Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 22, pp. 1–12.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.22

CONTINUOUS DEPENDENCE FOR SYSTEMS GOVERNED BY FRACTIONAL LAPLACIAN ON DATA AND PARAMETERS

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Abstract. We study a boundary value problem governed by nonlinear equations involving the fractional Laplacian with exterior Dirichlet conditions. We establish sufficient conditions for the existence of solutions as well as their continuous dependence on the data and parameters. The proof of the main result relies on the variational formulation of the problem and exploits a saddle point structure imposed by the assumptions, in the spirit of the Ky Fan theorem.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ for $n \geq 3$ be a bounded domain with a Lipschitz boundary. We consider a boundary value problem for a nonlinear system of equations involving the fractional Laplacian $(-\Delta)^{\alpha/2}$ with $\alpha \in (0,2)$ and the exterior condition $\mathrm{u} = \phi, \mathrm{v} = \psi$ on $\mathbb{R}^n \setminus \Omega$ of the form

$$\begin{gathered} -(-\Delta)^{\alpha/2}\mathrm{u} + G_{\mathrm{u}}(x,\mathrm{u},\mathrm{v},w) = 0 \quad \text{in } \Omega, \\ (-\Delta)^{\alpha/2}\mathrm{v} + G_{\mathrm{v}}(x,\mathrm{u},\mathrm{v},w) = 0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where ϕ and ψ belong to a fractional Sobolev space to be defined below, and parameter w is assumed to be Lebesgue integrable with an appropriate exponent.

Problems involving the fractional Laplacian have appeared in many aspects of applications, including financial mathematics [1], mechanics [6, 7, 19], hydrodynamics [8, 16, 17, 33, 35], elastostatics [6], and probability theory [1, 7, 18, 36].

The fractional Laplace operator can be defined for smooth and bounded function z as

$$(-\Delta)^{\alpha/2} z(x) = c(n,\alpha) \int_{\mathbb{R}^n} \frac{2z(x) - z(x+y) - z(x-y)}{|y|^{n+\alpha}} dy,$$

where

$$c(n,\alpha) = \frac{\Gamma((n+\alpha)/2)}{|\Gamma(-\alpha/2)|\pi^{n/2}2^{1-\alpha}}\,. \tag{1.2}$$

We shall prove that weak solutions of (1.1) depend continuously on the parameters and on the data defined outside the domain.

Weak solution of (1.1) in the appropriate fractional Sobolev space is a pair $(u,v) = (\mathrm{u}-\phi, \mathrm{v}-\psi)$ such that, for every test functions φ and ζ, the following equality holds

$$\begin{aligned} &\int_Q \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+\alpha}}\,dx\,dy - \int_Q \frac{(v(x)-v(y))(\zeta(x)-\zeta(y))}{|x-y|^{n+\alpha}}\,dx\,dy \\ &+ \int_Q \frac{(\phi(x)-\phi(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+\alpha}}\,dx\,dy - \int_Q \frac{(\psi(x)-\psi(y))(\zeta(x)-\zeta(y))}{|x-y|^{n+\alpha}}\,dx\,dy \\ &= \frac{1}{c(n,\alpha)} \int_\Omega G_u(x,(u+\phi)(x),(v+\psi)(x),w(x))\varphi(x)\,dx \\ &\quad + \frac{1}{c(n,\alpha)} \int_\Omega G_v(x,(u+\phi)(x),(v+\psi)(x),w(x))\zeta(x)\,dx. \end{aligned} \tag{1.3}$$

2020 *Mathematics Subject Classification.* 35A15, 35B30, 35R11, 49J20, 93D05.
Key words and phrases. Fractional Laplacian; minimax approach; Ky Fan theorem, continuous dependence.

Submitted October 4, 2025. Published March 18, 2026.

Here the set Q is

$$Q = \mathbb{R}^{2n} \backslash ((\mathbb{R}^n \backslash \Omega) \times (\mathbb{R}^n \backslash \Omega)) ,$$

so that integration is taken over all pairs $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ with at least one point inside the domain Ω.

In Section 3, we shall show that problem (1.1) possesses at least one weak solution satisfying (1.3) for any parameter w and the exterior data (ϕ, ψ) provided assumptions from Section 2 are satisfied. The results concerning the continuous dependence of weak solution on parameters and exterior data are presented in Section 4. For a fixed parameter w_k and exterior data (ϕ_k, ψ_k), we denote by (u_k, v_k) a weak solution of problem (1.1). We shall prove that if the sequences $\{w_k\}$ tends to w_0 in L^p and $\{(\phi_k, \psi_k)\}$ tends to (ϕ_0, ψ_0) in $Y^{\alpha/2} \times Y^{\alpha/2}$, then the corresponding sequence of solutions $\{(u_k, v_k)\}$ tends to (u_0, v_0) in $X_0^{\alpha/2} \times X_0^{\alpha/2}$. The precise functional framework will be introduce in detail in the sequel. In particular, we shall show that the exterior Dirichlet problem (1.1) is well posed; that is, it admits a solution which depends continuously on both the parameters and the exterior data.

The functional $\mathcal{F}$ is defined on the domain $X_0^{\alpha/2}$ specifies by (1.6) as

$$\begin{aligned}\mathcal{F}(u, v) = &-\frac{c(n,\alpha)}{2} \int_Q \frac{(u(x) - u(y))^2}{|x - y|^{n+\alpha}}\, dx\, dy + \frac{c(n,\alpha)}{2} \int_Q \frac{(v(x) - v(y))^2}{|x - y|^{n+\alpha}}\, dx\, dy \\ &- \int_Q \frac{(\phi(x) - \phi(y))(u(x) - u(y))}{c(n,\alpha)^{-1}|x - y|^{n+\alpha}}\, dx\, dy + \int_Q \frac{(\psi(x) - \psi(y))(v(x) - v(y))}{c(n,\alpha)^{-1}|x - y|^{n+\alpha}}\, dx\, dy \\ &+ \int_\Omega G(x, (u + \phi)(x), (v + \psi)(x), w(x))\, dx .\end{aligned} \tag{1.4}$$

The approach employed in this paper relies on a variational structure of minimax type; see, for instance, [28, 38] for the classical concave-convex framework. By invoking a Ky Fan-type minimax theorem, we obtain the existence of saddle-point solutions.

The functional setting adopted here is borrowed from [12], see also the related results in [30] and [21]. We define

$$X^{\alpha/2} = \big\{ z : \mathbb{R}^n \to \mathbb{R} : z|_\Omega \in L^2(\Omega) \text{ and } \frac{z(x) - z(y)}{|x - y|^{(n+\alpha)/2}} \in L^2(Q) \big\}$$

and equip this space with the norm

$$\|z\|_{X^{\alpha/2}} = \|z\|_{L^2(\Omega)} + [z]_{n,\alpha} = \|z\|_{L^2(\Omega)} + \Big(\int_Q \frac{|z(x) - z(y)|^2}{|x - y|^{n+\alpha}}\, dx\, dy \Big)^{1/2} . \tag{1.5}$$

For the proof that $\| \cdot \|_{X^{\alpha/2}}$ is indeed a norm on $X^{\alpha/2}$, we refer the reader to [30]. We also consider the following linear subspace of $X^{\alpha/2}$, defined by

$$X_0^{\alpha/2} = \big\{ z \in X^{\alpha/2} : z = 0 \text{ a.e. in } \mathbb{R}^n \backslash \Omega \big\} \tag{1.6}$$

and equipped with the norm

$$\|z\|_{X_0^{\alpha/2}} = \Big(\int_Q \frac{|z(x) - z(y)|^2}{|x - y|^{n+\alpha}}\, dx\, dy \Big)^{1/2} . \tag{1.7}$$

