Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 19, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.19

POSITIVE SOLUTIONS FOR GENERALIZED QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract. This article concerns the existence of solutions to the generalized quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $N \geq 1$, $g \in C^1(\mathbb{R})$ and the nonlinearity is asymptotically linear at infinity. Through the stretching transformation, we adjust the approach in [1] and overcome the impediment that there might not exist a $t > 0$ such that $tu_\infty(x-y)$ belongs to the Nahari manifold, where u_∞ is a ground state solution for the limiting problem.

1. Introduction

This article concerns the generalized quasilinear Schrödinger equation

$$-\operatorname{div}(g^2(u)\nabla u) + g(u)g'(u)|\nabla u|^2 + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, the limits of V and f exist as $|x| \to \infty$ and g is continuously differentiable with a nonnegative derivative on $[0,\infty)$.

Solutions to equation (1.1) are directly associated with the stationary wave solutions of a time-dependent quasilinear Schrödinger equation, which takes the form

$$i\Phi_t = -\Delta\Phi + W(x)\Phi - q(x,|\Phi|^2)\Phi - \Delta(l(|\Phi|^2))l'(|\Phi|^2)\Phi, \tag{1.2}$$

where $\Phi(x,t)$ denotes the wave function, $W(x)$ stands for the potential, while q and l are functions selected appropriately. This equation arises in the search for solitary wave solutions to a general quasilinear Schrödinger equation, which models several physical phenomena, and further details can be found in [17, 16, 18] for an explanation. By substituting the standing wave ansatz $\Phi(x,t) = e^{-i\lambda t}u(x)$ into equation (1.2), one obtains the time-independent elliptic equation

$$-\Delta u + V(x)u - \Delta\big(k(u^2)\big)k'(u^2)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

which constitutes a particular case of problem (1.1) with the coefficient function $g^2(u) = 1 + \frac{[(k(u^2))']^2}{2}$.

Two physically relevant specializations of this structure are noteworthy. First, choosing $g^2(u) := 1 + 2u^2$, which corresponds to $k(s) = s$, yields the superfluid film equation

$$-\Delta u + V(x)u - \Delta(u^2)u = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Second, the choice $g^2(u) := 1 + \frac{u^2}{2(1+u^2)}$ leads to

$$-\Delta u + V(x)u - \Big[\Delta\big(1+u^2\big)^{1/2}\Big]\frac{u}{2(1+u^2)^{1/2}} = f(x,u), \quad x \in \mathbb{R}^N. \tag{1.5}$$

The latter model describes the propagation and self-channeling of an intense ultrashort laser pulse inside a nonlinear optical medium.

2020 *Mathematics Subject Classification.* 35J20, 35J62, 49J35.
Key words and phrases. Quasilinear Schrödinger equation; asymptotically linear; Nehari manifold.

Submitted January 5, 2026. Published February 26, 2026.

There has been significant attention in the literature focused on (1.4). The pioneering work in this direction was conducted by Poppenberg et al. [18], leveraging variational methods. Subsequently, Wang [16] demonstrated that a positive solution exists, via a constrained minimization argument. A significant methodological advancement was made in [17], which was achieved by reformulating the original quasilinear equation as a semilinear problem through an Orlicz-space-based variable transformation. Later, Colin et al. [8] introduced a modified transformation of variables, which allowed the analysis to be carried out within the conventional Sobolev space setting. Further extending these results, Silva et al. [20] proved the presence of nonnegative solutions under asymptotically periodic and critical growth conditions.

Following these foundational works, considerable attention has been devoted to (1.4). For a broader overview of subsequent developments, we refer the reader to [11, 13, 14, 15, 26] and the references cited therein.

Our analysis focuses on the problem

$$-\operatorname{div}(g^2(u)\nabla u)+g(u)g'(u)|\nabla u|^2+V(x)u=f(x,u),\quad u\in H^1(\mathbb{R}^N). \tag{1.6}$$

We define E as the Sobolev space $H^1(\mathbb{R}^N)$ with the norm

$$\|u\|:=\Big(\int_{\mathbb{R}^N}|\nabla u|^2+V(x)u^2\Big)^{1/2}.$$

We set $G(t):=\int_0^t g(\tau)\,d\tau,\ F(x,t):=\int_0^t f(x,\tau)\,d\tau$ and $F_\infty(t):=\int_0^t f_\infty(\tau)\,d\tau$. Our analysis is based on the following assumptions: concerning V, g, and h:

- (A1) V is continuous and $0<\inf_{y\in\mathbb{R}^N}V(y)\le V(x)\le V_\infty$ for all $x\in\mathbb{R}^N$, where $V_\infty:=\lim_{|x|\to\infty}V(x)$.
- (A2) g is of class C^1 on $\mathbb{R}$, even, and increasing on $(0,\infty)$, with $g(0)=1$.
- (A3) $f(x,t)\in C(\mathbb{R}^N\times\mathbb{R}^+)$ and for every $t\ge0$, $\lim_{|x|\to\infty}f(x,t)=f_\infty(t)$, where $f_\infty(t)$ is continuous in $\mathbb{R}^+$.
- (A4) $\lim_{t\to0^+}\frac{f(x,t)}{t}=0$, uniformly for $x\in\mathbb{R}^N$.
- (A5) $\lim_{t\to+\infty}\frac{f(x,t)}{g(t)G(t)}=s(x)$, uniformly for $x\in\mathbb{R}^N$, where $s(x)$ is continuous, $\lim_{|x|\to\infty}s(x)=s_\infty$ and $s(x)>\frac{V(x)}{g^2(\infty)}$ for every $x\in\mathbb{R}^N$, with $\frac{1}{g(\infty)}:=\lim_{t\to\infty}\frac{1}{g(t)}$.
- (A6) $t\mapsto\frac{f(x,t)}{g(t)G(t)}$ and $t\mapsto\frac{f_\infty(t)}{g(t)G(t)}$ are increasing on $(0,\infty)$. Moreover, on the set where $g(t)=1$, the aforementioned monotonicity is strict.
- (A7) $\frac{f_\infty(t)-f(x,t)}{g(t)G(t)}\le s_\infty-s(x)$, for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}^+$.
- (A8) If $g\equiv1$, then $\lim_{t\to+\infty}\frac12 s(x)t^2-F(x,t)=\infty$, for a.e. $x\in\mathbb{R}^N$.
- (A9) For each $\delta>0$, $\lim_{|x|\to\infty}f(x,t)=f_\infty(t)$, uniformly for $0\le t\le\delta$.
- (A10) If $\omega_n\rightharpoonup0$ in E, then for every $\gamma>0$, there exists $y\in\mathbb{R}^N$ such that
$$\liminf_{l\to\infty}\int_{|x-y|>l,\,|\omega_n|<\gamma}F(x,G^{-1}(|\omega_n|))-F_\infty(G^{-1}(|\omega_n|))=0,$$
$$\lim_{l\to\infty}\int_{|x-y|>l,\,|\omega_n|<\gamma}\Big[\frac{f(x,G^{-1}(|\omega_n|))}{g(G^{-1}(\omega_n))}-\frac{f_\infty(G^{-1}(|\omega_n|))}{g(G^{-1}(\omega_n))}\Big]^2=0,$$
uniformly for n.
- (A11) $0<s_\infty-\frac{V_\infty}{g^2(\infty)}\le\inf_{x\in\mathbb{R}^N}[s(x)-\frac{V(x)}{g^2(\infty)}]+\mu(\inf_{x\in\mathbb{R}^N}[s(x)-\frac{V(x)}{g^2(\infty)}])$, i.e., $\sup_{x\in\mathbb{R}^N}[\frac{V(x)-V_\infty}{g^2(\infty)}+s_\infty-s(x)]\le\mu(\inf_{x\in\mathbb{R}^N}[s(x)-\frac{V(x)}{g^2(\infty)}])$, where $\mu(\cdot)$ is as in Lemma 3.19.

Theorem 1.1. *Under assumptions* (A1)–(A8), (A11) *and at least one of* (A9) *and* (A10), *we further impose the condition*

- (A12) *The minimal energy value c_∞ corresponding to equation* (2.4) *is an isolated critical level of the functional J_∞.*

Then (1.6) *admits a positive solution.*

Remark 1.2. To prove that (1.6) has a positive solution, we can let f be odd for t.

Remark 1.3. For $b > c_0 = c_\infty$, we follow the approach in [1]. Unlike [1], however, we cannot guarantee that $u_\infty(x-y) \in \mathcal{T}$ for arbitrary y under our conditions. To address this, we introduce a parameter θ_y and modify the definition of Γ, where θ_y is continuous in y.

Remark 1.4. In lemma 3.19, we prove that hypothesis (A11) is equivalent to $a^0 \le \inf_{x\in\mathbb{R}^N}[s(x) - \frac{V(x)}{g^2(\infty)}] \le s_\infty - \frac{V_\infty}{g^2(\infty)}$, where $a^0 > 0$ is independent on the specific forms of $V(x)$ and $s(x)$.

Remark 1.5. Assuming the decomposition $f(x,t) = s(x)f(t)$, hypotheses (A9) and (A10) are readily verified by direct computation. Furthermore, under the additional condition $s(x) \le s_\infty$, hypothesis (A7) is also satisfied. In the special case where $g \equiv 1$, $f(t) = t - \ln(t+1)$ fulfills our assumptions. Note that the choice of $f(t)$ is independent of that of $s(x)$. As an example, for suitably chosen functions $g(t)$ and $f(t)$, consider $V(x) \equiv V_\infty$ and $s(x) = s_\infty - (s_\infty - \frac{V_\infty}{g^2(\infty)} - a^0)e^{-|x|}$ where $s_\infty > \frac{V_\infty}{g^2(\infty)}$ and a^0 is as in the Lemma 3.19. This choice satisfies conditions (A1) and (A11).

