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NONLINEAR STABILITY AND OPTIMAL DECAY RATES OF INVISCID MAGNETIC BÉNARD FLUIDS

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Abstract. This article studies the nonlinear stability of two-dimensional (2D) incompressible magnetic Bénard fluids near hydrostatic equilibrium in the absence of viscosity. We prove the global well-posedness in Besov space by utilizing a frequency decomposition approach, based on the potential hyperbolic structure. Furthermore, under appropriate additional conditions on low-frequency part. We derive a Lyapunov-type differential inequality and establish the optimal temporal decay rates for the global solution, in the sense that the obtained decay rates coincide with those of the associated linear heat semigroup and therefore cannot be improved in general. Compared with the results in [30], our findings not only provide the precise decay rates but also demonstrate faster decay than those previously obtained.

1. Introduction

Magnetic Bénard fluids describe the dynamic phenomenon of the interaction between velocity field and magnetic field in conductive fluids, which plays a crucial role in thermal convection problems [8, 38]. In this article, we study the stability of 2D magnetic Bénard fluids, described by the following system:

$$\begin{gathered} u_t + u\cdot\nabla u + \nabla P - \eta\Delta u = (0,\vartheta)^{\mathrm{T}} + B\cdot\nabla B, \\ B_t + u\cdot\nabla B - \mu\Delta B = B\cdot\nabla u, \\ \vartheta_t + u\cdot\nabla\vartheta - \kappa\Delta\vartheta = 0, \\ \operatorname{div} u = \operatorname{div} B = 0, \end{gathered} \tag{1.1}$$

where $u := u(x,t)$, $B := B(x,t)$, $P := P(x,t)$ and $\vartheta := \vartheta(x,t)$ represent the velocity field, magnetic field, pressure and temperature of the fluid, respectively. The constants η, μ and κ denote the coefficients of viscosity, magnetic diffusivity and thermometric conductivity, respectively. The term $(0,\vartheta)^{\mathrm{T}}$ represents the buoyancy force acting due to the temperature variation.

The magnetic Bénard fluid is a classical model for studying heat convection in the presence of a magnetic field, and it plays a crucial role in Rayleigh–Bénard convection. This system has broad applications in both physics and geophysics [10, 34]. Significant progress has been made in understanding the global stability and large-time behavior of solution for viscous, thermally and electrically conducting fluids, as demonstrated in [5, 11, 21, 25, 28, 39, 45]. In particular, the well-posedness of the system under partial dissipation has been extensively studied in various settings, such as in [6, 7, 15, 23, 24, 28, 30, 31, 37].

When $\vartheta = 0$, equation (1.1) reduces to the classical MHD system. The global well-posedness of this system has been widely studied [1, 20, 36], particularly for the inviscid and non-resistive case ($\eta = \mu = 0$), as discussed in [4, 9, 22]. Initiated by Lin and Zhang [33], many papers [2, 9, 17, 26, 27, 32, 35, 40, 46] have focused on the global well-posedness of the MHD system with dissipation but without magnetic resistivity ($\eta > 0, \mu = 0$). For the inviscous and resistive MHD equations ($\eta = 0, \mu > 0$), the global regularity problem has attracted attention. Under the certain symmetry assumptions, Zhou–Zhu [47] studied the global existence of classical solutions

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near an equilibrium state in the periodic domain. Utilizing the assumption that $\int_{\mathbb{T}^2} b_0 \mathrm{d}x = 0$, Wei–Zhang [41] proved the global well-posedness of the system without a non-trivial background magnetic field in H^4. Later, Ye–Yin [43] conducted small solution with lower regularity in $H^s(s > 2)$. Assuming the initial magnetic field is close to a background magnetic field satisfying the Diophantine condition, Zhai [44] and Chen–Zhang–Zhou [14] proved the global well-posedness in the periodic domain $\mathbb{T}^2$ and $\mathbb{T}^3$, respectively. Based on these researches, all globally well-posed results have been established in bounded domains. However, constructing a global solution for this inviscid system, even with small initial data, in the whole space remains an extremely challenging open problem.

The goal of this article is to address this issue within the context of Besov spaces. When the inviscid fluid is coupled with the temperature through the magnetic Bénard system (1.3), which involves a thermal damping term instead of thermal diffusion, this paper aims to provide a new mathematical result in Besov spaces. Specifically, it shows that the temperature enhances dissipation and contributes to stabilizing the fluid. A previous study by Lai et al. [30] demonstrated the global existence and stability of the system (1.3) in the Sobolev space H^3. Furthermore, they also established large-time behavior for the solutions:

$$\begin{gathered} \|(u_2,\theta)\|_{L^2} \to 0, \quad \text{as } t \to \infty, \\ \|(\nabla u, \nabla b, \nabla\theta)\|_{H^2} \lesssim (1+t)^{-1/2}. \end{gathered} \tag{1.2}$$

However, important questions regarding large-time behavior, such as explicit decay rates for the solution itself and for its high-order derivatives, remains unresolved. The primary objective of this paper is to establish optimal decay rates for the solutions of the 2D magnetic Bénard problem (1.3), which align with the decay rates of the heat semi-group.

The purpose of this paper is to study the asymptotic stability near an equilibrium state. We note that if a solution $\theta_e(x_2)$ satisfies $\theta_e'(x_2^0) < 0$ for some $x_2^0 \in \mathbb{R}$, which implies that fluid with a higher temperature lies below that with a lower temperature, then the system is unstable-the Rayleigh–Bénard instability occurs, as discussed in [12, 19, 25], among other. Therefore, we focus on the opposite case, where $\theta_e'(x_2^0) > 0$, implying that the fluid with lower temperature lies below the fluid with higher temperature. More specifically, we consider the equilibrium state

$$B_e = (0,1)^T, \quad \vartheta_e = x_2, \quad P_e = \frac{1}{2}x_2^2.$$

The perturbations around this equilibrium state are defined by

$$b = B - B_e, \quad \theta = \vartheta - \vartheta_e, \quad p = P - P_e.$$

The evolution of the perturbations (u, b, θ) is governed by the following inviscid system of equations:

$$\begin{gathered} u_t + u\cdot\nabla u + \nabla p = (0,\theta)^{\mathrm{T}} + b\cdot\nabla b + \partial_2 b, \\ b_t + u\cdot\nabla b - \mu\Delta b = b\cdot\nabla u + \partial_2 u, \\ \theta_t + u\cdot\nabla\theta + \nu\theta = -u_2, \\ \operatorname{div} u = \operatorname{div} b = 0. \end{gathered} \tag{1.3}$$

The corresponding initial conditions are

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \quad \theta(x,0) = \theta_0(x). \tag{1.4}$$

Before presenting our main results, we clarify the decomposition into low-frequency and high-frequency components. For any $u \in \mathcal{S}'$, we define its low-frequency part u^L and high-frequency part u^H via the Littlewood–Paley decomposition as

$$u^L := \sum_{j\leqslant -1} \dot\Delta_j u, \quad u^H := \sum_{j\geqslant 0} \dot\Delta_j u.$$

To measure separately the contributions of low and high frequencies in Besov norms, we use the following notation:

$$\|u\|^L_{\dot B^s_{p,r}} := \|(2^{js}\|\dot\Delta_j u\|_{L^p})_{j\leqslant 0}\|_{l^r}, \quad \|u\|^H_{\dot B^s_{p,r}} := \|(2^{js}\|\dot\Delta_j u\|_{L^p})_{j\geqslant -1}\|_{l^r},$$

$$\|u\|^L_{\tilde{L}^q_t(\dot{B}^s_{p,r})} := \|(2^{js}\|\dot{\Delta}_j u\|_{L^q_t(L^p)})_{j\leqslant 0}\|_{l^r}, \quad \|u\|^H_{\tilde{L}^q_t(\dot{B}^s_{p,r})} := \|(2^{js}\|\dot{\Delta}_j u\|_{L^q_t(L^p)})_{j\geqslant -1}\|_{l^r}.$$

For additional background on Besov spaces and related notation, we refer the reader to Section 2. Now we state the nonlinear stability result of the magnetic Bénard problem.

Theorem 1.1. *Let* $(u_0^L, b_0^L, \theta_0^L) \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ *and* $(u_0^H, b_0^H, \theta_0^H) \in \dot{B}^2_{2,1}(\mathbb{R}^2)$*, with* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Then, there exists a constant* $\epsilon_0 > 0$ *such that if the initial data satisfies*

$$\mathcal{E}_0 := \|(u_0, b_0, \theta_0)\|^L_{\dot{B}^0_{2,1}} + \|(u_0, b_0, \theta_0)\|^H_{\dot{B}^2_{2,1}} \leqslant \epsilon_0, \tag{1.5}$$

system (1.3) *and* (1.4) *admit a unique global solution with*

$$\begin{gathered} (u^L, b^L, \theta^L) \in C(\mathbb{R}_+; \dot{B}^0_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^2_{2,1}), \\ b^H \in C(\mathbb{R}_+; \dot{B}^2_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^3_{2,1}), \\ (u^H, \theta^H) \in C(\mathbb{R}_+; \dot{B}^2_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^2_{2,1}). \end{gathered} \tag{1.6}$$

Moreover, for all $t > 0$*, the solution* (u, b, θ) *satisfies*

$$\begin{aligned} &\|(u,b,\theta)\|^L_{\tilde{L}^\infty_t(\dot{B}^0_{2,1})} + \|(u,b,\theta)\|^H_{\tilde{L}^\infty_t(\dot{B}^2_{2,1})} + \|(u,b,\theta)\|^L_{L^1_t(\dot{B}^2_{2,1})} + \|(u,\theta)\|^H_{L^1_t(\dot{B}^2_{2,1})} + \|b\|^H_{L^1_t(\dot{B}^3_{2,1})} \\ &\lesssim \mathcal{E}_0. \end{aligned} \tag{1.7}$$

Remark 1.2. If the fluid is unaffected by temperature and magnetic field, the system (1.3) reduces to the 2D incompressible Euler equation, where the vorticity gradient can grow double exponentially over time [18, 29, 42]. However, when considering the interaction between temperature and the magnetic field, the system (1.3) becomes stable.