Both spaces $X^{\alpha/2}$ and $X_0^{\alpha/2}$ are nonempty. Indeed, by [31, Lemma 11], we have $C_0^2(\Omega) \subseteq X_0^{\alpha/2}$. Moreover, the space $X_0^{\alpha/2}$ is a Hilbert space; see [21, Lemma 2.3] or [30, Lemma 7]. The associated inner product is

$$\langle z_1, z_2 \rangle_{X_0^{\alpha/2}} = \int_Q \frac{(z_1(x) - z_1(y))(z_2(x) - z_2(y))}{|x - y|^{n+\alpha}}\, dx\, dy.$$

Observe that the functional $\mathcal{F}$ defined in (1.4), may be rewritten, with $c = c(n, \alpha)$ and $G = G(x, u + \phi, v + \psi, w)$, as

$$\mathcal{F}(u, v) = \frac{c}{2} (\|v\|^2_{X_0^{\alpha/2}} - \|u\|^2_{X_0^{\alpha/2}}) + c\langle \psi, v \rangle_{X_0^{\alpha/2}} - c\langle \phi, u \rangle_{X_0^{\alpha/2}} + \int_\Omega G . \tag{1.8}$$

To incorporate non-homogeneous exterior data, we require that such values be given by functions defined on the whole space $\mathbb{R}^n$. This necessitates a modification of the space $X^{\alpha/2}$ so that it becomes a Hilbert space equipped with an appropriate inner product. For this purpose, we introduce the space

$$Y^{\alpha/2} = X^{\alpha/2} \cap L^2(\mathbb{R}^n) \tag{1.9}$$

endowed with the norm

$$\|z\|_{Y^{\alpha/2}} = \|z\|_{L^2(\mathbb{R}^n)} + \Big(\int_Q \frac{|z(x)-z(y)|^2}{|x-y|^{n+\alpha}}\,dx\,dy \Big)^{1/2}. \tag{1.10}$$

Arguing as in the proof of [21, Lemma 2.3] or [30, Lemma 7], one can show that $Y^{\alpha/2}$, equipped with the inner product

$$\langle z_1, z_2\rangle_{Y^{\alpha/2}} = \langle z_1, z_2\rangle_{L^2(\mathbb{R}^n)} + \langle z_1, z_2\rangle_{X_0^{\alpha/2}}, \tag{1.11}$$

is a separable Hilbert space. From the definition, we immediately obtain the inclusion

$$Y^{\alpha/2} \subset X^{\alpha/2}.$$

Note that, if $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary, then the space $X_0^{\alpha/2}$ can be compactly embedded into $L^s(\Omega)$ for every $s \in [1, 2^*_\alpha)$, where $2^*_\alpha = 2n/(n-\alpha)$. Moreover, for $n > \alpha$ and any $z \in X_0^{\alpha/2}$ the following inequality holds

$$\|z\|_{L^s(\Omega)} \le d_s \|z\|_{X_0^{\alpha/2}}, \tag{1.12}$$

see [30, Lemma 8] or [20, Corollary 7.2] when $s = 2$, and also the Poincaré inequality presented in [22]. Furthermore, the norm $\|z\|_{X_0^{\alpha/2}}$ is weakly lower semicontinuous, convex and coercive, as is the case of any norm on a reflexive Banach space; see [3, 22] for details. For additional background on the fractional Sobolev spaces, we refer to [20] and the references therein. Further properties of the spaces $X^{\alpha/2}$ and $X_0^{\alpha/2}$ can be found in [31], where these functional spaces were introduced and several of their fundamental characteristics were established.

2. Statement of the problem

Consider the system (1.1) of nonlinear equations, where $u \in X^{\alpha/2}$, $v \in X^{\alpha/2}$, and where G is a function defined on $\Omega \times \mathbb{R}^{2+m}$, with $m \ge 1$ and $w \in \mathcal{W}$, where

$$\mathcal{W} = \{w \in L^p(\Omega, \mathbb{R}^m) : w(x) \in M \text{ for a.e. } x \in \Omega\},$$

while the set $M \subset \mathbb{R}^m$ is assumed to be convex and bounded.

We investigate the problem of continuous dependence of weak solutions of (1.1) - that is solutions of (1.3) - on parameter the $w \in \mathcal{W}$ and on the data $(\phi, \psi) \in Y^{\alpha/2} \times Y^{\alpha/2}$. Under properly chosen assumptions on the function $G = G(x, u, v, w)$, we shall address the question of the continuous dependence on parameters of saddle points of the functional $\mathcal{F}(u, v)$ associated with the problem (1.3), defined on the product space $\mathbb{X}_0^{\alpha/2} = X_0^{\alpha/2} \times X_0^{\alpha/2}$ with the norm

$$\|(u,v)\|^2_{\mathbb{X}_0^{\alpha/2}} = \|u\|^2_{X_0^{\alpha/2}} + \|v\|^2_{X_0^{\alpha/2}}.$$

Let us recall that a pair $(u_0, v_0) \in \mathbb{X}_0^{\alpha/2}$ is called a saddle point of the functional $\mathcal{F}$, if

$$\mathcal{F}(u, v_0) \le \mathcal{F}(u_0, v_0) \le \mathcal{F}(u_0, v)$$

for any $u \in X_0^{\alpha/2}$ and any $v \in X_0^{\alpha/2}$. This condition is equivalent to the minimax identity

$$\sup_u \inf_v \mathcal{F}(u,v) = \inf_v \sup_u \mathcal{F}(u,v) = \mathcal{F}(u_0, v_0)$$

provided that both quantities $\sup_u \inf_v \mathcal{F}(u,v)$ and $\inf_v \sup_u \mathcal{F}(u,v)$ are finite and attained.

In the sequel, we use the following assumptions:

(A1) Let G, G_u, G_v be Carathéodory functions, that is measurable with respect to x for any $(u, v, w) \in \mathbb{R}^{2+m}$, and continuous with respect to (u, v, w) for a.e. $x \in \Omega$.

(A2) For $p = \infty$, there is a constant $c > 0$ such that, for any $z \in \{u, v\}$
$$|G(x, u, v, w)| \le c(1 + |u|^s + |v|^s),$$
$$|G_z(x, u, v, w)| \le c(1 + |u|^{s-1} + |v|^{s-1}),$$
where $s \in (1, 2^*_\alpha)$ for $n \ge 3$, a.e. $x \in \Omega$, $u \in \mathbb{R}$, $v \in \mathbb{R}$ and $w \in M$. For $p \in [1, \infty)$, there exists $c > 0$ such that, for all $z \in \{u, v\}$,
$$|G(x, u, v, w)| \le c(1 + |u|^s + |v|^s + |w|^p),$$
$$|G_z(x, u, v, w)| \le c(1 + |u|^{s-1} + |v|^{s-1} + |w|^{p-\frac{p}{s}}),$$
where $s \in (1, 2^*_\alpha)$ for $n \ge 3$, a.e. $x \in \Omega$, $u \in \mathbb{R}$, $v \in \mathbb{R}$ and $w \in \mathbb{R}^m$.

