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SOLUTIONS FOR ELLIPTIC EQUATIONS WITH COMPETING POTENTIALS IN 4 DIMENSIONS

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Abstract. We study the elliptic equation with competing potentials

$$-\Delta u + V(y)u = K(y)u^{2^*-1}, \quad \text{in } H^1(\mathbb{R}^N),$$
$$u > 0, \quad y \in \mathbb{R}^N,$$

where $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent, $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, $V(|y'|, y'')$ and $K(|y'|, y'')$ are nonnegative and bounded functions. Using a finite dimensional reduction argument and local Pohozaev identities, we prove the existence infinitely many solutions, when $N = 4$, $K(r, y'')$ has a stable critical point (r_0, y_0'') with $r_0 > 0$ and $K(r_0, y_0'') > 0$.

1. Introduction

This article concerns the existence of infinitely many solutions for the problem

$$\begin{gathered} -\Delta u + V(y)u = K(y)u^{2^*-1}, \\ u > 0, \quad u \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, $V(y) = V(|y'|, y'')$ and $K(y) = K(|y'|, y'')$ are nonnegative and bounded functions, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent. Since there are two potentials in problem (1.1), an interesting study is how the critical points of $V(y)$ and $K(y)$ affect the existence of solutions for problem (1.1), whose critical points of $V(y)$ and $K(y)$ are more important to guarantee the existence of solutions for problem (1.1). In [15], when $N \geq 5$, $V(y) = V(|y'|, y'')$ and $K(y) = K(|y'|, y'')$, He, Wang and Wang used a finite dimensional reduction argument and local Pohozaev identities to prove that if $K(r, y'')$ has a stable point (r_0, y_0'') with $K(r_0, y_0'') > 0$, then problem (1.1) has infinitely many solutions. This result implies that the role of stable critical points of $K(r, y'')$ in constructing bump solutions is more important than that of $V(r, y'')$. In this article, we consider problem (1.1) for $N = 4$. By using a finite dimensional reduction argument and local Pohozaev identities, we prove that the (1.1) has infinitely many solutions.

When $K(y)$ be a positive constant, problem (1.1) can be reduced to the following problem by a dilation:

$$-\Delta u + V(y)u = u^{2^*-1}, \quad u > 0, \quad u \in H^1(\mathbb{R}^N). \tag{1.2}$$

It is well known that problem (1.2) does not always have a solution. For example, it follows from the Pohozaev identity $\int_{\mathbb{R}^N} (V(|y|) + \frac{1}{2}|y|V'(|y|))u^2 = 0$ that if $r^2V(r)$ is always non-decreasing or non-increasing, problem (1.2) with $V(y) = V(|y|)$ has no solutions. Many scholars try to find the sufficient conditions on $V(y)$, under which problem (1.2) has a solution. If $V(y) \geq 0$ and $V(y) \not\equiv 0$, then the mountain pass value for (1.2) is not a critical value of the corresponding functional. Thus we cannot obtain an existence result of solutions for (1.2) by the concentration compactness principle [13, 14]. Benci and Cerami [3] proved that if $\|V(y)\|_{L^{N/2}(\mathbb{R}^N)}$ is suitably

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small, then the problem (1.2) has a solution. To our knowledge, this is the first existence result. When $V(y) \equiv 0$, problem (1.1) becomes

$$-\Delta u = K(y)u^{2^*-1}, \quad u > 0, \quad u \in D^{1,2}(\mathbb{R}^N), \tag{1.3}$$

which comes from the following prescribed curvature problem by using the stereo-graphic projection:

$$-\Delta_{\mathbb{S}^N} u + \frac{N(N-2)}{2} u = \widetilde{K}(y)u^{2^*-1}, \quad u > 0 \quad \text{on } \mathbb{S}^N. \tag{1.4}$$

To our knowledge, $\max_{\mathbb{S}^N} \widetilde{K} > 0$ is a necessary condition for the existence result. But there also exist some topological type obstructions. By Pohozaev identity, Kazdan-Warner condition $\int_{\mathbb{S}^N} \nabla \widetilde{K} \cdot \nabla_y u^{\frac{2N}{N-2}} = 0$ is a necessary condition. Some existence results have been proved under some assumptions on the critical point of $\widetilde{K}$, such as [2, 6, 22, 23, 30]. In [2], when $N = 3$, $\widetilde{K}$ is a positive Morse function with $\Delta \widetilde{K} \neq 0$ if $\nabla \widetilde{K} = 0$, Bahri and Coron proved that (1.4) has a solution. This type of results was extended by Li in [22, 23] to the case $N \geq 3$. He supposed that there exists $\alpha \in (N-2, N)$ satisfying

$$\widetilde{K}(y) = \widetilde{K}(y_0) + \sum_{j=1}^{N} c_j |y_j - y_{0,j}|^\alpha + h.o.t, \tag{1.5}$$

where $c_j \neq 0, \sum_{j=1}^N c_j \neq 0$.

To obtain our result, we assume that $V(y)$ and $K(y)$ satisfy the following assumptions:

- (A1) $0 \leq V(y), K(y) \in L^\infty(\mathbb{R}^N)$, $V(y) = V(|y'|, y'')$, $K(y) = K(|y'|, y'')$, where $y = (y', y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$;
- (A2) $K(r, y'')$ has a stable critical point (r_0, y_0'') satisfying $r_0 > 0, K(r_0, y_0'') > 0$ (Hereafter, w.l.g. $K(r_0, y_0'') = 1$), and
$$\deg\big(\nabla(K(r, y'')), (r_0, y_0'')\big) \neq 0;$$
- (A3) $V(r, y'') \in C^1(B_\vartheta(r_0, y_0''))$, $K(r, y'') \in C^2(B_\vartheta(r_0, y_0''))$, $\nabla V(r, y'')$ is bounded, $\vartheta > 0$ is a small constant.

Our main result reads as follows.

Theorem 1.1. *Suppose that* (A1)–(A3) *are satisfied. If $N = 4$, then problem* (1.1) *has infinitely many solutions, whose energy can be made arbitrarily large.*

We will prove Theorem 1.1 by a finite dimensional reduction argument and local Pohozaev identities. The finite dimensional reduction method has been extensively used to construct solutions. We referee to [2, 5, 6, 7, 8, 11, 12, 16, 17, 18, 19, 20, 23, 24, 26, 27, 29, 31, 30, 32, 33, 34, 35, 36] and references therein. Chen, Wei and Yan [4] proved that (1.2) has infinitely many nonradial solutions, if $V(y) > 0$ is radially symmetric and $r^2V(r)$ has a local maximum point or a local minimum point. By combining a finite reduction argument and local Pohozaev type of identities, Peng, Wang and Yan [28] proved that if $N \geq 5$ and $r^2V(r, y'')$ has a stable critical point, then problem (1.2) has infinitely many solutions. Li [21] proved the existence of infinitely many solutions of (1.3) with $N = 3$ and K being periodic in one variable. Yan [36] constructed solutions concentrating at 2 different local maximum points for $N \geq 3$, and constructed solutions concentrating at k different local maximum points for $N = 3$. Wei and Yan [35] constructed infinitely many non-radial solutions of (1.3), when $N \geq 5$, $K(y)$ is radially symmetric and satisfies: There is $r_0 > 0$, such that

$$K(r) = K(r_0) - c_0(r - r_0)^m + O(|r - r_0|^{m+\theta}), \quad r \in (r_0 - \delta, r_0 + \delta), \tag{1.6}$$

where $c_0 > 0$, $\theta > 0$ are some constants, and the constant m satisfies $m \in [2, N-2)$.

When $K(y)$ satisfies periodic conditions, Li, Wei and Xu [25] proved the existence of solutions with infinitely many bubbles for (1.3), where the centers of the bubbles can be placed on all the κ-dimensional lattice points with $\kappa < \frac{N-2}{2}$. In [10], Deng, Lin and Yan studied the local uniqueness and periodic property of the bubbling solutions obtained in [25]. In [29], when $N \geq 5$,

$K(y)$ satisfies a weaker radially symmetry condition, Peng, Wang and Wei proved the existence of infinitely many non-radial solutions of (1.3).