Remark 1.3. Theorem 1.1 shows that if the initial data is sufficiently small in $\dot{B}^0_{2,1}$ and $\dot{B}^2_{2,1}$, then system (1.3) admits a global solution. Since $\dot{H}^s(\mathbb{R}^2) \hookrightarrow \dot{B}^2_{2,1}(\mathbb{R}^2)$ for $s > 2$, we can replace $\dot{B}^0_{2,1}$ and $\dot{B}^2_{2,1}$ with H^3. This existence result can be found in [30], and here we extend it to the homogeneous Besov spaces.

Then we also obtain the optimal temporal decay rates of the solution obtained in Theorem 1.1.

Theorem 1.4. *Assume that the initial data* $(u_0^L, b_0^L, \theta_0^L) \in \dot{B}^{-\sigma}_{2,\infty}(\mathbb{R}^2)(0 < \sigma \leqslant 1)$. *Then, the corresponding solution* (u, b, θ) *obtained in Theorem 1.1 satisfies the following optimal temporal decay rates for all* $t \geqslant 0$:

$$\|\Lambda^s(u, b, \theta)\|_{L^2} \lesssim (1+t)^{-\frac{\sigma+s}{2}}, \tag{1.8}$$

where the pseudo-differential operator $\Lambda := (-\Delta)^{1/2}$ *and* $s \in (-\sigma, 0]$.

Remark 1.5. The temporal decay rates given in (1.8) are optimal in the sense that they coincide with that of the heat semi-group. In contrast to the large time behavior $\|(u_2, \theta)(t)\|_{L^2} \to 0$ as $t \to \infty$ described in [30], our result provides a precise decay rate.

If the damping term $\nu\theta$ in (1.3) is replaced by the thermal diffusivity term $\kappa\Delta\theta$, then (1.3) can be written as

$$\begin{gathered} u_t + u \cdot \nabla u + \nabla p = (0, \theta)^{\mathrm{T}} + b \cdot \nabla b + \partial_2 b, \\ b_t + u \cdot \nabla b - \mu\Delta b = b \cdot \nabla u + \partial_2 u, \\ \theta_t + u \cdot \nabla\theta - \kappa\Delta\theta = -u_2, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \quad \theta(x,0) = \theta_0(x). \end{gathered} \tag{1.9}$$

For the system in (1.9), the stability result and optimal temporal decay rates still hold. The proof of Theorem 1.6 is similar to that of Theorem 1.1 and 1.4, so we omit it here.

Theorem 1.6. *Let* $(u_0^L, b_0^L, \theta_0^L) \in \dot{B}^0_{2,1}(\mathbb{R}^2)$ *and* $(u_0^H, b_0^H, \theta_0^H) \in \dot{B}^2_{2,1}(\mathbb{R}^2)$*, with* $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. *Then, there exists a constant* $\epsilon_0 > 0$ *such that if the initial data satisfies*

$$\mathcal{E}_0 := \|(u_0, b_0, \theta_0)\|^L_{\dot{B}^0_{2,1}} + \|(u_0, b_0, \theta_0)\|^H_{\dot{B}^2_{2,1}} \leqslant \epsilon_0, \tag{1.10}$$

then system (1.9) *admits a unique global solution with*

$$\begin{aligned}(u^L, b^L, \theta^L) &\in C(\mathbb{R}_+; \dot{B}^0_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^2_{2,1}),\\ u^H &\in C(\mathbb{R}_+; \dot{B}^2_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^2_{2,1}),\\ (b^H, \theta^H) &\in C(\mathbb{R}_+; \dot{B}^2_{2,1}) \cap L^1(\mathbb{R}_+; \dot{B}^3_{2,1}).\end{aligned} \tag{1.11}$$

In addition, if the initial data $(u_0^L, b_0^L, \theta_0^L) \in \dot{B}^{-\sigma}_{2,\infty}(\mathbb{R}^2)$ *with* $0 < \sigma \leqslant 1$*, then the following optimal temporal decay rates hold for all* $t \geqslant 0$*:*

$$\|\Lambda^s(u, b, \theta)\|_{L^2} \lesssim (1+t)^{-\frac{\sigma+s}{2}}. \tag{1.12}$$

Now we outline the main idea of the proof. The key to proving the global existence of the solution is to establish uniform *a priori* estimates. A significant challenge arises in controlling the convection term $u\cdot\nabla u$ due to the absence of viscosity in the system. To overcome this, we exploit the potential hyperbolic structure of the system. By applying the Leray–Helmholtz projection $\mathbb{P} := I - \nabla\Delta^{-1}\mathrm{div}$ to the first equation of (1.3), we obtain the system

$$\begin{aligned}\partial_t u_1 + \partial_1\partial_2\Delta^{-1}\theta - \partial_2 b_1 &= \mathbb{P}(b\cdot\nabla b_1 - u\cdot\nabla u_1),\\ \partial_t u_2 - \partial_1^2\Delta^{-1}\theta - \partial_2 b_2 &= \mathbb{P}(b\cdot\nabla b_2 - u\cdot\nabla u_2),\end{aligned} \tag{1.13}$$

where I represents the identity matrix. If the temperature is neglected, differentiating (1.3) with respect to time and making several substitutions, using (1.13), we have

$$\begin{aligned}\partial_{tt}u_1 - \mu\partial_t\Delta u_1 - \partial_2^2 u_1 &= \mathcal{M}_1,\\ \partial_{tt}u_2 - \mu\partial_t\Delta u_2 - \partial_2^2 u_2 &= \mathcal{M}_2,\\ \partial_{tt}b - \mu\partial_t\Delta b - \partial_2^2 b &= \mathcal{M}_3,\end{aligned} \tag{1.14}$$

where

$$\begin{aligned}\mathcal{M}_1 &:= (\partial_t - \mu\Delta)\mathbb{P}(b\cdot\nabla b_1 - u\cdot\nabla u_1) + \partial_2(b\cdot\nabla u_1 - u\cdot\nabla b_1),\\ \mathcal{M}_2 &:= (\partial_t - \mu\Delta)\mathbb{P}(b\cdot\nabla b_2 - u\cdot\nabla u_2 + \theta) + \partial_2(b\cdot\nabla u_2 - u\cdot\nabla b_2),\\ \mathcal{M}_3 &:= \partial_t(b\cdot\nabla u - u\cdot\nabla b) + \partial_2\mathbb{P}(b\cdot\nabla b - u\cdot\nabla u + (0,\theta)^{\mathrm{T}}).\end{aligned}$$

Leveraging the hyperbolic smoothing effect of the above equations, we observe that the magnetic field captures a vertical dissipation effect on the velocity (see (3.8) for details), that is, $\partial_2 u$. Similarly, if the fluid is not affected by the magnetic field, differentiating (1.3) with respect to time and making several substitutions, we also obtain

$$\begin{aligned}\partial_{tt}u_1 + \nu\partial_t u_1 + \partial_1^2\Delta^{-1}u_1 &= \mathcal{M}_4,\\ \partial_{tt}u_2 + \nu\partial_t u_2 + \partial_1^2\Delta^{-1}u_2 &= \mathcal{M}_5,\\ \partial_{tt}\theta + \nu\partial_t\theta + \partial_1^2\Delta^{-1}\theta &= \mathcal{M}_6,\end{aligned} \tag{1.15}$$

where

$$\begin{aligned}\mathcal{M}_4 &:= (\partial_t + \nu)\mathbb{P}(b\cdot\nabla b_1 - u\cdot\nabla u_1 + \partial_2 b_1) + \partial_1\partial_2\Delta^{-1}(u\cdot\nabla\theta),\\ \mathcal{M}_5 &:= (\partial_t - \nu)\mathbb{P}(b\cdot\nabla b_2 - u\cdot\nabla u_2 + \partial_2 b_2) - \partial_1^2\Delta^{-1}(u\cdot\nabla\theta),\\ \mathcal{M}_6 &:= \mathbb{P}(u\cdot\nabla u_2 - b\cdot\nabla b_2) + \partial_t(u\cdot\nabla\theta) - \partial_2 b_2.\end{aligned}$$

This system also suggests a potential construction of a hyperbolic structure. It shows that the temperature contributes a horizontal dissipation effect on the velocity (see (3.6) for details), that is, $\partial_1 u$. Based on the above observations, to handle the nonlinear terms, we employ the technique of high and low frequency decomposition, separating the solution (u, b, θ) into its low-frequency and high-frequency parts. For the low-frequency part, we focus on ensuring the minimal regularity requirement, i.e., $(u^L, b^L, \theta^L) \in \dot{B}^0_{2,1}$. For the high-frequency part, we control the nonlinear interactions using appropriate product and commutator estimates. This leads to the functional framework $(u^H, b^H, \theta^H) \in \dot{B}^2_{2,1}$. Finally, by applying interpolation inequality and negative Besov norms, we derive a Lyapunov-type differential inequality, which leads to the optimal temporal decay rates.

