\lambda(t)}{\xi(t)}\\
&= -\left(kE(0)-\widehat{c}\varepsilon_0\right)\frac{E(t)}{E(0)}\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\frac{c\lambda(t)}{\xi(t)}.
\end{aligned}
$$

After fixing ε_0 much smaller (if needed), we arrive at

$$
F_1'(t)\leq -k_1\frac{E(t)}{E(0)}\overline{G}'\Big(\frac{\varepsilon_0 E(t)}{E(0)}\Big)+\frac{c\lambda(t)}{\xi(t)},
$$

where $k_1>0$.

Then, we multiply by $\xi(t)$ and take ε_0 so small that, $\varepsilon_0 \frac{E(t)}{E(0)} < r$, $\overline{G}'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) = G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big)$ to arrive at

$$\begin{aligned}\xi(t)F_1'(t) &\le -k_1\xi(t)\frac{E(t)}{E(0)}G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) + cq\lambda(t)\\ &\le -k_1\xi(t)\frac{E(t)}{E(0)}G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) - cE'(t).\end{aligned}$$

Let $F_2 = \xi F_1 + cE$ and recall that ξ is non-increasing, to obtain

$$F_2'(t) \le \xi(t)F_1'(t) + cE'(t) \le -k_1\xi(t)\frac{E(t)}{E(0)}G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big),$$

or

$$k_1\xi(t)\frac{E(t)}{E(0)}G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big) \le -F_2'(t). \tag{4.33}$$

Using the strictly increasing property of G', non-increasing properties of E and ξ, we obtain for $t \ge t_0$,

$$k_1\frac{\varepsilon_0 E(t)}{E(0)}G'\Big(\varepsilon_0 \frac{E(t)}{E(0)}\Big)\int_{t_0}^t \xi(s)ds \le \varepsilon_0 F_2(t_0).$$

Next, we set $G_2(\tau) = \tau G'(\tau)$ which is strictly increasing, then we obtain, for some fixed positive constants C and γ

$$E(t) \le CG_2^{-1}\Big(\frac{\gamma}{\int_{t_0}^t \xi(s)ds}\Big), \quad \forall t \ge t_0.$$

This complete the proof. □

Examples. Here we give several examples to illustrate our decay result given by Theorem 4.10.

(1) Consider the relaxation function

$$g(t) = \frac{a}{[\ln(e+t)]^p}, \quad a > 0,\ p > 2\,,$$

where a is chosen so that hypothesis (A1) is satisfied.

$$g'(t) = -\xi(t)G(g(t)), \quad \text{with} \quad \xi(t) = \frac{p}{(e+t)\ln(e+t)}, \quad G(s) = s.$$

Therefore, estimate (4.23) implies that

$$E(t) \le \frac{C}{\ln\ln(e+t) - \ln\ln(e+t_0)}, \quad \forall t \ge t_0.$$

(2) Consider the relaxation function

$$g(t) = e^{-at},$$

where a is chosen so that condition (A1) is satisfied, then

$$g'(t) = -\xi(t)G(g(t)), \quad \xi(t) = a, \quad G(s) = s.$$

Therefore, estimate (4.23) implies that

$$E(t) \le \frac{C}{t - t_0}, \quad \forall t \ge t_0.$$

(3) Consider the relaxation function

$$g(t) = ae^{-(1+t)^\nu}, \quad 0 < \nu < 1,$$

and a is chosen so that condition (A1) is satisfied. Then

$$g'(t) = -\xi(t)G(g(t)), \quad \xi(t) = \nu(1+t)^{\nu-1}, \quad G(s) = s$$

and estimate (4.23) entails

$$E(t) \le \frac{C}{(1+t)^\nu - (1+t_0)^\nu}, \quad \forall t \ge t_0.$$

(4) Consider the relaxation function, for $\nu > 1$,

$$g(t) = \frac{a}{(1+t)^\nu}$$

and a is chosen so that hypothesis (A1) remains valid. Then

$$g'(t) = -\xi(t)G(g(t)), \quad \text{with } \xi(t) = b,\ G(s) = s^p,$$

where b is a fixed constant, $p = \frac{1+\nu}{\nu}$, and $1 < p < 2$. Then $G_2(t) = pt^p$, and from (4.24), we deduce that

$$E(t) \le C\Big[(1+t)^\nu - (1+t_0)^\nu\Big]^{-\frac{\nu}{1+\nu}}, \quad \forall t > t_0.$$