Let us introduce some notation. It is well known that the functions

$$U_{x,\lambda}(y) = [N(N-2)]^{\frac{N-2}{4}}\Big(\frac{\lambda}{1+\lambda^2|y-x|^2}\Big)^{\frac{N-2}{2}}, \quad \lambda>0,\ x\in\mathbb{R}^N,$$

are the only solutions for the problem

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad u>0 \ \text{ in } \ \mathbb{R}^N. \tag{1.7}$$

We define

$$\begin{aligned} H_s = \Big\{u:\ & u\in H^{1,2}(\mathbb{R}^N),\ u(y_1,y_2,y'') = u(y_1,-y_2,y''),\\ & u\Big(r\cos\Big(\theta+\frac{2\pi j}{m}\Big), r\sin\Big(\theta+\frac{2\pi j}{m}\Big), y''\Big) = u(r\cos\theta, r\sin\theta, y'')\Big\}, \end{aligned}$$

and

$$x_j = \Big(\overline{r}\cos\frac{2(j-1)\pi}{m}, \overline{r}\sin\frac{2(j-1)\pi}{m}, \overline{y}''\Big), \quad j=1,\dots,m.$$

To construct the solution of (1.1), we want to use $U_{x_j,\lambda}$ as an approximate solution. However, the decay of this function is too slow when $N=4$. So we need to cut off this function. Let $\delta>0$ be a small constant, such that $K(r,y'')>0$ if $|(r,y'')-(r_0,y_0'')|\le 10\delta$. Let $\xi(y)=\xi(r,y'')$ be a smooth function satisfying $\xi=1$ if $|(r,y'')-(r_0,y_0'')|\le\delta$, $\xi=0$ if $|(r,y'')-(r_0,y_0'')|\ge 2\delta$, and $0\le\xi\le 1$. We denote

$$Z_{x_j,\lambda}(y) = \xi U_{x_j,\lambda}(y), \quad Z^*_{\overline{r},\overline{y}'',\lambda}(y) = \sum_{j=1}^m U_{x_j,\lambda}(y), \quad Z_{\overline{r},\overline{y}'',\lambda}(y) = \sum_{j=1}^m Z_{x_j,\lambda}(y).$$

By the weak symmetry of $V(y)$ and $K(y)$, we observe that $V(x_j)=V(\overline{r},\overline{y}'')$, $K(x_j)=K(\overline{r},\overline{y}'')$, $j=1,\dots,m$. Let

$$Z_{j,1} = \frac{\partial Z_{x_j,\lambda}}{\partial\lambda}, \quad Z_{j,2} = \frac{\partial Z_{x_j,\lambda}}{\partial\overline{r}}, \quad Z_{j,l} = \frac{\partial Z_{x_j,\lambda}}{\partial\overline{y}_l''}, \quad l=3,\dots,N.$$

We define the norms

$$\|u\|_* = \sup_{y\in\mathbb{R}^N}|u(y)|\Big(\sum_{j=1}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^{-1},$$

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^N}|f(y)|\Big(\sum_{j=1}^m\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}\Big)^{-1}.$$

To prove Theorem 1.1, we need the following result.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there is a positive integer $m_0>0$, such that for any integer $m\ge m_0$, problem (1.1) has a solution u_m of the form*

$$u_m = Z_{\overline{r}_m,\overline{y}''_m,\lambda_m} + \phi_{\overline{r}_m,\overline{y}''_m,\lambda_m} = \sum_{j=1}^m \xi U_{x_j,\lambda_m} + \phi_{\overline{r}_m,\overline{y}''_m,\lambda_m}, \tag{1.8}$$

where $\phi_{\overline{r}_m,\overline{y}''_m,\lambda_m}\in H_s$. Moreover, $\lambda_m\in[e^{L_0m^2}, e^{L_1m^2}]$ for some constants $L_1>L_0>0$, and, as $m\to+\infty$, $(\overline{r}_m,\overline{y}''_m)\to(r_0,y_0'')$, and $\lambda_m^{-\frac{N-2}{2}}\|\phi_m\|_{L^\infty}\to 0$.

To simplify the proof of Theorem 1.2, we assume that $m>0$ is a large integer, $\lambda\in[e^{L_0m^2}, e^{L_1m^2}]$ and

$$|(\overline{r},\overline{y}'')-(r_0,y_0'')|\le\frac{(\ln\lambda)^{1/3}}{\lambda}. \tag{1.9}$$

This article is organized as follows. In section 2, we perform a finite dimensional reduction. Then, we will study the reduced finite-dimensional problem and prove Theorem 1.2 in section 3. In the appendix, we give some essential estimates.

2. Finite dimensional reduction

In this section, we use $Z_{\overline{r},\overline{y}'',\lambda}$ as the approximate solution to perform a finite-dimensional reduction. We consider the linearized problem

$$
\begin{gathered}
-\Delta\phi + V(r,y'')\phi - (2^*-1)K(r,y'')Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi = h + \sum_{l=1}^{N} c_l \sum_{j=1}^{m} Z_{x_j,\lambda}^{2^*-2} Z_{j,l},\\
\phi \in H_s,\ \sum_{j=1}^{m}\int_{\mathbb{R}^N} Z_{x_j,\lambda}^{2^*-2} Z_{j,l}\phi dy = 0, \quad l = 1,2,\dots,N,
\end{gathered}
\tag{2.1}
$$

where c_l are some real numbers.

Lemma 2.1. *Suppose that ϕ_m is a solution of (2.1) with $h = h_m$. If $\|h_m\|_{**} \to 0$ as $m \to +\infty$, then $\|\phi_m\|_* \to 0$ as $m \to +\infty$.*

Proof. We prove this lemma by contradiction. Assume that there exist $m \to +\infty$, $\overline{r}_m \to r_0$, $\overline{y}''_m \to y''_0$, $\lambda_m \in [e^{L_0 m^2}, e^{L_1 m^2}]$ and ϕ_m solving (2.1) with $\|h_m\|_{**} \to 0$ and $\|\phi_m\|_* \geq c > 0$. We may assume that $\|\phi_m\|_* = 1$. For simplicity, we drop the subscript m.

First of all, we have

$$
\begin{aligned}
|\phi(y)| &\leq C\int_{\mathbb{R}^N} \frac{|K(|z'|,z'')|}{|z-y|^{N-2}} Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}(z)|\phi(z)|\,dz + C\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}}|h|\,dz\\
&\quad + C\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}}\Big|\sum_{l=1}^{N} c_l \sum_{j=1}^{m} Z_{x_j,\lambda}^{2^*-2} Z_{j,l}\Big|\,dz\\
&:= A_1 + A_2 + A_3.
\end{aligned}
$$

By Lemma 4.3, we obtain

$$
\begin{aligned}
|A_1| &\leq C\|\phi\|_* \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}(z)\Big(\lambda^{\frac{N-2}{2}} \sum_{j=1}^{m} \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}}\Big)\,dz\\
&\leq C\|\phi\|_* \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}+\iota}},
\end{aligned}
$$

where ι is a small constant. Using Lemma 4.2, we obtain

$$
\begin{aligned}
|A_2| &\leq C\|h\|_{**} \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}}\Big(\lambda^{\frac{N+2}{2}} \sum_{j=1}^{m} \frac{1}{(1+\lambda|z-x_j|)^{\frac{N+2}{2}}}\Big)\,dz\\
&\leq C\|h\|_{**} \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}},
\end{aligned}
$$

and

$$
\begin{aligned}
|A_3| &\leq C\sum_{l=1}^{N}|c_l| \sum_{j=1}^{m} \int_{\mathbb{R}^N} \frac{\lambda^{n_l}}{|z-y|^{N-2}} \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|z-x_j|)^{N+2}}\,dz\\
&\leq C\sum_{l=1}^{N}|c_l| \sum_{j=1}^{m} \frac{\lambda^{\frac{N-2}{2}+n_l}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}},
\end{aligned}
$$

where $n_l = 1, l = 2,\dots,N, n_1 = -1$. Thus we have

$$
\Big(\sum_{j=1}^{m} \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^{-1}|\phi| \leq C\|\phi\|_* \frac{\sum_{j=1}^{m} \frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}+\iota}}}{\sum_{j=1}^{m} \frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}} + C\|h\|_{**} + C\sum_{l=1}^{N}|c_l|\lambda^{n_l}. \tag{2.2}
$$

We need to estimate c_l, $l=1,2,\dots,N$. Multiplying (2.1) by $Z_{1,t}$ ($t=1,2,\dots,N$) and integrating, we obtain

$$\begin{aligned}&\sum_{l=1}^{N} c_l \sum_{j=1}^{m}\int_{\mathbb{R}^N} Z_{x_j,\lambda}^{2^*-2} Z_{j,l} Z_{1,t}\,dy\\ &= \Big\langle -\Delta\phi + V(r,y'')\phi - (2^*-1)K(r,y'')Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi, Z_{1,t}\Big\rangle - \langle h, Z_{1,t}\rangle.\end{aligned} \tag{2.3}$$