This article paper is organized as follows. In Section 2, we introduce the Littlewood-Paley decomposition, Besov spaces and other essential tools. In Section 3, we prove Theorem 1.1, and in Section 4, we establish the proof of Theorem 1.4.

Notation. For $1 < p \leqslant +\infty$, we simplify $\int_{\mathbb{R}^2}$, $\|f\|_{L^p(\mathbb{R}^2)}$ and $\dot{B}^p_{q,r}(\mathbb{R}^2)$ as $\int$, $\|f\|_{L^p}$ and $\dot{B}^p_{q,r}$, respectively. For a Banach space A, we use the shorthand notation $\|(f,g)\|_A := \|f\|_A + \|g\|_A$. The symbols C and C_i $(i = 1, 2, 3, 4)$ represent generic positive constants, which may vary from one line to another. For a uniform constant C, we write $f \lesssim g$ to mean $f \leqslant Cg$, and $f \approx g$ to indicate both $f \leqslant Cg$ and $g \leqslant Cf$. For two operators X and Y, we define the commutator as $[X, Y] = XY - YX$. $\mathcal{F}$ $(\mathcal{F}^{-1})$ is the Fourier (inverse Fourier) transform operator.

2. Preliminaries

In this section, we introduce the Littlewood-Paley decomposition, the definition of Besov spaces and some useful properties. For further details, we refer to [3].

Let χ be radial function such that $\chi \equiv 1$ in $\{\xi \in \mathbb{R}^2 : |\xi| \leqslant 3/4\}$ which supported in $\{\xi \in \mathbb{R}^2 : |\xi| \leqslant 1\}$. Then $\varphi := \chi(\xi) - \chi(2\xi)$ is supported in the annulus $\{\xi \in \mathbb{R}^2 : 3/4 \leqslant |\xi| \leqslant 8/3\}$ and satisfies

$$\chi(\xi) + \sum_{j\geqslant 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2,$$

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2 \setminus \{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ are defined by

$$\dot{\Delta}_j u := \varphi(2^{-j}D)u = 2^{2j} \int h(2^j y)u(x-y)\mathrm{d}y, \quad \forall j \in \mathbb{Z},$$

where $h := \mathcal{F}^{-1}\varphi$. For tempered distribution $u \in \mathcal{S}'$, we have the decomposition

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u.$$

According to the above decomposition, we have the definition of the homogeneous Besov spaces.

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leqslant p, r \leqslant \infty$, then the homogeneous Besov space $\dot{B}^s_{p,r}$ is defined as

$$\dot{B}^s_{p,r} := \{u \in \mathcal{S}' : \|u\|_{\dot{B}^s_{p,r}} < \infty\},$$

where

$$\|u\|_{\dot{B}^s_{p,r}} := \|(2^{js}\|\dot{\Delta}_j u\|_{L^p})_{j\in\mathbb{Z}}\|_{l^r}.$$

Next, we present Berstein's inequalities, which will be used frequently.

Lemma 2.2. *For any $r \in (0, R)$, nonnegative integer k and pair $(p, q) \in [1, \infty]^2$ with $p \in [1, q]$, there exists a constant $C > 0$ for $u \in L^p$ such that*

$$\operatorname{supp} \mathcal{F}u \subset \{\xi \in \mathbb{R}^2 : |\xi| \leqslant \lambda R\} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leqslant C^{k+1}\lambda^{k+2(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$

$$\operatorname{supp} \mathcal{F}u \subset \{\xi \in \mathbb{R}^2 : \lambda r \leqslant |\xi| \leqslant \lambda R\} \Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \leqslant \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leqslant C^{k+1}\lambda^k\|u\|_{L^p}.$$

Lemma 2.3. *Assume $s \in \mathbb{R}$ and $1 \leqslant p, r \leqslant \infty$, then we have the following properties:*

(1) *Embedding: for any $1 \leqslant r \leqslant \tilde{r} \leqslant \infty$ and $1 \leqslant p \leqslant \tilde{p} \leqslant \infty$, it holds*

$$\dot{B}^s_{p,r}(\mathbb{R}^2) \hookrightarrow \dot{B}^{s-2(\frac{1}{p}-\frac{1}{\tilde{p}})}_{\tilde{p},\tilde{r}}(\mathbb{R}^2), \quad \dot{B}^0_{p,1}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2), \quad \dot{B}^{\frac{2}{p}}_{p,1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2). \tag{2.1}$$

(2) *Interpolation: for any $0 < \theta < 1$ and $s_1 < s_2$, then it holds that*

$$\|u\|_{\dot{B}^{s_1\theta+s_2(1-\theta)}_{p,r}} \leqslant \|u\|^\theta_{\dot{B}^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}, \tag{2.2}$$

$$\|u\|_{\dot{B}^{s_1\theta+s_2(1-\theta)}_{p,1}} \leqslant \frac{C}{s_2 - s_1}\Big(\frac{1}{\theta} + \frac{1}{1-\theta}\Big)\|u\|^\theta_{\dot{B}^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}. \tag{2.3}$$

(3) *Derivatives: for any integer k, then it holds that*

$$\sup_{|\alpha|=k} \|\partial^\alpha u\|_{\dot{B}^s_{p,r}} \approx \|u\|_{\dot{B}^{s+k}_{p,r}}. \tag{2.4}$$

Next, for the sake of time integration, we provide the following mixed Besov space about time and space, which was introduced in [13].

Definition 2.4. Assume $s \in \mathbb{R}$, $t > 0$ and $1 \leqslant p, q, r \leqslant \infty$, the space $\tilde{L}^q(0,t;\dot{B}^s_{p,r})$ is defined as

$$\tilde{L}^q(0,t;\dot{B}^s_{p,r}) := \{u \in L^q(0,t;\mathcal{S}'(\mathbb{R}^2)) : \|u\|_{\tilde{L}^q_t(\dot{B}^s_{p,r})} < \infty\},$$

where

$$\|u\|_{\tilde{L}^q_t(\dot{B}^s_{p,r})} := \|(2^{js}\|\dot{\Delta}_j u\|_{L^q_t(L^p)})_{j\in\mathbb{Z}}\|_{l^r}.$$

Remark 2.5. One easily from Minkowski's inequality check that

$$\begin{aligned} \|u\|_{L^q_t(\dot{B}^s_{p,r})} &\leqslant \|u\|_{\tilde{L}^q_t(\dot{B}^s_{p,r})}, \quad \text{if } r \leqslant q, \\ \|u\|_{\tilde{L}^q_t(\dot{B}^s_{p,r})} &\leqslant \|u\|_{L^q_t(\dot{B}^s_{p,r})}, \quad \text{if } q \leqslant r. \end{aligned} \tag{2.5}$$

For any $s' > 0$, one observes that

$$\begin{aligned} \|u^L\|_{\dot{B}^s_{p,r}} &\lesssim \|u\|^L_{\dot{B}^s_{p,r}} \lesssim \|u\|^L_{\dot{B}^{s-s'}_{p,r}}, \\ \|u^H\|_{\dot{B}^s_{p,r}} &\lesssim \|u\|^H_{\dot{B}^s_{p,r}} \lesssim \|u\|^H_{\dot{B}^{s+s'}_{p,r}}. \end{aligned} \tag{2.6}$$

Finally, let us recall some product estimates and commutator estimates.

Lemma 2.6. *We have the following product estimates:*

- *For any $1 \leqslant p, r \leqslant \infty$ and $s > 0$, it holds that*

$$\|fg\|_{\dot{B}^s_{p,r}} \lesssim \|f\|_{\dot{B}^s_{p,r}}\|g\|_{L^\infty} + \|g\|_{\dot{B}^s_{p,r}}\|f\|_{L^\infty}. \tag{2.7}$$

- *For any $s_1 \leqslant \frac{2}{p}$, $s_2 \leqslant \frac{2}{p}$, $s_1 + s_2 > 0$ and $1 \leqslant p \leqslant \infty$, it holds that*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{2}{p}}_{p,1}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,1}}. \tag{2.8}$$

- *For any $s_1 \leqslant \frac{2}{p}$, $s_2 < \frac{2}{p}$, $s_1 + s_2 \geqslant 0$ and $2 \leqslant p \leqslant \infty$, it holds that*

$$\|fg\|_{\dot{B}^{s_1+s_2-\frac{2}{p}}_{p,\infty}} \lesssim \|f\|_{\dot{B}^{s_1}_{p,1}}\|g\|_{\dot{B}^{s_2}_{p,\infty}}. \tag{2.9}$$

Lemma 2.7. *For $1 \leqslant p \leqslant \infty$ and a vector field $X := (X_1, X_2)^T$, we have the following commutator estimates:*

- *For $-\frac{2}{p} < s \leqslant 1 + \frac{2}{p}$, one has*

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[\dot{\Delta}_j, X\cdot\nabla]f\|_{L^p} \lesssim \|f\|_{\dot{B}^s_{p,1}}\|X\|_{\dot{B}^{1+\frac{2}{p}}_{p,1}}. \tag{2.10}$$

- *For $s > 0$, one has*

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[\dot{\Delta}_j, X\cdot\nabla]f\|_{L^p} \lesssim \|f\|_{\dot{B}^s_{p,1}}\|\nabla X\|_{L^\infty} + \|X\|_{\dot{B}^s_{p,1}}\|\nabla f\|_{L^\infty}. \tag{2.11}$$