5. Numerical approach

In this section, we examine the computational behavior of the system (1.1) using the finite volume method by discretizing the system on the space-time domain $(0,1)\times(0,60)$ using second order finite difference method. For a similar construction, we refer to [22, 13]. Our numerical simulations are realized in the one-dimensional case. Then, we consider the system

$$\begin{gathered}
u_{tt} - u_{xx} + \int_0^t g(t-s)u_{xx}(s)\,ds + \alpha v = 0, \quad \text{in } (0,1)\times(0,T), \\
v_{tt} - v_{xx} + \alpha u = 0, \quad \text{in } (0,1)\times(0,T), \\
u(0,t) = u(1,t) = 0, \quad v(0,t) = v(1,t) = 0, \quad 0 < t < T, \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\
v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad 0 < x < 1.
\end{gathered}$$

5.1. **Explicit time integration method.** For our purpose, we set $\Delta x = L/(J+1)$ and $\Delta t = L/(N+1)$ for $J, N \in \mathbb{N}$, and consider the discretization

$$\begin{gathered}
x_0 = 0 < x_1 = \Delta x < \dots < x_J = J\Delta x < x_{J+1} = L, \\
t_0 = 0 < t_1 = \Delta t < \dots < t_N = N\Delta t < t_{N+1} = T,
\end{gathered}$$

where $x_j = j\Delta x$ and $t_n = n\Delta t$, for $j = 1,2,\dots,J$ and $n = 1,2,\dots,N$. For the memory term

$$\int_0^t g(t-s)\Delta u(s)\,ds,$$

we apply the trapezoidal rule

$$I_j^n = \sum_{i=0}^{n-1} \frac{\Delta t}{2}\Big[g(t_n - t_i)u_{xx_j}^i + g(t_n - t_{i+1})u_{xx_j}^{i+1}\Big], \quad \text{where } u_{xx_j}^i = \frac{u_{j+1}^i - 2u_j^i + u_{j-1}^i}{\Delta x^2}.$$

Thus, we obtain from the cumulative sum the numerical approximation of (5.1)

$$I_j^n = \frac{\Delta t}{2}\Big[g(t_n - t_0)u_{xx_j}^0 + \sum_{i=1}^{n-1}\Big(g(t_n - t_i)u_{xx_j}^i + g(t_n - t_{i+1})u_{xx_j}^{i+1}\Big) + g(t_n - t_n)u_{xx_j}^n\Big].$$

We consider the finite-difference discretization of (5)-(5)

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + I_j^n + \alpha v_j^n = 0, \tag{5.1}$$

$$\frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2} - \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2} + \alpha u_j^n = 0, \tag{5.2}$$

for all $j = 1,2,\dots,J$, $n = 1,2,\dots,N$. Such a scheme is in fact explicit and its implementation is straightforward. The boundary conditions are

$$u_0^n = u_{J+1}^n = 0, \quad v_0^n = v_{J+1}^n = 0, \quad \forall n = 1,2,\dots,N \tag{5.3}$$

and the initial conditions are

$$u_j^0 = u(x_j,0), \quad u_j^1 = u_j^0 + \Delta t\, u_t(x_j,0), \quad \forall j = 1,2,\dots,J, \tag{5.4}$$

$$v_j^0 = v(x_j, 0), \quad v_j^1 = v_j^0 + \Delta t\, v_t(x_j, 0), \quad \forall j = 1, 2, \dots, J. \tag{5.5}$$