(A3) For all $u \in X_0^{\alpha/2}(\Omega)$, there exist $b \in \mathbb{R}$, $\beta_1 \in L^2(\Omega)$, $\gamma_1 \in L^1(\Omega)$, such that
$$G(x, u(x), v, w) \ge -b|v|^2 - \beta_1(x)v - \gamma_1(x)$$
for any $v \in \mathbb{R}$, $w \in M$, a.e. $x \in \Omega$, and $2bd_2^2 < c(n, \alpha)$, where $c(n, \alpha)$ is the normalizing constant from (1.2) and d_2 is the Sobolev embedding constant from (1.12).

(A4) For all $v \in X_0^{\alpha/2}(\Omega)$, there exist $B \in \mathbb{R}$, $\beta_2 \in L^2(\Omega)$, $\gamma_2 \in L^1(\Omega)$, such that
$$G(x, u, v(x), w) \le B|u|^2 + \beta_2(x)u + \gamma_2(x)$$
for all $u \in \mathbb{R}$, $w \in M$, a.e. $x \in \Omega$, and $2Bd_2^2 < c(n, \alpha)$.

(A5) For all $w \in \mathcal{W}$ and $(\phi, \psi) \in \mathbb{Y}^{\alpha/2} = Y^{\alpha/2} \times Y^{\alpha/2}$, the functional $\mathcal{F}$ is concave with respect to u for any $v \in X_0^{\alpha/2}$, and convex with respect to v for any $u \in X_0^{\alpha/2}$. In short, for any $w \in \mathcal{W}$ and $(\phi, \psi) \in \mathbb{Y}^{\alpha/2}$, the functional $\mathcal{F}$ is concave-convex.

Remark 2.1. If assumptions (A1) and (A2) are satisfied, then for any $w \in \mathcal{W}$ and $(\phi, \psi) \in \mathbb{Y}^{\alpha/2}$, the functional $\mathcal{F}$ defined in (1.4) is well-defined and of class C^1 with respect to u and v; see, for example [34, Theorems C.1 and C.2]. The role of the optimality of the subcritical growth (A2) can be compared with the non-existence result presented in [5].

Remark 2.2. Assumptions (A1)–(A4) imply, by [34, Theorem 1.6], that for all $w \in \mathcal{W}$ and $(\phi, \psi) \in \mathbb{Y}^{\alpha/2}$, the functional $\mathcal{F}$ is weakly lower semicontinuous with respect to v for any $u \in X_0^{\alpha/2}$ and weakly upper semicontinuous with respect to u for any $v \in X_0^{\alpha/2}$. For general results on lower semicontiuity, see [24].

3. Existence of saddle points

In this section, we show that for any $w \in \mathcal{W}$ and $(\phi, \psi) \in \mathbb{Y}^{\alpha/2}$ there exists a saddle point of the functional $\mathcal{F}$ defined by (1.4). To establish the existence of a saddle point, we apply Ky Fan's minimax theorem; see [27, Theorem 5.2.2]. For general background on minimax methods and critical point theory, we refer to Willem's monograph [38] and Rabinowitz's classical work [28].

Moreover, we prove that the set of all saddle points is bounded, and that this bound may be chosen independently of the parameters. For any $w \in \mathcal{W}$ and $(\phi, \psi) \in \mathbb{Y}^{\alpha/2}$, let $S_{w,\phi,\psi}$ denote the set all saddle points of $\mathcal{F}$, i.e.,
$$S_{w,\phi,\psi} = \big\{(u_{w,\phi,\psi}, v_{w,\phi,\psi}) \in \mathbb{X}_0^{\alpha/2} : \mathcal{F}(u, v_{w,\phi,\psi}) \le \mathcal{F}(u_{w,\phi,\psi}, v_{w,\phi,\psi}) \le \mathcal{F}(u_{w,\phi,\psi}, v)\big\}.$$
We are now in a position to state the theorem regarding the existence of saddle points and their uniform a priori bound.

Theorem 3.1. *Assume that conditions* (A1)–(A5) *are satisfied. Then, for any* $w \in \mathcal{W}$ *and* $(\phi, \psi) \in \mathbb{Y}^{\alpha/2}$*, the functional* $\mathcal{F}$ *defined in* (1.4) *admits at least one saddle point*
$$(u_{w,\phi,\psi}, v_{w,\phi,\psi}) \in \mathbb{X}_0^{\alpha/2}.$$
Moreover, there exist radii $r_1, r_2 > 0$*, independent of* $w \in \mathcal{W}$ *and and* $(\phi, \psi) \in \mathbb{Y}^{\alpha/2}$ *such that*
$$S_{w,\phi,\psi} \subset B_1(0, r_1) \times B_2(0, r_2) \subset \mathbb{X}_0^{\alpha/2},$$

where $S_{w,\phi,\psi}$ denotes the set of all saddle points of $\mathcal{F}$. In particular, every saddle point is uniformly bounded with respect to the parameters and the exterior data. Finally, if the functional $\mathcal{F} = \mathcal{F}(u, v)$ is strictly concave in u - strictly convex in v, then the saddle point is unique.

Proof. Let $w \in \mathcal{W}$ be fixed. Notice that, for any $u \in X_0^{\alpha/2}$ the functional $\mathcal{F}(u, \cdot)$ is coercive. Indeed, by assumption $(A3)$, for any $u \in X_0^{\alpha/2}$, there exist a positive constant b and functions $\beta_1 \in L^2(\Omega)$, $\gamma_1 \in L^1(\Omega)$, such that, writing $c = c(n, \alpha)$ one has that $\mathcal{F}(u, v)$ is greater or equal than

$$\frac{c}{2}|v|^2_{X_0^{\alpha/2}} + c\langle \psi, v\rangle_{X_0^{\alpha/2}} - \int_\Omega (b|v(x)|^2 + 2bv(x)\psi(x) + \beta_1(x)v(x))\,dx + \gamma_0,$$

where

$$\gamma_0 = -\int_\Omega \gamma_1(x) + \beta_1(x)\psi(x) + \psi(x)^2 dx - \frac{c(n,\alpha)}{2}|u|^2_{X_0^{\alpha/2}} - \langle \phi, u\rangle_{X_0^{\alpha/2}}.$$

Applying the Sobolev embedding estimate (1.12) and the Cauchy-Schwartz inequality, we arrive at the inequality

$$\mathcal{F}(u, v) \ge \Big(\frac{c(n,\alpha)}{2} - bd_2^2\Big)\|v\|^2_{X_0^{\alpha/2}} - C\|v\|_{X_0^{\alpha/2}} + \gamma_0,$$

where C is a constant depending only on $\beta_1, c(n, \alpha), \psi$. In consequence, for any $u \in X_0^{\alpha/2}$, the functional $\mathcal{F}(u, \cdot)$ is coercive as $2bd_2^2 < c(n, \alpha)$. Since $\mathcal{F}(u, \cdot)$ is also weakly lower semicontinuous (cf. Remark 2.2), it follows that for each fixed $u \in X_0^{\alpha/2}$, the functional $\mathcal{F}(u, \cdot)$ attains its minimum. Subsequently, for any $u \in X_0^{\alpha/2}$, we define

$$\mathcal{F}^-(u) = \min_v \mathcal{F}(u, v).$$

From assumption (A4), and using the fact that $\mathcal{F}^-(u) \le \mathcal{F}(u, 0)$, we obtain similarly as before

$$\mathcal{F}^-(u) \le \Big(-\frac{c(n,\alpha)}{2} + Bd_2^2\Big)\|u\|^2_{X_0^{\alpha/2}} + D\|u\|_{X_0^{\alpha/2}} + \gamma_0 =: p(u), \tag{3.1}$$

where D, γ_0 are non-negative constants. We next show that the functional $\mathcal{F}^-$ is weakly upper semicontinuous. Let $\{u_k\}_{k\in\mathbb{N}}$ converge weakly to u_0 in $X_0^{\alpha/2}$, and let $\{v_k\}_{k\in\mathbb{N}_0}$ be such that $\mathcal{F}^-(u_k) = \mathcal{F}(u_k, v_k) = \min_v \mathcal{F}(u_k, v)$ for $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Then

$$\limsup_{k\to\infty} \mathcal{F}^-(u_k) = \limsup_{k\to\infty} \mathcal{F}(u_k, v_k) \le \limsup_{k\to\infty} \mathcal{F}(u_k, v_0) \le \mathcal{F}(u_0, v_0) = \mathcal{F}^-(u_0),$$

which proves the claimed weak upper semicontinuity.