Using Lemma 4.1, we see that

$$\begin{aligned}&|\langle h, Z_{1,t}\rangle|\\ &\le C\|h\|_{**}\int_{\mathbb{R}^N}\lambda^{n_t}\frac{\xi\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_1|)^{N-2}}\sum_{j=1}^{m}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}\,dy\\ &\le C\lambda^{n_t}\|h\|_{**}\Big(\int_{\mathbb{R}^N}\frac{\lambda^N}{(1+\lambda|y-x_1|)^{N+\frac{N-2}{2}}}\,dy\\ &\quad+\int_{\mathbb{R}^N}\Big(\frac{\lambda^N}{(1+\lambda|y-x_1|)^{N+\frac{N-2}{2}-\epsilon}}+\frac{\lambda^N}{(1+\lambda|y-x_j|)^{N+\frac{N-2}{2}-\epsilon}}\Big)\,dy\sum_{j=2}^{m}\frac{1}{(\lambda|x_j-x_1|)^{\epsilon}}\Big)\\ &\le C\lambda^{n_t}\|h\|_{**},\end{aligned} \tag{2.4}$$

where ϵ is a small constant. By direct computation, we have

$$\begin{aligned}&|\langle V(r,y'')\phi, Z_{1,t}\rangle|\\ &\le C\|\phi\|_*\int_{\mathbb{R}^N}\frac{\xi\lambda^{\frac{N-2}{2}+n_t}}{(1+\lambda|y-x_1|)^{N-2}}\sum_{j=1}^{m}\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\,dy\\ &\le C\lambda^{n_t}\|\phi\|_*\Big(\int_{\mathbb{R}^N}\frac{\xi\lambda^{N-2}}{(1+\lambda|y-x_1|)^{\frac{3N-6}{2}}}\,dy\\ &\quad+\int_{\mathbb{R}^N}\Big(\frac{\xi\lambda^{N-2}}{(1+\lambda|y-x_1|)^{\frac{3N-6}{2}-\epsilon}}+\frac{\xi\lambda^{N-2}}{(1+\lambda|y-x_j|)^{\frac{3N-6}{2}-\epsilon}}\Big)\,dy\sum_{j=2}^{m}\frac{1}{(\lambda|x_j-x_1|)^{\epsilon}}\Big)\\ &\le\frac{Cm^{\epsilon}\lambda^{n_t}\|\phi\|_*}{\lambda}.\end{aligned} \tag{2.5}$$

On the other hand, a direct computation shows that

$$|\langle -\Delta\phi-(2^*-1)K(r,y'')Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi, Z_{1,t}\rangle| = O\Big(\frac{m^{\epsilon}\lambda^{n_t}\|\phi\|_*}{\lambda}\Big). \tag{2.6}$$

Combining (2.4), (2.5) and (2.6), we have

$$\begin{aligned}&\langle -\Delta\phi+V(r,y'')\phi-(2^*-1)K(r,y'')Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi, Z_{1,t}\rangle-\langle h, Z_{1,t}\rangle\\ &= O\Big(\lambda^{n_t}\Big(\frac{m^{\epsilon}\|\phi\|_*}{\lambda}+\|h\|_{**}\Big)\Big).\end{aligned} \tag{2.7}$$

By the orthogonality, we obtain

$$\sum_{j=1}^{m}\langle Z_{x_j,\lambda}^{2^*-2}Z_{j,l}, Z_{1,t}\rangle\begin{cases}=(\overline{c}+o(1))\lambda^{2n_t} & l=t,\\ \le\frac{\overline{c}\lambda^{n_t}\lambda^{n_l}}{\lambda^N} & l\neq t,\end{cases} \tag{2.8}$$

for some constant $\overline{c}>0$.

Inserting (2.7) and (2.8) into (2.3), we conclude that

$$c_t=\frac{1}{\lambda^{n_t}}\big(o(\|\phi\|_*)+O(\|h\|_{**})\big). \tag{2.9}$$

Since $\|\phi\|_*=1$, from (2.2) there is $R>0$ such that

$$\big\|\lambda^{-\frac{N-2}{2}}\phi\big\|_{L^\infty(B_{\frac{R}{\lambda}}(x_j))}\ge a>0, \tag{2.10}$$

for some j. But $\widetilde{\phi}(y) = \lambda^{-\frac{N-2}{2}}\phi(\lambda^{-1}y + x_j)$ converges uniformly in any compact set to a solution u of

$$-\Delta u - (2^*-1)U_{0,\Lambda}^{2^*-2}u = 0, \quad in \quad \mathbb{R}^N, \tag{2.11}$$

for some $\Lambda \in [\Lambda_1, \Lambda_2]$ and u is perpendicular to the kernel of (2.11). So $u = 0$. This is a contraction to (2.10). We complete the proof. □

Using the same argument as in [9, Proposition 4.1], we obtain the following result.

Proposition 2.2. *There exist $m_0 > 0$ and a constant $C > 0$ independent of m, such that for all $m \geq m_0$ and $h \in L^\infty(\mathbb{R}^N)$, problem (2.1) has a unique solution $\phi \equiv L_m(h)$. Besides*

$$\|L_m(h)\|_* \leq C\|h\|_{**}, \quad |c_l| \leq \frac{C}{\lambda^{n_l}}\|h\|_{**}. \tag{2.12}$$

Now we consider the problem

$$\begin{aligned} &-\Delta(Z_{\overline{r},\overline{y}'',\lambda} + \phi) + V(r,y'')(Z_{\overline{r},\overline{y}'',\lambda} + \phi) \\ &= K(r,y'')(Z_{\overline{r},\overline{y}'',\lambda} + \phi)^{2^*-1} + \sum_{l=1}^{N} c_l \sum_{j=1}^{m} Z_{x_j,\lambda}^{2^*-2} Z_{j,l}, \quad y \in \mathbb{R}^N, \\ &\phi \in H_s, \sum_{j=1}^{m}\int_{\mathbb{R}^N} Z_{x_j,\lambda}^{2^*-2} Z_{j,l}\phi\, dy = 0, \quad l = 1,2,\dots,N. \end{aligned} \tag{2.13}$$

Now, we prove the following proposition by using the contraction mapping theorem.

Proposition 2.3. *If $N = 4$, there exist $m_0 > 0$ and constant $C > 0$ independent of m, such that for all $m \geq m_0$, then problem (2.13) has a unique solution $\phi = \phi_{\overline{r},\overline{y}'',\lambda} \in H_s$ satisfying*

$$\|\phi\|_* \leq \frac{C}{\lambda}, \quad |c_l| \leq \frac{C}{\lambda^{1+n_l}}. \tag{2.14}$$

We rewrite (2.13) as

$$-\Delta\phi + V(r,y'')\phi - (2^*-1)K(r,y'')Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi = \mathcal{F}(\phi) + l_m(y) + \sum_{l=1}^{N} c_l \sum_{j=1}^{m} Z_{x_j,\lambda}^{2^*-2} Z_{j,l}, \quad y \in \mathbb{R}^N \tag{2.15}$$

where

$$\mathcal{F}(\phi) = K(r,y'')\Big(\big(Z_{\overline{r},\overline{y}'',\lambda} + \phi\big)_+^{2^*-1} - Z_{\overline{r},\overline{y}'',\lambda}^{2^*-1} - (2^*-1)Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi\Big), \tag{2.16}$$

and

$$\begin{aligned} l_m(y) = &\Big(K(r,y'')Z_{\overline{r},\overline{y}'',\lambda}^{2^*-1} - \xi\sum_{j=1}^{m} U_{x_j,\lambda}^{2^*-1}\Big) - V(r,y'')Z_{\overline{r},\overline{y}'',\lambda} + Z_{\overline{r},\overline{y}'',\lambda}^{*}\Delta\xi + 2\nabla\xi\cdot\nabla Z_{\overline{r},\overline{y}'',\lambda}^{*} \\ &:= J_0 + J_1 + J_2 + J_3. \end{aligned} \tag{2.17}$$

To prove Proposition 2.3 by the contraction mapping theorem, we need to estimate $\mathcal{F}(\phi)$ and $l_m(y)$.

Lemma 2.4. *If $N = 4$, then*

$$\|\mathcal{F}(\phi)\|_{**} \leq C(\ln\lambda)\|\phi\|_*^2.$$

Proof. Since $K(y)$ is bounded and $2^* - 1 = 3$, we have

$$|\mathcal{F}(\phi)| \leq C|\phi|^3 + C|\phi|^2 Z_{\overline{r},\overline{y}'',\lambda}.$$

Then we have

$$\begin{aligned} |\mathcal{F}(\phi)| \leq{} & C\|\phi\|_*^3\Big(\sum_{j=1}^{m}\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^3 \\ &+ C\|\phi\|_*^2\Big(\sum_{j=1}^{m}\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^2\sum_{j=1}^{m}\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}} \end{aligned}$$

$$
\begin{aligned}
&\le C(\|\phi\|_*^3+\|\phi\|_*^2)\Big(\sum_{j=1}^m \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^3\\
&\le Cm^2\|\phi\|_*^2\sum_{j=1}^m \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}},
\end{aligned}
$$

where we have used that

$$
\Big(\sum_{i=1}^m a_i\Big)^\alpha \le m^{\alpha-1}\Big(\sum_{i=1}^m a_i^\alpha\Big), \quad (\alpha\ge 1 \text{ and } a_i\ge 0).
$$

So we have $\|\mathcal{F}(\phi)\|_{**}\le C(\ln\lambda)\|\phi\|_*^2$. □

Lemma 2.5. *When $N=4$, we have $\|l_m(y)\|_{**}\le \frac{C}{\lambda}$.*

Proof. We define

$$
\Omega_j=\Big\{y: y=(y',y'')\in\mathbb{R}^2\times\mathbb{R}^{N-2}, \langle \frac{y'}{|y'|},\frac{x_j'}{|x_j'|}\rangle\ge\cos\frac{\pi}{m}\Big\},\quad j=1,\dots,m.
$$

By symmetry, we can assume that $y\in\Omega_1$. So we have $|y-x_j|\ge|y-x_1|$. From the expression of l_m as in (2.17), we will estimate the term in l_m one by one.