Prior research has predominantly focused on nonlinearities of specific types. For instance, reference [19] investigated the case where the nonlinear term is independent of the spatial variable, i.e., $f(x,t) = f(t)$, and obtained a nontrivial Mountain Pass solution. Under our assumptions, the Mountain Pass Geometry can be established by Lemmas 3.3, 3.4 and the weak continuity of J'. However, the nontriviality of the solution cannot be guaranteed. To address this, we introduce additional hypotheses (A5) and (A6), which ensure that the Nehari manifold exhibits a suitable geometric structure.

Via (1.4), the problem studied in [11] can be recovered as a particular example of our model, with the condition $s(x) \le s_\infty$ there being equivalently replaced by $V(x) \le V_\infty$ here, under the substitution $g(t) \longleftrightarrow \sqrt{1+2t^2}$ and $f(x,t) \longleftrightarrow q(x)g(t)$ from [11].

Remark 1.6. To prevent our minimax value from being an accumulation point of critical levels of (2.4) we impose hypothesis (A12) as in [1] (see also in [4]).

Notation. Throughout this paper, we use the letters C, C_1, C_2, ... to denote positive constants whose values may change from line to line and play no essential role. For a center $y \in \mathbb{R}^N$ and a radius $r > 0$, the open ball $B_r(y)$ is defined as $\{x \in \mathbb{R}^N : |x-y| < r\}$. The symbol S^* stands for the unit sphere in the function space E. The usual L^p-norm is written as $|u|_p := \left(\int_{\mathbb{R}^N} |u|^p\right)^{1/p}$. Moreover, given a translation vector $y \in \mathbb{R}^N$ and a dilation parameter $\theta \in \mathbb{R}^+$, we define an action of $\mathbb{R}^N \times \mathbb{R}^+$ on E by setting $u^{y,\theta}(x) := u\left(\frac{x-y}{\theta}\right)$.

2. Preliminary results

The variational structure associated with equation (1.6) can be formally described by the energy functional

$$J_0(v) = \frac{1}{2}\int_{\mathbb{R}^N} g^2(v)|\nabla v|^2 + V(x)v^2 - \int_{\mathbb{R}^N} F(x,v).$$

However, J_0 may fail to be well-defined in the space E. We follow the approach introduced in [19] and perform a variable substitution,

$$w = G(v) = \int_0^v g(\tau)\,d\tau.$$

Hence, we obtain that

$$J(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + V(x)|G^{-1}(w)|^2 - \int_{\mathbb{R}^N} F(x, G^{-1}(w)). \tag{2.1}$$

From (A2), (A3) and (A4), we deduce that $J \in C^1(E,\mathbb{R})$. Obviously, we have

$$\langle J'(w), \phi\rangle = \int_{\mathbb{R}^N} \nabla w \nabla\phi + V(x)\frac{G^{-1}(w)}{g(G^{-1}(w))}\phi - \int_{\mathbb{R}^N} \frac{f(x,G^{-1}(w))}{g(G^{-1}(w))}\phi, \tag{2.2}$$

for every $w, \phi \in E$. Moreover, a function $u \in H^1(\mathbb{R}^N)$ is a critical point of J if and only if it is a weak solution of the equation

$$-\Delta u + V(x)\frac{G^{-1}(u)}{g(G^{-1}(u))} = \frac{f(x,G^{-1}(u))}{g(G^{-1}(u))}, \quad u \in H^1(\mathbb{R}^N). \tag{2.3}$$

It is established in [19] that (1.6) and (2.3) are equivalent in E.

We introduce the Nehari manifold associated with the functional J as

$$\mathcal{N} := \{u \in E \setminus \{0\} : \langle J'(u), u \rangle = 0\},$$

and we set the corresponding minimum level $c_0 := \inf_{\mathcal{N}} J$.

Next, consider the limiting problem

$$-\Delta u = \frac{f_\infty(G^{-1}(u))}{g(G^{-1}(u))} - V_\infty \frac{G^{-1}(u)}{g(G^{-1}(u))}. \tag{2.4}$$

The corresponding limiting functional and Nehari manifold are defined as

$$J_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V_\infty |G^{-1}(u)|^2 - \int_{\mathbb{R}^N} F_\infty(G^{-1}(u)), \tag{2.5}$$

$$\mathcal{N}_\infty := \{u \in E \setminus \{0\} : \langle J'_\infty(u), u \rangle = 0\}. \tag{2.6}$$

Equation (2.4) admits a positive radial ground state solution $u_\infty \in C^2(\mathbb{R}^N)$, since the function $g^*(u) := \frac{f_\infty(G^{-1}(u))}{g(G^{-1}(u))} - V_\infty \frac{G^{-1}(u)}{g(G^{-1}(u))}$ can be verified to satisfy all assumptions of [8, Theorem 3.1], which relies on the classical framework of [2] and [3] (cf. [8]). Set $c_\infty := J_\infty(u_\infty)$.

Let

$$\mathcal{T} := \Big\{u \in E : \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} \frac{V(x)}{g^2(\infty)} u^2 < \int_{\mathbb{R}^N} s(x) u^2\Big\},$$

$$\mathcal{T}_\infty := \Big\{u \in E : \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} \frac{V_\infty}{g^2(\infty)} u^2 < \int_{\mathbb{R}^N} s_\infty u^2\Big\}.$$

By conditions (A5) and (A11), we have $\mathcal{T} \neq \emptyset$ and $\mathcal{T}_\infty \neq \emptyset$. Although we cannot guarantee that all functions in $\mathcal{T}_\infty$ belong to $\mathcal{T}$, it is established in Lemma 3.11 that the modified functions do possess this property.

Given that our assumptions could not ensure that $\mathcal{N}$ forms a manifold of class C^1, we consequently adopt the framework established in [23, 24].

The following properties are consequences of assumption (A2). For their proofs, we refer the reader ito [19, 10].

Lemma 2.1. *The following statements hold for the function $G(t)$ and $G^{-1}(t)$:*

1. $G(t)$ *and* $G^{-1}(t)$ *are odd;*
2. $G^{-1}(t) \le t \le G(t) \le g(t)t$, *for any* $t > 0$;
3. $\frac{G^{-1}(t)}{t}$ *is decreasing on* $t \in (0, \infty)$;
4. $\lim_{t\to 0} \frac{G^{-1}(t)}{t} = 1$, $\lim_{t\to\infty} \frac{G^{-1}(t)}{t} = \frac{1}{g(\infty)}$.

3. Proof of Theorem 1.1

Lemma 3.1. *(1)* $0 \le F(x,t) \le \frac{f(x,t)G(t)}{2g(t)}$, *for all* $t \in \mathbb{R}$, $x \in \mathbb{R}^N$.

(2) For each $\varepsilon > 0$, $p \ge 1$, *there is* $C_{\varepsilon,p} > 0$ *such that* $|f(x,t)| \le \varepsilon|t| + C_{\varepsilon,p} g(t)|G(t)|^p$ *for all* $t \in \mathbb{R}$, $x \in \mathbb{R}^N$.

Proof. (1) It follows from (A4) and (A6) that $\frac{f(x,t)}{g(t)G(t)} \ge 0$ for every $t \neq 0$. It follows from (A2), Lemma 2.1-(1), (2) and the oddness of $f(x,\cdot)$ that $F(x,t) \ge 0$ for every $t \in \mathbb{R}$. By (A6), we obtain that

$$F(x,t) = \int_0^t f(x,s)ds \le \frac{f(x,t)}{2g(t)G(t)} \int_0^t \frac{dG^2(s)}{ds} ds = \frac{f(x,t)G(t)}{2g(t)}.$$

(2) Let $s^* := \sup_{x\in\mathbb{R}^N} s(x)$, it follows from (A2), (A5) and (A6) that

$$|f(x,t)| \le s^* g(t)|G(t)| \le C_1 + s^* g(t)|G(t)|^p \le s^*(C_2 + g(t)|G(t)|^p),$$

for every $p \ge 1$. Using (A4), we obtain the conclusion. □

For each element $u \in \mathcal{T}$, we introduce the auxiliary function Q_u by $Q_u(t) = J(tu)$ for $t > 0$.

Lemma 3.2. *(1) There is a unique $t_u > 0$ such that $Q'_u(t_u) = 0$. Furthermore, $t_0 u \in \mathcal{N}$ is equivalent to $t_0 = t_u$.*
(2) $\mathbb{R}^+\mathcal{N} = \mathcal{T}$.

