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NEW COST TERMS THROUGH HOMOGENIZATION OF AN OPTIMAL CONTROL PROBLEM UNDER DYNAMIC BOUNDARY CONDITIONS ON MICROSCOPIC PARTICLES

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Abstract. This article concerns optimal control problems in a heterogeneous body with a periodic structure of particles depending on a small parameter ε. We study the asymptotic behavior, as $\varepsilon \to 0$, of the optimal control functional and of the optimal state when the initial problem is of parabolic type. We assume a dynamic condition and the effect of some controls for some of the particles on the boundary. In the so-called "critical case", we show the appearance of some new non-local in time "strange terms", in the limit parabolic equation and in the limit cost functional. Microscopic localized controls generate peculiar terms in both the limit equation and the cost functional that do not appear when controls are applied to the entire set of particles, or when the boundary condition on the particles is of Robin type.

1. Introduction

The problem we consider here arises in many fields. For instance, it is well-known that many problems in chemical engineering lead to the optimization of some cost functionals [22, 29]. The same thing happens in porous media theory in which the word "particle" must be replaced by "perforation" (see, e.g. [1, 4, 16, 17, 18, 19, 23, 27] and many other references quoted in the monograph [6]). Simplified models in Climatology can be also modeled in terms very close to the ones we will study in this paper (see [11]).

The main goal of this article is to illustrate how the homogenization of some optimal control problems may give rise to new non-local in time "strange terms". This also happens in the limit parabolic equation and in the limit cost functional, assuming a dynamic boundary condition and the actuation of some controls on some subset of the particles. We will do that for the so-called "critical case", that is characterized by the relation between the structure's period, the diameter of the balls, and the growth coefficient in the particles' boundary condition. In this way, microscopic localized controls generate peculiar terms in both the limit equation and the cost function that do not appear, for instance, in the case of the Robin type boundary condition on the particles.

We give a detailed presentation of the heterogeneous domain Ω_ε in the next section. At the moment, we outline that since in the most of the cases it is impossible to act over the entire spatial domain Ω_ε, the control is applied only on the boundary of the particles contained in a small portion of the domain (ω such that $\overline{\omega} \subset \Omega$). Thus, the set of boundaries of the internal particles is constituted in the form $S_\varepsilon = S_\varepsilon^1 \cup S_\varepsilon^2$, where S_ε^2 is the set of boundaries of the controlling particles G_ε^2 and S_ε^1 is the set of boundaries of the particles G_ε^1 to which no control is implemented.

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The state of the control problem is given through

$$\begin{gathered}
\partial_t u_\varepsilon(v) - \Delta u_\varepsilon(v) = f, \quad (x,t) \in Q_\varepsilon^T, \\
\varepsilon^{-\gamma}\partial_t u_\varepsilon(v) + \partial_\nu u_\varepsilon(v) = \varepsilon^{-\gamma}\chi_{S_\varepsilon^{2,T}} v, \quad (x,t) \in S_\varepsilon^T, \\
u_\varepsilon(v)(x,0) = 0, \quad x \in \Omega_\varepsilon \cup S_\varepsilon, \\
u_\varepsilon(v)(x,t) = 0, \quad (x,t) \in \Gamma^T,
\end{gathered} \tag{1.1}$$

where $f \in L^2(Q^T)$ and $v \in L^2(S_\varepsilon^{2,T})$ is the control. Here, we are using the notation (considering $0 < T < \infty$)

$$\begin{gathered}
\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega, \quad Q_\varepsilon^T = \Omega_\varepsilon \times (0,T), \\
\Gamma^T = \partial\Omega \times (0,T), \quad S_\varepsilon^T = S_\varepsilon \times (0,T), \quad Q^T = \Omega \times (0,T),
\end{gathered} \tag{1.2}$$

which will be described in detail in the next section. We note that G_ε is the set of small particles (ε-periodically distributed and homothetic to a unit ball G_0) in an open bounded regular set Ω of $\mathbb{R}^n$, $n \geq 3$. By $\chi_{S_\varepsilon^{2,T}}$, we denote the characteristic function of the set $S_\varepsilon^{2,T} = S_\varepsilon^2 \times (0,T)$ that lies entirely in the set ω_ε^T defined below

$$\omega_\varepsilon = \omega \cap \Omega_\varepsilon, \quad \omega_\varepsilon^T = \omega_\varepsilon \times (0,T), \quad \omega^T = \omega \times (0,T).$$

The parameter $\gamma > 0$ plays a crucial role since in this paper we consider the so-called "critical case" governed by the size of particles that are translations of a small particle $a_\varepsilon G_0$, where G_0 is the unit ball with radius $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$, and C_0 is some positive constant.

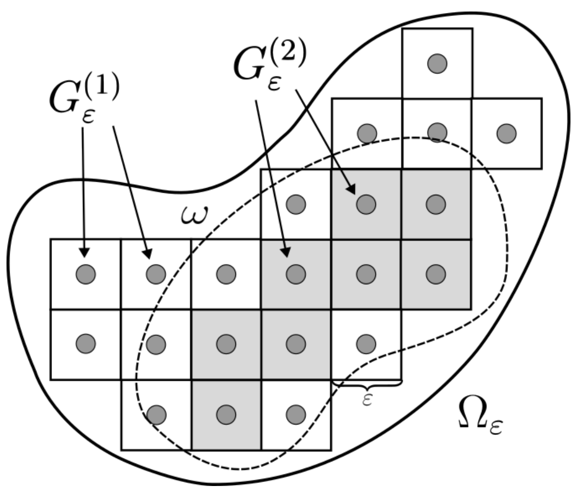

Figure 1. Perforated domain. The control is implemented only on S_ε^2, the boundary of some of the internal balls: the ones collected under the G_ε^2.

Notice that since problem (1.1) is linear, by some obvious change of variable, we can also consider the case of a non-zero initial datum. To finalize the statement of the optimal control problem, we introduce the cost functional $J_\varepsilon : L^2(0,T; L^2(S_\varepsilon^2)) \to \mathbb{R}$,

$$\begin{aligned}
&J_\varepsilon(v) \\
&= \frac{1}{2}\|\nabla u_\varepsilon(v)\|^2_{L^2(Q_\varepsilon^T)} + \frac{1}{2}\int_{\Omega_\varepsilon} u_\varepsilon^2(v)(x,T)dx + \frac{\varepsilon^{-\gamma}}{2}\int_{S_\varepsilon} u_\varepsilon^2(v)(x,T)ds + \varepsilon^{-\gamma}\frac{N}{2}\|v\|^2_{L^2(S_\varepsilon^{2,T})},
\end{aligned} \tag{1.3}$$

where $N > 0$. Then, the optimal control v_ε is

$$J_\varepsilon(v_\varepsilon) = \inf_{v \in L^2(0,T;L^2(S_\varepsilon^2))} J_\varepsilon(v). \tag{1.4}$$

In what follows, we will abuse the notation and simply write u_ε instead of $u_\varepsilon(v_\varepsilon)$. By applying different results in the literature (see, e.g., [21, 14, 28, 15]), it is well-known that there exists a unique optimal control $v_\varepsilon \in L^2(0,T; L^2(S_\varepsilon^2))$.

We point out that the consideration of a non-zero target function $u_T \in L^2(\Omega_\varepsilon)$, a given profile observed at the final time T, can be reduced to the above case of $u_T \equiv 0$ by a suitable change of variables, at least for a dense set of u_T in $L^2(\Omega)$ (see Remark 2.1 below).

The main goal of this article is to apply a homogenization process to the above optimal control problem when $\varepsilon \to 0$. As in many other formulations, the kind of the limit problem strongly depends on the size of the particles' radii $C_0\varepsilon^\alpha$, $C_0 > 0$ (see, e.g. [1, 27] and the exposition made in [6]). Here, we consider the critical case in which $\alpha = \gamma = n/(n-2)$. For different elliptic and parabolic problems, it is well-known that this critical choice leads to the emergence of a new local "strange" term (the naming is due to [3]) in the effective partial differential equation (see [3, 6, 18, 20, 30]).

It is well-known that the introduction of a dynamic boundary condition on the particle boundary causes the aforementioned "strange" term to become a " non-local" operator, derived by solving a suitable ordinary differential equation (we refer to [5, 13], for the case of an elliptic Poisson equation for the state). Also, it was shown that in the framework of optimal control problems there appear some new terms in the limit cost functional (in contrast with previous results in the literature for related formulations, (see, e.g., [7, 8, 9, 12, 24, 25, 31, 26] for the case of distributed controls appearing in the state equation). One of the major new features we will demonstrate in this article is that when the controls act on the boundary of some particles, some new terms appear in the cost functional, the non-local terms in time are of a different nature and some new non-local in time operators must be introduced.

To state the homogenization results, we need to introduce several auxiliary problems. On the uncontrolled particles, we use non-local operator $M(\varphi)$, arising in previous studies (see [12]), that is defined as a solution to

$$\begin{gathered} \partial_t M(\varphi) + \mathcal{B}_n M(\varphi) = \varphi, \quad t \in (0,T), \\ M(\varphi)(0) = 0, \end{gathered} \tag{1.5}$$

where $\mathcal{B}_n = (n-2)C_0^{-1}$ and $\varphi \in L^2(0,T)$ is a given function, and its adjoint operator M^*,

$$\begin{gathered} -\partial_t M^*(\varphi) + \mathcal{B}_n M^*(\varphi) = \varphi, \quad t \in (0,T), \\ M^*(\varphi)(T) = 0. \end{gathered} \tag{1.6}$$

A similar non-local operator, $G(\varphi)$, and its adjoint operator G^*, must be defined on the controlled particles

$$\begin{gathered} \partial_t G(\varphi) + (\mathcal{B}_n + N^{-1})G(\varphi) = \varphi, \quad t \in (0,T), \\ G(\varphi)(0) = 0, \end{gathered} \tag{1.7}$$

and

$$\begin{gathered} -\partial_t G^*(\varphi) + (\mathcal{B}_n + N^{-1})G^*(\varphi) = \varphi, \quad t \in (0,T), \\ G^*(\varphi)(T) = 0. \end{gathered} \tag{1.8}$$

Besides that, we will need to define some new operators H and H^*, coupled with G^* and G, respectively, by the problems

$$\begin{gathered} -\partial_t H^*(\varphi) + (\mathcal{B}_n + N^{-1})H^*(\varphi) - N^{-1}(\mathcal{B}_n + N^{-1})G(H^*(\varphi)) = \varphi, \quad t \in (0,T), \\ H^*(\varphi)(T) = 0, \end{gathered} \tag{1.9}$$

and

$$\begin{gathered} \partial_t H(\varphi) + (\mathcal{B}_n + N^{-1})H(\varphi) - N^{-1}(\mathcal{B}_n + N^{-1})G^*(H(\varphi)) = \varphi, \quad t \in (0,T), \\ H(\varphi)(0) = 0. \end{gathered} \tag{1.10}$$

Notice that the operators G, M, G^*, and M^* can be explicitly written. For instance

$$G(\varphi)(t) = \int_0^t e^{-(\mathcal{B}_n + N^{-1})(t-s)}\varphi(s)\,ds,$$

which show the non-local in time nature. Some useful properties of these operators will be shown later (see Section 4).

Although the detailed statements of our results will be presented later, we summarize now that we will prove the convergence of the optimal controls $v_\varepsilon \chi_{S^{2,T}_\varepsilon} \to v_0 \chi_{\omega^T}$ strongly in $L^2(\omega^T)$, the convergence of the corresponding states (extended to Ω) $\tilde{u}_\varepsilon \rightharpoonup u_0$ weakly in $L^2(0,T;H_0^1(\Omega,\partial\Omega))$ and $\partial_t\tilde{u}_\varepsilon \rightharpoonup \partial_t u_0$ weakly in $L^2(Q^T)$, and in some sense, that will be indicated later, the microscopic optimal control v_ε converges to the macroscopic optimal control $v_0 \in H^1(0,T;L^2(\omega))$, with the limit state problem given by

$$\begin{aligned}
&\partial_t u_0(v) - \Delta u_0(v) + \mathcal{A}_n(u_0(v) - \mathcal{B}_n H(u_0(v)))\chi_{\omega^T}\\
&+ \mathcal{A}_n(u_0(v) - \mathcal{B}_n M(u_0(v)))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)}\\
&= f + \mathcal{A}_n\mathcal{B}_n v\chi_{\omega^T}, \quad (x,t)\in Q^T,\\
&\qquad u_0(v)(x,0) = 0, \quad x\in\Omega,\\
&\qquad u_0(v)(x,t) = 0, \quad (x,t)\in\Gamma^T,
\end{aligned} \tag{1.11}$$

where $\mathcal{A}_n = (n-2)C_0^{n-2}\omega_n$, $v\in H^1(0,T;L^2(\omega))$ with $v(x,0)=0$ and the limit cost functional

$$\begin{aligned}
&J_0(v)\\
&= \frac{1}{2}\|\nabla u_0(v)\|^2_{L^2(Q^T)} + \frac{1}{2}\|u_0(v)(x,T)\|^2_{L^2(\Omega)} + \frac{\mathcal{A}_n\mathcal{B}_n}{2N}\int_{\omega^T}(\partial_t G^*(H(u_0(v))))^2\,dx\,dt\\
&+ \frac{\mathcal{A}_n\mathcal{B}_n}{2}\int_{\Omega\setminus\overline{\omega}}|M(u_0(v))(x,T)|^2dx + \frac{\mathcal{A}_n}{2}\int_{(\Omega\setminus\overline{\omega})\times(0,T)}|u_0(v) - \mathcal{B}_n M(u_0(v))|^2\,dx\,dt\\
&+ \frac{\mathcal{A}_n\mathcal{B}_n}{2}\int_{\omega}|H(u_0(v))(x,T)|^2dx + \frac{\mathcal{A}_n}{2}\int_{\omega^T}|u_0(v) - \mathcal{B}_n H(u_0(v))|^2\,dx\,dt\\
&+ \frac{N\mathcal{A}_n\mathcal{B}_n}{2}\int_{\omega^T}(\partial_t v)^2\,dx\,dt + \frac{N\mathcal{A}_n\mathcal{B}_n(\mathcal{B}_n+N^{-1})}{2}\int_{\omega}v^2(x,T)dx +\\
&+ \frac{N\mathcal{A}_n\mathcal{B}_n^2(\mathcal{B}_n+N^{-1})}{2}\int_{\omega^T}v^2\,dx\,dt.
\end{aligned} \tag{1.12}$$

of the optimal control problem

$$J_0(v_0) = \min_{v\in U_{\rm ad}} J_0(v), \tag{1.13}$$

where the set of admissible functions is now

$$U_{\rm ad} = \{\psi\in H^1(0,T;L^2(\omega)) | \psi(x,0)=0\}.$$

It can be seen that the first two terms and the last term of J_0 clearly correspond to the three terms present in J_ε, but the rest of the terms of J_0 are, in some way, unexpected. The terms of J_0 which are related to the final evaluation at time T are new, and two of them are actually non-local in time since they involve the operators M and H, respectively. The terms of J_0 which contain the operator M are integrals extended on the complementary of ω, and they are a consequence of the microscopic control v_ε being applied only at the boundary of some particles, S_ε^2, and not at all of them. The unexpected terms of J_0 appear as a consequence of several implicit relations that are justified in the proof of the Theorem 5.2 below. The last set of the terms that affect time derivatives of a function of u_0 and the control v are very surprising since nothing suggests their appearance when observing the expression for J_ε.