For J_0, we have

$$
J_0=K(r,y'')\Big(Z_{\bar r,\bar y'',\lambda}^{2^*-1}-\xi\sum_{j=1}^m U_{x_j,\lambda}^{2^*-1}\Big)+\Big(K(r,y'')-1\Big)\xi\sum_{j=1}^m U_{x_j,\lambda}^{2^*-1}:=J_{01}+J_{02}.
$$

To estimate J_{01}, we have

$$
\begin{aligned}
|J_{01}|&\le CU_{x_1,\lambda}^{2^*-2}\sum_{j=2}^m U_{x_j,\lambda}+C\Big(\sum_{j=2}^m U_{x_j,\lambda}\Big)^{2^*-1}\\
&\le C\frac{\lambda^2}{(1+\lambda|y-x_1|)^4}\sum_{j=2}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}+C\Big(\sum_{j=2}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\Big)^{2^*-1}.
\end{aligned}\tag{2.18}
$$

We have

$$
\begin{aligned}
\frac{\lambda^2}{(1+\lambda|y-x_1|)^4}\sum_{j=2}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}&\le C\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_1|)^{\frac{N+2}{2}}}\Big(\frac{m}{\lambda}\Big)^{N-2}\\
&\le\frac{Cm^2}{\lambda^2}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_1|)^{\frac{N+2}{2}}}.
\end{aligned}
$$

Using the Hölder inequality, we can obtain that

$$
\begin{aligned}
&\Big(\sum_{j=2}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\Big)^{2^*-1}\\
&=\Big(\sum_{j=2}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^{2^*-1}\\
&\le\sum_{j=2}^m\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}\Big(\sum_{j=2}^m\frac{1}{(1+\lambda|y-x_j|)^{\frac{(N-2)(N+2)}{8}}}\Big)^{2^*-2}\\
&\le\frac{Cm^3}{\lambda^3}\sum_{j=2}^m\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}.
\end{aligned}\tag{2.19}
$$

So we obtain

$$
|J_{01}|\le\frac{C}{\lambda}\sum_{j=1}^m\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}.\tag{2.20}
$$

Next we estimate J_{02}. Noting that $K(y_0)=1$ and y_0 is a stable critical point of $K(y)$, by the Taylor expansion, in a neighbourhood of y_0 we have

$$K(y)=1+\sum_{i,j=1}^{N}\frac{1}{2}\frac{\partial^2 K(y_0)}{\partial y_i\partial y_j}(y_i-y_{0i})(y_j-y_{0j})+o(|y-y_0|^2).$$

So we have

$$\begin{aligned}|J_{02}|&=\Big|\Big(\sum_{i,j=1}^{N}\frac{1}{2}\frac{\partial^2 K(y_0)}{\partial y_i\partial y_j}(y_i-y_{0i})(y_j-y_{0j})+o(|y-y_0|^2)\Big)\xi\sum_{j=1}^{m}U_{x_j,\lambda}^{2^*-1}\Big|\\&\le C\left(|y-x_j|^2+|(\bar r,\bar y'')-(r_0,y_0'')|^2\right)\sum_{j=1}^{m}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{N+2}}\\&\le\frac{C(\ln\lambda)^{2/3}}{\lambda^2}\sum_{j=1}^{m}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{N+2}}.\end{aligned}\tag{2.21}$$

Combining (2.20) and (2.21), we have

$$\|J_0\|_{**}\le\frac{C}{\lambda}.\tag{2.22}$$

Noting that when $|(r,y'')-(r_0,y_0'')|\le 2\delta$, we have $\frac{1}{\lambda}\le\frac{C}{1+\lambda|y-x_j|}$. So we obtain

$$|J_1|\le C\sum_{j=1}^{m}\frac{\lambda^{\frac{N-2}{2}}\xi}{(1+\lambda|y-x_j|)^{N-2}}\le\frac{C}{\lambda}\sum_{j=1}^{m}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}},\tag{2.23}$$

since when $N=4$, $N-1=\frac{N+2}{2}$. Then $\|J_1\|_{**}\le\frac{C}{\lambda}$.

For J_2, similarly we have

$$|J_2|\le C\sum_{j=1}^{m}\frac{\lambda^{\frac{N-2}{2}}|\Delta\xi|}{(1+\lambda|y-x_j|)^{N-2}}\le\frac{C}{\lambda}\sum_{j=1}^{m}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}.\tag{2.24}$$

Then we obtain $\|J_2\|_{**}\le\frac{C}{\lambda}$. Moreover, we can check that

$$|J_3|\le C\sum_{j=1}^{m}\frac{\lambda^{N/2}|\nabla\xi|}{(1+\lambda|y-x_j|)^{N-1}}\le\frac{C}{\lambda}\sum_{j=1}^{m}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}.\tag{2.25}$$

Hence we obtain $\|J_3\|_{**}\le\frac{C}{\lambda}$. As a result, we have $\|l_m\|_{**}\le\frac{C}{\lambda}$. We complete the proof. □

Proof of Proposition 2.3. We set $\mathcal{A}(\phi)=L_m\big(\mathcal{F}(\phi)\big)+L_m(l_m)$ and

$$\mathbb{E}=\Big\{u:u\in C(\mathbb{R}^N)\cap H_s,\|u\|_*\le\frac{C_0}{\lambda},\sum_{j=1}^{m}\int_{\mathbb{R}^N}Z_{x_j,\lambda}^{2^*-2}Z_{j,l}u\,dy=0,\ l=1,\dots,N\Big\},$$

where L_m is defined in Proposition 2.2, C_0 is a large constant such that

$$\|\mathcal{A}(\phi)\|_*\le C\|\mathcal{F}(\phi)\|_{**}+C\|l_m(y)\|_{**}\le C(\ln\lambda)\|\phi\|_*^2+\frac{C}{\lambda}\le\frac{C_0}{\lambda}.$$

By Proposition 2.2, ϕ being a solution to (2.15) is equivalent to the following fixed point problem $\phi=\mathcal{A}(\phi)$. It is sufficient to prove that $\mathcal{A}$ is a contraction map from $\mathbb{E}$ to itself. It is easy to see that

$$\|\mathcal{A}(\phi_1)-\mathcal{A}(\phi_2)\|_*=\|L_m(\mathcal{F}(\phi_1))-L_m(\mathcal{F}(\phi_2))\|_*\le C\|\mathcal{F}(\phi_1)-\mathcal{F}(\phi_2)\|_{**}.$$

We can check that

$$\begin{aligned}|\mathcal{F}(\phi_1)-\mathcal{F}(\phi_2)|&\le C(|\phi_1|^{2^*-2}+|\phi_2|^{2^*-2})|\phi_1-\phi_2|\\&\le C(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2})\|\phi_1-\phi_2\|_*\Big(\sum_{j=1}^{m}\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^{2^*-1}\end{aligned}$$

$$\le C(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2})\|\phi_1-\phi_2\|_* m^{2^*-2}\sum_{j=1}^m \frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}.$$

So,

$$\|\mathcal{A}(\phi_1)-\mathcal{A}(\phi_2)\|_* \le C(\|\phi_1\|_*^{2^*-2}+\|\phi_2\|_*^{2^*-2})m^{2^*-2}\|\phi_1-\phi_2\|_* \le \frac{1}{2}\|\phi_1-\phi_2\|_*.$$

Therefore, $\mathcal{A}$ is a contraction map. The Banach fixed point theorem shows that problem (2.15) has a unique solution $\phi\in\mathbb{E}$.

Finally, by Proposition 2.2, Lemma 2.4 and Lemma 2.5 we have

$$\|\phi\|_* \le \frac{C}{\lambda} \quad \text{and} \quad \|c_l| \le \frac{C}{\lambda^{1+n_l}}.$$

The proof is complete. □

3. Proof of the main theorem

Let

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla u|^2+V(y)u^2\big)\,dy-\frac{1}{2^*}\int_{\mathbb{R}^N}K(y)(u)_+^{2^*}\,dy.$$

In this section, we will choose suitable $(\overline{r},\overline{y}'',\lambda)$ so that $Z_{\overline{r},\overline{y}'',\lambda}+\phi_{\overline{r},\overline{y}'',\lambda}$ is a solution of (1.1). For this purpose, we need the following proposition.

Proposition 3.1. *Suppose that $(\overline{r},\overline{y}'',\lambda)$ satisfies*

$$\int_{D_\rho}\Big(-\Delta u_m+V(r,y'')u_m-K(r,y'')(u_m)_+^{2^*-1}\Big)\langle y,\nabla u_m\rangle\,dy=0, \tag{3.1}$$

$$\int_{D_\rho}\Big(-\Delta u_m+V(r,y'')u_m-K(r,y'')(u_m)_+^{2^*-1}\Big)\frac{\partial u_m}{\partial y_i}\,dy=0,\quad i=3,\dots,N, \tag{3.2}$$

$$\int_{\mathbb{R}^N}\Big(-\Delta u_m+V(r,y'')u_m-K(r,y'')(u_m)_+^{2^*-1}\Big)\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy=0, \tag{3.3}$$

where $u_m=Z_{\overline{r},\overline{y}'',\lambda}+\phi_{\overline{r},\overline{y}'',\lambda}$ and $D_\rho=\{(r,y''):|(r,y'')-(r_0,y_0'')|\le\rho\}$ with $\rho\in(3\delta,4\delta)$, then $c_l=0, l=1,\dots,N$.