- *For $-\frac{2}{p} \leqslant s \leqslant 1 + \frac{2}{p}$, one has*

$$\sum_{j\in\mathbb{Z}} 2^{js}\|[\dot{\Delta}_j, X\cdot\nabla]f\|_{L^p} \lesssim \|f\|_{\dot{B}^s_{p,\infty}}\|X\|_{\dot{B}^{1+\frac{2}{p}}_{p,1}}. \tag{2.12}$$

3. Proof of Theorem 1.1

In this section, we prove the existence of global solutions to system (1.3), that is, Theorem 1.1. Combining the local well-posedness with some a priori estimates, and then employing a standard continuity argument to extend the local solution to a global one. The proof of local existence for equations (1.3) is standard, and we omit the details here. Interested reads are referred to [16, 43]. With the local existence result in hand, the key step is to prove uniform a priori estimates of the solution, as stated in (1.7). To do so, we introduce the energy functionals

$$\begin{aligned} \mathcal{E}(t) &= \|(u,b,\theta)\|^L_{\dot B^0_{2,1}} + \|(u,b,\theta)\|^H_{\dot B^2_{2,1}},\\ \mathcal{D}(t) &= \|(u,b,\theta)\|^L_{\dot B^2_{2,1}} + \|(u,\theta)\|^H_{\dot B^2_{2,1}} + \|b\|^H_{\dot B^3_{2,1}}. \end{aligned} \tag{3.1}$$

First, we focus on the uniform stability estimate for the low-frequency part of (u,b,θ).

Lemma 3.1. *It holds that*

$$\|(u,b,\theta)\|^L_{\tilde L^\infty_t(\dot B^0_{2,1})} + \|(u,b,\theta)\|^L_{L^1_t(\dot B^2_{2,1})} \lesssim \|(u_0,b_0,\theta_0)\|^L_{\dot B^0_{2,1}} + \int_0^t \mathcal{E}(s)\mathcal{D}(s)\mathrm{d}s. \tag{3.2}$$

Proof. Applying the operator $\dot\Delta_j (j\in\mathbb{Z})$ to (1.3) yields

$$\begin{aligned} \partial_t\dot\Delta_j u + \dot\Delta_j\nabla p - \dot\Delta_j(0,\theta)^{\mathrm T} - \dot\Delta_j\partial_2 b &= \dot\Delta_j\mathcal{N}_1,\\ \partial_t\dot\Delta_j b - \mu\Delta\dot\Delta_j b - \dot\Delta_j\partial_2 u &= \dot\Delta_j\mathcal{N}_2,\\ \partial_t\dot\Delta_j\theta + \nu\dot\Delta_j\theta + \dot\Delta_j u_2 &= -\dot\Delta_j\mathcal{N}_3, \end{aligned} \tag{3.3}$$

where

$$\mathcal{N}_1 := b\cdot\nabla b - u\cdot\nabla u,\quad \mathcal{N}_2 := b\cdot\nabla u - u\cdot\nabla b,\quad \mathcal{N}_3 := u\cdot\nabla\theta.$$

Taking the L^2 inner product with $\dot\Delta_j u$, $\dot\Delta_j b$ and $\dot\Delta_j\theta$ to the first three equations of (3.3), respectively, then using integration by parts and summing up the resulting equality, we obtain

$$\begin{aligned} &\frac12\frac{\mathrm d}{\mathrm dt}\|(\dot\Delta_j u,\dot\Delta_j b,\dot\Delta_j\theta)\|^2_{L^2} + \mu\|\nabla\dot\Delta_j b\|^2_{L^2} + \nu\|\dot\Delta_j\theta\|^2_{L^2}\\ &= \int\Big(\dot\Delta_j\mathcal{N}_1\cdot\dot\Delta_j u + \dot\Delta_j\mathcal{N}_2\cdot\dot\Delta_j b - \dot\Delta_j\mathcal{N}_3\dot\Delta_j\theta\Big)\mathrm dx, \end{aligned} \tag{3.4}$$

where we have used the cancelation that

$$\int\dot\Delta_j(\nabla p - (0,\theta) - \partial_2 b)\cdot\dot\Delta_j u\mathrm dx - \int\dot\Delta_j\partial_2 u\cdot\dot\Delta_j b\mathrm dx + \int\dot\Delta_j u_2\dot\Delta_j\theta\mathrm dx = 0.$$

So as to get the horizontal dissipation of u, applying the operator $\dot\Delta_j\nabla$ to the second component of the momentum equation $(1.3)_1$ and $(1.3)_3$ lead to

$$\begin{aligned} \partial_t\dot\Delta_j\nabla u_2 + \dot\Delta_j\nabla\partial_2 p - \dot\Delta_j\nabla\theta - \dot\Delta_j\nabla\partial_2 b_2 &= \dot\Delta_j\nabla\mathcal{N}_1^2,\\ \partial_t\dot\Delta_j\nabla\theta + \nu\dot\Delta_j\nabla\theta + \dot\Delta_j\nabla u_2 &= -\dot\Delta_j\nabla\mathcal{N}_3, \end{aligned} \tag{3.5}$$

where $\mathcal{N}_1^2$ stands for the second component of $\mathcal{N}_1$.

Noting that $\mathrm{div}u = 0$, then we have $\|\dot\Delta_j\partial_1 u\|_{L^2} = \|\dot\Delta_j\nabla u_2\|_{L^2}$. Thus, multiplying $(3.5)_1$ and $(3.5)_2$ by $\dot\Delta_j\nabla\theta$ and $\dot\Delta_j\nabla u_2$ in L^2, respectively, we find that

$$\begin{aligned} &\frac{\mathrm d}{\mathrm dt}\int\dot\Delta_j\nabla u_2\cdot\dot\Delta_j\nabla\theta\mathrm dx + \|\dot\Delta_j\partial_1 u\|_{L^2} - \|\dot\Delta_j\nabla\theta\|_{L^2}\\ &+ \int\Big(\dot\Delta_j\nabla\partial_2 p - \dot\Delta_j\nabla\partial_2 b_2\Big)\cdot\dot\Delta_j\nabla\theta\mathrm dx + \nu\int\dot\Delta_j\nabla\theta\cdot\dot\Delta_j\nabla u_2\mathrm dx\\ &= \int\Big(\dot\Delta_j\nabla\mathcal{N}_1^2\cdot\dot\Delta_j\nabla\theta - \dot\Delta_j\nabla\mathcal{N}_3\cdot\dot\Delta_j\nabla u_2\Big)\mathrm dx. \end{aligned} \tag{3.6}$$

To control the term $\int\dot\Delta_j\nabla\partial_2 p\cdot\dot\Delta_j\nabla\theta\mathrm dx$, applying the operator div to $(1.3)_1$, by $(1.3)_4$ we check that

$$\Delta p = \nabla b : \nabla b - \nabla u : \nabla u + \partial_2\theta. \tag{3.7}$$

Next, we aim to obtain the vertical dissipation of u. Taking L^2 inner product of $(3.3)_1$ and $(3.3)_2$ with $\dot{\Delta}_j\partial_2 b$ and $-\dot{\Delta}_j\partial_2 u$, respectively, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \dot{\Delta}_j u\cdot\dot{\Delta}_j\partial_2 b\mathrm{d}x+\|\dot{\Delta}_j\partial_2 u\|_{L^2}-\|\dot{\Delta}_j\partial_2 b\|_{L^2}-\int\dot{\Delta}_j\theta\dot{\Delta}_j\partial_2 b_2\mathrm{d}x+\nu\int\Delta\dot{\Delta}_j b\cdot\dot{\Delta}_j\partial_2 u\mathrm{d}x\\&=\int\Big(\dot{\Delta}_j\mathcal{N}_1\cdot\dot{\Delta}_j\partial_2 b-\dot{\Delta}_j\mathcal{N}_2\cdot\dot{\Delta}_j\partial_2 u\Big)\mathrm{d}x,\end{aligned}\tag{3.8}$$

where we have used $\int\dot{\Delta}_j\nabla p\cdot\dot{\Delta}_j\partial_2 b\mathrm{d}x=0$.