To prove of these convergence results, we will use the extension of the Pontryagin's method to the case of boundary controls (see, e.g., [21]). In Section 2, we give the details of the formulation of the direct problem and of the coupled system arising in terms of the adjoint optimal state p_ε: we will show that the optimal control is given by $v_\varepsilon = -N^{-1}p_\varepsilon\chi_{S_\varepsilon^2,T}$. The a priori estimates allow passing to the limit in the couple $(u_\varepsilon,p_\varepsilon)$ (and thus in the controls v_ε) are obtained in Section 3. Some detailed statements of the main theorems of this paper are collected in Section 5, but before that, we present in Section 4 some properties of the auxiliary non-local in time operators G, H and M defined above. The proof characterizing the limit couple (u_0,p_0) from the microscopic couple $(u_\varepsilon,p_\varepsilon)$ is given in Section 6. Finally, the identification of the limit cost functional $J_0(v)$ from the microscopic cost functional $J_\varepsilon(v)$ is obtained in Section 7.

2. Problem statement and the adjoint problem

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with a smooth boundary $\partial\Omega$. We denote the unit cube $(-1/2, 1/2)^n$ centered at the coordinates origin as Y. Let G_0 be a ball of radius C_0 such that $\overline{G_0} \subset Y$. Next, given a set B of $\mathbb{R}^n$, by δB, $\delta > 0$, we denote the set $\{x \in \mathbb{R}^n | \delta^{-1} x \in B\}$. For $\varepsilon > 0$, we define $\widetilde{\Omega}_\varepsilon = \{x \in \Omega | \rho(x, \partial\Omega) > 2\varepsilon\}$, where ρ is the Euclidean distance. Let $a_\varepsilon = C_0 \varepsilon^\alpha$, where C_0 is a positive constant and $\alpha = \frac{n}{n-2}$. We define sets $G_\varepsilon^j = a_\varepsilon G_0 + j$, where $j \in \mathbb{Z}^n$, $\mathbb{Z}^n$ is the set of vectors in $\mathbb{R}^n$ with integer coordinates. Now, we introduce the set of indices $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j) \cap \widetilde{\Omega_\varepsilon} \neq \emptyset\}$, note that the cardinal of Υ_ε satisfies that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$ for some $d = const > 0$. Finally, we define the set

$$G_\varepsilon = \cup_{j \in \Upsilon_\varepsilon} G_\varepsilon^j.$$

Now, if we define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, $P_\varepsilon^j = \varepsilon j$, where $Y = (-1/2, 1/2)^n$, then it is easy to see that $\overline{G_\varepsilon^j} \subset Y_\varepsilon^j$ and the center of the ball $G_\varepsilon^j = a_\varepsilon G_0 + \varepsilon j$ coincides with the center of the cube Y_ε^j.

In the formulation of the optimal control problem, we will consider only some controllable region ω, $\overline{\omega} \subset \Omega$, in the whole domain Ω. Thus, we split indices of Υ_ε into two subsets $\Upsilon_\varepsilon^2 = \{j \in \Upsilon_\varepsilon : \overline{Y_\varepsilon^j} \subset \omega\}$ and $\Upsilon_\varepsilon^1 = \Upsilon_\varepsilon \setminus \Upsilon_\varepsilon^2$. Based on these sets, we will use the following notations

$$G_\varepsilon^1 = \cup_{j \in \Upsilon_\varepsilon^1} G_\varepsilon^j, \quad G_\varepsilon^2 = \cup_{j \in \Upsilon_\varepsilon^2} G_\varepsilon^j, \quad S_\varepsilon^1 = \partial G_\varepsilon^1, \quad S_\varepsilon^2 = \partial G_\varepsilon^2.$$

Further, we introduce the sets

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon, \quad S_\varepsilon = S_\varepsilon^1 \cup S_\varepsilon^2,$$

and, for $0 < T < \infty$, we define

$$Q_\varepsilon^T = \Omega_\varepsilon \times (0, T), \quad \omega^T = \omega \times (0, T),$$
$$S_\varepsilon^T = S_\varepsilon \times (0, T), \quad S_\varepsilon^{i,T} = S_\varepsilon^i \times (0, T), \quad i = 1, 2.$$

Now, we are in a position to formulate optimal control problem. Let $v \in L^2(0, T; S_\varepsilon^2)$. By $u_\varepsilon(v)$, we denote an element of $L^2(0, T; H^1(\Omega_\varepsilon, \partial\Omega))$ with the time derivative satisfying $\partial_t u_\varepsilon(v) \in L^2(0, T; L^2(\Omega_\varepsilon)) \cap L^2(0, T; L^2(S_\varepsilon))$ and $u_\varepsilon(x, 0) = 0$ for $x \in \Omega_\varepsilon \cup S_\varepsilon$, that is a solution to the parabolic problem with the internal dynamic boundary condition. By $H^1(\Omega_\varepsilon, \partial\Omega)$, we denote the closure with respect to the norm $H^1(\Omega_\varepsilon)$ of the set of infinitely differentiable in $\overline{\Omega}_\varepsilon$ functions vanishing near the boundary $\partial\Omega$. As a solution of (1.1), we will consider a function $u_\varepsilon(v)$ with the above-mentioned properties that satisfies the integral identity

$$\begin{aligned} &\int_{Q_\varepsilon^T} \partial_t u_\varepsilon \varphi \, dx \, dt + \int_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla \varphi \, dx \, dt + \varepsilon^{-\gamma} \int_{S_\varepsilon^T} \partial_t u_\varepsilon \varphi \, ds \, dt \\ &= \int_{Q_\varepsilon^T} f \varphi \, dx \, dt + \varepsilon^{-\gamma} \int_{S_\varepsilon^{2,T}} v \varphi \, ds \, dt \end{aligned} \tag{2.1}$$

for an arbitrary function $\varphi \in L^2(0, T; H^1(\Omega_\varepsilon, \partial\Omega))$. We consider now the optimal control problem stated in the Introduction (see (1.4)).

Remark 2.1. Our approach can be easily extended to the case of a non-zero target $u_T \in L^2(\Omega)$, at least for a dense set of u_T in $L^2(\Omega)$, i.e. the cost functional will be

$$\begin{aligned} J_\varepsilon(v) = &\frac{1}{2} \|\nabla u_\varepsilon(v)\|_{L^2(Q_\varepsilon^T)}^2 + \frac{1}{2} \int_{\Omega_\varepsilon} (u_\varepsilon(v)(x, T) - u_T)^2 \, dx \\ &+ \frac{\varepsilon^{-\gamma}}{2} \int_{S_\varepsilon} u_\varepsilon^2(v)(x, T) ds + \varepsilon^{-\gamma} \frac{N}{2} \|v\|_{L^2(S_\varepsilon^{2,T})}^2. \end{aligned} \tag{2.2}$$

Indeed, let us assume that $u_T \in L^2(\Omega)$ is such that there exists a converging as $\varepsilon \to 0$ sequence of functions $V_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon))$, i.e.

$$V_\varepsilon \rightharpoonup V_0 \quad \text{weakly in } L^2(Q^T), \text{ for some } V_0 \in L^2(Q^T), \tag{2.3}$$

such that for the unique solution U_ε of the auxiliary problem

$$\begin{gathered}\partial_t U_\varepsilon - \Delta U_\varepsilon = V_\varepsilon, \quad (x,t) \in Q_\varepsilon^T,\\ \varepsilon^{-\gamma}\partial_t U_\varepsilon + \partial_\nu U_\varepsilon = 0, \quad (x,t) \in S_\varepsilon^T,\\ U_\varepsilon(x,0) = 0, \quad x \in \Omega_\varepsilon \cup S_\varepsilon,\\ U_\varepsilon(x,t) = 0, \quad (x,t) \in \Gamma^T,\end{gathered} \tag{2.4}$$

we have

$$\begin{gathered}\widetilde{U}_\varepsilon \rightharpoonup U_0 \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)),\\ \partial_t \widetilde{U}_\varepsilon \rightharpoonup \partial_t U_0 \quad \text{weakly in } L^2(Q^T),\end{gathered}$$

and

$$U_0(x,T) = u_T(x) \quad \text{a.e. } x \in \Omega.$$

Then by defining the change of variables

$$w_\varepsilon(v) = u_\varepsilon(v) - U_\varepsilon,$$

where now $u_\varepsilon(v)$ is the optimal control associated to the cost functional (2.2), we find that $w_\varepsilon(v)$ is the optimal control associate to the previous cost functional (1.3) with $u_T \equiv 0$. Finally, by the arguments of Remark 7.1 of [12] (or Theorem 4 of [11]), it is easy to prove that the set of final data $u_T \in L^2(\Omega)$ satisfying the above mentioned conditions is a dense set of $L^2(\Omega)$. Then, the perturbed equation satisfied by $w_\varepsilon(v)$, i.e.

$$\partial_t w_\varepsilon(v) - \Delta w_\varepsilon(v) = f - V_\varepsilon,$$

does not add any difficulty, once we know that (2.3) holds.

To obtain a characterization of the optimal control, we consider the adjoint problem

$$\begin{gathered}-\partial_t p_\varepsilon - \Delta p_\varepsilon = -\Delta u_\varepsilon, \quad (x,t) \in Q_\varepsilon^T,\\ \partial_\nu p_\varepsilon - \varepsilon^{-\gamma}\partial_t p_\varepsilon = \partial_\nu u_\varepsilon, \quad (x,t) \in S_\varepsilon^T,\\ p_\varepsilon(x,T) = u_\varepsilon(x,T), \quad x \in \Omega_\varepsilon \cup S_\varepsilon,\\ p_\varepsilon(x,t) = 0, \quad (x,t) \in \Gamma^T.\end{gathered} \tag{2.5}$$

We say that a function $p_\varepsilon \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$, with $\partial_t p_\varepsilon \in L^2(0,T;L^2(\Omega_\varepsilon)) \cap L^2(0,T;L^2(S_\varepsilon))$, is a weak solution to (2.5) if $p_\varepsilon(x,T) = u_\varepsilon(x,T)$ for a.e. $x \in \Omega_\varepsilon$ and a.e. $x \in S_\varepsilon$ and if it satisfies the integral identity

$$-\int_{Q_\varepsilon^T} \partial_t p_\varepsilon \varphi \,dx\,dt + \int_{Q_\varepsilon^T} \nabla p_\varepsilon \nabla\varphi \,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^T} \partial_t p_\varepsilon \varphi \,ds\,dt = \int_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla \varphi \,dx\,dt, \tag{2.6}$$

for any test function $\varphi \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$. For a given u_ε (with the regularity of the weak solutions of (1.1)) it is well-known that there exists a unique solution to the problem (2.5) (see, e.g., [2] and its references).

The following theorem gives a characterization of the optimal control v_ε in terms of the adjoint state p_ε.

Theorem 2.2. *Let the pair of functions $(u_\varepsilon(v_\varepsilon), v_\varepsilon)$ be an optimal solution of the problem* (1.4)*, then $v_\varepsilon = -N^{-1} p_\varepsilon \chi_{S_\varepsilon^{2,T}}$, where p_ε is the solution to* (2.5)*. The converse is also true.*

Proof. Let v be an arbitrary function in $L^2(S_\varepsilon^{2,T})$ and $\lambda > 0$. By u_ε^λ, we denote the solution of (1.4) with the control $v_\varepsilon + \lambda v$, i.e. $u_\varepsilon^\lambda = u_\varepsilon(v_\varepsilon + \lambda v)$. We use $u_\varepsilon = u_\varepsilon(v_\varepsilon)$ to simplify the notation. Then we have

$$\begin{aligned}&J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon)\\ &= \frac{1}{2}\|\nabla u_\varepsilon^\lambda\|^2_{L^2(Q_\varepsilon^T)} + \frac{1}{2}\|u_\varepsilon^\lambda(x,T)\|^2_{L^2(\Omega_\varepsilon)} + \frac{\varepsilon^{-\gamma}}{2}\|u_\varepsilon^\lambda(x,T)\|^2_{L^2(S_\varepsilon)}\\ &\quad + \varepsilon^{-\gamma}\frac{N}{2}\|v_\varepsilon + \lambda v\|^2_{L^2(S_\varepsilon^{2,T})} - \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} - \frac{1}{2}\|u_\varepsilon(x,T)\|^2_{L^2(\Omega_\varepsilon)}\end{aligned}$$

$$
\begin{aligned}
&- \frac{\varepsilon^{-\gamma}}{2}\|u_\varepsilon(x,T)\|^2_{L^2(S_\varepsilon)} - \varepsilon^{-\gamma}\frac{N}{2}\|v_\varepsilon\|^2_{L^2(S_\varepsilon^{2,T})}\\
&= \frac{1}{2}\int_{Q_\varepsilon^T} \nabla(u_\varepsilon^\lambda - u_\varepsilon)\nabla(u_\varepsilon^\lambda + u_\varepsilon)\,dx\,dt + \frac{1}{2}\int_{\Omega_\varepsilon}(u_\varepsilon^\lambda - u_\varepsilon)(x,T)(u_\varepsilon^\lambda + u_\varepsilon)(x,T)dx\\
&\quad + \frac{\varepsilon^{-\gamma}}{2}\int_{S_\varepsilon}(u_\varepsilon^\lambda - u_\varepsilon)(x,T)(u_\varepsilon^\lambda + u_\varepsilon)(x,T)ds + \varepsilon^{-\gamma}\frac{N}{2}\int_{S_\varepsilon^{2,T}}(2\lambda v_\varepsilon v + \lambda^2 v^2)\,ds\,dt.
\end{aligned}
$$