Finally, we obtain this discrete system

$$\begin{aligned} u_j^{n+1} &= 2u_j^n - u_j^{n-1} + \frac{\Delta t^2}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) - \Delta t^2 I_j^n - \Delta t^2 \alpha v_j^n, \\ v_j^{n+1} &= 2v_j^n - v_j^{n-1} + \frac{\Delta t^2}{\Delta x^2}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right) - \Delta t^2 \alpha u_j^n. \end{aligned}$$

5.2. **Energy conserving scheme.** In this subsection, we construct the numerical energy associated with the system (5.1)-(5.2)

$$\begin{aligned} E^n := \frac{\Delta x}{2}\Big[&\sum_{j=0}^{J}\Big(\frac{u_j^{n+1} - u_j^n}{\Delta t}\Big)^2 + \sum_{j=0}^{J}\Big(\frac{(v_j^{n+1} - v_j^n)}{\Delta t}\Big)^2 + \Big(1 - \int_0^{t_n} g(s)\, ds\Big)\sum_{j=1}^{J-1}\Big(\frac{u_{j+1}^n - u_j^n}{\Delta x}\Big)^2 \\ &+ \sum_{j=1}^{J-1}\Big(\frac{v_{j+1}^n - v_j^n}{\Delta x}\Big)^2 + \sum_{k=0}^{n-1}\frac{\Delta t}{2}\sum_{j=1}^{J-1} g(t_n - t_k)\Big(\Big(\frac{(u_{j+1}^n - u_j^n)}{\Delta x}\Big)^2 - \Big(\frac{(u_{j+1}^k - u_j^k)}{\Delta x}\Big)^2\Big)^2 \\ &+ \sum_{j=0}^{J}\alpha v_j^n u_j^n\Big]. \end{aligned}$$

We note that E^n is the discrete version of the continuous energy (4.1) in the one-dimensional setting. Instead of computing the time derivative of the energy we can use the summation by parts. The discrete energy E^n is an important numerical instrument to certify our analytical results concerning the stabilization of system (5)-(5). Now, we present our tests.

Test 1. We present the decay case using the exponential function $g_1(t) = 0.5\, e^{-2t}$ and the polynomial function $g_2(t) = (1/(1+t)^{5/2})$, where we consider the following initial conditions

$$\begin{aligned} u(x,0) &= \sin(\pi x), \quad u_t(x,0) = 0, \quad x \in [0,1], \\ v(x,0) &= -\sin(\pi x), \quad v_t(x,0) = -\sin(\pi x), \quad x \in [0,1]. \end{aligned}$$

Test 2. We examine the energy decay using the function $g_2(t) = (1/(1+t)^{5/2})$ and taking the following initial conditions

$$\begin{aligned} u(x,0) &= \sin(\pi x), \quad u_t(x,0) = \sin(\pi x), \quad x \in [0,1], \\ v(x,0) &= 2x(1-x), \quad v_t(x,0) = 2x(1-x), \quad x \in [0,1]. \end{aligned}$$

To ensure numerical stability, we impose the Courant-Friedrichs-Lewy (CFL) condition, which requires the time step Δt and spatial step Δx to satisfy $\Delta t < \Delta x$. In our implementation, we set $\Delta t = 0.005$ and $\Delta x = 0.02$, ensuring the stability criterion is met. The spatial domain $[0,1]$ is discretized into 200 subintervals, and the temporal interval is determined based on the stability condition. The simulation is executed for 10,000 time steps with $\Delta t = 2 \times 10^{-3}$.

In Figures 1–3, we plot the damping behavior of the two waves, u and v, at the cross-sections $x = 0.25$, $x = 0.5$, and $x = 0.75$, and the decay behavior of the resulting energies. As a key conclusion, we conducted multiple tests for two types of decaying relaxation functions: exponential decay and polynomial decay. In each case, we observed that the energy decays at least at a polynomial rate, which aligns with our theoretical predictions.