Proof. (1) From (A4), we obtain that $\lim_{t\to 0} \frac{F(x,t)}{|G(t)|^2} = 0$. Consider that
$$\frac{Q_u(t)}{t^2} = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int_{u\neq 0} V(x)\frac{|G^{-1}(tu)|^2}{t^2u^2}u^2 - \int_{u\neq 0} \frac{F(x, G^{-1}(tu))}{t^2u^2}u^2.$$
Applying Lemma 2.1-(4) yields
$$\lim_{t\to 0} \frac{Q_u(t)}{t^2} = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 > 0.$$
Using (A5), we have $\lim_{t\to\infty} \frac{F(x,t)}{|G(t)|^2} = \frac{1}{2}s(x)$. Thus,
$$\lim_{t\to\infty} \frac{Q_u(t)}{t^2} = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{V(x)}{g^2(\infty)}u^2 - \frac{1}{2}\int_{\mathbb{R}^N} s(x)u^2 < 0.$$
Obviously, the condition $Q'_u(t) = 0$ is equivalent to
$$\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{u\neq 0} V(x)\frac{G^{-1}(tu)}{g(G^{-1}(tu))tu}u^2 = \int_{u\neq 0} \frac{f(x, G^{-1}(tu))}{g(G^{-1}(tu))tu}u^2.$$
It follows from (A2) and Lemma 2.1-(3) that
$$\frac{G^{-1}(\tau)}{g(G^{-1}(\tau))\tau} = \frac{G^{-1}(\tau)}{\tau}\cdot\frac{1}{g(G^{-1}(\tau))}$$
decreases for $\tau > 0$. Moreover, $\frac{d}{d\tau}(\frac{G^{-1}(\tau)}{\tau})|_{\tau=\tau_0} = 0$ if and only if $g(\tau) = 1$ for any $0 \le |\tau| \le |\tau_0|$. Using (A6), we obtain that $\frac{f(x,G^{-1}(\tau))}{g(G^{-1}(\tau))\tau} - \frac{V(x)G^{-1}(\tau)}{g(G^{-1}(\tau))\tau}$ is strictly increasing on $\tau \in (0,\infty)$ and strictly decreasing on $\tau \in (-\infty, 0)$, which yields the existence and uniqueness of t_u. The conclusion follows directly from the r elation $Q'_u(t) = \frac{\langle J'(tu), tu\rangle}{t}$.

(2) Obviously, $\mathcal{T} \subset \mathbb{R}^+\mathcal{N}$. Remember that $\frac{f(x,G^{-1}(\tau))}{g(G^{-1}(\tau))\tau} - \frac{V(x)G^{-1}(\tau)}{g(G^{-1}(\tau))\tau}$ is strictly increasing on $\tau \in (0,\infty)$ and strictly decreasing on $\tau \in (-\infty, 0)$, for each $u \in \mathcal{N}$, we obtain
$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u|^2 &= \int_{u\neq 0} \frac{f(x, G^{-1}(u))}{g(G^{-1}(u))u}u^2 - V(x)\frac{G^{-1}(u)}{g(G^{-1}(u))u}u^2 \\ &< \int_{\mathbb{R}^N} s(x)u^2 - \frac{V(x)}{g^2(\infty)}u^2.\end{aligned}$$
Hence $u \in \mathcal{T}$, which means $\mathcal{N} \subset \mathcal{T}$, i.e., $\mathbb{R}^+\mathcal{N} \subset \mathcal{T}$. □

Lemma 3.3. *(1) There exist positive constants r_0 and M such that $M \le \inf_{S_{r_0}} J \le c_0$, where $S_{r_0} := \{u \in E : \|u\| = r_0\}$.*
(2) Every $u \in \mathcal{N}$ satisfies $\|u\|^2 \ge 2c_0$.

Proof. (1) The estimate $\inf_{S_{r_0}} J \ge M$ for some $M > 0$ follows immediately from [19, Lemma 2.1], so its proof is omitted. Using Lemma 3.2, for every $u \in \mathcal{N}$, we have $\frac{r_0 u}{\|u\|} \in S_{r_0}$ and $J(u) \ge J(\frac{r_0 u}{\|u\|})$. Consequently, the first inequality holds.

(2) The conclusion is obtained directly from $|G^{-1}(u)|^2 \le u^2$ and $F(x,t) \ge 0$. □

Lemma 3.4. *For every compact subset $\mathcal{B} \subset \mathcal{T}$, there is $r_1 > 0$ such that $J(tu) \le 0$ for any $u \in \mathcal{B}$ and $t \ge r_1$.*

Proof. Arguing by contradiction, set $(u_n) \subset \mathcal{B}$ and $(t_n) \subset \mathbb{R}^+$ with $J(t_n u_n) \ge 0$ and $t_n \to \infty$. In addition, because of the compactness of $\mathcal{B}$, there exists $u \in \mathcal{B}$ such that, up to a subsequence, u_n converges to u. Then, applying (A5), Lemma 2.1-(4) and the Lebesgue dominated convergence theorem, we obtain
$$0 \le \frac{J(t_n u_n)}{t_n^2} = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{1}{2}\int_{u_n\neq 0} V(x)\frac{|G^{-1}(t_n u_n)|^2}{t_n^2u_n^2}u_n^2 - \int_{u_n\neq 0} \frac{F(x, G^{-1}(t_n u_n))}{t_n^2u_n^2}u_n^2$$

$$\to \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + \frac{V(x)}{g^2(\infty)}u^2 - \frac{1}{2}\int_{\mathbb{R}^N} s(x)u^2 < 0,$$

a contradiction. □

The preceding lemmas hold analogously for the limiting functional J_∞, with the replacements of $\mathcal{N}$, $\mathcal{T}$, and c_0 by $\mathcal{N}_\infty$, $\mathcal{T}_\infty$, and c_∞, respectively.

Lemma 3.5. *In the Nehari manifold $\mathcal{N}$, every Palais-Smale sequence (v_n) is bounded.*

Proof. Argument by contradiction. Suppose that there exists a Palais–Smale sequence (v_n) contained in $\mathcal{N}$ such that $J(v_n) \le C$ and $J'(v_n) \to 0$ while $\|v_n\| \to \infty$. Define the normalized sequence $w_n := v_n/\|v_n\|$. By extracting a suitable subsequence, w_n converges weakly in E to w, and converges almost everywhere in $\mathbb{R}^N$ t o w.

If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} w_n^2 \to 0,$$

then according to P.L. Lions' lemma [25, p.16], we obtain $w_n \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*)$, with $2^* := \frac{2N}{N-2}$ for $N \ge 3$ and $2^* := \infty$ otherwise. From Lemma 2.1-(2) and Lemma 3.1-(2), for every fixed $t > 0$, we find $\int_{\mathbb{R}^N} F(x, G^{-1}(tw_n)) \to 0$ as $n \to \infty$. Using Lemma 2.1-(4), we have

$$\lim_{\tau\to 0} \frac{\tau^2 - |G^{-1}(\tau)|^2}{\tau^2} = 0 \quad \text{and} \quad \lim_{\tau\to\infty} \frac{\tau^2 - |G^{-1}(\tau)|^2}{\tau^p} = 0.$$

Thus, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $\left|t^2w_n^2 - |G^{-1}(t^2w_n^2)|^2\right| \le \varepsilon|tw_n|^2 + C_\varepsilon|tw_n|^p$. Hence

$$\int_{\mathbb{R}^N} V(x)\left(t^2w_n^2 - |G^{-1}(tw_n)|^2\right) \to 0,$$

for every $t > 0$. Using Lemma 3.2, we obtain $J(tw_n) \le J(v_n) \le d$ and

$$\begin{aligned} J(tw_n) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla w_n|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(tw_n)|^2 - \int_{\mathbb{R}^N} F(x, G^{-1}(tw_n)) \\ &= \frac{t^2}{2}\|w_n\|^2 - \frac{1}{2}\int_{\mathbb{R}^N} V(x)\left(t^2w_n^2 - |G^{-1}(tw_n)|^2\right) - \int_{\mathbb{R}^N} F(x, G^{-1}(tw_n)). \end{aligned}$$

For sufficiently large t, this leads to a contradiction.

Consequently, taking a subsequence if necessary, we can find a constant $\eta > 0$ and a sequence $(y_n) \subset \mathbb{R}^N$ satisfying

$$\int_{B_1(0)} (w_n^{y_n,1})^2 \ge \eta > 0.$$

Provided that the sequence (y_n) is bounded, then $y_n \to y_0$ by extracting a suitable subsequence which informs $w_n^{y_n,1} \to w^{y_0,1}$ in $L^2(B_1(0))$ and $w(x) \not\equiv 0$. Furthermore, since $v_n(x) \to \infty$ on the set $\{x : w(x) \ne 0\}$, for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, it holds that $\left\langle \frac{J'(v_n)}{\|v_n\|}, \varphi \right\rangle \to 0$, i.e.,

$$\int_{\mathbb{R}^N} \nabla w_n \nabla\varphi + \int_{v_n\ne 0} \frac{V(x)G^{-1}(v_n)}{g(G^{-1}(v_n))v_n} w_n\varphi = \int_{v_n\ne 0} \frac{f(x, G^{-1}(v_n))}{g(G^{-1}(v_n))v_n} w_n\varphi + o(1),$$

where $o(1) \to 0$ as $n \to \infty$. Since w_n converges weakly in E to w, we obtain

$$\int_{\mathbb{R}^N} \nabla w\nabla\varphi + \frac{V(x)}{g^2(\infty)} w\varphi = \int_{\mathbb{R}^N} s(x)w\varphi.$$

Considering that the essential spectrum of $-\Delta - \left(s(x) - \frac{V(x)}{g^2(\infty)}\right)$ is $[-s_\infty + \frac{V_\infty}{g^2(\infty)}, \infty)$ (cf. Theorem 3.15 in [22]), the preceding result leads to a contradiction.

Then $|y_n| \to \infty$, it follows that, up to a subsequence, $w_n^{y_n,1} \rightharpoonup w^* \not\equiv 0$. Likewise, we have

$$\int_{\mathbb{R}^N} \nabla w^*\nabla\varphi + \frac{V_\infty}{g^2(\infty)} w^*\varphi = \int_{\mathbb{R}^N} s_\infty w^*\varphi,$$

and it is also a contradiction. The conclusion follows. □

Lemma 3.6. *If $g \not\equiv 1$, then* $\lim_{t\to\infty} t^2 - \frac{G^2(t)}{g^2(\infty)} = \infty$.