We define the function $\theta_\varepsilon = (u_\varepsilon^\lambda - u_\varepsilon)/\lambda$. It is easy to see that θ_ε is the unique solution to the problem

$$
\begin{gathered}
\partial_t\theta_\varepsilon - \Delta\theta_\varepsilon = 0, \quad (x,t)\in Q_\varepsilon^T,\\
\varepsilon^{-\gamma}\partial_t\theta_\varepsilon + \partial_\nu\theta_\varepsilon = \chi_{S_\varepsilon^{2,T}}\varepsilon^{-\gamma}v, \quad (x,t)\in S_\varepsilon^T,\\
\theta_\varepsilon(x,0) = 0, \quad x\in\Omega_\varepsilon\cup S_\varepsilon,\\
\theta_\varepsilon(x,t) = 0, \quad (x,t)\in\Gamma^T.
\end{gathered}
$$

Using the definition of θ_ε, we have

$$
\begin{aligned}
J'_\varepsilon(v_\varepsilon)v &= \lim_{\lambda\to 0}(J_\varepsilon(v_\varepsilon+\lambda v) - J_\varepsilon(v_\varepsilon))/\lambda\\
&= \int_{Q_\varepsilon^T}\nabla\theta_\varepsilon\nabla u_\varepsilon\,dx\,dt + \int_{\Omega_\varepsilon}\theta_\varepsilon(x,T)u_\varepsilon(x,T)dx\\
&\quad + \varepsilon^{-\gamma}\int_{S_\varepsilon}\theta_\varepsilon(x,T)u_\varepsilon(x,T)ds + \varepsilon^{-\gamma}N\int_{S_\varepsilon^{2,T}} v_\varepsilon v\,ds\,dt.
\end{aligned}
$$

Now, we use the definition of p_ε and derive from the last expression the identity

$$
J'_\varepsilon(v_\varepsilon)v = \varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} p_\varepsilon v\,dx\,dt + \varepsilon^{-\gamma}N\int_{S_\varepsilon^{2,T}} v_\varepsilon v\,ds\,dt.
$$

As v_ε is the optimal control, we should have $J'_\varepsilon(v_\varepsilon)\cdot v = 0$ for all $v\in L^2(0,T;L^2(S_\varepsilon^2))$. Hence, $v_\varepsilon = -N^{-1}p_\varepsilon$ for a.e. $(x,t)\in S_\varepsilon^{2,T}$. This completes the proof. $\square$

In consequence, by Theorem 2.2, the optimal control problem is characterized through the coupled system

$$
\begin{gathered}
\partial_t u_\varepsilon - \Delta u_\varepsilon = f, \quad (x,t)\in Q_\varepsilon^T,\\
-\partial_t p_\varepsilon - \Delta p_\varepsilon = -\Delta u_\varepsilon, \quad (x,t)\in Q_\varepsilon^T,\\
\partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\partial_t u_\varepsilon = -\varepsilon^{-\gamma}N^{-1}\chi_{S_\varepsilon^{2,T}}p_\varepsilon, \quad (x,t)\in S_\varepsilon^T,\\
\partial_\nu p_\varepsilon - \varepsilon^{-\gamma}\partial_t p_\varepsilon = \partial_\nu u_\varepsilon, \quad (x,t)\in S_\varepsilon^T,\\
u_\varepsilon(x,0) = 0, \quad x\in\Omega_\varepsilon\cup S_\varepsilon,\\
p_\varepsilon(x,T) = u_\varepsilon(x,T), \quad x\in\Omega_\varepsilon\cup S_\varepsilon,\\
u_\varepsilon(x,t) = p_\varepsilon(x,t) = 0, \quad (x,t)\in\Gamma^T.
\end{gathered}
\tag{2.7}
$$

3. A priori estimates

In this section, we obtain several a priori estimates of the state and adjoint state. Taking p_ε as a test function in the integral identity for u_ε, we obtain

$$
\begin{aligned}
&\int_{Q_\varepsilon^T}\partial_t u_\varepsilon p_\varepsilon\,dx\,dt + \varepsilon^{-\gamma}\int_{S_\varepsilon^T}\partial_t u_\varepsilon p_\varepsilon\,ds\,dt + \int_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla p_\varepsilon\,dx\,dt\\
&= \int_{Q_\varepsilon^T} f p_\varepsilon\,dx\,dt - N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} p_\varepsilon^2\,dx\,dt.
\end{aligned}
\tag{3.1}
$$

Now, taking u_ε as a test function in the integral identity for p_ε, we obtain

$$
-\int_{Q_\varepsilon^T}\partial_t p_\varepsilon u_\varepsilon\,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^T}\partial_t p_\varepsilon u_\varepsilon\,ds\,dt + \int_{Q_\varepsilon^T}\nabla p_\varepsilon\nabla u_\varepsilon\,dx\,dt = \int_{Q_\varepsilon^T}|\nabla u_\varepsilon|^2\,dx\,dt. \tag{3.2}
$$

Next, we subtract (3.1) from (3.2) and obtain the expression

$$\begin{aligned}&-\int_{Q_\varepsilon^T}\partial_t(u_\varepsilon p_\varepsilon)\,dx\,dt-\varepsilon^{-\gamma}\int_{S_\varepsilon^T}\partial_t(u_\varepsilon p_\varepsilon)\,ds\,dt-N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}p_\varepsilon^2\,dx\,dt\\&=\int_{Q_\varepsilon^T}|\nabla u_\varepsilon|^2\,dx\,dt-\int_{Q_\varepsilon^T}fp_\varepsilon\,dx\,dt.\end{aligned}$$

From this, we obtain

$$\begin{aligned}&\|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}+\|u_\varepsilon(x,T)\|^2_{L^2(\Omega_\varepsilon)}+\varepsilon^{-\gamma}\|u_\varepsilon(x,T)\|^2_{L^2(S_\varepsilon)}+N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}p_\varepsilon^2\,ds\,dt\\&\le\int_{Q_\varepsilon^T}|f||p_\varepsilon|\,dx\,dt.\end{aligned}\tag{3.3}$$

Then, we take p_ε as a test function in the integral identity (2.6), and obtain

$$-\frac{1}{2}\|u_\varepsilon(x,T)\|^2_{L^2(\Omega_\varepsilon)}-\frac{\varepsilon^{-\gamma}}{2}\|u_\varepsilon(x,T)\|^2_{L^2(S_\varepsilon)}+\|\nabla p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}\le\int_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla p_\varepsilon\,dx\,dt.$$

From here and (3.3), we conclude that

$$\begin{aligned}\|\nabla p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}&\le C(\|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}+\|u_\varepsilon(x,T)\|^2_{L^2(\Omega_\varepsilon)}+\varepsilon^{-\gamma}\|u_\varepsilon(x,T)\|^2_{L^2(S_\varepsilon)})\\&\le C\int_{Q_\varepsilon^T}|f||p_\varepsilon|\,dx\,dt.\end{aligned}\tag{3.4}$$

Here and below, constant C is independent from ε. As p_ε is in $H^1(\Omega_\varepsilon,\partial\Omega)$, we can apply Poincaré-Friedrichs's inequality

$$\|p_\varepsilon(\cdot,t)\|_{L^2(\Omega_\varepsilon)}\le K\|\nabla p_\varepsilon(\cdot,t)\|_{L^2(\Omega_\varepsilon)}.$$

Using this inequality in the previous estimate (3.4), we obtain

$$\|p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}\le C\|f\|^2_{L^2(Q^T)}.$$

Now, we substitute this estimate into (3.3), and derive the following estimate of u_ε,

$$\|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}+\|u_\varepsilon(x,T)\|^2_{L^2(\Omega_\varepsilon)}+\varepsilon^{-\gamma}\|u_\varepsilon(x,T)\|^2_{L^2(S_\varepsilon)}+N^{-1}\varepsilon^{-\gamma}\|p_\varepsilon\|^2_{L^2(S_\varepsilon^{2,T})}\le C\|f\|^2_{L^2(Q^T)}.$$

From this, by (3.4), we obtain the estimation of the gradient of p_ε,

$$\|\nabla p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}\le C\|f\|^2_{L^2(Q^T)}.$$

Now we derive some estimates on the time derivatives of u_ε and p_ε. We use Galerkin's approach and construct u_ε^m and p_ε^m, where $m=1,2,\dots$, that are approximations to u_ε and p_ε. Note that, for such approximations, we have the same estimates derived above on u_ε and p_ε. We take now $\partial_t u_\varepsilon^m$ as a test function in the equations for u_ε^m, and integrating from 0 to an arbitrary $\tau\in[0,T]$, we obtain

$$\begin{aligned}&\|\partial_t u_\varepsilon^m\|^2_{L^2(Q_\varepsilon^T)}+\varepsilon^{-\gamma}\|\partial_t u_\varepsilon^m\|^2_{L^2(S_\varepsilon^T)}+\max_{t\in[0,T]}\|\nabla u_\varepsilon^m\|^2_{L^2(\Omega_\varepsilon)}\\&\le K\Big(\int_{Q_\varepsilon^T}|f||\partial_t u_\varepsilon^m|\,dx\,dt+\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}|p_\varepsilon^m||\partial_t u_\varepsilon^m|\,ds\,dt\Big)\\&\le\frac{1}{2}\|\partial_t u_\varepsilon^m\|^2_{L^2(Q_\varepsilon^T)}+\frac{\varepsilon^{-\gamma}}{2}\|\partial_t u_\varepsilon^m\|^2_{L^2(S_\varepsilon^T)}+K(\varepsilon^{-\gamma}\|p_\varepsilon^m\|^2_{L^2(S_\varepsilon^{2,T})}+\|f\|^2_{L^2(Q^T)}),\end{aligned}$$

where constant K is independent of ε and m. From here, we immediately derive

$$\|\partial_t u_\varepsilon^m\|^2_{L^2(Q_\varepsilon^T)}+\varepsilon^{-\gamma}\|\partial_t u_\varepsilon^m\|^2_{L^2(S_\varepsilon^T)}+\max_{t\in[0,T]}\|\nabla u_\varepsilon^m\|^2_{L^2(\Omega_\varepsilon)}\le K\|f\|^2_{L^2(Q^T)}.$$

Then, passing to the limit, as $m\to\infty$, in this estimate we have

$$\|\partial_t u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}+\varepsilon^{-\gamma}\|\partial_t u_\varepsilon\|^2_{L^2(S_\varepsilon^T)}+\max_{t\in[0,T]}\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\le K\|f\|^2_{L^2(Q^T)}.$$

Moreover, if we use $\partial_t p_\varepsilon^m$ as a test function in the equation for p_ε^m, we obtain, for a.e. t

$$-\|\partial_t p_\varepsilon^m\|^2_{L^2(\Omega_\varepsilon)}-\varepsilon^{-\gamma}\|\partial_t p_\varepsilon^m\|^2_{L^2(S_\varepsilon)}+(\nabla p_\varepsilon^m,\partial_t\nabla p_\varepsilon^m)_{L^2(\Omega_\varepsilon)}$$

$$= -(\partial_t u_\varepsilon^m, \partial_t p_\varepsilon^m)_{L^2(\Omega_\varepsilon)} - \varepsilon^{-\gamma}(\partial_t u_\varepsilon^m, \partial_t p_\varepsilon^m)_{L^2(S_\varepsilon)} + (f, \partial_t p_\varepsilon^m)_{L^2(\Omega_\varepsilon)} - N^{-1}\varepsilon^{-\gamma}(p_\varepsilon^m, \partial_t p_\varepsilon^m)_{L^2(S_\varepsilon^2)}.$$

Integrating this equality with respect to t from 0 to T, we obtain

$$\begin{aligned}
&- \|\partial_t p_\varepsilon^m\|^2_{L^2(Q_\varepsilon^T)} + \frac{1}{2}\|\nabla p_\varepsilon^m(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} - \frac{1}{2}\|\nabla p_\varepsilon^m(\cdot,0)\|^2_{L^2(\Omega_\varepsilon)} - \varepsilon^{-\gamma}\|\partial_t p_\varepsilon^m\|^2_{L^2(S_\varepsilon^T)} \\
&= -\int_{Q_\varepsilon^T} \partial_t u_\varepsilon^m \partial_t p_\varepsilon^m \,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^T} \partial_t u_\varepsilon^m \partial_t p_\varepsilon^m \,ds\,dt \\
&\quad + \int_{Q_\varepsilon^T} f\partial_t p_\varepsilon^m \,dx\,dt - N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} p_\varepsilon^m \partial_t p_\varepsilon^m \,ds\,dt \\
&= -\int_{Q_\varepsilon^T} \partial_t u_\varepsilon^m \partial_t p_\varepsilon^m \,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^T} \partial_t u_\varepsilon^m \partial_t p_\varepsilon^m \,ds\,dt \\
&\quad + \int_{Q_\varepsilon^T} f\partial_t p_\varepsilon^m \,dx\,dt + N^{-1}\frac{\varepsilon^{-\gamma}}{2}\|p_\varepsilon^m(x,0)\|^2_{L^2(S_\varepsilon^2)} - N^{-1}\frac{\varepsilon^{-\gamma}}{2}\|u_\varepsilon^m(x,T)\|^2_{L^2(S_\varepsilon^2)}.
\end{aligned}$$

From this, we derive

$$\begin{aligned}
&\|\partial_t p_\varepsilon^m\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma}\|\partial_t p_\varepsilon^m\|^2_{L^2(S_\varepsilon^T)} + \frac{1}{2}\|\nabla p_\varepsilon^m(\cdot,0)\|^2_{L^2(\Omega_\varepsilon)} + \frac{\varepsilon^{-\gamma}}{2N}\|p_\varepsilon^m(x,0)\|^2_{L^2(\omega_\varepsilon)} \\
&\le \frac{1}{2}\|\nabla u_\varepsilon^m(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} + \|\partial_t u_\varepsilon^m\|_{L^2(Q_\varepsilon^T)}\|\partial_t p_\varepsilon^m\|_{L^2(Q_\varepsilon^T)} \\
&\quad + \varepsilon^{-\gamma}\|\partial_t u_\varepsilon^m\|_{L^2(S_\varepsilon^T)}\|\partial_t p_\varepsilon^m\|_{L^2(S_\varepsilon^T)} + \|f\|_{L^2(Q_\varepsilon^T)}\|\partial_t p_\varepsilon^m\|_{L^2(Q_\varepsilon^T)} + N^{-1}\frac{\varepsilon^{-\gamma}}{2}\|u_\varepsilon^m(x,T)\|^2_{L^2(S_\varepsilon^2)}.
\end{aligned}$$