The proof of the above proposition is similar to that of [15, Proposition 3.1] and [28, Lemma 3.1]. We omit the proof. This proposition tells us that if $(\overline{r},\overline{y}'',\lambda)$ satisfies (3.1), (3.2) and (3.3) then u_m is a solution of (1.1). Thus to prove Theorem 1.2, we need to solve (3.1), (3.2) and (3.3).

Lemma 3.2. *It holds*

$$\int_{D_{4\delta}\backslash D_{3\delta}}(|\nabla\phi|^2+|\phi|^2+|\phi|^{2^*})\,dy\le\frac{C\ln\lambda}{\lambda^2}. \tag{3.4}$$

Proof. Noting that $Z_{\overline{r},\overline{y}'',\lambda}=0$ in $D_{5\delta}\backslash D_{2\delta}$, we obtain from (2.14) that

$$\begin{aligned}\int_{D_{4\delta}\backslash D_{3\delta}}(|\nabla\phi|^2+|\phi|^2+|\phi|^{2^*})\,dy &\le C\int_{D_{5\delta}\backslash D_{2\delta}}(\phi^2+|\phi|^{2^*})\,dy\\ &\le C\|\phi\|_*^2\int_{D_{5\delta}\backslash D_{2\delta}}\Big(\sum_{j=1}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^2\,dy\\ &\le\frac{Cm^2}{\lambda^2}\le\frac{C\ln\lambda}{\lambda^2}.\end{aligned}$$

The proof is complete. □

By Lemma 3.2, we can find a $\rho\in(3\delta,4\delta)$ such that

$$\int_{\partial D_\rho}(|\nabla\phi|^2+|\phi|^2+|\phi|^{2^*})\,dy=O(\frac{\ln\lambda}{\lambda^2}).$$

Lemma 3.3. *It holds*

$$\int_{D_\rho}u_m^2\,dy\le\frac{C\ln\lambda}{\lambda^2}.$$

Proof. Since $u_m = Z_{\overline{r},\overline{y}'',\lambda} + \phi$, we have

$$\int_{D_\rho} u_m^2\,dy = \int_{D_\rho} Z^2_{\overline{r},\overline{y}'',\lambda}\,dy + 2\int_{D_\rho} Z_{\overline{r},\overline{y}'',\lambda}\phi\,dy + \int_{D_\rho}\phi^2\,dy.$$

Similar to (2.5), we have

$$\begin{aligned}\big|2\int_{D_\rho} Z_{\overline{r},\overline{y}'',\lambda}\phi\,dy\big| &\le C\|\phi\|_*\int_{D_\rho}\sum_{j=1}^m \frac{\xi\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\sum_{i=1}^m \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2}{2}}}\,dy\\ &\le \frac{Cm^{1+\epsilon}}{\lambda^2},\end{aligned} \tag{3.5}$$

where ϵ is a small constant. Direct calculations show that

$$\begin{aligned}&\int_{D_\rho} Z^2_{\overline{r},\overline{y}'',\lambda}\,dy\\ &\le Cm\Big(\int_{D_\rho}\frac{\lambda^{N-2}}{(1+\lambda|y-x_1|)^{2N-4}}\,dy\\ &\quad+\int_{D_\rho}\frac{\lambda^{N-2}}{(1+\lambda|y-x_1|)^{N-2}}\sum_{j=2}^m\frac{1}{(1+\lambda|y-x_j|)^{N-2}}\,dy\Big)\\ &\le \frac{Cm}{\lambda^{2+\epsilon}}\Big(\int_{D_\rho}\frac{\lambda^N}{(1+\lambda|y-x_1|)^{2N-4-\epsilon}}\,dy\\ &\quad+\int_{D_\rho}\Big(\frac{\lambda^N}{(1+\lambda|y-x_1|)^{2N-4-2\epsilon}}+\frac{\lambda^N}{(1+\lambda|y-x_j|)^{2N-4-2\epsilon}}\Big)\,dy\sum_{j=2}^m\frac{1}{(\lambda|x_j-x_1|)^\epsilon}\Big)\\ &\le \frac{Cm^{1+\epsilon}}{\lambda^2}.\end{aligned} \tag{3.6}$$

So by Lemma 3.2, (3.5) and (3.6), we can obtain the result. ☐

Lemma 3.4. *For any C^1 bounded function $g(r,y'')$, it holds*

$$\int_{D_\rho} g(r,y'')|u_m|^{2^*}\,dy = m\Big(g(\overline{r},\overline{y}'')\int_{\mathbb{R}^N}U_{0,1}^{2^*}\,dy + O\big(\frac{1}{\lambda}\big)\Big).$$

Proof. Since $u_m = Z_{\overline{r},\overline{y}'',\lambda} + \phi$, we have

$$\begin{aligned}\int_{D_\rho} g(r,y'')|u_m|^{2^*}\,dy &= \int_{D_\rho} g(r,y'')|Z_{\overline{r},\overline{y}'',\lambda}|^{2^*}\,dy + \int_{D_\rho} g(r,y'')|\phi|^{2^*}\,dy\\ &\quad + O\Big(\int_{D_\rho}|Z_{\overline{r},\overline{y}'',\lambda}||\phi|^{2^*-1}\,dy + \int_{D_\rho}|Z_{\overline{r},\overline{y}'',\lambda}|^{2^*-1}|\phi|\,dy\Big).\end{aligned}$$

By symmetry and direct computations, we have

$$\begin{aligned}&\int_{D_\rho}|Z_{\overline{r},\overline{y}'',\lambda}||\phi|^{2^*-1}\,dy\\ &\le C\|\phi\|_*^{2^*-1}\int_{D_\rho}\sum_{j=1}^m\frac{\xi\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\Big(\sum_{i=1}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2}{2}}}\Big)^{2^*-1}\,dy\\ &\le C\|\phi\|_*^{2^*-1}\int_{D_\rho} m^{2^*-2}\sum_{j=1}^m\frac{\xi\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\sum_{i=1}^m\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N+2}{2}}}\,dy\\ &\le Cm^3\|\phi\|_*^3 + Cm^3\|\phi\|_*^3\int_{D_\rho}\frac{\xi\lambda^N}{(1+\lambda|y-x_1|)^{\frac{3N-2}{2}-\epsilon}}\sum_{j=2}^m\frac{1}{(\lambda|x_j-x_1|)^\epsilon}\,dy\\ &\le \frac{Cm\ln\lambda}{\lambda^3}.\end{aligned} \tag{3.7}$$

As in (2.19) we can obtain that

$$
\begin{aligned}
&\int_{D_\rho} |Z_{\overline{r},\overline{y}'',\lambda}|^{2^*-1}|\phi|\,dy \\
&\le Cm\|\phi\|_* \int_{D_\rho\cap\Omega_1} \xi^{2^*-1}\Big(\sum_{j=1}^m \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\Big)^{2^*-1}\Big(\sum_{i=1}^m \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2}{2}}}\Big)\,dy \\
&\le Cm\|\phi\|_* + Cm\|\phi\|_* \int_{D_\rho\cap\Omega_1} (\frac{m}{\lambda})^3 \sum_{j=2}^m \frac{\xi^{2^*-1}\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N+2}{2}}}\Big(\sum_{i=1}^m \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2}{2}}}\Big)\,dy \\
&\le Cm\|\phi\|_* + \frac{Cm^5\ln\lambda}{\lambda^4} \le \frac{Cm}{\lambda}.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
\int_{D_\rho} g(r,y'')|\phi|^{2^*}\,dy &\le C\|\phi\|^{2^*}\int_{D_\rho}\Big(\sum_{j=1}^m \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}}}\Big)^{2^*}\,dy \\
&\le C\|\phi\|^{2^*} m^{2^*}\ln\lambda \le \frac{C(\ln\lambda)^3}{\lambda^4}.
\end{aligned}
$$

Next we have the estimate

$$
\int_{D_\rho} g(r,y'')|Z_{\overline{r},\overline{y}'',\lambda}|^{2^*}\,dy = m\Big(\int_{D_\rho} g(r,y'')\xi^{2^*}U_{x_j,\lambda}^{2^*}\,dy + \int_{D_\rho} g(r,y'')\xi^{2^*}\sum_{i\ne j} U_{x_j,\lambda}U_{x_i,\lambda}^{2^*-1}\,dy\Big).
$$