Proof. We shall only consider $t > 0$. By (A2), we have

$$t^2 - \frac{G^2(t)}{g^2(\infty)} \geq t^2 - \frac{tG(t)}{g(\infty)} = t\int_0^t [1 - \frac{g(s)}{g(\infty)}]ds.$$

If $g \not\equiv 1$, then there is a positive constant C such that $t^2 - \frac{G^2(t)}{g^2(\infty)} \geq Ct$ for any $t > 1$. The conclusion follows. □

We define $\mathcal{N}_0 := \mathcal{T} \cap S^*$. We introduce a map $n : \mathcal{N}_0 \mapsto \mathcal{N}$ defined by $n(u) := t_u u$, where t_u is given by Lemma 3.2-(1). The openness of $\mathcal{T}$ in E implies the openness of $\mathcal{N}_0$ in S^*.

Lemma 3.7. *If $(u_n) \subset \mathcal{N}_0$, $t_n u_n \in \mathcal{N}$ and $u_n \to u_0 \in \partial\mathcal{N}_0$, then $t_n \to \infty$ and $J(t_n u_n) \to \infty$.*

Proof. Suppose $t_n \not\to \infty$, then we can assume $t_n \to t_0 \geq 2c_0 > 0$ from Lemma 3.3-(2). It follows from $0 = \langle J'(t_n u_n), t_n u_n\rangle \to \langle J'(t_0 u_0), t_0 u_0\rangle$ that $t_0 u_0 \in \mathcal{N}$, which is contrary to $u_0 \in \partial\mathcal{N}_0$.

Note that

$$\int_{\mathbb{R}^N} |\nabla u_0|^2 + \int_{\mathbb{R}^N} \frac{V(x)}{g^2(\infty)} u_0^2 = \int_{\mathbb{R}^N} s(x) u_0^2.$$

Thus, we have

$$\begin{aligned} J(tu_0) &= \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|G^{-1}(tu_0)|^2 - \int_{\mathbb{R}^N} F(x, G^{-1}(tu_0)) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} V(x)\Big[|G^{-1}(tu_0)|^2 - \frac{t^2u_0^2}{g^2(\infty)}\Big] + \int_{\mathbb{R}^N} \frac{1}{2}s(x)t^2u_0^2 - F(x, G^{-1}(tu_0)). \end{aligned}$$

Using (A8), Lemma 3.6 and Fatou's lemma, we have $J(tu_0) \to \infty$, as $t \to \infty$. For any positive integer k, choose $t_k > 0$ that satisfies $J(t_k u_0) \geq k$. From Lemma 3.2-(1), we have $J(t_n u_n) \geq J(t_k u_n) \to J(t_k u_0) \geq k$. Since k was arbitrary, it follows that $J(t_n u_n) \to \infty$, as $n \to \infty$. □

While the functional J is not coercive on $\mathcal{N}$ under our assumptions (whereas coercivity would imply the continuity of n by [23, Lemma 2.8]), the following lemma states that n is continuous.

Lemma 3.8. *The map $n : \mathcal{N}_0 \mapsto \mathcal{N}$ is continuous.*

Proof. Set $(u_n) \subset \mathcal{N}_0$ satisfy $u_n \to u_0$ and let $n(u_n) = t_n u_n$. The sequence (t_n) is bounded. Indeed, if it were unbounded, we would have (after passing to a subsequence) $t_n \to \infty$. It follows from (A5), Lemma 2.1-(4) and the Lebesgue dominated convergence theorem that

$$\begin{aligned} 0 &\leq \frac{J(t_n u_n)}{t_n^2} \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n|^2 + \frac{1}{2}\int_{u_n\neq 0} V(x)\frac{|G^{-1}(t_n u_n)|^2}{t_n^2 u_n^2} u_n^2 - \int_{u_n\neq 0} \frac{F(x, G^{-1}(t_n u_n))}{t_n^2 u_n^2} u_n^2 \\ &\to \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_0|^2 + \frac{V(x)}{g^2(\infty)} u_0^2 - \frac{1}{2}\int_{\mathbb{R}^N} s(x) u_0^2 < 0. \end{aligned}$$

Thus, along a subsequence, we obtain $t_n \to t^*$. From Lemma 3.3-(2), we have $t^* \geq 2c_0 > 0$. Hence

$$0 = \langle J'(t_n u_n), t_n u_n\rangle \to \langle J'(t^* u_0), t^* u_0\rangle,$$

which implies $t^* u_0 = n(u_0)$. □

Lemma 3.9. *The map $n : \mathcal{N}_0 \mapsto \mathcal{N}$ is a homeomorphism whose inverse is $n^{-1}(u) = \frac{u}{\|u\|}$.*

Proof. The invertibility of n follows directly from Lemma 3.2. Using Lemma 3.3-(2), we have

$$\|n^{-1}(w_1) - n^{-1}(w_2)\| = \|\frac{w_1 - w_2}{\|w_1\|} + \frac{(\|w_2\| - \|w_1\|)w_2}{\|w_1\|\|w_2\|}\| \leq \sqrt{\frac{2}{c_0}}\|w_1 - w_2\|,$$

for every $w_1, w_2 \in \mathcal{N}$. Then n^{-1} is Lipschitz continuous. □

To study the functional J on the Nehari manifold $\mathcal{N}$, we define $\Upsilon : \mathcal{N}_0 \to \mathbb{R}$ by $\Upsilon(w) := J(n(w))$, which leads to the following lemma on the correspondence between the critical values and Palais-Smale sequences of Υ and J. A key property of Υ and J is given in the next lemma (cf. [11, Lemma 3.9]).

Lemma 3.10. *(1) $\Upsilon \in C^1(\mathcal{N}_0, \mathbb{R})$ and*

$$\langle \Upsilon'(u), v\rangle = \|n(u)\|\langle J'(n(u)), v\rangle \quad \text{for all } v \in T_u(\mathcal{N}_0).$$

(2) If $(v_n) \subset \mathcal{N}_0$ is a Palais-Smale sequence for Υ, then $(n(v_n)) \subset \mathcal{N}$ is a Palais-Smale sequence for J. On the other hand, if $(u_n) \subset \mathcal{N}$ is a bounded Palais-Smale sequence for J, then $(n^{-1}(u_n)) \subset \mathcal{N}_0$ is a Palais-Smale sequence for Υ.

(3) The critical points of Υ and the nontrivial critical points of J are in a one-to-one correspondence via the map n: v is a critical point of Υ if and only if $n(v)$ is a nontrivial critical point of J. Furthermore, the corresponding critical values coincide, and we have $\inf_{\mathcal{N}_0} \Upsilon = \inf_{\mathcal{N}} J$.

Lemma 3.11. $c_0 \le c_\infty$.

Proof. Note that $u_\infty^{y,1}(x) := u_\infty(x - y)$. Using Lemma 2.1-(4), (A3), (A5) and an integral Transform, we have

$$\begin{aligned}\frac{\langle J'_\infty(ru_\infty^{y,1}), ru_\infty^{y,1}\rangle}{r^2} &= \int_{\mathbb{R}^N} |\nabla u_\infty^{y,1}|^2 + \int_{\mathbb{R}^N} V_\infty \frac{G^{-1}(ru_\infty^{y,1})}{g(G^{-1}(ru_\infty^{y,1}))ru_\infty^{y,1}}(u_\infty^{y,1})^2 \\ &\quad - \int_{\mathbb{R}^N} \frac{f_\infty(G^{-1}(ru_\infty^{y,1}))}{g(G^{-1}(ru_\infty^{y,1}))ru_\infty^{y,1}}(u_\infty^{y,1})^2 \\ &\to \int_{\mathbb{R}^N} |\nabla u_\infty|^2 + \int_{\mathbb{R}^N} \frac{V_\infty}{g^2(\infty)} u_\infty^2 - \int_{\mathbb{R}^N} s_\infty u_\infty^2 < 0,\end{aligned}$$

as $r \to \infty$, for every $y \in \mathbb{R}^N$. Consequently, for some $\alpha < 0$ and $R \ge 0$, the inequality $\frac{\langle J'_\infty(ru_\infty^{y,1}), ru_\infty^{y,1}\rangle}{r^2} \le \alpha$ holds for any $r \ge R$. Applying (A1) and Lemma 2.1-(3), (4), we obtain

$$\lim_{|y|\to\infty} \int_{\mathbb{R}^N} [V(x+y) - V_\infty] \frac{G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty} u_\infty^2 = 0.$$

A similar argument shows that

$$\lim_{|y|\to\infty} \int_{\mathbb{R}^N} \frac{f_\infty(G^{-1}(ru_\infty)) - f(x+y, G^{-1}(ru_\infty))}{g(G^{-1}(ru_\infty))ru_\infty} u_\infty^2 = 0,$$

by (A3), (A5) and (A6). Then $\frac{\langle J'(ru_\infty^{y,1}), ru_\infty^{y,1}\rangle}{r^2} = \frac{\langle J'_\infty(ru_\infty^{y,1}), ru_\infty^{y,1}\rangle}{r^2} + o(1)$, where $o(1) \to 0$, as $|y| \to \infty$. Hence, for some $S > 0$, the inequality $\frac{\langle J'(ru_\infty^{y,1}), ru_\infty^{y,1}\rangle}{r^2} \le \frac{\alpha}{2} < 0$ holds whenever $r \ge R$ and $|y| \ge S$.