Finally, using the estimates obtained for u_ε^m, we conclude that

$$\|\partial_t p_\varepsilon^m\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma}\|\partial_t p_\varepsilon^m\|^2_{L^2(S_\varepsilon^T)} \le K\|f\|^2_{L^2(Q^T)}. \tag{3.5}$$

Passing to the limit, as $m \to \infty$, we obtain the estimation of $\partial_t p_\varepsilon$

$$\|\partial_t p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma}\|\partial_t p_\varepsilon\|^2_{L^2(S_\varepsilon^T)} \le K\|f\|^2_{L^2(Q^T)}. \tag{3.6}$$

Having proved some a priori estimates of u_ε and v_ε, we proceed with the extension of these solution to the whole cylinder Q^T. We know (see, e.g. [6] and its references) that there exists an extension operator $P_\varepsilon : H^1(Q_\varepsilon^T) \to H^1(Q^T)$ such that

$$\|P_\varepsilon(u)\|_{H^1(Q^T)} \le \|u\|_{H^1(Q_\varepsilon^T)}.$$

Let $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ be the extensions of the functions u_ε, p_ε. Then we obtain the following estimates

$$\|\partial_t \tilde{u}_\varepsilon\|^2_{L^2(Q^T)} + \|\nabla \tilde{u}_\varepsilon\|^2_{L^2(Q^T)} \le K(\|\partial_t u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \|\nabla u_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}), \tag{3.7}$$

$$\|\partial_t \tilde{p}_\varepsilon\|^2_{L^2(Q^T)} + \|\nabla \tilde{p}_\varepsilon\|^2_{L^2(Q^T)} \le K(\|\partial_t p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \|\nabla p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)}). \tag{3.8}$$

The obtained estimates imply that there exist some subsequences (still denoted as the original) and some limit functions, u_0 and p_0, such that

$$\begin{aligned}
&\tilde{u}_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \quad \partial_t \tilde{u}_\varepsilon \rightharpoonup \partial_t u_0 \quad \text{weakly in } L^2(Q^T), \\
&\tilde{p}_\varepsilon \rightharpoonup p_0 \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \quad \partial_t \tilde{p}_\varepsilon \rightharpoonup \partial_t p_0 \quad \text{weakly in } L^2(Q^T).
\end{aligned} \tag{3.9}$$

Moreover, the embedding theorem implies also that $\tilde{u}_\varepsilon \to u_0$ and $\tilde{p}_\varepsilon \to p_0$ in $L^2(Q^T)$.

In the rest of the paper we give the characterization of these limit functions and derive a formulation of the homogenized optimal control problem.

4. Auxiliary non-local in time operators G, H and M

As already noted in the introduction, to state the homogenization results we need to introduce some auxiliary problems. The first non-local operator $M(\varphi)$ already was used in our previous study related to the case of distributed controls ([12]),

$$\begin{aligned}
\partial_t M(\varphi) + \mathcal{B}_n M(\varphi) = \varphi, \quad t \in (0,T), \\
M(\varphi)(0) = 0,
\end{aligned} \tag{4.1}$$

where $\mathcal{B}_n = (n-2)C_0^{-1}$, and $\varphi \in L^2(0,T)$ is given. This operator is related to the region that is the complementary to the one were the controls are localized. We also consider the adjoint operator $M^*(\varphi)$ given by

$$\begin{gathered} -\partial_t M^*(\varphi) + \mathcal{B}_n M^*(\varphi) = \varphi, \quad t \in (0,T), \\ M^*(\varphi)(T) = 0. \end{gathered} \tag{4.2}$$

For the domain that contains particles to which the controls are applied, we introduce operator $G(\varphi)$ that satisfies the similar problem to (4.1), but with a different coefficient

$$\begin{gathered} \partial_t G(\varphi) + (\mathcal{B}_n + N^{-1})G(\varphi) = \varphi, \quad t \in (0,T), \\ G(\varphi)(0) = 0. \end{gathered} \tag{4.3}$$

This operator $G(\varphi)$ can be explicitly written as

$$G(\varphi)(t) = \int_0^t e^{-(\mathcal{B}_n + N^{-1})(t-s)}\varphi(s)ds,$$

which show the non-local in time nature. We define its adjoint operator G^* as the solution of the problem adjoint to (4.3)

$$\begin{gathered} -\partial_t G^*(\varphi) + (\mathcal{B}_n + N^{-1})G^*(\varphi) = \varphi, \quad t \in (0,T), \\ G^*(\varphi)(x,T) = 0. \end{gathered} \tag{4.4}$$

Nevertheless, it turns out that for the case of boundary controls, as we are assuming in problem (1.1), we will need to define some new operators H, and H^*, coupled with G^* and G, respectively, in the following way

$$\begin{gathered} -\partial_t H^*(\varphi) + (\mathcal{B}_n + N^{-1})H^*(\varphi) - N^{-1}(\mathcal{B}_n + N^{-1})G(H^*(\varphi)) = \varphi, \quad t \in (0,T), \\ H^*(\varphi)(T) = 0, \end{gathered} \tag{4.5}$$

and

$$\begin{gathered} \partial_t H(\varphi) + (\mathcal{B}_n + N^{-1})H(\varphi) - N^{-1}(\mathcal{B}_n + N^{-1})G^*(H(\varphi)) = \varphi, \quad t \in (0,T), \\ H(\varphi)(0) = 0. \end{gathered} \tag{4.6}$$

Notice that both problems are now non-local in time, but since the operators G and G^* are globally Lipschitz continuous on $L^2(0,T)$, we obtain the existence and uniqueness of the associate solutions once $\varphi \in L^2(0,T)$ is given.

It is straightforward to show that the operator G is the adjoint operator to G^*, i.e.

$$\int_0^T G(\varphi)\psi dt = \int_0^T \varphi G^*(\psi)dt, \tag{4.7}$$

for any arbitrary functions φ, $\psi \in L^2(0,T)$. Indeed, we have

$$\begin{aligned} \int_0^T G(\varphi)\psi dt &= \int_0^T G(\varphi)(-\partial_t G^*(\psi) + (\mathcal{B}_n + N^{-1})G^*(\psi))dt \\ &= \int_0^T (\mathcal{B}_n + N^{-1})G(\varphi)G^*(\psi)dt - G(\varphi)G^*(\varphi)\Big|_0^T + \int_0^T \partial_t G(\varphi)G^*(\psi)dt \\ &= \int_0^T (\partial_t G(\varphi) + (\mathcal{B}_n + N^{-1})G(\varphi))G^*(\psi)dt = \int_0^T \varphi G^*(\psi)dt. \end{aligned}$$

Similarly to the above argument we know that M is the adjoint operator to M^*,

$$\int_0^T M(\varphi)\psi dt = \int_0^T \varphi M^*(\psi)dt.$$

The case of operators H and H^* is less trivial. Nevertheless, we also have that, for any arbitrary functions φ, $\psi \in L^2(0,T)$,

$$\int_0^T H(\varphi)\psi dt = \int_0^T \varphi H^*(\psi)dt. \tag{4.8}$$

Indeed, we make the following transformations

$$\begin{aligned}\int_0^T H(\varphi)\psi dt &= \int_0^T H(\varphi)(-\partial_t H^*(\psi) + (\mathcal{B}_n + N^{-1})H^*(\psi) - N^{-1}(\mathcal{B}_n + N^{-1})G(H^*(\psi)))dt \\ &= \int_0^T \partial_t H(\varphi)H^*(\psi)dt + (\mathcal{B}_n + N^{-1})\int_0^T H(\varphi)H^*(\psi)dt \\ &\quad - N^{-1}(\mathcal{B}_n + N^{-1})\int_0^T G^*(H(\varphi))H^*(\psi)dt \\ &= \int_0^T (\partial_t H(\varphi) + (\mathcal{B}_n + N^{-1})H(\varphi) - N^{-1}(\mathcal{B}_n + N^{-1})G^*(H(\varphi)))H^*(\psi)dt \\ &= \int_0^T \varphi H^*(\psi)dt.\end{aligned}$$

In addition to the above properties it will be useful to get some other relations among the above operators.

Lemma 4.1. *For the functions G, G^*, H and H^* introduced in* (4.3)-(4.5)*, we have the following relations*

(i) $G^*(H(\varphi)) = H^*(G(\varphi))$, *and* $G(H^*(\varphi)) = H(G^*(\varphi))$,

(ii) $H(\varphi) = G(\varphi) + N^{-1}(\mathcal{B}_n + N^{-1})G(H^*(G(\varphi)))$, *and we also have the adjoint version* $H^*(\varphi) = G^*(\varphi) + N^{-1}(\mathcal{B}_n + N^{-1})G^*(H(G^*(\varphi)))$.

Proof. We start with the proof of (i). The relations given in (ii) are direct consequences of the ones (i)). We consider two coupled auxiliary linear systems. The first one is a system coupling the functions $A(t) = G^*(H(\varphi))$ and $B(t) = H(\varphi)$. From (4.4) and (4.6), we obtain

$$\begin{gathered} -\partial_t A + (\mathcal{B}_n + N^{-1})A = B, \quad t \in (0,T), \\ \partial_t B + (\mathcal{B}_n + N^{-1})B - N^{-1}(\mathcal{B}_n + N^{-1})A = \varphi, \quad t \in (0,T), \\ A(T) = 0, \quad B(0) = 0. \end{gathered} \tag{4.9}$$

Analogously, from (4.3) and (4.5), we obtain a second system coupling the functions, $\tilde{A}(t) = G(H^*(G(\varphi)))$ and $\tilde{B}(t) = H^*(G(\varphi))$:

$$\begin{gathered} \partial_t \tilde{A} + (\mathcal{B}_n + N^{-1})\tilde{A} = \tilde{B}, \quad t \in (0,T), \\ -\partial_t \tilde{B} + (\mathcal{B}_n + N^{-1})\tilde{B} - N^{-1}(\mathcal{B}_n + N^{-1})\tilde{A} = G(\varphi), \quad t \in (0,T), \\ \tilde{A}(0) = 0, \quad \tilde{B}(T) = 0. \end{gathered} \tag{4.10}$$

From (4.9), we can derive a linear second order ODE problem on the function A. To do this, we substitute the expression for B from the first equation of the system (4.9) into the second one. Thus, we obtain

$$-\partial_{tt}^2 A + (\mathcal{B}_n + N^{-1})\partial_t A - (\mathcal{B}_n + N^{-1})\partial_t A + (\mathcal{B}_n + N^{-1})^2 A - N^{-1}(\mathcal{B}_n + N^{-1})A = \varphi,$$

and, simplifying it, we obtain

$$-\partial_{tt}^2 A + \mathcal{B}_n(\mathcal{B}_n + N^{-1})A = \varphi. \tag{4.11}$$

Also, substituting B written in terms of A into $B(0) = 0$, we obtain a condition on $A(0)$ (recall that we already have the condition at $t = T$ from the definition of A). Hence, the two boundary conditions are

$$A(T) = 0, \quad -\partial_t A(0) + (\mathcal{B}_n + N^{-1})A(0) = 0. \tag{4.12}$$

This is a linear coercive equation which has uniqueness of solutions. For instance, if we consider the homogeneous case ($\varphi \equiv 0$) in the equation (4.11) we obtain that, obviously, the trivial solution satisfies this problem. Moreover, by multiplying the equation by A and integrating from 0 to T, we obtain

$$\int_0^T |\partial_t A|^2 dt + \mathcal{B}_n(\mathcal{B}_n + N^{-1})\int_0^T A^2 dt + (\mathcal{B}_n + N^{-1})A^2(0) = 0$$

and we immediately conclude that, necessarily, $A \equiv 0$. Thus, for any $\varphi \in L^2(0,T)$, the inhomogeneous problem has also a unique solution.

Let us obtain now the ODE satisfied by the function $\tilde{B}(t)$. From the second equation of (4.10), we write $\tilde{A}$ in terms of $\tilde{B}$

$$\tilde{A} = -N(\mathcal{B}_n + N^{-1})^{-1}\partial_t \tilde{B} + N\tilde{B} - N(\mathcal{B}_n + N^{-1})^{-1} G(\varphi).$$

Substituting this expression into the first equation of (4.10), we derive

$$\begin{aligned} &- N(\mathcal{B}_n + N^{-1})^{-1}\partial^2_{tt}\tilde{B} + N\partial_t\tilde{B} - N(\mathcal{B}_n + N^{-1})^{-1}\partial_t G(\varphi) \\ &- N\partial_t\tilde{B} + (\mathcal{B}_n + N^{-1})N\tilde{B} - NG(\varphi) = \tilde{B}. \end{aligned}$$

Combining similar terms and using the definition of $G(\varphi)$, we obtain

$$-\partial^2_{tt}\tilde{B} + \mathcal{B}_n(\mathcal{B}_n + N^{-1})\tilde{B} = \partial_t G(\varphi) + (\mathcal{B}_n + N^{-1})G(\varphi) = \varphi.$$

From condition $\tilde{A}(0) = 0$, we conclude that

$$-\partial_t\tilde{B}(0) + (\mathcal{B}_n + N^{-1})\tilde{B}(0) = 0, \quad \tilde{B}(T) = 0.$$

Therefore, we obtain exactly the same linear problem as for the function A. But, since the solution to this problem is unique, we obtain that $A = \tilde{B}$, or in other terms $G^*(H(\varphi)) = H^*(G(\varphi))$. This completes the proof of the first relation in *(i)*.

Part (ii) can be proved using some similar arguments. Using the definition of H^*, we have that

$$G(\varphi) + N^{-1}(\mathcal{B}_n + N^{-1})G(H^*(G(\varphi))) = -\partial_t H^*(G(\varphi)) + (\mathcal{B}_n + N^{-1})H^*(G(\varphi)).$$

We use (i) and substitute $H^*(G(\varphi))$ with $G^*(H(\varphi))$ in the right-hand side, and using the definition of G^*, we obtain

$$G(\varphi) + N^{-1}(\mathcal{B}_n + N^{-1})G(H^*(G(\varphi))) = -\partial_t G^*(H(\varphi)) + (\mathcal{B}_n + N^{-1})G^*(H(\varphi)) = H(\varphi).$$

The second relation is proved in a similar way. This completes the proof. □

5. Statement of the homogenization theorems

Now, we are in a position to state the main theorem that characterizes the pair of functions (u_0, p_0) given by (3.9). The homogenized problem contains auxiliary functions defined in Section 4.