We have

$$
\begin{aligned}
\Big|\int_{D_\rho} g(r,y'')\xi^{2^*}\sum_{i\ne j} U_{x_j,\lambda}U_{x_i,\lambda}^{2^*-1}\,dy\Big| &\le C\sum_{i\ne j}\int_{D_\rho} \frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\frac{\lambda^{\frac{N+2}{2}}}{(1+\lambda|y-x_i|)^{N+2}}\,dy \\
&\le \frac{Cm^2}{\lambda^2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{D_\rho} g(r,y'')\xi^{2^*}U_{x_j,\lambda}^{2^*}\,dy \\
&= \int_{D_\rho} g(\overline{r},\overline{y})\xi^{2^*}U_{x_j,\lambda}^{2^*}\,dy + \int_{D_\rho}\big(g(r,y'')-g(\overline{r},\overline{y})\big)\xi^{2^*}U_{x_j,\lambda}^{2^*}\,dy \\
&= g(\overline{r},\overline{y})\int_{\mathbb{R}^N} U_{0,1}^{2^*}\,dy + \int_{D_\rho}\big(g(r,y'')-g(\overline{r},\overline{y})\big)\xi^{2^*}U_{x_j,\lambda}^{2^*}\,dy + O\Big(\frac{1}{\lambda^{N-\epsilon}}\Big) \\
&= g(\overline{r},\overline{y})\int_{\mathbb{R}^N} U_{0,1}^{2^*}\,dy + O\Big(\int_{D_\rho}\big|(r,y'')-(\overline{r},\overline{y})\big|U_{x_j,\lambda}^{2^*}\,dy\Big) + O\Big(\frac{1}{\lambda^{N-\epsilon}}\Big) \\
&= g(\overline{r},\overline{y})\int_{\mathbb{R}^N} U_{0,1}^{2^*}\,dy + O(\frac{1}{\lambda}).
\end{aligned}
$$

Thus, we have

$$
\int_{D_\rho} g(r,y'')|u_m|^{2^*}\,dy = m\Big(g(\overline{r},\overline{y}'')\int_{\mathbb{R}^N} U_{0,1}^{2^*}\,dy + O(\frac{1}{\lambda})\Big).
$$

This completes the proof. □

Lemma 3.5. *Equation* (3.1) *is equivalent to*

$$
\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{r}} = O(\frac{1}{\lambda}).
$$

Proof. Integrating by parts, (3.1) is equivalent to

$$\begin{aligned} &\frac{2-N}{2}\int_{D_\rho}|\nabla u_m|^2\,dy-\frac{1}{2}\int_{D_\rho}\Big(NV(y)+\langle y,\nabla V(y)\rangle\Big)|u_m|^2\,dy\\ &+\frac{1}{2^*}\int_{D_\rho}\Big(NK(y)+\langle y,\nabla K(y)\rangle\Big)|u_m|^{2^*}\,dy \\ &=O\Big(\int_{\partial D_\rho}(|\nabla\phi|^2+|\phi|^2+|\phi|^{2^*})\,dy\Big). \end{aligned}\tag{3.8}$$

From (2.13), we obtain

$$\begin{aligned} \int_{D_\rho}|\nabla u_m|^2\,dy&=-\int_{D_\rho}V(y)u_m^2\,dy+\int_{D_\rho}K(y)(u_m)_+^{2^*}\,dy\\ &\quad+\sum_{l=1}^N c_l\sum_{j=1}^m\int_{\mathbb{R}^N}Z_{x_j,\lambda}^{2^*-2}Z_{\overline{r},\overline{y}'',\lambda}Z_{j,l}\,dy+O\Big(\int_{\partial D_\rho}(|\nabla\phi|^2+\phi^2)\,dy\Big), \end{aligned}\tag{3.9}$$

since

$$\int_{\mathbb{R}^N}\sum_{j=1}^m Z_{x_j,\lambda}^{2^*-2}Z_{j,l}\phi\,dy=0.$$

Inserting (3.9) into (3.8), we have

$$\begin{aligned} &-\int_{D_\rho}\big(V(y)+\langle y,\nabla V(y)\rangle\big)|u_m|^2\,dy+\frac{1}{2^*}\int_{D_\rho}\langle y,\nabla K(y)\rangle|u_m|^{2^*}\,dy\\ &=\frac{2-N}{2}\sum_{l=1}^N c_l\sum_{j=1}^m\int_{\mathbb{R}^N}Z_{x_j,\lambda}^{2^*-2}Z_{\overline{r},\overline{y}'',\lambda}Z_{j,l}\,dy+O\Big(\int_{\partial D_\rho}(|\nabla\phi|^2+|\phi|^2+|\phi|^{2^*})\,dy\Big). \end{aligned}\tag{3.10}$$

Since $c_i=O\big(\frac{1}{\lambda^2}\big)$, $i=2,\dots,N$ and $c_1=O(1)$, and

$$\int_{\mathbb{R}^N}Z_{x_j,\lambda}^{2^*-1}Z_{j,l}\,dy=O\Big(\frac{1}{\lambda^{N-n_l}}\Big),$$

we find that (3.10) is equivalent to

$$\begin{aligned} &-\int_{D_\rho}(V(y)+\langle y,\nabla V(y)\rangle)|u_m|^2\,dy+\frac{1}{2^*}\int_{D_\rho}\langle y,\nabla K(y)\rangle|u_m|^{2^*}\,dy\\ &=O\Big(\frac{m}{\lambda^2}\Big)+O\Big(\int_{\partial D_\rho}(|\nabla\phi|^2+|\phi|^2+|\phi|^{2^*})\,dy\Big). \end{aligned}$$

Since $V(y)$ and $\nabla V(y)$ are bounded, it follows from Lemmas 3.2-3.4 that

$$m\Big(\frac{1}{2^*}\overline{r}\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{r}}\int_{\mathbb{R}^N}U_{0,1}^{2^*}\,dy+O\big(\frac{1}{\lambda}\big)\Big)=O\Big(\frac{\ln\lambda}{\lambda^2}\Big),\tag{3.11}$$

that is

$$\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{r}}=O\big(\frac{1}{\lambda}\big).\tag{3.12}$$

The proof is complete. □

Lemma 3.6. *Equation* (3.2) *is equivalent to*

$$\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{y}_i}=O\big(\frac{1}{\lambda}\big),\ \ i=3,\dots,N.$$

Proof. For $i=3,\dots,N$, integrating by parts we have

$$\begin{aligned} 0&=\int_{D_\rho}\Big(-\Delta u_m+V(r,y'')u_m-K(r,y'')(u_m)_+^{2*-1}\Big)\frac{\partial u_m}{\partial y_i}\,dy\\ &=\int_{D_\rho}\frac{1}{2}\frac{\partial V(r,y'')}{\partial y_i}|u_m|^2\,dy-\int_{D_\rho}\frac{1}{2^*}\frac{\partial K(r,y'')}{\partial y_i}|u_m|^{2^*}\,dy \end{aligned}$$

$$
\begin{aligned}
&+O\Big(\int_{\partial D_\rho}(|\nabla\phi|^2+|\phi|^2+|\phi|^{2^*})\,dy\Big)\\
&=m\Big(-\frac{1}{2^*}\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{y}_i}\int_{\mathbb{R}^N}U_{0,1}^{2^*}\,dy+O\big(\frac{1}{\lambda}\big)\Big),
\end{aligned}
$$

where we used Lemmas 3.2-3.4. This implies that

$$
\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{y}_i}=O\big(\frac{1}{\lambda}\big).
$$

The proof is complete. □

Lemma 3.7. *If $N=4$, then*

$$
\frac{\partial I(Z_{\overline{r},\overline{y}'',\lambda})}{\partial\lambda}=m\Big(-\frac{B_1\ln\lambda}{\lambda^3}+\sum_{j=2}^m\frac{B_2}{\lambda^{N-1}|x_1-x_j|^{N-2}}+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big),
$$

where B_1,B_2 are positive constants.