Observe that $\langle J'(ru_\infty^{y,1}), ru_\infty^{y,1}\rangle$ is positive definite for small $r > 0$. Following the argument in Lemma 3.2-(1), we find that $\frac{\langle J'(ru_\infty^{y,1}), ru_\infty^{y,1}\rangle}{r^2}$ decreases strictly on $r \in (0, \infty)$. Consequently, for every $|y| \ge S$, the interval $(0, R)$ contains precisely one element T^y satisfying $\langle J'(T^y u_\infty^{y,1}), T^y u_\infty^{y,1}\rangle = 0$, which means $T^y u_\infty^{y,1} \in \mathcal{N}$. Thus,

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_\infty|^2 &= \int_{\mathbb{R}^N} \Big[\frac{f(x+y, G^{-1}(T^y u_\infty))}{g(G^{-1}(T^y u_\infty))T^y u_\infty} - \frac{V(x+y)G^{-1}(T^y u_\infty)}{g(G^{-1}(T^y u_\infty))T^y u_\infty}\Big] u_\infty^2 \\ &= \int_{\mathbb{R}^N} \Big[\frac{f_\infty(G^{-1}(T^y u_\infty))}{g(G^{-1}(T^y u_\infty))T^y u_\infty} - \frac{V_\infty G^{-1}(T^y u_\infty)}{g(G^{-1}(T^y u_\infty))T^y u_\infty}\Big] u_\infty^2 + o(1).\end{aligned}$$

Given that $\frac{f_\infty(G^{-1}(s))}{g(G^{-1}(s))s} - \frac{V_\infty G^{-1}(s)}{g(G^{-1}(s))s}$ is strictly increasing on $s \in (0, \infty)$, we have $T^y \to 1$ and then $c_0 \le J(T^y u_\infty^{y,1}) \to c_\infty$. □

Lemma 3.12. *For each Palais-Smale sequence $(u_n) \subset \mathcal{N}$ with $J(u_n) \to C$, after passing to a suitable subsequence, there exists a solution $u^0 \in E$ to*

$$-\Delta w + V(x) \frac{G^{-1}(w)}{g(G^{-1}(w))} = \frac{f(x, G^{-1}(w))}{g(G^{-1}(w))},$$

a finite sequence of solutions $u^1, \dots, u^k \in E$ *to*

$$-\Delta w + V_\infty \frac{G^{-1}(w)}{g(G^{-1}(w))} = \frac{f_\infty(G^{-1}(w))}{g(G^{-1}(w))},$$

and k *sequences* $(y_n^j) \subset \mathbb{R}^N$ *satisfying*

$$|y_n^j| \to \infty, \quad |y_n^j - y_n^{j'}| \to \infty, \quad j \neq j', \quad n \to \infty,$$

$$\int_{\mathbb{R}^N} |\nabla(u_n - u^0 - \sum_{j=1}^k u^j(\cdot - y_n^j))|^2 + |G^{-1}(u_n - u^0 - \sum_{j=1}^k u^j(\cdot - y_n^j))|^2 \to 0,$$

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 + |G^{-1}(u_n)|^2 \to \sum_{j=0}^k \int_{\mathbb{R}^N} |\nabla u^j|^2 + |G^{-1}(u^j)|^2,$$

$$J(u^0) + \sum_{j=1}^k J_\infty(u^j) = C.$$

Proof. Using Lemma 3.5, selecting a subsequence if necessary, $u_n \rightharpoonup u^0$ in E. We claim that $\int_{\mathbb{R}^N} |G^{-1}(u_n)|^2 - |G^{-1}(u_n - u^0)|^2 - |G^{-1}(u^0)|^2 = o(1)$ and $\int_{\mathbb{R}^N} [\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(u_n - u^0)}{g(G^{-1}(u_n - u^0))} - \frac{G^{-1}(u^0)}{g(G^{-1}(u^0))}]\varphi = o(1)$ where $o(1) \to 0$, as $n \to \infty$, uniformly for $\|\varphi\| \le 1$.

We now establish the second assertion. Since $\frac{G^{-1}(t)}{g(G^{-1}(t))t}$ is nonincreasing on $t \in (0, \infty)$, we have $[\frac{G^{-1}(t)}{g(G^{-1}(t))}]' \le \frac{G^{-1}(t)}{g(G^{-1}(t))t} \le 1$, for any $t > 0$. An application of the mean value theorem and Hölder's inequality shows that

$$\int_{|x|>r} [\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(u_n - u^0)}{g(G^{-1}(u_n - u^0))}]\varphi \le |\varphi|_2 [\int_{|x|>r} |u^0|^2]^{\frac{1}{2}}.$$

Thus, for each $\varepsilon > 0$, there is a corresponding $r_1 > 0$ satisfying

$$\int_{|x|>r_1} [\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(u_n - u^0)}{g(G^{-1}(u_n - u^0))} - \frac{G^{-1}(u^0)}{g(G^{-1}(u^0))}]\varphi \le \varepsilon \|\varphi\|.$$

Then, using the Rellich theorem, we obtain that

$$\int_{|x|\le r_1} [\frac{G^{-1}(u_n)}{g(G^{-1}(u_n))} - \frac{G^{-1}(u_n - u^0)}{g(G^{-1}(u_n - u^0))} - \frac{G^{-1}(u^0)}{g(G^{-1}(u^0))}]\varphi \le \varepsilon \|\varphi\|.$$

The first statement can be proved similarly.

Following the argument in Lemmas 3.4 and 3.5 of [27], we obtain, by (A2) and the proof of Lemma 3.1-(2), that $\int_{\mathbb{R}^N} F(x, G^{-1}(u_n)) - F(x, G^{-1}(u_n - u^0)) - F(x, G^{-1}(u^0)) = o(1)$ and

$$\int_{\mathbb{R}^N} [\frac{f(x, G^{-1}(u_n))}{g(G^{-1}(u_n))} - \frac{f(x, G^{-1}(u_n - u^0))}{g(G^{-1}(u_n - u^0))} - \frac{f(x, G^{-1}(u^0))}{g(G^{-1}(u^0))}]\varphi = o(1),$$

where $o(1) \to 0$, as $n \to \infty$, uniformly for $\|\varphi\| \le 1$. Hence, $J(u_n - u^0) = J(u_n) - J(u^0) + o(1)$ and $J'(u_n - u^0) = J'(u_n) - J'(u^0) + o(1)$. Similarly, we obtain $J_\infty(u_n - u^0) = J_\infty(u_n) - J_\infty(u^0) + o(1)$ and $J'_\infty(u_n - u^0) = J'_\infty(u_n) - J'_\infty(u^0) + o(1)$.

Let $u_n^1 := u_n - u^0$. Then $u_n^1 \to 0$ in $L^2_{loc}(\mathbb{R}^N)$ by the Rellich theorem. We assert that $J_\infty(u_n^1) = J(u_n) - J(u^0) + o(1)$ and $J'_\infty(u_n - u^0) = J'(u_n) - J'(u^0) + o(1)$.

We now prove the second assertion.

$$\begin{aligned} |\langle J'_\infty(u_n^1) - J'(u_n^1), \varphi\rangle| &\le \int_{\mathbb{R}^N} |V(x) - V_\infty| \cdot |\frac{G^{-1}(u_n^1)}{g(G^{-1}(u_n^1))}| \cdot |\varphi| \\ &\quad + \int_{\mathbb{R}^N} |\frac{f(x, G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))} - \frac{f_\infty(G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))}| \cdot |\varphi|. \end{aligned}$$

For each $\varepsilon > 0$, a positive number r_2 can be found such that $|V(x) - V_\infty| < \varepsilon$ whenever $|x| > r_2$. Employing Hölder's inequality under this condition produces

$$\int_{|x|>r_2} |V(x) - V_\infty| \cdot |\frac{G^{-1}(u_n^1)}{g(G^{-1}(u_n^1))}| \cdot |\varphi| \le \varepsilon C |\varphi|_2.$$

From the Rellich theorem, we obtain

$$\int_{|x|\le r_2} |V(x) - V_\infty| \cdot |\frac{G^{-1}(u_n^1)}{g(G^{-1}(u_n^1))}| \cdot |\varphi| \le \varepsilon \|\varphi\|,$$

for n large enough. Note that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\frac{f(x, G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))} - \frac{f_\infty(G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))}| \cdot |\varphi| \\
&\le \int_{|x-y|\le r_3} |\frac{f(x, G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))}| \cdot |\varphi| + |\frac{f_\infty(G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))}| \cdot |\varphi| \\
&\quad + \int_{|x-y|>r_3,\, |G^{-1}(u_n^1)|>T} |\frac{f(x, G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))u_n^1} - s(x)| \cdot |u_n^1| \cdot |\varphi| \\
&\quad + \int_{|x-y|>r_3,\, |G^{-1}(u_n^1)|>T} |\frac{f_\infty(G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))u_n^1} - s_\infty| \cdot |u_n^1| \cdot |\varphi| + |s_\infty - s(x)| \cdot |\varphi| \\
&\quad + \int_{|x-y|>r_3,\, |u_n^1|\le G(T)} |\frac{f(x, G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))} - \frac{f_\infty(G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))}| \cdot |u_n^1| \cdot |\varphi|.
\end{aligned}$$

Using (A3), (A5), (A9) (or (A10)), the Rellich theorem and Hölder inequality, there exist some $T > 0$ large enough (if (A9) holds, $y = 0$; if (A10) holds, y is as in (A10)) and $r_3 > 0$ large enough such that $|\langle J'_\infty(u_n^1) - J'(u_n^1), \varphi\rangle| \le \varepsilon C\|\varphi\|$, for n large enough. A similar argument applies to the first statement.