Theorem 5.1. *Let $n \geq 3$, $a_\varepsilon = C_0\varepsilon^\gamma$, where $C_0 > 0$, $\gamma = \frac{n}{n-2}$. If the pair $(u_\varepsilon, p_\varepsilon)$ is the solution to the problem* (2.7)*, then (u_0, p_0), defined in* (3.9)*, is a solution to the system*

$$\begin{gathered} \begin{aligned} &\partial_t u_0 - \Delta u_0 + \mathcal{A}_n(u_0 - \mathcal{B}_n H(u_0))\chi_{\omega^T} + \mathcal{A}_n(u_0 - \mathcal{B}_n M(u_0))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} \\ &= f - N^{-1}\mathcal{A}_n\mathcal{B}_n H(G^*(p_0))\chi_{\omega^T}, \quad (x,t) \in Q^T, \\ &- \partial_t p_0 - \Delta p_0 + \mathcal{A}_n(p_0 - \mathcal{B}_n H^*(p_0))\chi_{\omega^T} + \mathcal{A}_n(p_0 - \mathcal{B}_n M^*(p_0))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} \\ &= -\Delta u_0 + \mathcal{A}_n(u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0))\chi_{\omega^T} \\ &\quad + \mathcal{A}_n(u_0 - \mathcal{B}_n^2 M^*(M(u_0)))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)}, \quad (x,t) \in Q^T \end{aligned} \\ u_0(x,0) = 0, \quad x \in \Omega, \\ u_0(x,t) = 0, \quad (x,t) \in \Gamma^T, \\ p(x,T) = u_0(x,T), \quad x \in \Omega, \\ p(x,t) = 0, \quad (x,t) \in \Gamma^T, \end{gathered} \tag{5.1}$$

where, $\mathcal{A}_n = (n-2)C_0^{n-2}\omega_n$*,* $\mathcal{B}_n = (n-2)C_0^{-1}$*. Moreover, if we introduce the limit state problem*

$$\begin{aligned}
&\partial_t u_0(v) - \Delta u_0(v) + \mathcal{A}_n(u_0(v) - \mathcal{B}_n H(u_0(v)))\chi_{\omega^T} \\
&+ \mathcal{A}_n(u_0(v) - \mathcal{B}_n M(u_0(v)))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} \\
&= f + \mathcal{A}_n\mathcal{B}_n v\chi_{\omega^T}, \quad (x,t) \in Q^T, \\
&u_0(v)(x,0) = 0, \quad x \in \Omega, \\
&u_0(v)(x,t) = 0, \quad (x,t) \in \Gamma^T,
\end{aligned} \tag{5.2}$$

where $v \in H^1(0,T;L^2(\omega))$*, with* $v(x,0) = 0$*, and if we define the cost functional* $J_0(v)$ *given by* (1.12)*, then, from* (5.2) *and* (3.9)*, we have that* $u_0(v)$ *is the state associated to the optimal control problem*

$$J_0(v_0) = \min_{v\in U_{\rm ad}} J_0(v), \tag{5.3}$$

where the set of admissible functions is

$$U_{\rm ad} = \{\psi \in H^1(0,T;L^2(\omega)) |\ \psi(x,0) = 0\}.$$

In Section 7 we will prove the convergence of the sequence of functionals J_ε to the limit functional J_0 given by (1.12). This will show that the system (5.1) characterizes the optimal control problem (5.3), i.e. that $v_0 = -N^{-1}H(G^*(p_0))\chi_{\omega^T}$. Then we have the following result.

Theorem 5.2. *Under the conditions of Theorem 5.1, we have*

$$\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0),$$

where v_ε *is the optimal control of the* (1.1)*,* (1.3)*, and* v_0 *is the optimal control of the limit optimal control problem* (5.3)*.*

Lastly, we will show that system (5.1) is related to the limit functional and the limit optimal control v_0.

Theorem 5.3. *Let the pair of functions* $(u_0(v_0), v_0)$ *be an optimal solution of problem* (5.3)*, then* $v_0 = -N^{-1}H(G^*(p_0))\chi_{\omega^T}$*, where* p_0 *is the solution to* (6.19)*.*

6. Proof of the homogenization theorem

Proof. The main idea is to adapt to our setting the main lines of the so called *alternating test functions* (initially due to Luc Tartar to some simple framework and then extended by many different authors: see, e.g., the monograph [6]). We introduce auxiliary functions w_ε^j, $j \in \mathbb{Z}^n$, that are solutions to the boundary-value problems

$$\begin{aligned}
\Delta w_\varepsilon^j &= 0, \quad x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j}, \\
w_\varepsilon^j &= 1, \quad x \in \partial G_\varepsilon^j, \\
w_\varepsilon^j &= 0, \quad x \in \partial T_{\varepsilon/4}^j,
\end{aligned} \tag{6.1}$$

where $T_{\varepsilon/4}^j$ denotes the ball centered in P_ε^j of $\varepsilon/4$ radii. Based on these functions, we construct auxiliary functions in the whole domain Ω (where $i = 1,2$)

$$W_{i,\varepsilon} = \begin{cases} w_\varepsilon^j(x), & x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j},\ j \in \Upsilon_\varepsilon^i, \\ 1, & x \in \overline{G_\varepsilon^j},\ j \in \Upsilon_\varepsilon^i, \\ 0, & x \in \Omega \setminus \overline{\cup_{j\in\Upsilon_\varepsilon^i} T_{\varepsilon/4}^j}. \end{cases} \tag{6.2}$$

They are related to controllable and uncontrollable sets of particles. Note that $W_{i,\varepsilon} \in H_0^1(\Omega)$ and $W_{i,\varepsilon} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$. By the embedding theorems, for some subsequence for which we preserve the notation of the original, we have $W_{i,\varepsilon} \to 0$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$.

We will structure this long proof in a series of different steps.

Step A.1. We take $W_{2,\varepsilon}H^*(\varphi)$, where $\varphi=\psi(x)\eta(t)$ with $\psi(x)\in C_0^\infty(\Omega)$, $\eta(t)\in C^1([0,T])$, as a test function in the integral identity (2.1) and obtain

$$\begin{aligned}&\int_{Q_\varepsilon^T}\partial_t u_\varepsilon W_{2,\varepsilon}H^*(\varphi)\,dx\,dt+\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}\partial_t u_\varepsilon H^*(\varphi)\,ds\,dt+\int_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla(W_{2,\varepsilon}H^*(\varphi))\,dx\,dt\\&=\int_{Q_\varepsilon^T}fW_{2,\varepsilon}H^*(\varphi)\,dx\,dt-N^{-1}\int_{S_\varepsilon^{2,T}}p_\varepsilon H^*(\varphi)\,ds\,dt.\end{aligned}\tag{6.3}$$

Using the convergence (3.9) and the properties of $W_{2,\varepsilon}$, we have

$$\begin{gathered}\lim_{\varepsilon\to0}\int_{Q_\varepsilon^T}\partial_t u_\varepsilon W_{2,\varepsilon}H^*(\varphi)\,dx\,dt=0,\quad\lim_{\varepsilon\to0}\int_{Q_\varepsilon^T}fW_{2,\varepsilon}H^*(\varphi)\,dx\,dt=0,\\\lim_{\varepsilon\to0}\int_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla(W_{2,\varepsilon}H^*(\varphi))\,dx\,dt=\lim_{\varepsilon\to0}\int_{Q_\varepsilon^T}\nabla(u_\varepsilon H^*(\varphi))\nabla W_{2,\varepsilon}\,dx\,dt.\end{gathered}$$

Thus, from (6.3), we derive

$$\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}\partial_t u_\varepsilon H^*(\varphi)\,ds\,dt=-\int_{Q_\varepsilon^T}\nabla W_{2,\varepsilon}\nabla(u_\varepsilon H^*(\varphi))\,dx\,dt-N^{-1}\int_{S_\varepsilon^{2,T}}p_\varepsilon H^*(\varphi)\,ds\,dt+\zeta_\varepsilon,$$

where, here and below, $\zeta_\varepsilon\to0$ as $\varepsilon\to0$ (we will abuse the notation and will always use ζ_ε for the terms converging to zero).

Next, we use the following fundamental relation (calling it "from surface to volume averaging convergence principle" [6, Theorem 4.5] also applied, under different formulations, by many authors, see [23, 30]). We have

$$\int_{\Omega_\varepsilon}\nabla W_{i,\varepsilon}\nabla\eta dx=-\mathcal{A}_n\int_{\Omega^i}\eta dx+\mathcal{B}_n\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon^i}\int_{\partial G_\varepsilon^j}\eta ds+\zeta_\varepsilon,\tag{6.4}$$

here $\Omega^1=\Omega\setminus\omega$, $\Omega^2=\omega$, and then

$$\begin{aligned}&\lim_{\varepsilon\to0}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}\partial_t u_\varepsilon H^*(\varphi)\,ds\,dt\\&=\mathcal{A}_n\int_{\omega^T}H(u_0)\varphi\,dx\,dt-\lim_{\varepsilon\to0}\varepsilon^{-\gamma}\mathcal{B}_n\int_{S_\varepsilon^{2,T}}u_\varepsilon H^*(\varphi)\,ds\,dt-N^{-1}\lim_{\varepsilon\to0}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}p_\varepsilon H^*(\varphi)\,ds\,dt.\end{aligned}$$

Using that $u_\varepsilon(x,0)=0$ and $H^*(\varphi)(x,T)=0$, we integrate by parts the integral in the right-hand side, and further transform the previous equality to obtain

$$\begin{aligned}&\lim_{\varepsilon\to0}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}u_\varepsilon(-\partial_t H^*(\varphi)+\mathcal{B}_nH^*(\varphi))\,ds\,dt\\&=\mathcal{A}_n\int_{\omega^T}H(u_0)\varphi\,dx\,dt-\lim_{\varepsilon\to0}N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}p_\varepsilon H^*(\varphi)\,ds\,dt.\end{aligned}\tag{6.5}$$

Step A.2. We take $\varphi=W_{2,\varepsilon}G(\varphi)$, where $\varphi=\psi(x)\eta(t)$ with $\psi\in C_0^\infty(\Omega)$, $\eta\in C^1([0,T])$ as a test function in the integral identity (2.6), and obtain

$$\begin{aligned}&-\int_{Q_\varepsilon^T}\partial_t p_\varepsilon W_{2,\varepsilon}G(\varphi)\,dx\,dt-\int_{S_\varepsilon^{2,T}}\partial_t p_\varepsilon G(\varphi)\,ds\,dt+\int_{Q_\varepsilon^T}\nabla p_\varepsilon\nabla(W_{2,\varepsilon}G(\varphi))\,dx\,dt\\&=\int_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla(W_{2,\varepsilon}G(\varphi))\,dx\,dt.\end{aligned}\tag{6.6}$$

Taking $W_{2,\varepsilon}G(\varphi)$ as a test function in (2.1), we have (using the same arguments as a above)

$$\int_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla(W_{2,\varepsilon}G(\varphi))\,dx\,dt=-\int_{S_\varepsilon^{2,T}}\partial_t u_\varepsilon G(\varphi)\,ds\,dt-N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}p_\varepsilon G(\varphi)\,ds\,dt+\zeta_\varepsilon.$$

Substituting this relation into (6.6), we derive

$$\begin{aligned}&\int_{Q_\varepsilon^T} \nabla W_{2,\varepsilon}\nabla(p_\varepsilon G(\varphi))\,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t(p_\varepsilon - u_\varepsilon)G(\varphi)\,ds\,dt\\ &= \int_{Q_\varepsilon^T} \partial_t p_\varepsilon W_{2,\varepsilon} G(\varphi)\,dx\,dt - N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} p_\varepsilon G(\varphi)\,ds\,dt + \zeta_\varepsilon.\end{aligned}\tag{6.7}$$

Using the properties of $W_{2,\varepsilon}$ and a priori estimates of u_ε and p_ε, we conclude that the first integral in the right-hand side of the above equality converges to zero as $\varepsilon \to 0$. Also, using the relation (6.4), we derive

$$\begin{aligned}&-\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t(p_\varepsilon - u_\varepsilon)G(\varphi)\,ds\,dt + \varepsilon^{-\gamma}\mathcal{B}_n\int_{S_\varepsilon^{2,T}} p_\varepsilon G(\varphi)\,ds\,dt\\ &= \mathcal{A}_n\int_{\omega^T} p_0 G(\varphi)\,dx\,dt - N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} p_\varepsilon G(\varphi)\,ds\,dt + \zeta_\varepsilon.\end{aligned}\tag{6.8}$$

Note, that $p_\varepsilon(x,T) = u_\varepsilon(x,T)$ and $G(\varphi)(0) = 0$, therefore,

$$-\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t(p_\varepsilon - u_\varepsilon)G(\varphi)\,ds\,dt = \varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t G(\varphi)(p_\varepsilon - u_\varepsilon)\,ds\,dt.$$

Thus, from (6.8), we obtain

$$\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} p_\varepsilon(\partial_t G(\varphi)+(\mathcal{B}_n+N^{-1})G(\varphi))\,ds\,dt = \mathcal{A}_n\int_{\omega^T} p_0 G(\varphi)\,dx\,dt+\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} u_\varepsilon\partial_t G(\varphi)\,ds\,dt+\zeta_\varepsilon.$$

Using the definition of $G(\varphi)$, we conclude that

$$\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} p_\varepsilon\varphi\,ds\,dt = \mathcal{A}_n\int_{\omega^T} p_0 G(\varphi)\,dx\,dt + \varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} u_\varepsilon(\varphi - (\mathcal{B}_n + N^{-1})G(\varphi))\,ds\,dt + \zeta_\varepsilon.\tag{6.9}$$

Then, we substitute (6.9) (with $\varphi = H^*(\varphi)$) into (6.5), and obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} u_\varepsilon(-\partial_t H^*(\varphi) + (\mathcal{B}_n + N^{-1})H^*(\varphi) - N^{-1}(\mathcal{B}_n + N^{-1})G(H^*(\varphi)))\,ds\,dt\\ &= \mathcal{A}_n\int_{\omega^T} H(u_0)\varphi\,dx\,dt - \mathcal{A}_n N^{-1}\int_{\omega^T} p_0 G(H^*(\varphi))\,dx\,dt.\end{aligned}$$

Using the definition of H^*, we derive

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} u_\varepsilon\varphi\,ds\,dt = \mathcal{A}_n\int_{\omega^T} H(u_0)\varphi\,dx\,dt - \mathcal{A}_n N^{-1}\int_{\omega^T} H(G^*(p_0))\varphi\,dx\,dt.\tag{6.10}$$

Step A.3. Now, we can find the limit of the integrals taken over $S_\varepsilon^{2,T}$ in the integral identity (2.1). Indeed, we take $W_{2,\varepsilon}\varphi$ as a test function in (2.1), and obtain

$$\begin{aligned}&\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t u_\varepsilon\varphi\,ds\,dt + \varepsilon^{-\gamma}N^{-1}\int_{S_\varepsilon^{2,T}} p_\varepsilon\varphi\,ds\,dt\\ &= -\int_{\Omega_\varepsilon^T} \nabla u_\varepsilon\nabla(W_{2,\varepsilon}\varphi)\,dx\,dt + \int_{Q_\varepsilon^T} fW_{2,\varepsilon}\varphi\,dx\,dt\\ &= \mathcal{A}_n\int_{\omega^T} u_0\varphi\,dx\,dt - \varepsilon^{-\gamma}\mathcal{B}_n\int_{S_\varepsilon^{2,T}} u_\varepsilon\varphi\,ds\,dt + \zeta_\varepsilon\\ &= \mathcal{A}_n\int_{\omega^T} u_0\varphi\,dx\,dt - \mathcal{A}_n\mathcal{B}_n\int_{\omega^T} H(u_0)\varphi\,dx\,dt + \mathcal{A}_n\mathcal{B}_n N^{-1}\int_{\omega^T} H(G^*(p_0))\varphi\,dx\,dt + \zeta_\varepsilon.\end{aligned}\tag{6.11}$$

Thus, we found the limit of the terms related to $S_\varepsilon^{2,T}$.