Then combining (3.4), (3.6), (3.7) and (3.8), for a large constant C_1 we can obtain the estimate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L^2+\tilde{\mathcal{D}}_L^2&=C_1\int\dot{\Delta}_j\mathcal{N}_1\cdot\dot{\Delta}_j u\mathrm{d}x+C_1\int\dot{\Delta}_j\mathcal{N}_2\cdot\dot{\Delta}_j b\mathrm{d}x-C_1\int\dot{\Delta}_j\mathcal{N}_3\cdot\dot{\Delta}_j\theta\mathrm{d}x\\&\quad+\int\dot{\Delta}_j\nabla\mathcal{N}_1^2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x-\int\dot{\Delta}_j\nabla\mathcal{N}_3\cdot\dot{\Delta}_j\nabla u_2\mathrm{d}x+\int\dot{\Delta}_j\mathcal{N}_1\cdot\dot{\Delta}_j\partial_2 b\mathrm{d}x\\&\quad-\int\dot{\Delta}_j\mathcal{N}_2\cdot\dot{\Delta}_j\partial_2 u\mathrm{d}x+\int\dot{\Delta}_j(\nabla u:\nabla u-\nabla b:\nabla b)\dot{\Delta}_j\partial_2\theta\mathrm{d}x,\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}\tilde{\mathcal{E}}_L^2&:=\frac{1}{2}C_1\|(\dot{\Delta}_j u,\dot{\Delta}_j b,\dot{\Delta}_j\theta)\|_{L^2}^2+\int\dot{\Delta}_j\nabla u_2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x+\int\dot{\Delta}_j u\cdot\dot{\Delta}_j\partial_2 b\mathrm{d}x,\\ \tilde{\mathcal{D}}_L^2&:=C_1\mu\|\nabla\dot{\Delta}_j b\|_{L^2}^2+C_1\nu\|\dot{\Delta}_j\theta\|_{L^2}^2+\|\dot{\Delta}_j\partial_1 u\|_{L^2}^2+\|\dot{\Delta}_j\partial_2 u\|_{L^2}^2+\|\dot{\Delta}_j\partial_2\theta\|_{L^2}^2\\&\quad-\|\dot{\Delta}_j\partial_2 b\|_{L^2}^2-\|\dot{\Delta}_j\nabla\theta\|_{L^2}^2-\int\dot{\Delta}_j\nabla\partial_2 b_2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x+\nu\int\dot{\Delta}_j\nabla\theta\cdot\dot{\Delta}_j\nabla u_2\mathrm{d}x\\&\quad-\int\dot{\Delta}_j\theta\dot{\Delta}_j\partial_2 b_2\mathrm{d}x+\nu\int\Delta\dot{\Delta}_j b\cdot\dot{\Delta}_j\partial_2 u\mathrm{d}x.\end{aligned}$$

By using Berstein's inequality, for $j\leqslant 0$, one has

$$\|\nabla\dot{\Delta}_j f\|_{L^2}\lesssim 2^j\|\dot{\Delta}_j f\|_{L^2}\lesssim\|\dot{\Delta}_j f\|_{L^2}.$$

Then taking C_1 large enough and using the above result, we have

$$\begin{aligned}\tilde{\mathcal{E}}_L^2&\approx\|(\dot{\Delta}_j u,\dot{\Delta}_j b,\dot{\Delta}_j\theta)\|_{L^2}^2,\\ \tilde{\mathcal{D}}_L^2&\approx\|(\nabla\dot{\Delta}_j u,\nabla\dot{\Delta}_j b,\dot{\Delta}_j\theta)\|_{L^2}^2\gtrsim 2^{2j}\|(\dot{\Delta}_j u,\dot{\Delta}_j b,\dot{\Delta}_j\theta)\|_{L^2}^2.\end{aligned}\tag{3.10}$$

Integration by parts,and using Hölder's inequality and Berstein's inequality, for $j\leqslant 0$, (3.9) and (3.10) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L^2+2^{2j}\tilde{\mathcal{E}}_L^2\lesssim\|(\dot{\Delta}_j\mathcal{N}_1,\dot{\Delta}_j\mathcal{N}_2,\dot{\Delta}_j\mathcal{N}_3,\dot{\Delta}_j(\nabla u:\nabla u-\nabla b:\nabla b))\|_{L^2}\tilde{\mathcal{E}}_L,$$

which together with $\|\dot{\Delta}_j(\nabla u:\nabla u-\nabla b:\nabla b))\|_{L^2}\lesssim\|\dot{\Delta}_j\mathcal{N}_1\|_{L^2}$ yields

$$\tilde{\mathcal{E}}_L(t)+2^{2j}\int_0^t\tilde{\mathcal{E}}_L(s)\mathrm{d}s\lesssim\tilde{\mathcal{E}}_L(0)+\int_0^t\|(\dot{\Delta}_j\mathcal{N}_1,\dot{\Delta}_j\mathcal{N}_2,\dot{\Delta}_j\mathcal{N}_3)\|_{L^2}\mathrm{d}s.\tag{3.11}$$

Summing over $j\leqslant 0$, together with (3.10) one obtains

$$\begin{aligned}&\|(u,b,\theta)\|^L_{\tilde{L}_t^\infty(\dot{B}^0_{2,1})}+\|(u,b,\theta)\|^L_{L_t^1(\dot{B}^2_{2,1})}\\&\lesssim\|(u_0,b_0,\theta_0)\|^L_{\dot{B}^0_{2,1}}+\int_0^t\|(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3)\|^L_{\dot{B}^0_{2,1}}\mathrm{d}s.\end{aligned}\tag{3.12}$$

Now, we estimate the nonlinear terms in the right-hand side of (3.12). According to (2.4), (2.6) and (2.8), we obtain

$$\|(u\cdot\nabla u,u\cdot\nabla b,u\cdot\nabla\theta)\|^L_{\dot{B}^0_{2,1}}\lesssim\|u\|_{\dot{B}^0_{2,1}}\|\nabla(u,b,\theta)\|_{\dot{B}^1_{2,1}}\lesssim\|u\|_{\dot{B}^0_{2,1}}\|(u,b,\theta)\|_{\dot{B}^2_{2,1}}\lesssim\mathcal{E}(t)\mathcal{D}(t),\tag{3.13}$$

and

$$\|(b\cdot\nabla b,b\cdot\nabla u)\|^L_{\dot{B}^0_{2,1}}\lesssim\|b\|_{\dot{B}^0_{2,1}}\|\nabla(b,u)\|_{\dot{B}^1_{2,1}}\lesssim\|b\|_{\dot{B}^0_{2,1}}\|(b,u)\|_{\dot{B}^2_{2,1}}\lesssim\mathcal{E}(t)\mathcal{D}(t).\tag{3.14}$$

In summary, putting the nonlinear estimates (3.13) and (3.14) into (3.12), we obtain the desired estimate (3.2). The proof is complete. □

Then we begin to control the stability estimate for the high-frequency part of (u, b, θ).

Lemma 3.2. *It holds*

$$\begin{aligned}&\|(u,b,\theta)\|^H_{\tilde{L}^\infty_t(\dot{B}^2_{2,1})}+\|(u,\theta)\|^H_{L^1_t(\dot{B}^2_{2,1})}+\|b\|^H_{L^1_t(\dot{B}^3_{2,1})}\\&\lesssim\|(u_0,b_0,\theta_0)\|^H_{\dot{B}^2_{2,1}}+\int_0^t\mathcal{E}(s)\mathcal{D}(s)\mathrm{d}s.\end{aligned}\tag{3.15}$$

Proof. Multiplying $(3.3)_1$, $(3.3)_2$ and $(3.3)_3$ by $-\Delta\dot{\Delta}_j u$, $-\Delta\dot{\Delta}_j b$ and $-\Delta\dot{\Delta}_j\theta$ in L^2, respectively, by integrating by parts, the fact that

$$\int\dot{\Delta}_j(\partial_2 b+(0,\theta)-\nabla p)\cdot\dot{\Delta}\Delta_j u\mathrm{d}x+\int\dot{\Delta}_j\partial_2 u\cdot\Delta\dot{\Delta}_j b\mathrm{d}x-\int\dot{\Delta}_j u_2\Delta\dot{\Delta}_j\theta\mathrm{d}x=0,$$

and summing up the resulting equality, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\nabla\dot{\Delta}_j u,\nabla\dot{\Delta}_j b,\nabla\dot{\Delta}_j\theta)\|^2_{L^2}+\mu\|\Delta\dot{\Delta}_j b\|^2_{L^2}+\nu\|\nabla\dot{\Delta}_j\theta\|^2_{L^2}\\&=\int\Big(\nabla\dot{\Delta}_j\mathcal{N}_1\cdot\nabla\dot{\Delta}_j u+\nabla\dot{\Delta}_j\mathcal{N}_2\cdot\nabla\dot{\Delta}_j b-\nabla\dot{\Delta}_j\mathcal{N}_3\cdot\nabla\dot{\Delta}_j\theta\Big)\mathrm{d}x.\end{aligned}\tag{3.16}$$

Multiplying (3.16) by a large constant C_2 and adding to (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^2_H+\tilde{\mathcal{D}}^2_H\\&=C_2\int\nabla\dot{\Delta}_j\mathcal{N}_1\cdot\nabla\dot{\Delta}_j u\mathrm{d}x+C_2\int\nabla\dot{\Delta}_j\mathcal{N}_2\cdot\nabla\dot{\Delta}_j b\mathrm{d}x-C_2\int\nabla\dot{\Delta}_j\mathcal{N}_3\cdot\nabla\dot{\Delta}_j\theta\mathrm{d}x\\&\quad+\int\dot{\Delta}_j\nabla\mathcal{N}_1^2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x-\int\dot{\Delta}_j\nabla\mathcal{N}_3\cdot\dot{\Delta}_j\nabla u_2\mathrm{d}x+\int\dot{\Delta}_j\mathcal{N}_1\cdot\dot{\Delta}_j\partial_2 b\mathrm{d}x\\&\quad-\int\dot{\Delta}_j\mathcal{N}_2\cdot\dot{\Delta}_j\partial_2 u\mathrm{d}x+\int\dot{\Delta}_j(\nabla u:\nabla u-\nabla b:\nabla b)\dot{\Delta}_j\partial_2\theta\mathrm{d}x,\end{aligned}\tag{3.17}$$

where

$$\begin{aligned}\tilde{\mathcal{E}}^2_H:=&\frac{1}{2}C_2\|(\nabla\dot{\Delta}_j u,\nabla\dot{\Delta}_j b,\nabla\dot{\Delta}_j\theta)\|^2_{L^2}+\int\dot{\Delta}_j\nabla u_2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x+\int\dot{\Delta}_j u\cdot\dot{\Delta}_j\partial_2 b\mathrm{d}x,\\\tilde{\mathcal{D}}^2_H:=&C_2\mu\|\Delta\dot{\Delta}_j b\|^2_{L^2}+C_2\nu\|\nabla\dot{\Delta}_j\theta\|^2_{L^2}+\|\dot{\Delta}_j\partial_1 u\|^2_{L^2}+\|\dot{\Delta}_j\partial_2 u\|^2_{L^2}+\|\dot{\Delta}_j\partial_2\theta\|^2_{L^2}\\&-\|\dot{\Delta}_j\partial_2 b\|^2_{L^2}-\|\dot{\Delta}_j\nabla\theta\|^2_{L^2}-\int\dot{\Delta}_j\nabla\partial_2 b_2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x+\nu\int\dot{\Delta}_j\nabla\theta\cdot\dot{\Delta}_j\nabla u_2\mathrm{d}x\\&-\int\dot{\Delta}_j\theta\dot{\Delta}_j\partial_2 b_2\mathrm{d}x+\nu\int\Delta\dot{\Delta}_j b\cdot\dot{\Delta}_j\partial_2 u\mathrm{d}x.\end{aligned}$$