Note that, since $-c(n, \alpha)/2 + Bd_2^2 < 0$, it follows from (3.1) that, for each $w \in \mathcal{W}$, the functional $\mathcal{F}^-$ attains its maximum at some point $u_{w,\phi,\psi} \in X_0^{\alpha/2}$. For any $u_{w,\phi,\psi}$ such that

$$\mathcal{F}^-(u_{w,\phi,\psi}) = \max_u \mathcal{F}^-(u), \tag{3.2}$$

from (A3) we obtain

$$\begin{aligned} \mathcal{F}^-(u_{w,\phi,\psi}) &\ge \mathcal{F}^-(0) = \min_v \mathcal{F}(0, v) \\ &\ge \min_v\Big(\Big(\frac{c(n,\alpha)}{2} - bd_2^2\Big)\|v\|^2_{X_0^{\alpha/2}} - C\|v\|_{X_0^{\alpha/2}} + \gamma_0\Big) \\ &=: \eta > -\infty, \end{aligned} \tag{3.3}$$

where b, C, γ_0, η are some constants and $c(n, \alpha)/2 - bd_2^2 > 0$. Importantly, the quantity η does not depend on w and (ϕ, ψ). Furthermore, for any maximizer $u_{w,\phi,\psi}$ satisfying (3.2), there exists $r_1 > 0$ such that for any $w \in \mathcal{W}$ and $\phi, \psi \in Y^{\alpha/2}$

$$u_{w,\phi,\psi} \in \{u : \mathcal{F}^-(u) \ge \eta\} \subset \{u : p(u) \ge \eta\} \subset B_1(0, r_1), \tag{3.4}$$

where p is the quadratic upper bound defined in (3.1) and η is from (3.3). We have thus shown that, for any $w \in \mathcal{W}$ and $\phi, \psi \in Y^{\alpha/2}$, there exists at least one $u_{w,\phi,\psi} \in X_0^{\alpha/2}$ such that

$$\mathcal{F}^-(u_{w,\phi,\psi}) = \max_u \mathcal{F}^-(u) = \max_u \big[\min_v \mathcal{F}(u, v)\big].$$

A completely analogous argument shows that, for any $w \in \mathcal{W}$ and $\phi, \psi \in Y^{\alpha/2}$, there exists at least one $v_{w,\phi,\psi} \in X_0^{\alpha/2}$ such that

$$\mathcal{F}^+(v_{w,\phi,\psi}) = \min_v \mathcal{F}^+(v) = \min_v \big[\max_u \mathcal{F}(u,v)\big], \tag{3.5}$$

where $\mathcal{F}^+(v) = \max_u \mathcal{F}(u,v)$. Additionally, there is $r_2 > 0$ such that for $v_{w,\phi,\psi}$ satisfying (3.5) one has

$$v_{w,\phi,\psi} \in B_2(0, r_2). \tag{3.6}$$

Moreover, the function $v \to \max_u \mathcal{F}(u,v)$ attains its minimum over $X_0^{\alpha/2}$, hence there is a real number λ such that

$$\lambda < \min_v \max_u \mathcal{F}(u,v) \le \max_u \mathcal{F}(u,0).$$

Consequently,

$$\{u \in X_0^{\alpha/2} : \mathcal{F}(u,0) \ge \lambda\} \subset \{u \in X_0^{\alpha/2} : p(u) \ge \lambda\} =: A_0,$$

where p is defined in (3.1). Furthermore, since $p(u) \ge \lambda$ implies a uniform bound on $\|u\|_{X_0^{\alpha/2}}$, the set A_0 is bounded in $X_0^{\alpha/2}$. Because $X_0^{\alpha/2}$ is a reflexive Banach space, every bounded subset is relatively compact in the weak topology. Therefore, the set $\{u \in X_0^{\alpha/2} : \mathcal{F}(u,0) \ge \lambda\}$ is weakly compact in $X_0^{\alpha/2}$. Next, by $(A5)$, $\mathcal{F}$ is concave in u - convex in v for any $w \in \mathcal{W}$ and $\phi, \psi \in Y^{\alpha/2}$. Thus, we have verified that all assumptions of Ky Fan's minimax theorem are satisfied. Therefore, $\max_u \min_v \mathcal{F}(u,v) = \min_v \max_u \mathcal{F}(u,v)$ for any $w \in \mathcal{W}$ and $\phi, \psi \in Y^{\alpha/2}$. As a consequence, for any $v \in X_0^{\alpha/2}$, we obtain

$$\begin{aligned}
\mathcal{F}(u_{w,\phi,\psi}, v_{w,\phi,\psi}) &\le \max_u \mathcal{F}(u, v_{w,\phi,\psi}) = \mathcal{F}^+(v_{w,\phi,\psi}) \\
&= \min_v \mathcal{F}^+(v) = \min_v [\max_u \mathcal{F}(u,v)] \\
&= \max_u [\min_v \mathcal{F}(u,v)] = \max_u \mathcal{F}^-(u) \\
&= \mathcal{F}^-(u_{w,\phi,\psi}) = \min_v \mathcal{F}(u_{w,\phi,\psi}, v) \\
&\le \mathcal{F}(u_{w,\phi,\psi}, v).
\end{aligned}$$

Similarly, one can show that for any $u \in X_0^{\alpha/2}$,

$$\mathcal{F}(u_{w,\phi,\psi}, v_{w,\phi,\psi}) \ge \mathcal{F}(u, v_{w,\phi,\psi}).$$

Whence, for any $u \in X_0^{\alpha/2}$ and $v \in X_0^{\alpha/2}$, we have

$$\mathcal{F}(u, v_{w,\phi,\psi}) \le \mathcal{F}(u_{w,\phi,\psi}, v_{w,\phi,\psi}) \le \mathcal{F}(u_{w,\phi,\psi}, v).$$

Thus, for any $w \in \mathcal{W}$ and $\phi, \psi \in Y^{\alpha/2}$, there exists at least one saddle point of $\mathcal{F}$. Finally, using (3.4) and (3.6), we conclude that $S_{w,\phi,\psi} \subset B_1(0, r_1) \times B_2(0, r_2)$, so every saddle point is uniformly bounded with respect to parameter w and exterior data (ϕ, ψ). $\square$

Remark 3.2. The set of solutions of (1.3) coincides with the set of saddle points of the functional $\mathcal{F}$ defined in (1.4), provided that $\mathcal{F}$ is concave in u and convex in v. Moreover, if $\mathcal{F}$ is strictly concave in u and strictly convex in v, then the problem (1.3) admits a unique solution.