Step A.4. To complete the derivation of the limit equation, we proceed with the terms related to $S_\varepsilon^{1,T}$. We take $W_{1,\varepsilon}M^*(\varphi)$ as a test function in the integral identity (2.1), and obtain (again using the properties of the function $W_{1,\varepsilon}$, we conclude that the integral with f converges to zero)

$$\int_{Q_\varepsilon^T} \nabla W_{1,\varepsilon}\nabla(u_\varepsilon M^*(\varphi))\,dx\,dt + \varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} \partial_t u_\varepsilon M^*(\varphi)\,ds\,dt = \zeta_\varepsilon,$$

where $\zeta_\varepsilon \to 0$ as $\varepsilon \to 0$. Using (6.4), we derive

$$\varepsilon^{-\gamma}\mathcal{B}_n \int_{S_\varepsilon^{1,T}} u_\varepsilon M^*(\varphi)\,ds\,dt + \varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} \partial_t u_\varepsilon M^*(\varphi)\,ds\,dt = \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} u_0 M^*(\varphi)\,dx\,dt + \zeta_\varepsilon.$$

As $M^*(\varphi)(x,T) = 0$ and $u_\varepsilon(x,0) = 0$, we have

$$\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} \partial_t u_\varepsilon M^*(\varphi)\,ds\,dt = -\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} u_\varepsilon \partial_t M^*(\varphi)\,ds\,dt.$$

Thus,

$$\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} u_\varepsilon(-\partial_t M^*(\varphi) + \mathcal{B}_n M^*(\varphi))\,ds\,dt = \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} u_0 M^*(\varphi)\,dx\,dt + \zeta_\varepsilon.$$

Using the definition of $M^*(\varphi)$, we obtain

$$\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} u_\varepsilon \varphi\,ds\,dt = \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} M(u_0)\varphi\,dx\,dt + \zeta_\varepsilon. \tag{6.12}$$

Now, we take $W_{1,\varepsilon}\varphi$ as a test function in the integral identity (2.1), and using similar arguments as above, we derive

$$\int_{Q_\varepsilon^T} \nabla W_{1,\varepsilon}\nabla(u_\varepsilon\varphi)\,dx\,dt + \varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} \partial_t u_\varepsilon\varphi\,ds\,dt = \zeta_\varepsilon.$$

We transform this identity using relation (6.4) and obtain

$$\begin{aligned}\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} \partial_t u_\varepsilon\varphi\,ds\,dt &= \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} u_0\varphi\,dx\,dt - \mathcal{B}_n\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}} u_\varepsilon\varphi\,ds\,dt + \zeta_\varepsilon\\ &= \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n M(u_0))\varphi\,dx\,dt + \zeta_\varepsilon.\end{aligned} \tag{6.13}$$

Finally, using the convergence (6.11) and (6.13), we can pass to the limit in the integral identity for u_ε and obtain the integral identity for u_0,

$$\begin{aligned}&\int_{Q^T} \partial_t u_0\varphi\,dx\,dt + \int_{Q^T} \nabla u_0\nabla\varphi\,dx\,dt\\ &+ \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n M(u_0))\varphi\,dx\,dt + \mathcal{A}_n \int_{w^T} (u_0 - \mathcal{B}_n H(u_0))\varphi\,dx\,dt\\ &= \int_{Q^T} f\varphi\,dx\,dt - N^{-1}\mathcal{A}_n\mathcal{B}_n \int_{w^T} H(G^*(p_0))\varphi\,dx\,dt.\end{aligned} \tag{6.14}$$

Thus, u_0 is a solution to the problem

$$\begin{gathered}\partial_t u_0 - \Delta u_0 + \mathcal{A}_n(u_0 - \mathcal{B}_n H(u_0))\chi_{\omega^T} + \mathcal{A}_n(u_0 - \mathcal{B}_n M(u_0))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)}\\ = f - N^{-1}\mathcal{A}_n\mathcal{B}_n H(G^*(p_0))\chi_{\omega^T}, \quad (x,t)\in Q^T,\\ u_0(x,0) = 0, \quad x\in\Omega,\\ u_0(x,t) = 0, \quad (x,t)\in\Gamma^T.\end{gathered}$$

Step B.1. Let us find the limit equation for p_0. We take $W_{2,\varepsilon}\varphi$ as a test function in the adjoint problem's integral identity (2.6), and obtain

$$-\int_{Q_\varepsilon^T} \partial_t p_\varepsilon W_{2,\varepsilon}\varphi\,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t p_\varepsilon\varphi\,ds\,dt + \int_{Q_\varepsilon^T} \nabla p_\varepsilon\nabla(W_{2,\varepsilon}\varphi)\,dx\,dt = \int_{Q_\varepsilon^T} \nabla u_\varepsilon\nabla(W_{2,\varepsilon}\varphi)\,dx\,dt.$$

The first integral in the left-hand side converges to zero because of the properties of $W_{2,\varepsilon}$, thus,

$$-\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t p_\varepsilon\varphi\,ds\,dt = \int_{Q_\varepsilon^T} \nabla(u_\varepsilon - p_\varepsilon)\nabla(W_{2,\varepsilon}\varphi)\,dx\,dt + \zeta_\varepsilon,$$

Using (6.4), (6.9) and (6.10), we derive

$$-\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}} \partial_t p_\varepsilon\varphi\,ds\,dt$$

$$= -\mathcal{A}_n \int_{w^T} (u_0 - p_0)\varphi \, dx \, dt + \varepsilon^{-\gamma}\mathcal{B}_n \int_{S_\varepsilon^{2,T}} (u_\varepsilon - p_\varepsilon)\varphi \, ds \, dt + \zeta_\varepsilon$$
$$= -\mathcal{A}_n \int_{w^T} (u_0 - p_0)\varphi \, dx \, dt - \mathcal{A}_n\mathcal{B}_n \int_{w^T} p_0 G(\varphi) \, dx \, dt + \varepsilon^{-\gamma}\mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{S_\varepsilon^{2,T}} u_\varepsilon G(\varphi) \, ds \, dt + \zeta_\varepsilon$$
$$= \mathcal{A}_n \int_{w^T} p_0(\varphi - \mathcal{B}_n G(\varphi)) \, dx \, dt - \mathcal{A}_n \int_{w^T} u_0 \varphi \, dx \, dt + \mathcal{A}_n\mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{Q^T} H(u_0) G(\varphi) \, dx \, dt$$
$$- N^{-1}\mathcal{A}_n\mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{w^T} p_0 G(H^*(G(\varphi))) \, dx \, dt + \zeta_\varepsilon.$$

Combining the terms with p_0, we obtain the expression

$$\varphi - \mathcal{B}_n G(\varphi) - N^{-1}\mathcal{B}_n(\mathcal{B}_n + N^{-1}) G(H^*(G(\varphi))) \equiv E(\varphi),$$

Using relation (ii) from Lemma 4.1, we derive

$$E(\varphi) = \varphi - \mathcal{B}_n H(\varphi).$$

Therefore, the term with p_0 is

$$\mathcal{A}_n \int_{\omega^T} p_0(\varphi - \mathcal{B}_n H(\varphi)) \, dx \, dt = \mathcal{A}_n \int_{\omega^T} (p_0 - \mathcal{B}_n H^*(p_0))\varphi \, dx \, dt.$$

Putting it all together, we derive

$$\begin{aligned} &- \varepsilon^{-\gamma} \int_{S_\varepsilon^{2,T}} \partial_t p_\varepsilon \varphi \, ds \, dt \\ &= \mathcal{A}_n \int_{w^T} (p_0 - \mathcal{B}_n H^*(p_0))\varphi \, dx \, dt - \mathcal{A}_n \int_{w^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1}) H^*(G(u_0)))\varphi \, dx \, dt. \end{aligned} \tag{6.15}$$

Step B.2. Next, we deal with the parts related to $S_\varepsilon^{1,T}$. We take $W_{1,\varepsilon} M(\varphi)$ as a test function in the integral identity (2.6), and obtain

$$\begin{aligned} &- \int_{Q_\varepsilon^T} \partial_t p_\varepsilon W_{1,\varepsilon} M(\varphi) \, dx \, dt - \varepsilon^{-\gamma} \int_{S_\varepsilon^{1,T}} \partial_t p_\varepsilon M(\varphi) \, ds \, dt + \int_{Q_\varepsilon^T} \nabla W_{1,\varepsilon} \nabla(p_\varepsilon M(\varphi)) \, dx \, dt \\ &= \int_{Q_\varepsilon^T} \nabla W_{1,\varepsilon} \nabla(u_\varepsilon M(\varphi)) \, dx \, dt + \zeta_\varepsilon. \end{aligned}$$

Using integral identity (2.1) taken with the same test function, we transform the last relation to the form

$$\begin{aligned} &\int_{Q_\varepsilon^T} \nabla W_{1,\varepsilon} \nabla(p_\varepsilon M(\varphi)) \, dx \, dt - \varepsilon^{-\gamma} \int_{S_\varepsilon^{1,T}} \partial_t (p_\varepsilon - u_\varepsilon) M(\varphi) \, ds \, dt \\ &= \int_{Q_\varepsilon^T} \partial_t (p_\varepsilon - u_\varepsilon) W_{1,\varepsilon} M(\varphi) \, dx \, dt + \int_{Q_\varepsilon^T} f W_{1,\varepsilon} M(\varphi) \, dx \, dt + \zeta_\varepsilon. \end{aligned}$$

Properties of $W_{1,\varepsilon}$ imply that the terms at the right-hand side converges to zero as $\varepsilon \to 0$. We use (6.4) and transform the first term in the left-hand side and finally obtain

$$-\varepsilon^{-\gamma} \int_{S_\varepsilon^{1,T}} \partial_t (p_\varepsilon - u_\varepsilon) M(\varphi) \, ds \, dt + \varepsilon^{-\gamma}\mathcal{B}_n \int_{S_\varepsilon^{1,T}} p_\varepsilon M(\varphi) \, ds \, dt = \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} p_0 M(\varphi) \, dx \, dt + \zeta_\varepsilon.$$

Integrating by parts in the first term, we conclude that

$$\varepsilon^{-\gamma} \int_{S_\varepsilon^{1,T}} (p_\varepsilon - u_\varepsilon) \partial_t M(\varphi) \, ds \, dt + \varepsilon^{-\gamma}\mathcal{B}_n \int_{S_\varepsilon^{1,T}} p_\varepsilon M(\varphi) \, ds \, dt = \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} p_0 M(\varphi) \, dx \, dt + \zeta_\varepsilon.$$

Using the definition of M^* and the convergence (6.12), we derive

$$\begin{aligned} &\varepsilon^{-\gamma} \int_{S_\varepsilon^{1,T}} p_\varepsilon \varphi \, ds \, dt = \varepsilon^{-\gamma} \int_{S_\varepsilon^{1,T}} p_\varepsilon (\partial_t M(\varphi) + \mathcal{B}_n M(\varphi)) \, ds \, dt \\ &= \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} p_0 M(\varphi) \, dx \, dt + \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} M(u_0)(\varphi - \mathcal{B}_n M(\varphi)) \, dx \, dt + \zeta_\varepsilon. \end{aligned} \tag{6.16}$$

Lastly, we take $W_{1,\varepsilon}\varphi$ as a test function in the integral identity (2.6), and using (6.13) and (6.16), we obtain

$$\begin{aligned}
&-\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}}\partial_t p_\varepsilon\varphi\,ds\,dt\\
&=\int_{Q_\varepsilon^T}\nabla(u_\varepsilon-p_\varepsilon)\nabla(W_{1,\varepsilon}\varphi)\,dx\,dt+\zeta_\varepsilon\\
&=\mathcal{A}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}(p_0-u_0)\varphi\,dx\,dt+\varepsilon^{-\gamma}\mathcal{B}_n\int_{S_\varepsilon^{1,T}}(u_\varepsilon-p_\varepsilon)\varphi\,ds\,dt+\zeta_\varepsilon\\
&=\mathcal{A}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}(p_0-u_0)\varphi\,dx\,dt+\mathcal{A}_n\mathcal{B}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}M(u_0)\varphi\,dx\,dt\\
&\quad-\mathcal{A}_n\mathcal{B}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}p_0M(\varphi)\,dx\,dt-\mathcal{A}_n\mathcal{B}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}M(u_0)(\varphi-\mathcal{B}_nM(\varphi))\,dx\,dt+\zeta_\varepsilon.
\end{aligned}$$

Grouping similar terms, we derive

$$\begin{aligned}
-\varepsilon^{-\gamma}\int_{S_\varepsilon^{1,T}}\partial_t p_\varepsilon\varphi\,ds\,dt&=\mathcal{A}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}(p_0-\mathcal{B}_nM^*(p_0))\varphi\,dx\,dt\\
&\quad-\mathcal{A}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}(u_0-\mathcal{B}_n^2M^*(M(u_0)))\varphi\,dx\,dt+\zeta_\varepsilon.
\end{aligned}\tag{6.17}$$