By Berstein's inequality, for $j\geqslant -1$, we have

$$\|\dot{\Delta}_j f\|_{L^2}\lesssim 2^{-j}\|\nabla\dot{\Delta}_j f\|_{L^2}\lesssim\|\nabla\dot{\Delta}_j f\|_{L^2}.\tag{3.18}$$

Taking C_2 large enough, then using (3.18) and Young's inequality, it follows for $j\geqslant -1$ that

$$\begin{aligned}\tilde{\mathcal{E}}^2_H&\approx\|(\nabla\dot{\Delta}_j u,\nabla\dot{\Delta}_j b,\nabla\dot{\Delta}_j\theta)\|^2_{L^2},\\\tilde{\mathcal{D}}^2_H&\approx\|(\nabla\dot{\Delta}_j u,\Delta\dot{\Delta}_j b,\nabla\dot{\Delta}_j\theta)\|^2_{L^2},\\\tilde{\mathcal{D}}^2_H&\gtrsim\|(\nabla\dot{\Delta}_j u,\nabla\dot{\Delta}_j b,\nabla\dot{\Delta}_j\theta)\|_{L^2}\|(\nabla\dot{\Delta}_j u,\Delta\dot{\Delta}_j b,\nabla\dot{\Delta}_j\theta)\|_{L^2}.\end{aligned}\tag{3.19}$$

Noting that

$$\begin{aligned}&\int\nabla\dot{\Delta}_j\mathcal{N}_1\cdot\nabla\dot{\Delta}_j u\mathrm{d}x+\int\nabla\dot{\Delta}_j\mathcal{N}_2\cdot\nabla\dot{\Delta}_j b\mathrm{d}x-\int\nabla\dot{\Delta}_j\mathcal{N}_3\cdot\nabla\dot{\Delta}_j\theta\mathrm{d}x\\&=\int\dot{\Delta}_j(\nabla(b\cdot\nabla)b)\cdot\dot{\Delta}_j\nabla u\mathrm{d}x+\int\dot{\Delta}_j(\nabla(b\cdot\nabla)u)\cdot\dot{\Delta}_j\nabla b\mathrm{d}x-\int\dot{\Delta}_j(\nabla(u\cdot\nabla)u)\cdot\dot{\Delta}_j\nabla u\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&-\int \dot{\Delta}_j(\nabla(u\cdot\nabla)b)\cdot\dot{\Delta}_j\nabla b\mathrm{d}x-\int \dot{\Delta}_j(\nabla(u\cdot\nabla)\theta)\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x+\int[\dot{\Delta}_j,b\cdot\nabla]\nabla b\cdot\dot{\Delta}_j\nabla u\mathrm{d}x\\
&+\int[\dot{\Delta}_j,b\cdot\nabla]\nabla u\cdot\dot{\Delta}_j\nabla b\mathrm{d}x-\int[\dot{\Delta}_j,u\cdot\nabla]\nabla b\cdot\dot{\Delta}_j\nabla b\mathrm{d}x-\int[\dot{\Delta}_j,u\cdot\nabla]\nabla u\cdot\dot{\Delta}_j\nabla u\mathrm{d}x\\
&-\int[\dot{\Delta}_j,u\cdot\nabla]\nabla\theta\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
&\int \dot{\Delta}_j\nabla\mathcal{N}_1^2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x-\int\dot{\Delta}_j\nabla\mathcal{N}_3\cdot\dot{\Delta}_j\nabla u_2\mathrm{d}x\\
&=\int\dot{\Delta}_j\nabla(b\cdot\nabla b_2)\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x-\int\dot{\Delta}_j(\nabla(u\cdot\nabla)u_2)\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x\\
&\quad-\int\dot{\Delta}_j(\nabla(u\cdot\nabla)\theta)\cdot\dot{\Delta}_j\nabla u_2\mathrm{d}x-\int[\dot{\Delta}_j,u\cdot\nabla]\nabla u_2\cdot\dot{\Delta}_j\nabla\theta\mathrm{d}x\\
&\quad-\int[\dot{\Delta}_j,u\cdot\nabla]\nabla\theta\cdot\dot{\Delta}_j\nabla u_2\mathrm{d}x,
\end{aligned}\tag{3.20}
$$

then by Hölder's inequality, (3.17)-(3.20), for $j\geqslant -1$, one has

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_H^2+\tilde{\mathcal{E}}_H\tilde{\mathcal{D}}_H\lesssim\Big(&\|\dot{\Delta}_j(\nabla b\cdot\nabla b,\nabla b\cdot\nabla u,\nabla u\cdot\nabla u,\nabla u\cdot\nabla b,\nabla u\cdot\nabla\theta)\|_{L^2}\\
&+\|([\dot{\Delta}_j,b\cdot\nabla](\nabla b,\nabla u),[\dot{\Delta}_j,u\cdot\nabla](\nabla b,\nabla u,\nabla\theta))\|_{L^2}+\|(\dot{\Delta}_j\mathcal{N}_1,\dot{\Delta}_j\mathcal{N}_2)\|_{L^2}\Big)\tilde{\mathcal{E}}_H,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\tilde{\mathcal{E}}_H(t)+\int_0^t\tilde{\mathcal{D}}_H(s)\mathrm{d}s\\
&\lesssim\tilde{\mathcal{E}}_H(0)+\int_0^t\Big(\|\dot{\Delta}_j(\nabla b\cdot\nabla b,\nabla b\cdot\nabla u,\nabla u\cdot\nabla u,\nabla u\cdot\nabla b,\nabla u\cdot\nabla\theta)\|_{L^2}\\
&\quad+\|\Big([\dot{\Delta}_j,b\cdot\nabla](\nabla b,\nabla u),[\dot{\Delta}_j,u\cdot\nabla](\nabla b,\nabla u,\nabla\theta)\Big)\|_{L^2}+\|(\dot{\Delta}_j\mathcal{N}_1,\dot{\Delta}_j\mathcal{N}_2)\|_{L^2})\mathrm{d}s.
\end{aligned}\tag{3.21}
$$

By using the Calderon-Zygmund inequality, (2.4), (3.19) and summing inequality (3.21) over $j\geqslant -1$, we have

$$
\begin{aligned}
&\|(u,b,\theta)\|^H_{\tilde{L}^\infty_t(\dot{B}^2_{2,1})}+\|(u,\theta)\|^H_{L^1_t(\dot{B}^2_{2,1})}+\|b\|^H_{L^1_t(\dot{B}^3_{2,1})}\\
&\lesssim\|(u_0,b_0,\theta_0)\|^H_{\dot{B}^2_{2,1}}+\int_0^t\Big(\|(\mathcal{N}_1,\mathcal{N}_2)\|_{\dot{B}^1_{2,1}}\\
&\quad+\sum_{j\geqslant-1}2^j\|([\dot{\Delta}_j,b\cdot\nabla](\nabla b,\nabla u),[\dot{\Delta}_j,u\cdot\nabla](\nabla b,\nabla u,\nabla\theta))\|_{L^2}\\
&\quad+\|(\nabla b\cdot\nabla b,\nabla b\cdot\nabla u,\nabla u\cdot\nabla u,\nabla u\cdot\nabla b,\nabla u\cdot\nabla\theta)\|_{\dot{B}^1_{2,1}}\Big)\mathrm{d}s.
\end{aligned}\tag{3.22}
$$

Next we turn to estimate the nonlinear terms of (3.22) in sequence. First, by (2.7) and (2.1), we obtain

$$
\begin{aligned}
&\|(\nabla b\cdot\nabla b,\nabla b\cdot\nabla u,\nabla u\cdot\nabla u,\nabla u\cdot\nabla b,\nabla u\cdot\nabla\theta)\|_{\dot{B}^1_{2,1}}\\
&\lesssim\|\nabla b\|_{L^\infty}\|(\nabla b,\nabla u)\|_{\dot{B}^1_{2,1}}+\|\nabla u\|_{L^\infty}\|(\nabla b,\nabla u,\nabla\theta)\|_{\dot{B}^1_{2,1}}+\|\nabla\theta\|_{L^\infty}\|\nabla u\|_{\dot{B}^1_{2,1}}\\
&\lesssim\|(\nabla b,\nabla u,\nabla\theta)\|^2_{\dot{B}^1_{2,1}}\\
&\lesssim\|(b,u,\theta)\|^2_{\dot{B}^2_{2,1}}\\
&\lesssim\mathcal{E}(t)\mathcal{D}(t).
\end{aligned}\tag{3.23}
$$