Remark 3.3. For $\alpha = 2$, the degenerate coercive case was treated in [10, 11, 37], while the saddle-point approach was considered in [14]. The case $\alpha = 1$ was treated in [15].

4. Stability by continuous dependence

We now formulate the assumptions under which the solutions of the variational problem exhibit stability. By stability we mean the continuous dependence of the saddle points of the functional on both the functional parameters and the exterior data.

Let $\{w_k\} \subset \mathcal{W}$ and $(\phi_k, \psi_k) \in \mathbb{Y}^{\alpha/2}$ for any $k \in \mathbb{N}_0$. For each k, we denote by $\{\mathcal{F}_k\}_{k\in\mathbb{N}_0}$ the sequence of functionals, emphasizing the dependence on w_k, ϕ_k, ψ_k. Let S_k be the set of saddle points of $\mathcal{F}_k$ for $k \in \mathbb{N}_0$, that is,

$$S_k = \big\{(\bar{u}, \bar{v}) \in \mathbb{X}_0^{\alpha/2} : \mathcal{F}_k(\bar{u}, \bar{v}) = \max_u \min_v \mathcal{F}_k(u, v) = \min_v \max_u \mathcal{F}_k(u, v)\big\}. \tag{4.1}$$

By Theorem 3.1, each $\mathcal{F}_k$ admits at least one saddle point, so S_k is nonempty. Moreover, there exist positive constants r_1, r_2, independent of k such that $S_k \subset B_1(0, r_1) \times B_2(0, r_2) \subset \mathbb{X}_0^{\alpha/2}$.

We now employ the concept of the upper limit of a sequence of sets in the sense of Kuratowski-Painlevé, see [2]. This will allow us to study the stability of saddle points by examining the limiting behavior of the sets S_k.

Proposition 4.1. *Assume that conditions* (A1)–(A5) *are satisfied and a sequence of data* $\{(\phi_k, \psi_k)\}$ *tends to* (ϕ_0, ψ_0) *in* $\mathbb{Y}^{\alpha/2}$ *and a sequence of parameters* $\{w_k\}$ *tends to* w_0 *in* $L^p(\Omega, \mathbb{R}^m)$ *for some* $p \geq 1$. *Then the weak Kuratowski-Painlevé upper limit of the sequence of sets* $S_k \subset \mathbb{X}_0^{\alpha/2}$ *satisfies:* $\operatorname{Lim\,sup} S_k \neq \emptyset$ *and* $\operatorname{Lim\,sup} S_k \subset S_0$*, where* S_k *is the set of saddle points defined in* (4.1).

Proof. We begin by proving that $\mathcal{F}_k$ converges uniformly to $\mathcal{F}_0$ on $B_1(0, r_1) \times B_2(0, r_2)$, where the balls $B_1(0, r_1)$, $B_2(0, r_2)$ are those furnished by Theorem 3.1, so that $S_k \subset B_1(0, r_1) \times B_2(0, r_2)$ for all k. Fix an arbitrary $v \in X_0^{\alpha/2}$. Suppose, on the contrary, that the sequence $\{\mathcal{F}_k(\cdot, v)\}_{k\in\mathbb{N}}$ does not converge uniformly to $\mathcal{F}_0(\cdot, v)$ on $B_1(0, r_1)$. Thus there exists a constant $\varepsilon > 0$ and a sequence $\{u_l\} \subset B_1(0, r_1)$ such that for sufficiently large k

$$|\mathcal{F}_k(u_l, v) - \mathcal{F}_0(u_l, v)| \geq \varepsilon.$$

Up to a subsequence, we may assume that $u_l \rightharpoonup u_0 \in B_1(0, r_1)$ weakly in $X_0^{\alpha/2}$ hence (by compact embedding of $X_0^{\alpha/2}$ into L^s for $1 < s < 2_\alpha^*$), we have, again up to a subsequence, $u_l \to u_0$ in L^s. Hence, for any $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
|\mathcal{F}_k(u_l, v) - \mathcal{F}_0(u_l, v)| \leq &\int_\Omega |G(x, u_l + \phi_k, v + \psi_k, w_k) - G(x, u_l + \phi_k, v + \psi_k, w_0)| dx \\
&+ \int_\Omega |G(x, u_l + \phi_k, v + \psi_k, w_0) - G(x, u_l + \phi_k, v + \psi_0, w_0)| dx \\
&+ \int_\Omega |G(x, u_l + \phi_k, v + \psi_0, w_0) - G(x, u_l + \phi_0, v + \psi_0, w_0)| dx \\
&+ c(n, \alpha)|\langle \psi_k - \psi_0, v\rangle_{X_0^{\alpha/2}}| + c(n, \alpha)|\langle \phi_k - \phi_0, u_l\rangle_{X_0^{\alpha/2}}|.
\end{aligned}$$

The lower estimate by ε contradicts the upper bound, since all the terms on the right-hand side of the above inequality tend to zero. Indeed, by Krasnoselskii's theorem on the continuity of superposition operators (see [23, Theorem 2]) the mapping: $L^s(\Omega) \times L^s(\Omega) \times L^p(\Omega, \mathbb{R}^m) \ni (u, v, w) \mapsto G(\cdot, u(\cdot), v(\cdot), w(\cdot)) \in L^1(\Omega)$, is continuous under assumption $(A2)$. Using the same argument - with the roles of u and v interchanged - we obtain uniform convergence of the sequence $\{\mathcal{F}_k(u, \cdot)\}$ on the ball $B_2(0, r_2)$. Hence, we conclude that $\mathcal{F}_k \rightrightarrows \mathcal{F}_0$ uniformly on $B_1(0, r_1) \times B_2(0, r_2)$. Let us denote

$$m_k = \max_u \min_v \mathcal{F}_k(u, v) = \max_{u\in B_1(0,r_1)} \min_{v\in B_2(0,r_2)} \mathcal{F}_k(u, v) \text{ for } k \in \mathbb{N}_0.$$

Since $\mathcal{F}_k \rightrightarrows \mathcal{F}_0$ uniformly on $B_1(0, r_1) \times B_2(0, r_2)$, it follows that for any $\varepsilon > 0$, there exists K_0 such that

$$\mathcal{F}_k(u, v) \leq \mathcal{F}_0(u, v) + \varepsilon$$

for any $(u, v) \in B_1(0, r_1) \times B_2(0, r_2)$ and $k > K_0$. This implies that

$$\min_{v\in B_2(0,r_2)} \mathcal{F}_k(u, v) \leq \min_{v\in B_2(0,r_2)} \mathcal{F}_0(u, v) + \varepsilon$$

for any $u \in B_2(0, r_2)$ and $k > K_0$. Consequently,

$$\max_{u\in B_1(0,r_1)} \min_{v\in B_2(0,r_2)} \mathcal{F}_k(u, v) \leq \max_{u\in B_1(0,r_1)} \min_{v\in B_2(0,r_2)} \mathcal{F}_0(u, v) + \varepsilon$$

for $k > K_0$. Thus $m_k - m_0 \leq \varepsilon$ for sufficiently large k. Analogously, it is possible to show that $-\varepsilon \leq m_k - m_0$ for sufficiently large k. We thus have proved that m_k tends to m_0 as $k \to \infty$.