Step B.3. Now, using convergence (6.15) and (6.17), we can pass to the limit as $\varepsilon\to0$ in (2.6), and obtain the integral identity for p_0,

$$\begin{aligned}
&-\int_{Q^T}\partial_t p_0\varphi\,dx\,dt+\int_{Q^T}\nabla p_0\nabla\varphi\,dx\,dt+\mathcal{A}_n\int_{\omega^T}(p_0-\mathcal{B}_nH^*(p_0))\varphi\,dx\,dt\\
&+\mathcal{A}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}(p_0-\mathcal{B}_nM^*(p_0))\varphi\,dx\,dt\\
&=\int_{Q^T}\nabla u_0\nabla\varphi\,dx\,dt+\mathcal{A}_n\int_{Q^T}(u_0-\mathcal{B}_n(\mathcal{B}_n+N^{-1})G^*(H(u_0))\varphi\,dx\,dt\\
&\quad+\mathcal{A}_n\int_{(\Omega\setminus\overline{\omega})\times(0,T)}(u_0-\mathcal{B}_n^2M^*(M(u_0)))\varphi\,dx\,dt,
\end{aligned}\tag{6.18}$$

that is valid for an arbitrary function $\varphi\in L^2(0,T;H_0^1(\Omega))$. Thus, the limit adjoint problem is

$$\begin{gathered}
-\partial_t p_0-\Delta p_0+\mathcal{A}_n(p_0-\mathcal{B}_nH^*(p_0))\chi_{\omega^T}+\mathcal{A}_n(p_0-\mathcal{B}_nM^*(p_0))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)}\\
=-\Delta u_0+\mathcal{A}_n(u_0-\mathcal{B}_n(\mathcal{B}_n+N^{-1})G^*(H(u_0))\chi_{\omega^T}\\
+\mathcal{A}_n(u_0-\mathcal{B}_n^2M^*(M(u_0)))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)},\quad (x,t)\in Q^T,\\
p(x,T)=u_0(x,T),\quad x\in\Omega,\\
p(x,t)=0,\quad (x,t)\in\Gamma^T.
\end{gathered}\tag{6.19}$$

Notice that, due to the well-posedness of the limit problem, all the convergences hold for the whole sequences and not only for the considered subsequences.This completes the proof of the homogenization theorem. ☐

7. Proof of the limit cost functional and controls convergence theorems

In this Section we will complete the characterization of the limit optimal control.

Proof of Theorem 5.2. We start with the expression of J_ε taken at $v_\varepsilon=-N^{-1}p_\varepsilon\chi_{S_\varepsilon^{2,T}}$:

$$\begin{aligned}
&2J_\varepsilon(v_\varepsilon)\\
&=\int_{Q_\varepsilon^T}|\nabla u_\varepsilon|^2\,dx\,dt+\int_{\Omega_\varepsilon}|u_\varepsilon(x,T)|^2dx+\varepsilon^{-\gamma}\int_{S_\varepsilon}|u_\varepsilon(x,T)|^2ds+N^{-1}\varepsilon^{-\gamma}\int_{S_\varepsilon^{2,T}}p_\varepsilon^2\,ds\,dt.
\end{aligned}$$

Using the integral identity (2.1), we obtain

$$\begin{aligned} 2J_\varepsilon(v_\varepsilon) = &\int_{Q_\varepsilon^T} |\nabla u_\varepsilon|^2\,dx\,dt + \int_{\Omega_\varepsilon} |u_\varepsilon(x,T)|^2 dx + \varepsilon^{-\gamma}\int_{S_\varepsilon} |u_\varepsilon(x,T)|^2 ds + \int_{Q_\varepsilon^T} f p_\varepsilon\,dx\,dt \\ &- \int_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla p_\varepsilon\,dx\,dt - \int_{Q_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon\,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon\,ds\,dt. \end{aligned} \tag{7.1}$$

From the integral identity (2.6), we have

$$\begin{aligned} &-\int_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla p_\varepsilon\,dx\,dt - \int_{Q_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon\,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon^T} \partial_t u_\varepsilon p_\varepsilon\,ds\,dt \\ &= -\int_{Q_\varepsilon^T} \nabla p_\varepsilon \nabla u_\varepsilon\,dx\,dt + \int_{Q_\varepsilon^T} \partial_t p_\varepsilon u_\varepsilon\,dx\,dt + \varepsilon^{-\gamma}\int_{S_\varepsilon^T} \partial_t p_\varepsilon u_\varepsilon\,ds\,dt \\ &\quad - \varepsilon^{-\gamma}\int_{S_\varepsilon} |u_\varepsilon(x,T)|^2 ds - \int_{\Omega_\varepsilon} |u_\varepsilon(x,T)|^2 dx \\ &= -\int_{Q_\varepsilon^T} |\nabla u_\varepsilon|^2\,dx\,dt - \varepsilon^{-\gamma}\int_{S_\varepsilon} |u_\varepsilon(x,T)|^2 ds - \int_{\Omega_\varepsilon} |u_\varepsilon(x,T)|^2 dx. \end{aligned}$$

Substituting this equality into (7.1), we obtain

$$2J_\varepsilon(v_\varepsilon) = \int_{Q_\varepsilon^T} f p_\varepsilon\,dx\,dt.$$

Then, passing to the limit as $\varepsilon \to 0$ we have

$$\lim_{\varepsilon\to 0} 2J_\varepsilon(v_\varepsilon) = \int_{Q^T} f p_0\,dx\,dt.$$

Using the integral identity (6.14) for the function u_0, we obtain

$$\begin{aligned} \int_{Q^T} f p_0\,dx\,dt &= \int_{Q^T} \partial_t u_0 p_0\,dx\,dt + \int_{Q^T} \nabla u_0 \nabla p_0\,dx\,dt + \mathcal{A}_n \int_{\omega^T} (u_0 - \mathcal{B}_n H(u_0)) p_0\,dx\,dt \\ &\quad + \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n M(u_0)) p_0\,dx\,dt + N^{-1}\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} H(G^*(p_0)) p_0\,dx\,dt \\ &= -\int_{Q^T} \partial_t p_0 u_0\,dx\,dt + \int_{\Omega} |u_0(x,T)|^2 dx + \int_{Q^T} \nabla p_0 \nabla u_0\,dx\,dt \\ &\quad + \mathcal{A}_n \int_{\omega^T} (p_0 - \mathcal{B}_n H^*(p_0)) u_0\,dx\,dt + \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (p_0 - \mathcal{B}_n M^*(p_0)) u_0\,dx\,dt \\ &\quad + N^{-1}\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} H(G^*(p_0)) p_0\,dx\,dt. \end{aligned}$$

Then, we use the integral identity (6.18) for the function p_0 and duality relation for the operators H and M, to conclude that

$$\begin{aligned} \int_{Q^T} f p_0\,dx\,dt = &\int_{\Omega} |u_0(x,T)|^2\,dx\,dt + \int_{Q^T} |\nabla u_0|^2\,dx\,dt \\ &+ \mathcal{A}_n \int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1}) G^*(H(u_0))) u_0\,dx\,dt \\ &+ \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n^2 M^*(M(u_0))) u_0\,dx\,dt \\ &+ N^{-1}\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} H(G^*(p_0)) p_0\,dx\,dt. \end{aligned}$$

Let us show that the derived expression is non-negative. According to the definition of M and M^*, we have

$$\int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n^2 M^*(M(u_0))) u_0\,dx\,dt$$

$$
\begin{aligned}
&= \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0^2 - \mathcal{B}_n^2 M^2(u_0))\,dx\,dt\\
&= \int_{(\Omega\setminus\overline{\omega})\times(0,T)} ((\partial_t M(u_0) + \mathcal{B}_n M(u_0))^2 - \mathcal{B}_n^2 M^2(u_0))\,dx\,dt\\
&= \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (\partial_t M(u_0))^2\,dx\,dt + \mathcal{B}_n \int_{\Omega\setminus\overline{\omega}} |M(u_0)(x,T)|^2 dx\\
&= \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n M(u_0))^2\,dx\,dt + \mathcal{B}_n \int_{\Omega\setminus\overline{\omega}} |M(u_0)(x,T)|^2 dx \geq 0.
\end{aligned}
$$

Using that $G^*(H(u_0))$ is the solution to the problem (4.11) with boundary conditions (4.12), we obtain

$$
\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))u_0\,dx\,dt\\
&= \int_{\omega^T} (u_0 - \partial_{tt}^2 G^*(H(u_0)) - u_0)u_0\,dx\,dt\\
&= -\int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))u_0\,dx\,dt\\
&= \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))(\partial_{tt}^2 G^*(H(u_0)) - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))\,dx\,dt\\
&= \int_{\omega^T} (\partial_{tt}^2 G^*(H(u_0)))^2\,dx\,dt - \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))G^*(H(u_0))\,dx\,dt\\
&= \int_{\omega^T} (\partial_{tt}^2 G^*(H(u_0)))^2\,dx\,dt + \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} (\partial_t G^*(H(u_0)))^2\,dx\,dt\\
&\quad + \mathcal{B}_n \int_{\omega} (\partial_t G^*(H(u_0))(x,0))^2 dx \geq 0.
\end{aligned}
$$

From this, we y see that the expression is non-zero, however, we convert it to a form that is similar to the one derived for the terms with M. We have

$$
\begin{aligned}
&\mathcal{B}_n \int_{\omega} (\partial_t G^*(H(u_0))(x,0))^2 dx\\
&= \mathcal{B}_n \int_{\omega} ((\partial_t G^*(H(u_0))(x,0))^2 - (\partial_t G^*(H(u_0))(x,T))^2) dx + \mathcal{B}_n \int_{\omega} (\partial_t G^*(H(u_0))(x,T))^2 dx\\
&= -\mathcal{B}_n \int_{\omega^T} \partial_t |\partial_t G^*(H(u_0))|^2\,dx\,dt + \mathcal{B}_n \int_{\omega} (\partial_t G^*(H(u_0))(x,T))^2 dx\\
&= -2\mathcal{B}_n \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))\partial_t G^*(H(u_0))\,dx\,dt + \mathcal{B}_n \int_{\omega} (\partial_t G^*(H(u_0))(x,T))^2 dx.
\end{aligned}
$$

Note, that from the definition of $G^*(H(u_0))$, we have

$$
\partial_t G^*(H(u_0))(x,T) = (\mathcal{B}_n + N^{-1})G^*(H(u_0))(x,T) - H(u_0)(x,T) = -H(u_0)(x,T).
$$

Thus, we transform the last equality to

$$
\begin{aligned}
&\mathcal{B}_n \int_{\omega} (\partial_t G^*(H(u_0))(x,0))^2 dx\\
&= -2\mathcal{B}_n \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))\partial_t G^*(H(u_0))\,dx\,dt + \mathcal{B}_n \int_{\omega} |H(u_0)(x,T)|^2 dx.
\end{aligned}
$$

Now, we put this into the original expression and obtain

$$
\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))u_0\,dx\,dt =\\
&= \int_{\omega^T} (\partial_{tt}^2 G^*(H(u_0)))^2\,dx\,dt + \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} (\partial_t G^*(H(u_0)))^2\,dx\,dt
\end{aligned}
$$

$$- 2\mathcal{B}_n \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))\partial_t G^*(H(u_0))\,dx\,dt + \mathcal{B}_n \int_\omega |H(u_0)(x,T)|^2 dx.$$

We transform it as follows,

$$\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))u_0\,dx\,dt \\
&= \int_{\omega^T} (\mathcal{B}_n\partial_t G^*(H(u_0)) - \partial_{tt}^2 G^*(H(u_0)))^2\,dx\,dt + N^{-1}\mathcal{B}_n \int_{\omega^T} (\partial_t G^*(H(u_0)))^2\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\omega |H(u_0)(x,T)|^2 dx = \int_{\omega^T} (\partial_t H(u_0) - N^{-1}\partial_t G^*(H(u_0)))^2\,dx\,dt \\
&\quad + N^{-1}\mathcal{B}_n \int_{\omega^T} (\partial_t G^*(H(u_0)))^2\,dx\,dt + \mathcal{B}_n \int_\omega |H(u_0)(x,T)|^2 dx.
\end{aligned}$$

Using again the definition of operators G^* and H, we have

$$\begin{aligned}
\partial_t H(u_0) - N^{-1}\partial_t G^*(H(u_0)) &= u_0 - (\mathcal{B}_n + N^{-1})H(u_0) + N^{-1}(\mathcal{B}_n + N^{-1})G^*(H(u_0)) \\
&\quad + N^{-1}H(u_0) - N^{-1}(\mathcal{B}_n + N^{-1})G^*(H(u_0)) = u_0 - \mathcal{B}_n H(u_0).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))u_0\,dx\,dt \\
&= \int_{\omega^T} |u_0 - \mathcal{B}_n H(u_0)|^2\,dx\,dt + N^{-1}\mathcal{B}_n \int_{\omega^T} (\partial_t G^*(H(u_0)))^2\,dx\,dt + \mathcal{B}_n \int_\omega |H(u_0)(x,T)|^2 dx.
\end{aligned}$$

Using that $H(G^*(p_0))$ is a solution (4.11) we obtain

$$\begin{aligned}
\int_{\omega^T} H(G^*(p_0))p_0\,dx\,dt &= -\int_{\omega^T} H(G^*(p_0))(\partial_{tt}^2 H(G^*(p_0)) - \mathcal{B}_n(\mathcal{B}_n + N^{-1})H(G^*(p_0)))\,dx\,dt \\
&= \int_{\omega^T} (\partial_t H(G^*(p_0)))^2\,dx\,dt - \int_\omega \partial_t(H(G^*(p_0)))(x,T)H(G^*(p_0))(x,T)dx \\
&\quad + \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} (H(G^*(p_0)))^2\,dx\,dt \\
&= \int_{\omega^T} (\partial_t H(G^*(p_0)))^2\,dx\,dt + \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} (H(G^*(p_0)))^2\,dx\,dt \\
&\quad + (\mathcal{B}_n + N^{-1}) \int_\omega (H(G^*(p_0))(x,T))^2 dx.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{\varepsilon\to 0} 2J_\varepsilon(v_\varepsilon) &= \|\nabla u_0\|^2_{L^2(Q^T)} + \|u_0(x,T)\|^2_{L^2(\Omega)} \\
&\quad + \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} |u_0 - \mathcal{B}_n M(u_0)|^2\,dx\,dt + \mathcal{A}_n\mathcal{B}_n \int_{\Omega\setminus\overline{\omega}} |M(u_0)(x,T)|^2 dx \\
&\quad + \mathcal{A}_n \int_{\omega^T} |u_0 - \mathcal{B}_n H(u_0)|^2\,dx\,dt + \mathcal{A}_n\mathcal{B}_n \int_\omega |H(u_0)(x,T)|^2 dx \\
&\quad + N^{-1}\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} (\partial_t G^*(H(u_0)))^2\,dx\,dt + N^{-1}\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} (\partial_t H(G^*(p_0)))^2\,dx\,dt \\
&\quad + N^{-1}\mathcal{A}_n\mathcal{B}_n^2(\mathcal{B}_n + N^{-1}) \int_{\omega^T} (H(G^*(p_0)))^2\,dx\,dt \\
&\quad + N^{-1}\mathcal{A}_n\mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_\omega (H(G^*(p_0))(x,T))^2 dx \\
&= J_0(-N^{-1}H(G^*(p_0))).
\end{aligned}$$

This completes the proof. □

Finally, we will show that system (5.1) is related to the limit functional and the limit optimal control v_0.