For the term concerning the commutator, we use (2.11) and (2.1) to obtain

$$\begin{aligned}&\sum_{j\geqslant -1} 2^j\|([\dot\Delta_j, b\cdot\nabla](\nabla b,\nabla u),[\dot\Delta_j, u\cdot\nabla](\nabla b,\nabla u,\nabla\theta))\|_{L^2}\\ &\lesssim \|(\nabla b,\nabla u,\nabla\theta)\|_{L^\infty}\|(\nabla b,\nabla u,\nabla\theta)\|_{\dot B^1_{2,1}}\\ &\lesssim \|(\nabla b,\nabla u,\nabla\theta)\|^2_{\dot B^1_{2,1}}\\ &\lesssim \|(b,u,\theta)\|^2_{\dot B^2_{2,1}}\\ &\lesssim \mathcal{E}(t)\mathcal{D}(t).\end{aligned} \tag{3.24}$$

For the last term, it follows from (2.8) and (2.1) that

$$\|(\mathcal{N}_1,\mathcal{N}_2)\|_{\dot B^1_{2,1}} \lesssim \|(b,u)\|_{\dot B^1_{2,1}}\|(\nabla b,\nabla u)\|_{\dot B^1_{2,1}} \lesssim \|(b,u)\|_{\dot B^1_{2,1}}\|(b,u,\theta)\|_{\dot B^2_{2,1}} \lesssim \mathcal{E}(t)\mathcal{D}(t). \tag{3.25}$$

Thus, inserting the estimates (3.23)-(3.25) into (3.22), we obtain (3.15) immediately, and complete the proof. □

Proof of Theorem 1.1. Here T^* denotes the life-span (maximal existence time) of the local solution (u,b,θ). Next, we need to proof that (1.7) holds and $T^*=\infty$. It infers from (3.2) and (3.15) that

$$\mathcal{E}(t)+\int_0^t \mathcal{D}(s)\mathrm{d}s \leqslant C_3\mathcal{E}_0 + C_4\sup_{0\leqslant s\leqslant t}\mathcal{E}(s)\int_0^t \mathcal{D}(s)\mathrm{d}s, \tag{3.26}$$

where $\mathcal{E}_0$ is defined in (1.5).

For any $t\in(0,T^*)$, we have

$$\sup_{0\leqslant s\leqslant t}\mathcal{E}(s)\leqslant \frac{1}{2C_4}.$$

Then (3.26) yields

$$\mathcal{E}(t)+\frac12\int_0^t \mathcal{D}(s)\mathrm{d}s \leqslant C_3\mathcal{E}_0. \tag{3.27}$$

Finally, choosing sufficiently small ϵ, by using the local existence result together with a standard continuity argument (sometimes referred to as a bootstrapping argument), we have $T^*=\infty$. This completes the proof. □

4. Proof of Theorem 1.4

This section is devoted to proving the optimal temporal decay rates for (u,b,θ) in Theorem 1.4. First, we prove that the negative Besov norms of (u,b,θ) in low-frequency are bounded along time evolution.

Lemma 4.1. *Assume $\sigma\in(0,1]$ and $(u_0^L,b_0^L,\theta_0^L)\in\dot B^{-\sigma}_{2,\infty}$, for any $t\geqslant 0$, one has*

$$\|(u,b,\theta)\|^L_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|(u_0,b_0,\theta_0)\|^L_{\dot B^{-\sigma}_{2,\infty}} + \mathcal{E}_0. \tag{4.1}$$

Proof. Just as we have done in the proof of (3.11) that for $j\leqslant 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L \lesssim \|(\dot\Delta_j\mathcal{N}_1,\dot\Delta_j\mathcal{N}_2,\dot\Delta_j\mathcal{N}_3)\|_{L^2},$$

which gives rise to

$$\|(u,b,\theta)\|^L_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|(u_0,b_0,\theta_0)\|^L_{\dot B^{-\sigma}_{2,\infty}} + \int_0^t \|(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3)\|_{\dot B^{-\sigma}_{2,\infty}}\mathrm{d}s. \tag{4.2}$$

To estimate the right-hand side of (4.2), it follows from (2.9), (2.4) and (2.1) that

$$\|(u\cdot\nabla u, u\cdot\nabla b, u\cdot\nabla\theta)\|_{\dot B^{-\sigma}_{2,\infty}} \lesssim \|u\|_{\dot B^{-\sigma}_{2,\infty}}\|(\nabla u,\nabla b,\nabla\theta)\|_{\dot B^1_{2,1}} \lesssim \|u\|_{\dot B^{-\sigma}_{2,\infty}}\|(u,b,\theta)\|_{\dot B^2_{2,1}}, \tag{4.3}$$

and

$$\|(b\cdot\nabla b, b\cdot\nabla u)\|_{\dot B^{-1}_{2,\infty}} \lesssim \|b\|_{\dot B^{-1}_{2,\infty}}\|(\nabla b,\nabla u)\|_{\dot B^1_{2,1}} \lesssim \|b\|_{\dot B^{-1}_{2,\infty}}\|(b,u)\|_{\dot B^2_{2,1}}. \tag{4.4}$$

Noting that

$$\begin{aligned}\|(u,b)\|_{\dot{B}^{-\sigma}_{2,\infty}} &\lesssim \|(u,b)\|^L_{\dot{B}^{-\sigma}_{2,\infty}} + \|(u,b)\|^H_{\dot{B}^{-1}_{2,\infty}}\\ &\lesssim \|(u,b)\|^L_{\dot{B}^{-\sigma}_{2,\infty}} + \|u\|^H_{\dot{B}^{2}_{2,\infty}} + \|b\|^H_{\dot{B}^{3}_{2,\infty}}\\ &\lesssim \|(u,b)\|^L_{\dot{B}^{-\sigma}_{2,\infty}} + \|u\|^H_{\dot{B}^{2}_{2,1}} + \|b\|^H_{\dot{B}^{3}_{2,1}},\end{aligned}$$

then putting (4.3), (4.4) and above inequality into (4.2), together with (2.1), (1.7) and (1.5), we obtain (4.1) immediately. □

Proof of Theorem 1.4. For the low-frequency part, following a similar approach to the proof of (3.9)-(3.11), and using $(3.10)_1$, we obtain the result for $j \leqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L + 2^{2j}\tilde{\mathcal{E}}_L \lesssim \|(\dot{\Delta}_j\mathcal{N}_1, \dot{\Delta}_j\mathcal{N}_2, \dot{\Delta}_j\mathcal{N}_3)\|_{L^2},$$

where $\tilde{\mathcal{E}}_L$ and $\mathcal{N}_i (i = 1, 2, 3)$ are defined in (3.9) and (3.3), respectively. Summing up on $j \leqslant 0$, by (3.10) and (2.4) one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\leqslant 0}\tilde{\mathcal{E}}_L + \|(u,b,\theta)\|^L_{\dot{B}^2_{2,1}} \lesssim \|(\mathcal{N}_1,\mathcal{N}_2,\mathcal{N}_3)\|^L_{\dot{B}^0_{2,1}}. \tag{4.5}$$

Next let us bound the nonlinear terms $\mathcal{N}_i (i = 1, 2, 3)$. Thanks to (2.4), (2.8), (1.7) and (1.5), we find that

$$\|(u\cdot\nabla u, u\cdot\nabla b, u\cdot\nabla\theta)\|^L_{\dot{B}^0_{2,1}} \lesssim \|u\|^L_{\dot{B}^0_{2,1}}\|\nabla(u,b,\theta)\|^L_{\dot{B}^1_{2,1}} \lesssim \|u\|^L_{\dot{B}^0_{2,1}}\|(u,b,\theta)\|^L_{\dot{B}^2_{2,1}} \lesssim \epsilon_0\|(u,b,\theta)\|^L_{\dot{B}^2_{2,1}},$$

and

$$\|(b\cdot\nabla b, b\cdot\nabla u)\|^L_{\dot{B}^0_{2,1}} \lesssim \|b\|^L_{\dot{B}^0_{2,1}}\|\nabla(b,u)\|^L_{\dot{B}^1_{2,1}} \lesssim \|b\|^L_{\dot{B}^0_{2,1}}\|(b,u)\|^L_{\dot{B}^2_{2,1}} \lesssim \epsilon_0\|(u,b,\theta)\|^L_{\dot{B}^2_{2,1}}.$$

Then, for suitable small ϵ_0, (4.5) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\leqslant 0}\tilde{\mathcal{E}}_L + \|(u,b,\theta)\|^L_{\dot{B}^2_{2,1}} \lesssim 0. \tag{4.6}$$

For the high-frequency part, a similar way as the proof of (3.17)-(3.21), by (3.19) and $j \geqslant -1$, we derive