Subsequently, let $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ be an arbitrary sequence of saddle points, such that $(u_k, v_k) \in S_k$ for $k \in \mathbb{N}$. By Theorem 3.1, for any $k \in \mathbb{N}$, each set S_k is nonempty, and there exist radii $r_1 > 0$ and $r_2 > 0$ such that $S_k \subset B_1(0, r_1) \times B_2(0, r_2)$ for every k, that is the sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{X}_0^{\alpha/2}$. Since $\mathbb{X}_0^{\alpha/2}$ is reflexive, the sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is weakly compact. Therefore, the set of its cluster points with respect of weak topology of $\mathbb{X}_0^{\alpha/2}$ is nonempty, and consequently $\operatorname{Lim\,sup} S_k \neq \emptyset$. Let $(u_0, v_0) \in B_1(0, r_1) \times B_2(0, r_2)$ be any cluster point of the sequence $\{(u_k, v_k)\}_{k\in\mathbb{N}}$. Passing to a subsequence if necessary, we may assume that $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ tends to (u_0, v_0) weakly in $\mathbb{X}_0^{\alpha/2}$. We now show that $(u_0, v_0) \in S_0$. Suppose, for contradiction, that (u_0, v_0) does not belong to S_0. Let $(\tilde{u}, \tilde{v})$ be an element of S_0. Then $\mathcal{F}_0(u_0, v_0) \neq \mathcal{F}_0(\tilde{u}, \tilde{v})$. Consider the case when $\mathcal{F}_0(\tilde{u}, \tilde{v}) - \mathcal{F}_0(u_0, v_0) = \lambda < 0$. In that situation we have

$$\begin{aligned}
m_k - m_0 &= \mathcal{F}_k(u_k, v_k) - \mathcal{F}_0(u_0, v_0)\\
&\leq \mathcal{F}_k(u_k, \tilde{v}) - \mathcal{F}_0(u_0, v_0)\\
&= (\mathcal{F}_k(u_k, \tilde{v}) - \mathcal{F}_0(u_k, \tilde{v})) + (\mathcal{F}_0(u_k, \tilde{v}) - \mathcal{F}_0(\tilde{u}, \tilde{v})) + (\mathcal{F}_0(\tilde{u}, \tilde{v}) - \mathcal{F}_0(u_0, v_0)).
\end{aligned}$$

Using the uniform convergence of $\mathcal{F}_k$ to $\mathcal{F}_0$ on $B_1(0, r_1) \times B_2(0, r_2)$ and the weak upper semicontinuity of $\mathcal{F}_0(\cdot, v)$ we obtain $\lim_{k\to\infty} \big[\mathcal{F}_k(u_k, \tilde{v}) - \mathcal{F}_0(u_k, \tilde{v})\big] = 0$ and $\limsup_{k\to\infty} \big[\mathcal{F}_0(u_k, \tilde{v}) - \mathcal{F}_0(\tilde{u}, \tilde{v})\big] \leq 0$. This gives $\limsup_{k\to\infty}(m_k - m_0) \leq \lambda < 0$. We have arrived at contradiction with the limit $m_k \to m_0$ as $k \to \infty$. An analogous contradiction arises when $\lambda > 0$. Hence $\lambda = 0$, which means that our assumption was false and $(u_0, v_0) \in S_0$. Therefore, $\operatorname{Lim\,sup} S_k \subset S_0$ in the weak topology of $\mathbb{X}_0^{\alpha/2}$, which completes the proof. □

Proposition 4.2. *Assume that conditions* (A1)–(A5) *hold, that* $w_k \to w_0$ *in* $L^p(\Omega, \mathbb{R}^m)$ *for some* $p \geq 1$, *and that* $(\phi_k, \psi_k) \to (\phi_0, \psi_0)$ *in* $\mathbb{Y}^{\alpha/2}$. *Then* $\operatorname{Lim\,sup} S_k \neq \emptyset$ *and* $\operatorname{Lim\,sup} S_k \subset S_0$ *in* $\mathbb{X}_0^{\alpha/2}$.

Proof. We commence with a verification of the uniform convergence of the derivatives $\mathcal{F}_k'$ to $\mathcal{F}_0'$ on $B_1(0, r_1) \times B_2(0, r_2)$, where the balls $B_1(0, r_1)$, $B_2(0, r_2)$ are those provided by Theorem 3.1 such that for every $w_k \in \mathcal{W}$ and $(\phi_k, \psi_k) \in \mathbb{Y}^{\alpha/2}$, we have $S_k \subset B_1(0, r_1) \times B_2(0, r_2)$.

Let $v \in X_0^{\alpha/2}$ be an arbitrary point. In the beginning, suppose that the sequence of derivatives $\{\frac{\partial \mathcal{F}_k}{\partial u}(\cdot, v)\}_{k\in\mathbb{N}}$ does not converge uniformly to $\frac{\partial \mathcal{F}_0}{\partial u}(\cdot, v)$ on $B_1(0, r_1)$. Then there exists a sequence $\{u_l\} \subset B_1(0, r_1)$ and a positive constant ε such that

$$|\langle \frac{\partial \mathcal{F}_k}{\partial u}(u_l, v) - \frac{\partial \mathcal{F}_0}{\partial u}(u_l, v), g_l \rangle| \geq \varepsilon \text{ for any } k \in \mathbb{N}$$

and $\{g_l\} \subset B_1(0, r_1)$. Passing to a subsequence if necessary, we may assume that $u_l \rightharpoonup u_0 \in B_1(0, r_1)$. For any $k \in \mathbb{N}$, we have

$$\begin{aligned}
&|\langle \frac{\partial \mathcal{F}_k}{\partial u}(u_l, v) - \frac{\partial \mathcal{F}_0}{\partial u}(u_l, v), g_l \rangle|\\
&\leq c(n, \alpha) |\langle \phi_k - \phi_0, g_l \rangle_{X_0^{\alpha/2}}|\\
&\quad + \int_\Omega |(G_u(x, u_l + \phi_k, v + \psi_k, w_k) - G_u(x, u_l + \phi_k, v + \psi_k, w_0))||g_l(x)| dx\\
&\quad + \int_\Omega |G_u(x, u_l + \phi_k, v + \psi_k, w_0) - G_u(x, u_l + \phi_k, v + \psi_0, w_0)||g_l(x)| dx\\
&\quad + \int_\Omega |(G_u(x, u_l + \phi_k, v + \psi_0, w_0) - G_u(x, u_l + \phi_0, v + \psi_0, w_0))||g_l(x)| dx\,.
\end{aligned}$$

Each of the above terms tends to zero as $l, k \to \infty$. Indeed, by assumption (A2) and Krasnoselskii's continuity theorem for superposition operators (see [23, Theorem 2]), the mapping $L^s(\Omega) \times L^s(\Omega) \times L^p(\Omega, \mathbb{R}^m) \ni (u, v, w) \mapsto G_u(\cdot, u(\cdot), v(\cdot), w(\cdot)) \in L^{s/(s-1)}(\Omega)$ is continuous. Since $(\phi_k, \psi_k) \to (\phi_0, \psi_0)$ in $\mathbb{Y}^{\alpha/2}$, $w_k \to w_0$ in $L^p(\Omega, \mathbb{R}^m)$ and the sequences $\{u_l\}$, $\{g_l\}$ remain in bounded subset of $\mathbb{X}_0^{\alpha/2}$, all three integrals converge to zero. Likewise, $\langle \phi_k - \phi_0, g_l \rangle_{X_0^{\alpha/2}} \to 0$. Using the same reasoning, we obtain uniform convergence of the sequence $\{\frac{\partial \mathcal{F}_k}{\partial u}(u,\cdot)\}_{k\in\mathbb{N}}$ on a ball $B_2(0,r_2)$. In consequence, $\mathcal{F}_k' \rightrightarrows \mathcal{F}_0'$ uniformly on $B_1(0,r_1)\times B_2(0,r_2)$ as claimed.