Proof of Theorem 5.3. If v_0 is the optimal control, then for any admissible function $v \in U_{\rm ad}$ we have

$$J_0'(v_0)v = \lim_{\lambda\to 0} \frac{J_0(v_0+\lambda v) - J_0(v_0)}{\lambda} = 0.$$

We set $\theta = (u_0(v_0+\lambda v) - u_0(v_0))/\lambda$. The function θ is a solution to the problem

$$\begin{gathered}
\partial_t\theta - \Delta\theta + \mathcal{A}_n(\theta - \mathcal{B}_n H(\theta))\chi_{\omega^T} + \mathcal{A}_n(\theta - \mathcal{B}_n M(\theta))\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} \\
= \mathcal{A}_n\mathcal{B}_n v\chi_{\omega^T}, \quad (x,t) \in Q^T, \\
\theta(x,0) = 0, \quad x \in \Omega, \\
\theta(x,t) = 0, \quad (x,t) \in \Gamma^T.
\end{gathered} \tag{7.2}$$

Note that because of the linearity of the operator M and H, we have

$$\begin{gathered}
(H(u_0(v_0+\lambda v) - H(u_0(v_0)))/\lambda = H(\theta), \\
(M(u_0(v_0+\lambda v) - M(u_0(v_0)))/\lambda = M(\theta).
\end{gathered}$$

Thus,

$$\begin{aligned}
J_0'(v_0)v &= \int_{Q^T} \nabla\theta\nabla u_0(v_0)\,dx\,dt + \int_\Omega \theta(x,T)u_0(v_0)(x,T)dx \\
&\quad + \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (\theta - \mathcal{B}_n M(\theta))(u_0(v_0) - \mathcal{B}_n M(u_0(v_0)))\,dx\,dt \\
&\quad + \mathcal{A}_n\mathcal{B}_n \int_{\Omega\setminus\overline{\omega}} M(\theta)(x,T)M(u_0(v_0))(x,T)dx \\
&\quad + \mathcal{A}_n \int_{\omega^T} (\theta - \mathcal{B}_n H(\theta))(u_0(v_0) - \mathcal{B}_n H(u_0(v_0)))\,dx\,dt \\
&\quad + \mathcal{A}_n\mathcal{B}_n \int_\omega H(\theta)(x,T)H(u_0(v_0))(x,T)dx \\
&\quad + N^{-1}\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} \partial_t G(H^*(\theta))\partial_t G(H^*(u_0(v_0)))\,dx\,dt + N\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} \partial_t v_0 \partial_t v\,dx\,dt \\
&\quad + N\mathcal{A}_n\mathcal{B}_n^2(\mathcal{B}_n + N^{-1}) \int_{\omega^T} v_0 v\,dx\,dt + N\mathcal{A}_n\mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_\omega v_0(x,T)v(x,T)dx.
\end{aligned} \tag{7.3}$$

Now, we use that p_0 is a solution to the problem (6.19), that is adjoint to (5.2), and obtain

$$\begin{aligned}
&\int_{Q^T} \nabla\theta\nabla u_0(v_0)\,dx\,dt \\
&= -\int_{Q^T} \partial_t p_0\theta\,dx\,dt + \int_{Q^T} \nabla p_0\nabla\theta\,dx\,dt \\
&\quad + \mathcal{A}_n \int_{\omega^T} (p_0 - \mathcal{B}_n H^*(p_0))\theta\,dx\,dt + \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (p_0 - \mathcal{B}_n M^*(p_0))\theta\,dx\,dt \\
&\quad - \mathcal{A}_n \int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))\theta\,dx\,dt \\
&\quad - \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n^2 M^*(M(u_0)))\theta\,dx\,dt.
\end{aligned}$$

As θ is a solution to problem (7.2), we have

$$\begin{aligned}
\int_{Q^T} \nabla\theta\nabla u_0(v_0)\,dx\,dt &= -\int_\Omega u_0(v_0)(x,T)\theta(x,T)dx + \mathcal{A}_n\mathcal{B}_n \int_{\omega^T} vp_0\,dx\,dt \\
&\quad - \mathcal{A}_n \int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0))\theta\,dx\,dt \\
&\quad - \mathcal{A}_n \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0 - \mathcal{B}_n^2 M^*(M(u_0)))\theta\,dx\,dt.
\end{aligned} \tag{7.4}$$

Using the same transformations as in the proof of the theorem above, we have

$$\begin{aligned}
&\int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0(v_0) - \mathcal{B}_n^2 M^*(M(u_0(v_0)))\theta \,dx\,dt \\
&= \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0(v_0)\theta - \mathcal{B}_n^2 M(u_0(v_0)))M(\theta)\,dx\,dt \\
&= \int_{(\Omega\setminus\overline{\omega})\times(0,T)} ((\partial_t M(u_0(v_0)) + \mathcal{B}_n M(u_0(v_0)))((\partial_t M(\theta) \\
&\quad + \mathcal{B}_n M(\theta)) - \mathcal{B}_n^2 M(u_0(v_0))M(\theta)\,dx\,dt \\
&= \int_{(\Omega\setminus\overline{\omega})\times(0,T)} (u_0(v_0) - \mathcal{B}_n M(u_0(v_0)))(\theta - \mathcal{B}_n M(\theta))\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\Omega M(u_0(v_0))(x,T)M(\theta)(x,T)dx.
\end{aligned} \tag{7.5}$$

And, similarly as in the proof of the theorem above, we obtain

$$\begin{aligned}
&\int_{\omega^T} (u_0(v_0) - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0(v_0))))\theta\,dx\,dt \\
&= \int_{\omega^T} (u_0(v_0) - \mathcal{B}_n H(u_0(v_0)))(\theta - \mathcal{B}_n H(\theta))\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\omega H(u_0(v_0))(x,T)H(\theta)(x,T)dx \\
&\quad + N^{-1}\mathcal{B}_n \int_{\omega^T} \partial_t G(H^*(u_0(v_0)))\partial_t G(H^*(\theta))\,dx\,dt.
\end{aligned}$$

Indeed, we use that $G^*(H(u_0))$ is the solution to problem (4.11) with boundary conditions (4.12), and derive that

$$\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))\theta\,dx\,dt \\
&= \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))(\partial_{tt}^2 G^*(H(\theta)) - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(\theta)))\,dx\,dt \\
&= \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))\partial_{tt}^2 G^*(H(\theta))\,dx\,dt - \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))G^*(H(\theta))\,dx\,dt \\
&= \int_{\omega^T} \partial_{tt}^2 G^*(H(u_0))\partial_{tt}^2 G^*(H(\theta))\,dx\,dt + \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} \partial_t G^*(H(u_0))\partial_t G^*(H(\theta))\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\omega \partial_t G^*(H(u_0))(x,0)\partial_t G^*(H(\theta))(x,0)dx.
\end{aligned}$$

We transform the last term as follows,

$$\begin{aligned}
&\mathcal{B}_n \int_\omega \partial_t G^*(H(u_0))(x,0)\partial_t G^*(H(\theta))(x,0)dx \\
&= -\mathcal{B}_n \int_{\omega^T} \partial_t(\partial_t G^*(H(u_0))G^*(H(\theta)))\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\omega \partial_t G^*(H(u_0))(x,T)\partial_t G^*(H(\theta))(x,T)dx \\
&= -\mathcal{B}_n \int_{\omega^T} (\partial_{tt}^2 G^*(H(u_0))\partial_t G^*(H(\theta)) + \partial_t G^*(H(u_0))\partial_{tt}^2 G^*(H(\theta)))\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\omega \partial_t G^*(H(u_0))(x,T)\partial_t G^*(H(\theta))(x,T)dx.
\end{aligned}$$

Note, that from the definition of $G^*(H(u_0))$, we have

$$\partial_t G^*(H(u_0))(x,T) = -H(u_0)(x,T), \quad \partial_t G^*(H(\theta))(x,T) = -H(\theta)(x,T).$$

Thus, we transform the last equality to

$$\begin{aligned}
&\mathcal{B}_n \int_\omega \partial_t G^*(H(u_0))(x,0)\partial_t G^*(H(\theta))(x,0)dx \\
&= -\mathcal{B}_n \int_{\omega^T} (\partial^2_{tt} G^*(H(u_0))\partial_t G^*(H(\theta)) + \partial_t G^*(H(u_0))\partial^2_{tt} G^*(H(\theta)))\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\omega H(u_0)(x,T)H(\theta)(x,T)dx.
\end{aligned}$$

Now, we put this into the original expression and obtain

$$\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))\theta\,dx\,dt \\
&= \int_{\omega^T} \partial^2_{tt} G^*(H(u_0))\partial^2_{tt} G^*(H(\theta))\,dx\,dt + \mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_{\omega^T} \partial_t G^*(H(u_0))\partial_t G^*(H(\theta))\,dx\,dt \\
&\quad - \mathcal{B}_n \int_{\omega^T} (\partial^2_{tt} G^*(H(u_0))\partial_t G^*(H(\theta)) + \partial_t G^*(H(u_0))\partial^2_{tt} G^*(H(\theta)))\,dx\,dt \\
&\quad + \mathcal{B}_n \int_\omega H(u_0)(x,T)H(\theta)(x,T)dx.
\end{aligned}$$

We transform it in as follows,

$$\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))\theta\,dx\,dt \\
&= \int_{\omega^T} (\mathcal{B}_n\partial_t G^*(H(u_0)) - \partial^2_{tt} G^*(H(u_0)))(\mathcal{B}_n\partial_t G^*(H(\theta)) - \partial^2_{tt} G^*(H(\theta)))\,dx\,dt \\
&\quad + N^{-1}\mathcal{B}_n \int_{\omega^T} \partial_t G^*(H(u_0))\partial_t G^*(H(\theta))\,dx\,dt + \mathcal{B}_n \int_\omega H(u_0)(x,T)H(\theta)(x,T)dx \\
&= \int_{\omega^T} (\partial_t H(u_0) - N^{-1}\partial_t G^*(H(\theta)))(\partial_t H(\theta) - N^{-1}\partial_t G^*(H(\theta)))\,dx\,dt \\
&\quad + N^{-1}\mathcal{B}_n \int_{\omega^T} \partial_t G^*(H(u_0))\partial_t G^*(H(\theta))\,dx\,dt + \mathcal{B}_n \int_\omega H(u_0)(x,T)H(\theta)(x,T)dx.
\end{aligned}$$

Using again the definition of operators G^* and H, we have

$$\begin{gathered}
\partial_t H(u_0) - N^{-1}\partial_t G^*(H(u_0)) = u_0 - \mathcal{B}_n H(u_0), \\
\partial_t H(\theta) - N^{-1}\partial_t G^*(H(\theta)) = u_0 - \mathcal{B}_n H(\theta).
\end{gathered}$$

Thus,

$$\begin{aligned}
&\int_{\omega^T} (u_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})G^*(H(u_0)))\theta\,dx\,dt \\
&= \int_{\omega^T} (u_0 - \mathcal{B}_n H(u_0))(\theta - \mathcal{B}_n H(\theta))\,dx\,dt \\
&\quad + N^{-1}\mathcal{B}_n \int_{\omega^T} \partial_t G^*(H(u_0))\partial_t G^*(H(\theta))\,dx\,dt + \mathcal{B}_n \int_\omega H(u_0)(x,T)H(\theta)(x,T)dx.
\end{aligned} \tag{7.6}$$

Substituting relations (7.5) and (7.6) into (7.4), and then, putting it into (7.3), we derive

$$\begin{aligned}
J_0'(v_0)v &= \mathcal{A}_n\mathcal{B}_n \int_{\omega^T} p_0 v\,dx\,dt + N\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} \partial_t v_0 \partial_t v\,dx\,dt \\
&\quad + N\mathcal{A}_n\mathcal{B}_n^2(\mathcal{B}_n + N^{-1}) \int_{\omega^T} v_0 v\,dx\,dt + N\mathcal{A}_n\mathcal{B}_n(\mathcal{B}_n + N^{-1}) \int_\omega v_0(x,T)v(x,T)dx.
\end{aligned}$$

Integrating by parts in the second integral, we obtain

$$\begin{aligned}
J_0'(v_0)v &= \mathcal{A}_n\mathcal{B}_n \int_{\omega^T} p_0 v\,dx\,dt + N\mathcal{A}_n\mathcal{B}_n \int_\omega (\partial_t v_0(x,T) + (\mathcal{B}_n + N^{-1})v_0(x,T))v(x,T)dx \\
&\quad + N\mathcal{A}_n\mathcal{B}_n \int_{\omega^T} (-\partial^2_{tt} v_0 + \mathcal{B}_n(\mathcal{B}_n + N^{-1})v_0)v\,dx\,dt = 0,
\end{aligned}$$

for an arbitrary admissible function $v \in U_{\rm ad}$. Hence, we obtain that v_0 must be a solution of the problem

$$\partial^2_{tt} v_0 - \mathcal{B}_n(\mathcal{B}_n + N^{-1})v_0 = N^{-1}p_0$$
$$v_0(x,0) = 0, \quad \partial_t v_0(x,T) + (\mathcal{B}_n + N^{-1})v_0(x,T) = 0.$$

But, this is related to the problem satisfied by $H(G^*(-N^{-1}p_0))$. Because of the uniqueness of the solution of the problem we conclude that $v_0 = -N^{-1}H(G^*(p_0))\chi_{\omega^T}$. □

Remark 7.1. As in [11, 12], thanks to Remark 2.1, by simplifying the cost functional J_ε as in [11], by making the parameter $N \to 0$, it seems possible to show the approximate controllability with final observation of solutions of the limit problem.

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