Proof. First, we have

$$
\begin{aligned}
&\frac{\partial I(Z_{\overline{r},\overline{y}'',\lambda})}{\partial\lambda}\\
&=\int_{\mathbb{R}^N}\Big(\nabla Z_{\overline{r},\overline{y}'',\lambda}\nabla\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}+V(y)Z_{\overline{r},\overline{y}'',\lambda}\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\Big)\,dy-\int_{\mathbb{R}^N}K(y)Z_{\overline{r},\overline{y}'',\lambda}^{2^*-1}\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy\\
&=\int_{\mathbb{R}^N}V(y)Z_{\overline{r},\overline{y}'',\lambda}\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy+\int_{\mathbb{R}^N}\big(1-K(y)\big)Z_{\overline{r},\overline{y}'',\lambda}^{2^*-1}\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy\\
&\quad+\int_{\mathbb{R}^N}\Big(\nabla Z_{\overline{r},\overline{y}'',\lambda}\nabla\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}-Z_{\overline{r},\overline{y}'',\lambda}^{2^*-1}\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\Big)\,dy\\
&=\int_{\mathbb{R}^N}V(y)Z_{\overline{r},\overline{y}'',\lambda}\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy+\int_{\mathbb{R}^N}\big(1-K(y)\big)Z_{\overline{r},\overline{y}'',\lambda}^{2^*-1}\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy\\
&\quad-\int_{\mathbb{R}^N}\Big((Z^*_{\overline{r},\overline{y}'',\lambda})^{2^*-1}-\sum_{j=1}^m U_{x_j,\lambda}^{2^*-1}\Big)\frac{\partial Z^*_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy\\
&\quad+\int_{\mathbb{R}^N}\Big((\xi^2-1)\nabla Z^*_{\overline{r},\overline{y}'',\lambda}\nabla\frac{\partial Z^*_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}+\xi\nabla\xi\nabla Z^*_{\overline{r},\overline{y}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\\
&\quad+\xi Z^*_{\overline{r},\overline{y}'',\lambda}\nabla\xi\nabla\frac{\partial Z^*_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}+|\nabla\xi|^2Z^*_{\overline{r},\overline{y}'',\lambda}\frac{\partial Z^*_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\Big)\,dy\\
&:=\widetilde{I}_1+\widetilde{I}_2-\widetilde{I}_3+\widetilde{I}_4.
\end{aligned}
$$

Now we estimate $\widetilde{I}_1,\widetilde{I}_2,\widetilde{I}_3,\widetilde{I}_4$ one by one.

$$
\begin{aligned}
\widetilde{I}_1&=m\Big(\int_{B_\rho(x_1)}V(y)U_{x_1,\lambda}\frac{\partial U_{x_1,\lambda}}{\partial\lambda}\,dy+O\Big(\frac{1}{\lambda}\int_{B_\rho(x_1)}U_{x_1,\lambda}\sum_{j=2}^m U_{x_j,\lambda}\,dy\Big)+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big)\\
&=m\Big(\int_{B_\rho(x_1)}V(r_0,y_0'')U_{x_1,\lambda}\frac{\partial U_{x_1,\lambda}}{\partial\lambda}\,dy+O\Big(\frac{1}{\lambda}\int_{B_\rho(x_1)}|V(r,y'')-V(r_0,y_0'')|U_{x_1,\lambda}^2\,dy\Big)+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big)\\
&=m\Big(\frac{8}{\lambda^3}V(r_0,y_0'')\int_{B_\rho(x_1)}\frac{\lambda^4(1-\lambda^2|y-x_1|^2)}{(1+\lambda^2|y-x_1|^2)^3}\,dy+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big)\\
&=m\Big(\frac{16\pi^2}{\lambda^3}V(r_0,y_0'')\int_0^{\lambda\rho}\frac{(1-|r|^2)r^3}{(1+|r|^2)^3}\,dr+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big)\\
&=m\Big(\frac{-B_1\ln\lambda}{\lambda^3}+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big),
\end{aligned}
$$

where $B_1=16\pi^2V(r_0,y_0'')$ and $\lim_{\lambda\to\infty}\frac{\int_0^{\lambda\rho}\frac{(1-|r|^2)r^3}{(1+|r|^2)^3}\,dr}{\ln\lambda}=-1$.

Noting that $K(y_0)=1$ and $|(\overline{r},\overline{y}'')-(r_0,y_0'')|\le \frac{(\ln\lambda)^{1/3}}{\lambda}$, we have

$$
\begin{aligned}
|\widetilde{I}_2| &\le \frac{Cm}{\lambda}\Big(\int_{\mathbb{R}^N}\big(1-K(y)\big)\xi^{2^*}U_{x_1,\lambda}^{2^*-1}\sum_{j=1}^m U_{x_j,\lambda}\,dy\Big)\\
&\le \frac{Cm}{\lambda}\Big(\int_{B_{2\delta}(y_0)}\big(1-K(y)\big)U_{x_1,\lambda}^{2^*}\,dy+\int_{B_{2\delta}(y_0)}\big(1-K(y)\big)U_{x_1,\lambda}^{2^*-1}\sum_{j=2}^m U_{x_j,\lambda}\,dy\Big)\\
&\le \frac{Cm}{\lambda}\Big(\int_{B_{2\delta}(y_0)}|y-y_0|^2\frac{\lambda^N}{(1+\lambda|y-x_1|)^{2N}}\,dy\\
&\quad+\int_{B_{2\delta}(y_0)}|y-y_0|^2\frac{\lambda^N}{(1+\lambda|y-x_1|)^{N+2}}\,dy\sum_{j=2}^m\frac{1}{(\lambda|x_j-x_1|)^{N-2}}\Big)\\
&\le \frac{Cm(\ln\lambda)^{2/3}}{\lambda^3}.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
\widetilde{I}_3 &= \int_{\mathbb{R}^N}\Big((Z^*_{\overline{r},\overline{y}'',\lambda})^{2^*-1}-\sum_{j=1}^m U_{x_j,\lambda}^{2^*-1}\Big)\frac{\partial Z^*_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy\\
&= \int_{\mathbb{R}^N}(2^*-1)U_{x_1,\lambda}^{2^*-2}\sum_{j=2}^m U_{x_j,\lambda}\frac{\partial U_{x_1,\lambda}}{\partial\lambda}\,dy+O\Big(\int_{\mathbb{R}^N}\frac{1}{\lambda}U_{x_1,\lambda}^{2^*-2}(\sum_{j=2}^m U_{x_j,\lambda})^2\,dy\Big)\\
&\quad+O\Big(\int_{\mathbb{R}^N}\frac{1}{\lambda}U_{x_1,\lambda}(\sum_{j=2}^m U_{x_j,\lambda})^{2^*-1}\,dy\Big)+O\Big(\int_{\mathbb{R}^N}\frac{1}{\lambda}(\sum_{j=2}^m U_{x_j,\lambda})^{2^*}\,dy\Big)\\
&= m\Big(\int_{\Omega_1}(2^*-1)U_{x_1,\lambda}^{2^*-2}\sum_{j=2}^m U_{x_j,\lambda}\frac{\partial U_{x_1,\lambda}}{\partial\lambda}\,dy+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big)\\
&= m\Big(-\sum_{j=2}^m\frac{B_2}{\lambda^{N-1}|x_1-x_j|^{N-2}}+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big),
\end{aligned}
$$

for a constant $B_2>0$.

Similar to the proof of [15, Lemma A.1], we have $|\widetilde{I}_4|\le \frac{Cm^{1+\epsilon}}{\lambda^3}$.

It follows from all the estimates above that

$$
\frac{\partial I(Z_{\overline{r},\overline{y}'',\lambda})}{\partial\lambda}=m\Big(-\frac{B_1\ln\lambda}{\lambda^3}+\sum_{j=2}^m\frac{B_2}{\lambda^{N-1}|x_1-x_j|^{N-2}}+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big).
$$

The proof is complete. □

Lemma 3.8. *Equation* (3.3) *is equivalent to*

$$
m\Big(-\frac{B_1\ln\lambda}{\lambda^3}+\frac{B_3m^{N-2}}{\lambda^{N-1}}+o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big)=0,
$$

where B_1,B_3 are positive constants.

Proof. Direct computations show that

$$
\begin{aligned}
0 &= \int_{\mathbb{R}^N}\Big(-\Delta u_m+V(r,y'')u_m-K(r,y'')(u_m)_+^{2^*-1}\Big)\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy\\
&= \Big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}),\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\Big\rangle\\
&\quad+m\Big\langle-\Delta\phi+V(r,y'')\phi-(2^*-1)K(r,y'')Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi,\frac{\partial Z_{x_1,\lambda}}{\partial\lambda}\Big\rangle\\
&\quad-\int_{\mathbb{R}^N}K(r,y'')\Big((Z_{\overline{r},\overline{y}'',\lambda}+\phi)_+^{2^*-1}-Z_{\overline{r},\overline{y}'',\lambda}^{2^*-1}-(2^*-1)Z_{\overline{r},\overline{y}'',\lambda}^{2^*-2}\phi\Big)\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\,dy
\end{aligned}
$$

$$:= \big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\big\rangle + mI_1 - I_2.$$

Using (2.5) and (2.6), we have

$$|I_1| = O\Big(\frac{\lambda^{n_1} m^{\epsilon}\|\phi\|_*}{\lambda}\Big) = O\Big(\frac{m^{\epsilon}}{\lambda^3}\Big).$$