Let $\{(u_k,v_k)\}\subset \mathbb{X}_0^{\alpha/2}$ be a sequence such that $(u_k,v_k)\in S_k$ for $k\in\mathbb{N}$. By Theorem 3.1, there exist r_1, $r_2>0$ such that for any $k\in\mathbb{N}$, $S_k\subset B_1(0,r_1)\times B_2(0,r_2)$. By reflexivity, the sequence $\{(u_k,v_k)\}$ admits a weakly convergent subsequence; without loss of generality we assume that $(u_k,v_k)\rightharpoonup(u_0,v_0)\in B_1(0,r_1)\times B_2(0,r_2)$ weakly in $\mathbb{X}_0^{\alpha/2}$. We now prove that $(u_k,v_k)\to(u_0,v_0)$ strongly in $\mathbb{X}_0^{\alpha/2}$. Let us observe that for any k, we have

$$\begin{aligned}
&\langle \mathcal{F}_0'(u_k,v_k)-\mathcal{F}_0'(u_0,v_0),(u_0-u_k,v_k-v_0)\rangle\\
&= c(n,\alpha)\|u_k-u_0\|^2_{X_0^{\alpha/2}}+c(n,\alpha)\|v_k-v_0\|^2_{X_0^{\alpha/2}}\\
&\quad+\int_\Omega (G_u(x,u_k+\phi_0,v_k+\psi_0,w_0)-G_u(x,u_0+\phi_0,v_0+\psi_0,w_0))(u_0(x)-u_k(x))\,dx\\
&\quad+\int_\Omega (G_v(x,u_k+\phi_0,v_k+\psi_0,w_0)-G_v(x,u_0+\phi_0,v_0+\psi_0,w_0))(v_k(x)-v_0(x))\,dx.
\end{aligned}$$

Because $\mathcal{F}_k'\rightrightarrows\mathcal{F}_0'$ on $B_1(0,r_1)\times B_2(0,r_2)$, and since $(u_k,v_k)\in S_k$, we have $\mathcal{F}_0'(u_k,v_k)\to 0$, so the left-hand side tends to 0. We now show that the nonlinear integral terms also tend to zero. The assumption (A2) and the Hölder inequality lead to the estimate

$$\begin{aligned}
&\Big|\int_\Omega (G_u(x,u_k+\phi_0,v_k+\psi_0,w_0)-G_u(x,u_0+\phi_0,v_0+\psi_0,w_0))(u_0(x)-u_k(x))\,dx\Big|\\
&\le \Big(\int_\Omega |G_u(\cdot,u_k+\phi_0,v_k+\psi_0,w_0)-G_u(\cdot,u_0+\phi_0,v_0+\psi_0,w_0)|^{\frac{s}{s-1}}\Big)^{\frac{s-1}{s}}\|u_k-u_0\|_{L^s}.
\end{aligned}$$

An analogous estimate holds for G_v.

Since $X_0^{\alpha/2}$ embeds compactly into $L^s(\Omega)$ for $s\in[1,2^*_\alpha)$ and $n\ge 3$, $u_k\to u_0$, $v_k\to v_0$ in $L^s(\Omega)$. By Krasnoselskii's theorem [23, Theorem 2] and assumption (A2), $G_u(\cdot,u_k+\phi_0,v_k+\psi_0,w_0)$ tends to $G_u(\cdot,u_0+\phi_0,v_0+\psi_0,w_0)$ in $L^{s/(s-1)}(\Omega)$, and similarly for G_v. So, both nonlinear terms tend to zero. Finally, $(u_k,v_k)\to(u_0,v_0)\in S_0$ in the strong topology of $\mathbb{X}_0^{\alpha/2}$. This shows that $\operatorname{Lim\,sup} S_k \neq\emptyset$ and $\operatorname{Lim\,sup} S_k\subset S_0$ in the strong topology of $\mathbb{X}_0^{\alpha/2}$. Since strong convergence implies weak convergence, this also strengthens the result $\operatorname{Lim\,sup} S_k\subset S_0$ obtained earlier in the weak topology of $\mathbb{X}_0^{\alpha/2}$ (see Proposition 4.1). ☐

Remark 4.3. Rephrasing Preposition 4.2, we state that set-valued mapping $\mathbb{Y}^{\alpha/2}\times L^p(\Omega,\mathbb{R}^m)\ni(\phi_k,\psi_k,w_k)\mapsto S_k\subset\mathbb{X}_0^{\alpha/2}$ is well-defined and upper semicontinuous with respect to the topology of $\mathbb{Y}^{\alpha/2}\times L^p(\Omega,\mathbb{R}^m)$ on domain and the topology of $\mathbb{X}_0^{\alpha/2}$ on the codomain. If, in addition, each set S_k is a singleton, that is, $S_k=\{(u_k,v_k)\}$, then one has the convergence $(u_k,v_k)\to(u_0,v_0)$ in $\mathbb{X}_0^{\alpha/2}$, whenever $w_k\to w_0$ in $L^p(\Omega,\mathbb{R}^m)$ and $(\phi_k,\psi_k)\to(\phi_0,\psi_0)$ in $\mathbb{Y}^{\alpha/2}$.

Remark 4.4. One can verify directly that the functional

$$\hat{\mathcal{F}}(u,v)=\frac{c(n,\alpha)}{2}\Big(|v|^2_{\mathbb{X}_0^{\alpha/2}}-|u|^2_{\mathbb{X}_0^{\alpha/2}}\Big)+\frac12\int_\Omega\Big(-\xi_1|v(x)|^2+\xi_2|u(x)|^2\Big)\,dx$$

is strictly concave in u and strictly convex in v for $\xi_1<d_2^2c(n,\alpha)$, $\xi_2<d_2^2c(n,\alpha)$ and concave in u and convex in v under the weaker conditions without strict inequalities. Here d_2 is the Sobolev embedding constant appearing in (1.12). Furthermore, the difference $\mathcal{F}(u,v)-\hat{\mathcal{F}}(u,v)$ can be rewritten in the form

$$\int_\Omega\Big(\frac{\xi_1}{2}|v(x)|^2-\frac{\xi_2}{2}|u(x)|^2+G(x,u+\phi,v+\psi,w)\Big)\,dx+c\langle\psi,v\rangle_{X_0^{\alpha/2}}-c\langle\phi,u\rangle_{X_0^{\alpha/2}}$$

for all $(u,v)\in\mathbb{X}_0^{\alpha/2}$. This observation shows that assumption (A5) can be relaxed. Indeed, it suffices to assume that the function

$$H(x,u,v,w)=\frac{\xi_1}{2}|v|^2-\frac{\xi_2}{2}|u|^2+G(x,u+\phi,v+\psi,w)$$

is concave in u and convex in v. Under these conditions, the full functional $\mathcal{F}$ inherits the required concave-convex structure necessary for the minimax-saddle point argument.