Set

$$\delta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n^1|^2.$$

In the case $\delta = 0$, we have $u_n^1 \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$, by the P.L. Lions lemma [25, Lemma 1.21]). This convergence, combined with assumptions (h_1), (g) and the results of Lemma 2.1-(2) and Lemma 3.1-(2), implies that for any $\varepsilon > 0$ one can find $C_\varepsilon > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\frac{f(x, G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))} u_n^1| &\le \int_{\mathbb{R}^N} \varepsilon |u_n^1|^2 + C_\varepsilon |u_n^1|^p, \\
\int_{\mathbb{R}^N} |\frac{f_\infty(G^{-1}(u_n^1))}{g(G^{-1}(u_n^1))} u_n^1| &\le \int_{\mathbb{R}^N} \varepsilon |u_n^1|^2 + C_\varepsilon |u_n^1|^p.
\end{aligned}$$

From the identity $\langle J'_\infty(u_n^1), u_n^1\rangle = \langle J'(u_n), u_n^1\rangle + \langle J'(u^0), u_n^1\rangle + o(1) = o(1)$, together with the estimate provided in Lemma 2.1-(2), we obtain that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n^1|^2 + V_\infty |G^{-1}(u_n^1)|^2 \\
&\le \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_n^1|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty |G^{-1}(u_n^1)|^p + C\int_{|G^{-1}(u_n^1)|\le 1} V_\infty \frac{G^{-1}(u_n^1)u_n^1}{g(G^{-1}(u_n^1))} \\
&\le C\int_{\mathbb{R}^N} |\nabla u_n^1|^2 + V_\infty \frac{G^{-1}(u_n^1)u_n^1}{g(G^{-1}(u_n^1))} + \frac{V_\infty}{2}\int_{\mathbb{R}^N} |u_n^1|^p \to 0,
\end{aligned}$$

as $n \to \infty$. The conclusion follows.

If $\delta > 0$, then there is a sequence (y_n^1) in $\mathbb{R}^N$ for which $\int_{B_1(0)} |(u_n^1)^{-y_n,1}|^2 > \frac{\delta}{2}$. By the local compactness of the Sobolev embedding and after passing to a subsequence, we may assume that $(u_n^1)^{-y_n,1} \rightharpoonup u^1 \ne 0$. The weak convergence $u_n^1 \rightharpoonup 0$ implies that (y_n^1) is unbounded. Therefore, we may extract a subsequence such that $y_n^1 \to \infty$, as $n \to \infty$. The identity $J'_\infty(u^1) = 0 \in H^{-1}(\mathbb{R}^N)$ holds due to the weak sequential continuity and translation invariance of J'_∞.

Define $u_n^2 := u_n^1 - (u^1)^{y_n,1}$, then $u_n^2 \rightharpoonup 0$ and $(u_n^2)^{-y_n,1} \rightharpoonup 0$ in E. Take the same argument as above and repeat the process, except that J should be replaced by J_∞. Since $(J(u_n))$ is bounded, the fact that every nontrivial critical point of J_∞ have energy at least c_∞ yields that the iteration terminates finitely. □

Lemma 3.13. *For each $b > 0$, there exist a constant $C_b \in (0, 1]$ such that*

$$C_b\|u\|^2 \le \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|G^{-1}(u)|^2 \le \|u\|^2,$$

for all $\|u\| \le b$.

Proof. The detailed argument can be found in [12, Lemma 3.2]. Nevertheless, we provide a proof sketch for the sake of completeness. An application of Lemma 2.1(2) yields that

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|G^{-1}(u)|^2 \le \|u\|^2.$$

Suppose, for a contradiction, that there is a sequence (u_n) of nonzero elements in E satisfying $\|u_n\| \le b$ and

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 + \int_{u_n\neq 0} V(x)\frac{|G^{-1}(u_n)|^2}{u_n^2} w_n^2 \to 0,$$

where $w_n := \frac{u_n}{\|u_n\|}$. This leads to $\int_{\mathbb{R}^N} |\nabla w_n|^2 \to 0$, whereas

$$\int_{\mathbb{R}^N} V(x)w_n^2 \to 1. \tag{3.1}$$

For a given constant $C_1 > 0$, let $M_n^1 := \{x \in \mathbb{R}^N : |u_n(x)| \ge C_1\}$ and M_n^2 denote its complement. Since $\|u_n\| \le b$, it follows that for an arbitrary $\varepsilon > 0$, there exists $C_1 = C_\varepsilon$ satisfying $|M_n^1| \le \varepsilon$. Additionally, Lemma 2.1-(3) implies that $\frac{G^{-1}(t)}{t}$ decreases on $t \in (0, \infty)$. Thus,

$$\frac{|G^{-1}(C_1)|^2}{C_1^2}\int_{M_n^2} V(x)w_n^2 \le \int_{M_n^2} V(x)\frac{|G^{-1}(u_n)|^2}{u_n^2} w_n^2 \to 0.$$

For sufficiently small ε, combining Hölder's inequality with assumption (A1), we obtain that

$$\int_{M_n^1} V(x)w_n^2 \le C_2\varepsilon^{(2^*-2)/2^*} \le \frac{1}{2},$$

which contradicts (3.1). □

Proposition 3.14. *Suppose conditions* (A1)–(A8), (A11) *hold, and at least one of* (A9) *and* (A10) *holds. Then, the strict inequality $c_0 < c_\infty$ guarantees the existence of a nontrivial ground state solution to* (2.3).

Proof. Although Ekeland's variational principle (see [6, Theorem 4.8.1]) is stated for a complete metric space, Lemma 3.7 ensures its validity in $\mathcal{N}_0$, as it precludes the limiting point from reaching the boundary. So, there exists $(w_n) \subset \mathcal{N}$ such that $\Upsilon(w_n) \to c_0$ and $\Upsilon'(w_n) \to 0$. Let $u_n := n(w_n)$. Then, by Lemmas 3.5 and 3.10, the sequence (u_n) constitutes a bounded Palais-Smale sequence for the functional J at level c_0. Since $c_0 < c_\infty$, Lemma 3.12 implies $\int_{\mathbb{R}^N} |\nabla(u_n - u^0)|^2 + |G^{-1}(u_n - u^0)|^2 \to 0$ and thus $\|u_n - u^0\| \to 0$ by Lemma 3.13. Using $J(n(w_n)) \to J(u^0)$ and Lemma 3.7, we obtain $w_n \to w^0 \in \mathcal{N}_0$ and $u^0 = n(w^0) \in \mathcal{N}$. □

We now consider the case $c_0 = c_\infty$. Following [1, 5], we define the barycenter map β for $u \in E \setminus \{0\}$. First, define $\tau(u)(x) := \frac{1}{|B_1|}\int_{B_1(x)} |u(y)|\,dy$, which is a continuous function and bounded. Then, set $\hat{u}(x) := \big[\tau(u)(x) - \frac{1}{2}\max\tau(u)\big]^+$. The barycenter of u is

$$\beta(u) := \frac{1}{|\hat{u}|_1}\int_{\mathbb{R}^N} x\hat{u}(x)\,dx \in \mathbb{R}^N.$$

Owing to the compact support of $\hat{u}$, the map $\beta(u)$ defined above is well-defined and verifies the following:

(1) β is continuous in $E \setminus \{0\}$.
(2) $\beta(u) = 0$ whenever u is a radial function.
(3) $\beta(tu) = \beta(u)$, for any $t > 0$.
(4) $\beta(u^{y,1}) = \beta(u) + y$, for any $y \in \mathbb{R}^N$.

Set $b := \inf_{u\in\mathcal{N},\beta(u)=0} J(u) = \inf_{v\in\mathcal{N}_0,\beta(n(v))=0} \Upsilon(v)$. Directly from the definitions, it follows that $b \geq c_0 = c_\infty$.

Proposition 3.15. *Suppose conditions* (A1)–(A8), (A11) *hold, and at least one of* (A9) *and* (A10) *holds. Then, the equality $b = c_0 = c_\infty$ implies the existence of a nontrivial ground state solution to* (2.3).

Proof. Consider a minimizing sequence $(v_n) \subset \mathcal{N}_0$ for Υ satisfying $\beta(n(v_n)) = 0$ and $\Upsilon(v_n) \to b$. An application of Ekeland's variational principle gives another sequence $(w_n) \subset \mathcal{N}_0$ with $\Upsilon(w_n) \to b$, $\Upsilon'(w_n) \to 0$ and $\|w_n - v_n\| \to 0$. Define $u_n := n(w_n) = t_n w_n$, then $(u_n) \subset \mathcal{N}$ constitutes a Palais-Smale sequence for J at the level b. It follows from Lemma 3.5 that (u_n) is bounded. Thus, we may assume (up to a subsequence) that $u_n \rightharpoonup u^0$.

We proceed by contradiction and assume that $u^0 \equiv 0$. Then it follows from Lemma 3.12 that

$$\int_{\mathbb{R}^N} \left|\nabla \left(u_n - u^1(\cdot - y_n^1)\right)\right|^2 + \left|G^{-1}\left(u_n - u^1(\cdot - y_n^1)\right)\right|^2 \to 0.$$

It follows from Lemma 3.13 that $\|u_n - u^1(\cdot - y_n^1)\| \to 0$, then $\|t_n v_n(\cdot + y_n^1) - u^1\| \to 0$. Since $|\beta(t_n v_n(\cdot + y_n^1))| = |-y_n^1| \to \infty$, a contradiction.