If $N=4$, then $2^*-1=3$. As a result, by (3.7) we have

$$\begin{aligned}
|I_2| &\le C\int_{\mathbb{R}^N} |\phi|^2 |Z_{\overline{r},\overline{y}'',\lambda}|^{2^*-3}\Big|\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\Big|\,dy + C\int_{\mathbb{R}^N}|\phi|^3\Big|\frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\Big|\,dy\\
&\le \frac{C}{\lambda}\int_{\mathbb{R}^N}|\phi|^2\Big(\xi\sum_{j=1}^m U_{x_j,\lambda}\Big)^{2^*-2}dy + \frac{C}{\lambda}\int_{\mathbb{R}^N}|\phi|^3\xi\sum_{j=1}^m U_{x_j,\lambda}\,dy\\
&\le \frac{C\|\phi\|_*^2}{\lambda}\int_{\mathbb{R}^N}\Big(\xi\sum_{j=1}^m U_{x_j,\lambda}\Big)^{2^*-2}\Big(\sum_{i=1}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2}{2}}}\Big)^2 dy\\
&\quad + \frac{C\|\phi\|_*^3}{\lambda}\int_{\mathbb{R}^N}\xi\sum_{j=1}^m U_{x_j,\lambda}\Big(\sum_{i=1}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2}{2}}}\Big)^3 dy\\
&\le \frac{C\|\phi\|_*^2}{\lambda}\int_{\mathbb{R}^N}\xi^{2^*-2}\Big(\sum_{j=1}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_j|)^{N-2}}\Big)^{2^*-2}\Big(\sum_{i=1}^m\frac{\lambda^{\frac{N-2}{2}}}{(1+\lambda|y-x_i|)^{\frac{N-2}{2}}}\Big)^2 dy\\
&\le \frac{Cm^2\|\phi\|_*^2}{\lambda},
\end{aligned}$$

since

$$\begin{aligned}
\Big(\sum_{j=1}^m\frac{1}{(1+\lambda|y-x_j|)^{N-2}}\Big)^{2^*-2} &\le \Big(\sum_{i=1}^m\frac{1}{(1+\lambda|y-x_i|)^{4-\epsilon}}\Big)\Big(\sum_{j=1}^m\frac{1}{(1+\lambda|y-x_j|)^{\frac{N-2}{6-N}\epsilon}}\Big)^{2^*-3}\\
&\le C\sum_{i=1}^m\frac{1}{(1+\lambda|y-x_i|)^{4-\epsilon}}.
\end{aligned}$$

So we have proved that

$$\big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}+\phi), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\big\rangle = \big\langle I'(Z_{\overline{r},\overline{y}'',\lambda}), \frac{\partial Z_{\overline{r},\overline{y}'',\lambda}}{\partial\lambda}\big\rangle + o\Big(\frac{m\ln\lambda}{\lambda^3}\Big).$$

Using Lemma 3.7, we obtain the result. □

Proof of Theorem 1.2. It follows from Lemmas 3.5, 3.6 and 3.8 that (3.1), (3.2) and (3.3) are equivalent to

$$\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{r}} = O\big(\frac{1}{\lambda}\big), \tag{3.13}$$

$$\frac{\partial K(\overline{r},\overline{y}'')}{\partial\overline{y}_i} = O\big(\frac{1}{\lambda}\big),\ i=3,\dots,N, \tag{3.14}$$

$$m\Big(-\frac{B_1\ln\lambda}{\lambda^3} + \frac{B_3 m^{N-2}}{\lambda^{N-1}} + o\Big(\frac{\ln\lambda}{\lambda^3}\Big)\Big) = 0. \tag{3.15}$$

Let $\lambda = e^{tm^2}$, then $t\in[L_0,L_1]$ since $\lambda\in[e^{L_0m^2}, e^{L_1m^2}]$. Then, from (3.15), we obtain

$$-B_1 t + B_3 = o(1),\ t\in[L_0,L_1]. \tag{3.16}$$

Let

$$F(t,\overline{r},\overline{y}'') = \Big(\nabla_{\overline{r},\overline{y}''}(K(\overline{r},\overline{y}'')), -B_1t+B_3\Big).$$

Then

$$\begin{aligned}
&\deg\Big(F(t,\overline{r},\overline{y}''), [L_0,L_1]\times B_{\frac{(\ln\lambda)^{1/3}}{\lambda}}\big((r_0,y_0'')\big)\Big)\\
&= -\deg\Big(\nabla_{\overline{r},\overline{y}''}\big(K(\overline{r},\overline{y}'')\big), B_{\frac{(\ln\lambda)^{1/3}}{\lambda}}\big((r_0,y_0'')\big)\Big) \ne 0.
\end{aligned}$$

So (3.13), (3.14) and (3.16) have a solution $t_m \in [L_0, L_1], (\overline{r}_m, \overline{y}''_m) \in B_{\frac{(\ln\lambda)^{1/3}}{\lambda}}\big((r_0, y''_0)\big)$. We completed the proof. □

4. Appendix: Some estimates

Here, we give some essential estimates. For $x_i, x_j, y \in \mathbb{R}^N$, set $g_{ij}(y) = \frac{1}{(1+|y-x_i|)^\mu(1+|y-x_j|)^\nu}$, where $x_i \neq x_j$, $\mu \geq 1$ and $\nu \geq 1$ are constants.

Lemma 4.1. *For any constant $\gamma \in (0, min(\mu,\nu)]$, we have*

$$g_{ij}(y) \leq \frac{C}{(|x_i - x_j|)^\gamma}\Big(\frac{1}{(1+|y-x_i|)^{\mu+\nu-\gamma}} + \frac{1}{(1+|y-x_j|)^{\mu+\nu-\gamma}}\Big).$$

For a proof of the above lemma see [35, Lemma B.1] and [25, Lemma A.1].

Lemma 4.2. *For any constant $0 < \vartheta < N-2$, there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} \frac{1}{(1+|z|)^{2+\vartheta}}\, dz \leq \frac{C}{(1+|y|)^\vartheta}$$

For a proof of the above lemma see [35, Lemma B.2] and [25, Lemma A.2] .

Lemma 4.3. *Suppose that $N \geq 4$. Then there is a small constant $\iota > 0$, such that*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} Z^{\frac{4}{N-2}}_{\overline{r},\overline{y}'',\lambda}(z) \sum_{j=1}^m \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}}\, dz \leq \sum_{j=1}^m \frac{C}{(1+\lambda|y-x_j|)^{\frac{N-2}{2}+\iota}}.$$

Proof. Note that for small $\tau_1 > 0$,

$$\sum_{j=2}^m \frac{1}{(\lambda|x_j - x_1|)^{\tau_1}} \sim \big(\frac{m}{\lambda}\big)^{\tau_1} \leq C.$$

For $z \in \Omega_1$, we have $|z - x_j| \geq |z - x_1|$. Using Lemma 4.1, we have

$$\begin{aligned}
\sum_{j=2}^m \frac{1}{(1+\lambda|z-x_j|)^{N-2}} &\leq \frac{1}{(1+\lambda|z-x_1|)^{\frac{N-2}{2}}} \sum_{j=2}^m \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}} \\
&\leq \frac{C}{(1+\lambda|z-x_1|)^{N-2-\tau_1}} \sum_{j=2}^m \frac{1}{(\lambda|x_j-x_1|)^{\tau_1}} \\
&\leq \frac{C}{(1+\lambda|z-x_1|)^{N-2-\tau_1}}.
\end{aligned}$$

Thus,

$$Z^{\frac{4}{N-2}}_{\overline{r},\overline{y}'',\lambda}(z) \leq \frac{C\lambda^2}{(1+\lambda|z-x_1|)^{4-\frac{4\tau_1}{N-2}}}.$$

So for $z \in \Omega_1$, using Lemma 4.1 again, we find that

$$\begin{aligned}
Z^{\frac{4}{N-2}}_{\overline{r},\overline{y}'',\lambda}(z) \sum_{j=1}^m \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}} &\leq \frac{C\lambda^2}{(1+\lambda|z-x_1|)^{4-\frac{4\tau_1}{N-2}}} \sum_{j=1}^m \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}} \\
&\leq \frac{C\lambda^2}{(1+\lambda|z-x_1|)^{\frac{N+6}{2}-\frac{4\tau_1}{N-2}-\tau_1}}.
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
&\int_{\Omega_1} \frac{1}{|y-z|^{N-2}} Z^{\frac{4}{N-2}}_{\overline{r},\overline{y}'',\lambda}(z) \sum_{j=1}^m \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}}\, dz \\
&\leq \int_{\Omega_1} \frac{1}{|y-z|^{N-2}} \frac{C\lambda^2}{(1+\lambda|z-x_1|)^{\min(N,\frac{N+6}{2})-\frac{4\tau_1}{N-2}-\tau_1}}\, dz
\end{aligned}$$

$$\leq \frac{C}{(1+\lambda|y-x_1|)^{\min(N-2,\frac{N+2}{2})-\frac{4\tau_1}{N-2}-\tau_1}},$$

which gives

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} Z^{\frac{4}{N-2}}_{\bar{r},\bar{y}'',\lambda}(z) \sum_{j=1}^{m} \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}}\, dz \\
&\leq \sum_{j=1}^{m} \int_{\Omega_j} \frac{1}{|y-z|^{N-2}} Z^{\frac{4}{N-2}}_{\bar{r},\bar{y}'',\lambda}(z) \sum_{j=1}^{m} \frac{1}{(1+\lambda|z-x_j|)^{\frac{N-2}{2}}}\, dz \\
&\leq \sum_{j=1}^{m} \frac{C}{(1+\lambda|y-x_j|)^{\min(N-2,\frac{N+2}{2})-\frac{4\tau_1}{N-2}-\tau_1}}.
\end{aligned}$$

Since $\min(\frac{N-2}{2},2)-\frac{4\tau_1}{N-2}-\tau_1>0$, we can obtain the result. □

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