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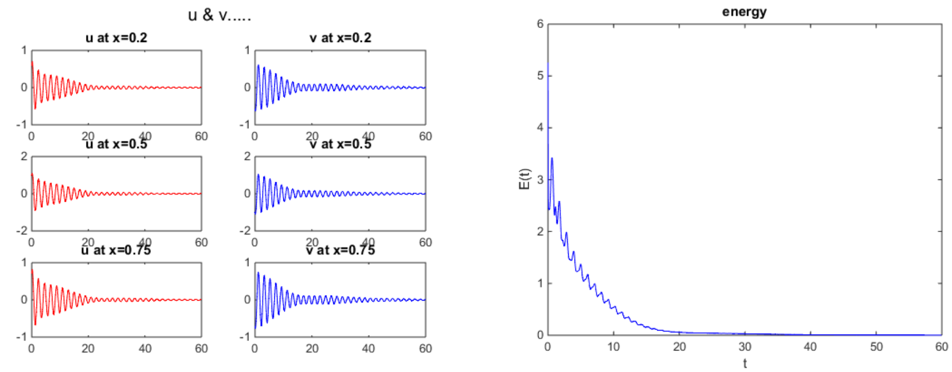

FIGURE 1. Test 1: Wave propagation of u and v over time and energy decay, with $g_1(t) = 0.5\, e^{-2t}$.

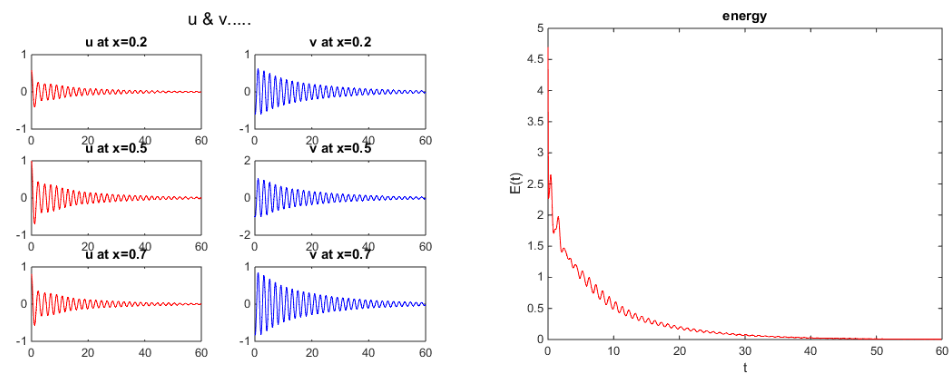

FIGURE 2. Test 1: Wave propagation of u and v over time and energy decay, with $g_2(t) = (1/(1+t)^{5/2})$.

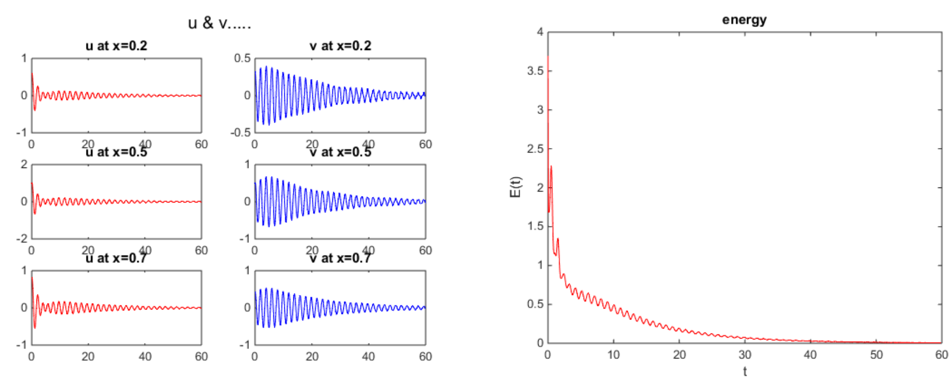

FIGURE 3. Test 2: Wave propagation of u and v over time and energy decay, with $g_2(t) = (1/(1+t)^{5/2})$.

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