It then follows from Lemmas 3.12 and 3.13 that $u_n \to u^0$ in E. Arguing as in Proposition 3.14, we finally obtain $u^0 \in \mathcal{N}$. ☐

It remains to analyze the case $b > c_0 = c_\infty$. The proof of Lemma 3.11 ensures the existence of $S > 0$ and $R > 1$ such that for each $|y| \geq S$, one can find $T^y \in (0, R)$ satisfying $T^y u_\infty^{y,1} \in \mathcal{N}$. The map $y \mapsto T^y$ is continuous and satisfies $T^y \to 1$ as $|y| \to \infty$. However, we cannot guarantee that $u^{y,1} \in \mathcal{T}$ for $|y| < S$, and therefore a modification of our approach is necessary.

Lemma 3.16. *There exist $\theta_y \geq 1$ and $T^y > 0$, which are bounded and continuous with respect to y, such that $T^y u_\infty^{y,\theta_y} \in \mathcal{N}$ for every $y \in \mathbb{R}^N$. Moreover, $\theta_y = 1$ for all $|y| \geq S' + 1$ with some $S' > 0$.*

Proof. We define $a := \inf_{x\in\mathbb{R}^N}[s(x) - \frac{V(x)}{g^2(\infty)}]$, $a^* := s_\infty - \frac{V_\infty}{g^2(\infty)}$ and

$$\mathcal{K}(y,\theta) := \frac{1}{\theta^2}\int_{\mathbb{R}^N} |\nabla u_\infty|^2 - \int_{\mathbb{R}^N}[s(\theta x + y) - \frac{V(\theta x + y)}{g^2(\infty)}]u_\infty^2.$$

It follows from (A5) that

$$\mathcal{K}(y,\theta) \leq \frac{1}{\theta^2}\int_{\mathbb{R}^N} |\nabla u_\infty|^2 - a\int_{\mathbb{R}^N} u_\infty^2,$$

with equality if and only if $s(x) - \frac{V(x)}{g^2(\infty)} \equiv a$, then $a = a^*$. Given any $c > 1$, let $\theta_0 := \max\{\sqrt{\frac{c}{a}} \cdot \frac{|\nabla u_\infty|_2}{|u_\infty|_2}, 1\}$, then $\mathcal{K}(y,\theta_0) \leq (\frac{1}{c} - 1)a\int_{\mathbb{R}^N} u_\infty^2 < 0$.

Using (A3), (A5), (A7) and Lemma 2.1-(4), we have

$$\begin{aligned}
\frac{\langle J'(ru_\infty^{y,\theta_0}), ru_\infty^{y,\theta_0}\rangle}{\theta_0^N r^2} &= \frac{1}{\theta_0^2}\int_{\mathbb{R}^N} |\nabla u_\infty|^2 + \int_{\mathbb{R}^N} \frac{V(\theta_0 x + y)G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty}u_\infty^2 \\
&\quad - \int_{\mathbb{R}^N} \frac{f(\theta_0 x + y, G^{-1}(ru_\infty))}{g(G^{-1}(ru_\infty))ru_\infty}u_\infty^2 \\
&= \mathcal{K}(y,\theta_0) + \int_{\mathbb{R}^N} V(\theta_0 x + y)[\frac{G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty} - \frac{1}{g^2(\infty)}]u_\infty^2 \\
&\quad + \int_{\mathbb{R}^N}[s(\theta_0 x + y) - \frac{f(\theta_0 x + y, G^{-1}(ru_\infty))}{g(G^{-1}(ru_\infty))ru_\infty}]u_\infty^2 \\
&\leq (\frac{1}{c} - 1)a\int_{\mathbb{R}^N} u_\infty^2 + V_\infty\int_{\mathbb{R}^N}[\frac{G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty} - \frac{1}{g^2(\infty)}]u_\infty^2
\end{aligned}$$

$$+\int_{\mathbb{R}^N}[s_\infty - \frac{f_\infty(G^{-1}(ru_\infty))}{g(G^{-1}(ru_\infty))ru_\infty}]u_\infty^2.$$

Let

$$F_1(r)=\int_{\mathbb{R}^N}[\frac{V_\infty G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty}-\frac{V_\infty}{g^2(\infty)}+s_\infty-\frac{f_\infty(G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty}]u_\infty^2.$$

Hence, by a method analogous to Lemma 3.2-(1), the function $F_1(r)$ is strictly decreasing for $r>0$ and $\lim_{r\to\infty}F_1(r)=0$. Note that $\lim_{r\to 0}F_1(r)\geq s_\infty\int_{\mathbb{R}^N}u_\infty^2$, then there is one and only one $R_1>0$ for which $F_1(R_1)=(1-\frac{1}{c})a\int_{\mathbb{R}^N}u_\infty^2$, i.e., $R_1=F_1^{-1}((1-\frac{1}{c})a\int_{\mathbb{R}^N}u_\infty^2)$. Thus, $\frac{\langle J'(ru_\infty^{y,\theta_0}),ru_\infty^{y,\theta_0}\rangle}{\theta_0^N r^2}<0$, for $r>R_1$, uniformly for $y\in\mathbb{R}^N$. Similarly, we have $\frac{\langle J'(ru_\infty^{y,\theta_0}),ru_\infty^{y,\theta_0}\rangle}{\theta_0^N r^2}>0$ for r small enough and $\frac{\langle J'(ru_\infty^{y,\theta_0}),ru_\infty^{y,\theta_0}\rangle}{\theta_0^N r^2}$ is strictly decreasing on $r\in(0,\infty)$. Hence, there exists a unique $0<T^y\leq R_1$ such that $\frac{\langle J'(T^yu_\infty^{y,\theta_0}),T^yu_\infty^{y,\theta_0}\rangle}{\theta_0^N (T^y)^2}=0$, i.e., $T^yu_\infty^{y,\theta_0}\in\mathcal{N}$, for any $y\in\mathbb{R}^N$.

The argument in Lemma 3.11 yields

$$\begin{aligned}&\frac{\langle J_\infty'(ru_\infty^{y,\theta}),ru_\infty^{y,\theta}\rangle}{\theta^N r^2}\\&=\frac{1}{\theta^2}\int_{\mathbb{R}^N}|\nabla u_\infty|^2+\int_{\mathbb{R}^N}\frac{V_\infty G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty}u_\infty^2-\int_{\mathbb{R}^N}\frac{f_\infty(G^{-1}(ru_\infty))}{g(G^{-1}(ru_\infty))ru_\infty}u_\infty^2\leq\alpha<0,\end{aligned}$$

for $r>R$ where α and R are as in Lemma 3.11, uniformly for $\theta\geq 1$. Then

$$\begin{aligned}\frac{\langle J'(ru_\infty^{y,\theta}),ru_\infty^{y,\theta}\rangle}{\theta^N r^2}&=\frac{\langle J_\infty'(ru_\infty^{y,\theta}),ru_\infty^{y,\theta}\rangle}{\theta^N r^2}+\int_{\mathbb{R}^N}\frac{[V(\theta x+y)-V_\infty]G^{-1}(ru_\infty)}{g(G^{-1}(ru_\infty))ru_\infty}u_\infty^2\\&\quad+\int_{\mathbb{R}^N}\frac{[f_\infty(G^{-1}(ru_\infty))-f(\theta x+y,G^{-1}(ru_\infty))]}{g(G^{-1}(ru_\infty))ru_\infty}u_\infty^2\\&\leq\frac{\langle J_\infty'(ru_\infty^{y,\theta}),ru_\infty^{y,\theta}\rangle}{\theta^N r^2}+\int_{\mathbb{R}^N}[s_\infty-s(\theta x+y)]u_\infty^2.\end{aligned}$$

Using the Lebesgue dominated convergence theorem, we obtain a constant $S'>0$ such that $\frac{\langle J'(ru_\infty^{y,\theta}),ru_\infty^{y,\theta}\rangle}{\theta^N r^2}\leq\frac{\alpha}{2}<0$ for $|y|\geq S'$ and $r>R$, uniformly for $1\leq\theta\leq\theta_0$. Similarly, we have $\frac{\langle J'(ru_\infty^{y,\theta}),ru_\infty^{y,\theta}\rangle}{\theta^N r^2}$ is strictly decreasing on $r\in(0,\infty)$ and $\frac{\langle J'(ru_\infty^{y,\theta}),ru_\infty^{y,\theta}\rangle}{\theta^N r^2}>0$ for r small enough, uniformly for $1\leq\theta\leq\theta_0$.

Set $\theta_y:=\theta_0$, if $|y|<S'$; $\theta_y:=|y|-S'+\theta_0(S'+1-|y|)$, if $S'\leq|y|\leq S'+1$; $\theta_y:=1$, if $|y|>S'+1$. So, there exist a unique $0<T^y\leq R_2:=\max\{R_1,R\}$ such that $\frac{\langle J'(T^yu_\infty^{y,\theta_y}),T^yu_\infty^{y,\theta_y}\rangle}{\theta_y^N(T^y)^2}=0$, i.e., $T^yu_\infty^{y,\theta_y}\in\mathcal{N}$, for any $y\in\mathbb{R}^N$. (Because the choice of S' does not affect the proof of the subsequent lemmas, we can arbitrarily choose $\alpha>0$ sufficiently small, i.e., $R>1$ sufficiently small in the following.)

Obviously, θ_y is continuous with respect to y. Assume $y_n\to y^0$ as $n\to\infty$. Since (T^{y_n}) is bounded, by extracting a suitable subsequence, we find $T^{y_n}\to T^0$. Then

$$\langle J'(T^0u_\infty^{y^0,\theta_{y^0}}),T^0u_\infty^{y^0,\theta_{y^0}}\rangle=\lim_{n\to\infty}\langle J'(T^{y_n}u_\infty^{y_n,\theta_{y_n}}),T^{y_n}u_\infty^{y_n,\theta_{y_n}}\rangle=0.$$

By the uniqueness of T^{y^0}, we have $T^{y^0}=T^0$. So, we obtain that T^y depends continuously on y from Heine theorem. □

We set $\Gamma[y]:=T^yu_\infty^{y,\theta_y}$ for $y\in\mathbb{R}^N$. This defines a continuous operator $\Gamma:\mathbb{R}^N\to\mathcal{N}$. The properties of β implies

$$\beta(\Gamma[y])=y, \tag{3.2}$$

which is given by [15, Lemma 3.13].