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_H + \tilde{\mathcal{D}}_H &\lesssim \|\dot{\Delta}_j(\nabla b\cdot\nabla b, \nabla b\cdot\nabla u, \nabla u\cdot\nabla u, \nabla u\cdot\nabla b, \nabla u\cdot\nabla\theta)\|_{L^2}\\ &\quad + \|([\dot{\Delta}_j, b\cdot\nabla](\nabla b,\nabla u), [\dot{\Delta}_j, u\cdot\nabla](\nabla b,\nabla u,\nabla\theta))\|_{L^2} + \|(\dot{\Delta}_j\mathcal{N}_1, \dot{\Delta}_j\mathcal{N}_2)\|_{L^2},\end{aligned}$$

which, by summing over $j \geqslant -1$, yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j\geqslant -1}\tilde{\mathcal{E}}_H + \|(u,\theta)\|^H_{\dot{B}^1_{2,1}} + \|b\|^H_{\dot{B}^2_{2,1}}\\ &\lesssim \|(\nabla b\cdot\nabla b, \nabla b\cdot\nabla u, \nabla u\cdot\nabla u, \nabla u\cdot\nabla b, \nabla u\cdot\nabla\theta)\|^H_{\dot{B}^0_{2,1}}\\ &\quad + \|(\mathcal{N}_1,\mathcal{N}_2)\|^H_{\dot{B}^0_{2,1}} + \sum_{j\geqslant -1}\|([\dot{\Delta}_j, b\cdot\nabla](\nabla b,\nabla u), [\dot{\Delta}_j, u\cdot\nabla](\nabla b,\nabla u,\nabla\theta))\|_{L^2}.\end{aligned} \tag{4.7}$$

Similar to (3.23)-(3.25), the nonlinear terms in (4.7) can be estimated as follows. For the first term, by (2.8) and (2.4), we infer that

$$\begin{aligned}&\|(\nabla b\cdot\nabla b, \nabla b\cdot\nabla u, \nabla u\cdot\nabla u, \nabla u\cdot\nabla b, \nabla u\cdot\nabla\theta)\|^H_{\dot{B}^0_{2,1}}\\ &\lesssim \|(\nabla b,\nabla u)\|^H_{\dot{B}^0_{2,1}}\|(\nabla b,\nabla u,\nabla\theta)\|^H_{\dot{B}^1_{2,1}}\\ &\lesssim \|(b,u)\|^H_{\dot{B}^1_{2,1}}\|(b,u,\theta)\|^H_{\dot{B}^2_{2,1}}\\ &\lesssim \epsilon_0(\|u\|^H_{\dot{B}^1_{2,1}} + \|b\|^H_{\dot{B}^2_{2,1}}).\end{aligned}$$

For the term concerning commutator, by using (2.10) and (2.4), we obtain

$$\begin{aligned}&\sum_{j\geqslant -1}\|([\dot\Delta_j, b\cdot\nabla](\nabla b,\nabla u),[\dot\Delta_j,u\cdot\nabla](\nabla b,\nabla u,\nabla\theta))\|_{L^2}\\&\lesssim \|(\nabla b,\nabla u,\nabla\theta)\|^H_{\dot B^0_{2,1}}\|(b,u)\|^H_{\dot B^2_{2,1}}\\&\lesssim \|(b,u,\theta)\|^H_{\dot B^1_{2,1}}\|(b,u)\|^H_{\dot B^2_{2,1}}\\&\lesssim \epsilon_0(\|(u,\theta)\|^H_{\dot B^1_{2,1}}+\|b\|^H_{\dot B^2_{2,1}}).\end{aligned}$$

For the last term, it follows from (2.8), (2.1) and (2.6) that

$$\|(\mathcal N_1,\mathcal N_2)\|^H_{\dot B^0_{2,1}}\lesssim \|(b,u)\|^H_{\dot B^1_{2,1}}\|(\nabla b,\nabla u)\|^H_{\dot B^0_{2,1}}\lesssim \|(b,u)\|^H_{\dot B^1_{2,1}}\|(b,u)\|^H_{\dot B^1_{2,1}}\lesssim \epsilon_0(\|u\|^H_{\dot B^1_{2,1}}+\|b\|^H_{\dot B^2_{2,1}}).$$

For suitable small ϵ_0, putting the above three results into (4.7), it gives

$$\frac{\mathrm d}{\mathrm dt}\sum_{j\geqslant -1}\tilde{\mathcal E}_H+\|(u,\theta)\|^H_{\dot B^1_{2,1}}+\|b\|^H_{\dot B^2_{2,1}}\lesssim 0. \tag{4.8}$$

Combining (4.6) and (4.8), we obtain

$$\frac{\mathrm d}{\mathrm dt}\tilde{\mathcal E}_s+\|(u,b,\theta)\|^L_{\dot B^2_{2,1}}+\|(u,\theta)\|^H_{\dot B^1_{2,1}}+\|b\|^H_{\dot B^2_{2,1}}\lesssim 0, \tag{4.9}$$

where $\tilde{\mathcal E}_s := \big(\sum_{j\leqslant 0}\tilde{\mathcal E}_L+\sum_{j\geqslant -1}\tilde{\mathcal E}_H\big)$.

It is easy to check that

$$\sum_{j\leqslant 0}\tilde{\mathcal E}_L\approx \|(u,b,\theta)\|^L_{\dot B^0_{2,1}},\quad \sum_{j\geqslant -1}\tilde{\mathcal E}_H\approx \|(u,b,\theta)\|^H_{\dot B^1_{2,1}}. \tag{4.10}$$

By the interpolation inequality (2.3) and (4.1), one finds that

$$\|(u,b,\theta)\|^L_{\dot B^0_{2,1}}\lesssim \Big(\|(u,b,\theta)\|^L_{\dot B^{-\sigma}_{2,\infty}}\Big)^{\frac{2}{2+\sigma}}\Big(\|(u,b,\theta)\|^L_{\dot B^2_{2,1}}\Big)^{\frac{\sigma}{2+\sigma}}$$

which gives

$$\Big(\|(u,b,\theta)\|^L_{\dot B^0_{2,1}}\Big)^{1+\frac{2}{\sigma}}\lesssim \|(u,b,\theta)\|^L_{\dot B^2_{2,1}}, \tag{4.11}$$

By the same method, we also have

$$\begin{aligned}\|(u,b,\theta)\|^H_{\dot B^1_{2,1}}&\lesssim \Big(\|(u,b,\theta)\|^H_{\dot B^{\frac{\sigma-2}{\sigma}}_{2,\infty}}\Big)^{\frac{\sigma}{2+\sigma}}\Big(\|(u,b,\theta)\|^H_{\dot B^2_{2,\infty}}\Big)^{\frac{2}{2+\sigma}}\\&\lesssim \Big(\|(u,b,\theta)\|^H_{\dot B^1_{2,1}}\Big)^{\frac{\sigma}{2+\sigma}}\Big(\|(u,b,\theta)\|^H_{\dot B^2_{2,1}}\Big)^{\frac{2}{2+\sigma}}\\&\lesssim \Big(\|(u,b,\theta)\|^H_{\dot B^1_{2,1}}\Big)^{\frac{\sigma}{2+\sigma}},\end{aligned}$$

which implies that

$$\Big(\|(u,b,\theta)\|^H_{\dot B^1_{2,1}}\Big)^{1+\frac{2}{\sigma}}\lesssim \|(u,b,\theta)\|^H_{\dot B^1_{2,1}}. \tag{4.12}$$

Thus, (4.9)-(4.12), we conclude that the Lyapunov-type inequality holds

$$\frac{\mathrm d}{\mathrm dt}\tilde{\mathcal E}_s(t)+\Big(\tilde{\mathcal E}_s(t)\Big)^{1+\frac{2}{\sigma}}\leqslant 0.$$

Furthermore, solving this differential inequality and using (4.10), we deduce that

$$\|(u,b,\theta)\|^L_{\dot B^0_{2,1}}+\|(u,b,\theta)\|^H_{\dot B^1_{2,1}}\lesssim (1+t)^{-\sigma/2}. \tag{4.13}$$

By using (2.6), (4.13) and (2.1), we obtain (1.8) for $s=0$. Next, for $-\sigma<s<0$, making use of (2.3), (2.6), (4.13) and (1.7), we derive the inequality

$$\begin{aligned}&\|(u,b,\theta)\|_{\dot B^s_{2,1}}\\&\lesssim \Big(\|(u,b,\theta)\|_{\dot B^{-\sigma}_{2,\infty}}\Big)^{-s/\sigma}\Big(\|(u,b,\theta)\|_{\dot B^0_{2,1}}\Big)^{1+\frac{s}{\sigma}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \Big(\|(u,b,\theta)\|^L_{\dot B^{-\sigma}_{2,\infty}} + \|(u,\theta)\|^H_{\dot B^2_{2,1}} + \|b\|^H_{\dot B^3_{2,1}}\Big)^{-s/\sigma} \Big(\|(u,b,\theta)\|^L_{\dot B^0_{2,1}} + \|(u,b,\theta)\|^H_{\dot B^1_{2,1}}\Big)^{1+\frac{s}{\sigma}} \\
&\lesssim \Big(\|(u,b,\theta)\|^L_{\dot B^{-\sigma}_{2,\infty}} + \|(u,\theta)\|^H_{\dot B^2_{2,1}} + \|b\|^H_{\dot B^3_{2,1}}\Big)^{-s/\sigma} (1+t)^{-\frac{\sigma+s}{2}} \\
&\lesssim (1+t)^{-\frac{\sigma+s}{2}}.
\end{aligned}
$$

The above inequality, combined with the embedding inequality $\dot B^0_{2,1}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$ and (2.4), leads to (1.8) for $-\sigma < s < 0$. This completes the proof. □

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