Example 4.5. Let $\Omega = P^3(0,\pi) = \{x \in \mathbb{R}^3 : 0 < x_i < \pi,\ i = 1,2,3\}$. Recall that $u_l = \sin x_1 \sin x_2 \sin x_3$ is the first eigenfunction of the classical Laplacian on $H_0^1(\Omega)$, with corresponding eigenvalue $\rho_1 = 3$, since $-\Delta u_l = 3u_l$. For the fractional Laplacian, by [18], the first eigenvalue λ_1 satisfies the bounds

$$\frac{3^{\alpha/2}}{2} \le \lambda_1 \le 3^{\alpha/2}\,.$$

We consider the system involving the fractional Laplacian

$$\begin{gathered}
-(-\Delta)^{\alpha/2}u + \beta_1 u + w_1 v + l_1(x) = 0 \quad \text{in } \Omega,\\
(-\Delta)^{\alpha/2}v - \beta_2 v + w_2 u + l_2(x) = 0 \quad \text{in } \Omega,\\
u = 0, \quad v = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{gathered} \tag{4.2}$$

where the coefficients satisfy $\beta_i < \frac{3^{\alpha/2}}{2}$, $l_i \in L^2(\Omega)$, $w_i \in \mathcal{W}$ for $i = 1,2$, a.e. $x \in \Omega$, and

$$\mathcal{W} = \{w \in L^p(\Omega,\mathbb{R}^2) : w(x) \in [0,1]\times[0,1] \text{ for a.e. } x \in \Omega\}$$

with $p \in (3/\alpha,\infty)$. The associated energy functional corresponding to (4.2) is

$$\begin{aligned}
\mathcal{F}(u,v) = \frac{c(n,\alpha)}{2}(\|v\|^2_{X_0^{\alpha/2}} - \|u\|^2_{X_0^{\alpha/2}}) + \int_\Omega \frac{\beta_1}{2}|u(x)|^2 - \frac{\beta_2}{2}|v(x)|^2 dx\\
+ \int_\Omega (w_1(x) + w_2(x))u(x)v(x) + l_1(x)u(x) + l_2(x)v(x)dx.
\end{aligned}$$

One can verify that

$$G(x,u,v,w) = \frac{\beta_1}{2}u^2 - \frac{\beta_2}{2}v^2 + (w_1 + w_2)uv + l_1(x)u + l_2(x)v$$

satisfies conditions (A1)–(A4) with $b = \beta_1/2$, $B = \beta_2/2$. Furthermore, the functional $\mathcal{F}$ is strictly concave in u and strictly convex in v. Hence, for any $w_k = (w_{1k}, w_{2k}) \in \mathcal{W}$, there exists a unique weak solution (u_k, v_k) of problem (4.2), cf. Theorem 3.1 and Remark 3.2. If, in addition, $w_k \to w_0$ in $L^p(\Omega,\mathbb{R}^2)$ for some $p \in (3/\alpha,\infty)$, then the corresponding solutions $(u_k,v_k) \to (u_0,v_0)$ in $\mathbb{X}_0^{\alpha/2}$, as stated in Proposition 4.2.

Example 4.6. Let $\Omega = P^3(0,\pi)$. We consider the nonlinear coupled fractional system

$$\begin{gathered}
-(-\Delta)^{\alpha/2}u + bu - s|x|^2u^{s-1}w_1 - |x|w_2 + v = 0 \quad \text{in } \Omega,\\
(-\Delta)^{\alpha/2}v - av + s|x|^2v^{s-1}w_1 - |x|w_2 + u = 0 \quad \text{in } \Omega,\\
u = 0, \quad v = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{gathered} \tag{4.3}$$

The exponent s and the integrability exponent p satisfy the admissibility ranges $1 + 1/(p-1) < s < 6/(3-\alpha)$, $p \in (6/(3+\alpha),\infty)$ or $1 < s < 6/(3-\alpha)$, $p = \infty$. These conditions ensure that the nonlinear terms $|x|^2u^{s-1}w_1$ and $|x|^2v^{s-1}w_1$ are well-defined and compatible with the fractional Sobolev embeddings $X_0^{\alpha/2} \hookrightarrow L^s(\Omega)$. The associated functional of action for system (4.3) is given by

$$\begin{aligned}
\mathcal{F}(u,v) = -\frac{c(n,\alpha)}{2}\int_Q \frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy + \frac{c(n,\alpha)}{2}\int_Q \frac{(v(x)-v(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy\\
+ \int_\Omega \Big[-\frac{a}{2}v^2(x) + \frac{b}{2}u^2(x) + |x|^2v^s(x)w_1(x) - |x|^2u^s(x)w_1(x)\\
- u(x)|x|w_2(x) - v(x)|x|w_2(x) + u(x)v(x)\Big]dx.
\end{aligned}$$

One can verify that the functional $\mathcal{F}$ satisfies all assumptions required by Theorem 3.1 as well as Propositions: 4.1 and 4.2. Moreover, by Remark 4.4, the functional $\mathcal{F}$ is strictly concave in u and strictly convex in v. Therefore, from Theorem 3.1 together with Remark 3.2, it follows that for any parameter w there exists exactly one weak solution (u_w, v_w) of the problem (4.3).

Example 4.7. Consider the exterior Dirichlet problem involving the fractional Laplace operator

$$\begin{gathered}(-\Delta)^{\alpha/2}u(x)=\varepsilon,\quad |x|\le 1\\ u(x)=0,\quad |x|\ge 1\,.\end{gathered}$$

It is well known, see [26], that for any $\varepsilon>0$ the unique solution u_ε can be found for some constant $D>0$ in the explicit form

$$u_\varepsilon(x)=\varepsilon D(1-|x|^2)_+^{\alpha/2}$$

and obviously it converges to zero as $\varepsilon\to 0^+$. This straightforward example naturally embeds into our formalism of the continuous dependence of the unique solution with respect to the right-hand side data.

Example 4.8. Consider the non-homogeneous exterior value problem involving the fractional Laplace operator

$$\begin{gathered}(-\Delta)^{\alpha/2}u=0,\quad 1/4<|x|<1\\ u(x)=0,\quad |x|\ge 1\\ u(x)=1,\quad |x|\le 1/4\,.\end{gathered}$$

It is well known (see [29]) that the unique solution u satisfies, for some constant $C>0$, the estimate

$$u(x)\ge C(1-|x|)^{\alpha/2}$$

for all $x<1$.

5. Concluding remarks

The continuous dependence of solutions on functional parameters or data for exterior problems involving the fractional Laplacian remains underexplored in the literature. In this work, we establish both the existence of weak solutions and their continuous dependence on parameters and exterior data. Notably, the weak solutions emerge as saddle critical points of a suitably defined action functional. The primary novelty of the results lies in the nonlocal structure of the action functional, which depends on the exterior Dirichlet data associated with the fractional Laplace operator.

It is worth noting that the author previously considered the fractional spectral Laplacian within a minimax framework in [13]. However, as observed in [4, 32] spectral and integral fractional Laplacian operators exhibit fundamentally different properties. Furthermore, one of the earliest applications of the Ky-Fan minimax principle to problems involving the classical Laplace operator was presented in [25]. Those results, which also investigated dependence on functional parameters, were built upon prior findings from the author's PhD thesis [9]. In the present paper, we extend these foundational results not only to the fractional Laplacian framework, but we also account for dependence on data defined on the exterior of the domain.

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