Lemma 3.17. *$J(\Gamma[y])\to c_\infty$, as $|y|\to\infty$.*

Proof. Since the limiting functional J_∞ is translation-invariant, we have $J(\Gamma[y])\to J_\infty(u_\infty)=c_\infty$. □

First, define $c_* := \inf\{c > c_\infty : c \text{ is a critical value of } J_\infty\}$. Then, set $\tilde{c} := \min\{c_*, 2c_\infty\}$. From (A12) we obtain $c_\infty < \tilde{c} \le 2c_\infty$.

Lemma 3.18. *Let $J^c := \{u \in E : J(u) \le c\}$. There exists $\delta > 0$ satisfying $c_\infty + \delta < \min\{b, \tilde{c}\}$ such that $\beta(u) \neq 0$ holds for all $u \in \mathcal{N} \cap J^{c_\infty+\delta}$.*

The definition of b immediately yields the conclusion of the above lemma.

Lemma 3.19. *Set $\mu(a) := \frac{2\theta_0^{-N}\tilde{c}-2c_\infty}{R_2^2|u_\infty|_2^2}$. Under this definition, hypothesis* (A11) *is well-defined.*

Proof. From Lemma 3.16, we obtain that $R_2 = \max\{F_1^{-1}((1-\frac{1}{c})a|u_\infty|_2^2), R\}$ and $\theta_0 := \max\{\sqrt{\frac{c}{a}} \cdot \frac{|\nabla u_\infty|_2}{|u_\infty|_2}, 1\}$. Then (A11) implies that $a > \frac{|\nabla u_\infty|_2^2}{|u_\infty|_2^2} \cdot (\frac{c_\infty}{\tilde{c}})^{\frac{2}{N}}$. Define $c(a) := \frac{a|u_\infty|_2^2}{|\nabla u_\infty|_2^2} \cdot (\frac{\tilde{c}}{c_\infty})^{\frac{2}{N}}$, for given $1 < c \le c(a)$, set

$$F^c(s) := a^* - s - \frac{2[\min\{\sqrt{\frac{s}{c}} \cdot \frac{|u_\infty|_2}{|\nabla u_\infty|_2}, 1\}]^N \cdot \tilde{c} - 2c_\infty}{[\max\{F_1^{-1}((1-\frac{1}{c})s|u_\infty|_2^2), R\}]^2|u_\infty|_2^2}.$$

Clearly, $F^c(a^*) \le 0$, $\lim_{s\to 0} F^c(s) = a^* > 0$ and $F^c(s)$ is strictly decreasing on $s \in (0, a^*]$. Thus, there exist a unique $0 < a^c \le a^*$ such that $F^c(a^c) = 0$. Since we can arbitrarily choose $R > 1$ sufficiently small, then (A11) holds exactly when there exists $1 < c \le c(a)$ such that $a \ge a^c$, i.e., $a = a^*$ or

$$2[\min\{\sqrt{\frac{a}{c}} \cdot \frac{|u_\infty|_2}{|\nabla u_\infty|_2}, 1\}]^N \cdot \tilde{c} - 2c_\infty > (a^* - a)|u_\infty|_2^2,$$

and

$$F_1\Big(\sqrt{\frac{2[\min\{\sqrt{\frac{a}{c}} \cdot \frac{|u_\infty|_2}{|\nabla u_\infty|_2}, 1\}]^N \cdot \tilde{c} - 2c_\infty}{(a^*-a)|u_\infty|_2^2}}\Big) \le (1 - \frac{1}{c})a|u_\infty|_2^2.$$

Note that $a^c \to a^*$ as $c \to 1$, then (A11) is equivalent to $a \ge \inf_{1<c\le C(a)}\{a^c\}$. Condition (A11) is monotonic in the sense that if it holds for some a_1, then it holds for all $a \in [a_1, a^*]$. Hence, there exists $a^0 \in (\frac{|\nabla u_\infty|_2^2}{|u_\infty|_2^2} \cdot (\frac{c_\infty}{\tilde{c}})^{\frac{2}{N}}, a^*)$ such that (A11) holds if and only if $a^0 \le a \le a^*$ (in fact, a^0 is the solution of $a = \inf_{1<c\le C(a)}\{a^c\}$). In hypothesis (A11), we choose $c = c_0$ such that $a^{c_0} = \inf_{1<c\le C(a)}\{a^c\}$. ☐

Lemma 3.20. *If* (A11) *holds, then $J(\Gamma[y]) < \tilde{c}$.*

Proof. Using (A7), (A11) and Lemma 2.1-(3), (4), we find that

$$\begin{aligned}
J(\Gamma[y]) &= J_\infty(\Gamma[y]) + \frac{1}{2}\int_{\mathbb{R}^N}[V(x) - V_\infty]|G^{-1}(T^y u_\infty^{y,\theta_y})|^2 \\
&\quad + \int_{\mathbb{R}^N} F_\infty(G^{-1}(T^y u_\infty^{y,\theta_y})) - F(x, G^{-1}(T^y u_\infty^{y,\theta_y})) \\
&\le \theta_0^N c_\infty + \frac{1}{2}\int_{\mathbb{R}^N}[\frac{V(x)}{g^2(\infty)} - \frac{V_\infty}{g^2(\infty)}](T^y u_\infty^{y,\theta_y})^2 + \frac{1}{2}\int_{\mathbb{R}^N}[s_\infty - s(x)](T^y u_\infty^{y,\theta_y})^2 \\
&\le \theta_0^N c_\infty + \frac{\theta_0^{-N}\tilde{c} - c_\infty}{R_2^2|u_\infty|_2^2}\int_{\mathbb{R}^N}(T^y u_\infty^{y,\theta_y})^2 \le \tilde{c}.
\end{aligned}$$

We claim that $J(\Gamma[y]) < \tilde{c}$. Assume, for a contradiction, that there is some $y_0 \in \mathbb{R}^N$ such that equality holds in all three inequalities above. The equality in the first relation implies that $\theta_0 = 1$. From the third equality holding, we can conclude that $T^{y_0} = R_2$, i.e., $T^{y_0} = R_1 \ge R$ which leads to $s(x) - \frac{V(x)}{g^2(\infty)} \equiv a$ by the proof of Lemma 3.16. Thus,

$$\begin{aligned}
J(\Gamma[y_0]) &= \theta_0^N c_\infty + \frac{1}{2}\int_{\mathbb{R}^N}[\frac{V(x)}{g^2(\infty)} - \frac{V_\infty}{g^2(\infty)}](T^{y_0} u_\infty^{y_0,\theta_{y_0}})^2 + \frac{1}{2}\int_{\mathbb{R}^N}[s_\infty - s(x)](T^y u_\infty^{y_0,\theta_{y_0}})^2 \\
&= c_\infty < \tilde{c},
\end{aligned}$$

a contradiction. ☐

Proof of Theorem 1.1. Lemma 3.11 implies $c_0 \leq c_\infty$. In the case where $c_0 < c_\infty$, Proposition 3.14 directly yields a ground state solution of (2.3). Also, if $b = c_0 = c_\infty$, the same conclusion follows readily from Proposition 3.15. Therefore, it remains to consider that $b > c_0 = c_\infty$.

Assume by contradiction that the functional J admits no critical value in the interval $(c_\infty, \tilde{c})$. Applying the deformation lemma (cf. [21, p.86]), we obtain a deformation η mapping $\Upsilon^{\tilde{c}} \setminus K_{\tilde{c}}$ into $\Upsilon^{c_\infty+\delta}$. Here, $\Upsilon^c := \{w \in \mathcal{N}_0 : \Upsilon(w) \leq c\}$ and K_c denotes the set of all critical points v of Υ with $\Upsilon(v) = c$, while δ is as in Lemma 3.18. By Lemmas 3.7 and 3.10, the flow η does not r each the boundary of $\mathcal{N}_0$, and it fixes every point in $\Upsilon^{c_\infty+\delta}$, i.e., $\eta(u) = u$ for all $u \in \Upsilon^{c_\infty+\delta}$.

Lemmas 3.17 and 3.18 imply the existence of $\rho_1 \geq S' > 0$ with the property that for any $\rho > \rho_1$,

$$c_\infty < \max_{|y|=\rho} J(\Gamma[y]) = \max_{|y|=\rho} \Upsilon(n^{-1}(\Gamma[y])) < c_\infty + \delta < b.$$

Let $\zeta : B_\rho(0) \mapsto \partial B_\rho(0)$ be a continuous map defined by

$$\zeta(y) := \rho \cdot \frac{\beta \circ n \circ \eta \circ n^{-1}(\Gamma[y])}{|\beta \circ n \circ \eta \circ n^{-1}(\Gamma[y])|}.$$

From (3.2), we have $\zeta(y) = y$ for all $y \in \partial B_\rho(0)$, which contradicts [25, D.11]. Furthermore, following the argument in [7, Lemma 2.4], we obtain that u^0 is sign-definite. Then, by elliptic regularity theory, we deduce that $u^0 \in C^{1,\alpha}(\mathbb{R}^N)$. Finally, the maximum principle guarantees that u^0 is positive. □

Acknowledgments. The authors thank the anonymous referees for their thorough review and constructive feedback.

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