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GLOBAL WELL-POSEDNESS OF 3D INHOMOGENEOUS INCOMPRESSIBLE LIQUID CRYSTAL SYSTEMS WITH DENSITY-DEPENDENT VISCOSITY IN CRITICAL SOBOLEV SPACES

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Abstract. This article concerns the well posedness of solutions for model of a three-dimensional non-homogeneous incompressible nematic liquid crystal flows with density-dependent viscosity. We establish the existence of global strong solutions when the initial data satisfies $(\rho_0, u_0, \nabla d_0) \in L^\infty(\mathbb{R}^3) \times \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{1/2}(\mathbb{R}^3)$, and uniqueness when $(\rho_0, u_0, \nabla d_0) \in \dot{B}^{3/q}_{q,1}(\mathbb{R}^3) \times \dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \times \dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. These results refines the corresponding results obtained by Ye and Zhang [23].

1. Introduction

This article studies the global well posedness for the inhomogeneous incompressible liquid crystal system with density-dependent viscosity

$$\begin{gathered}
\partial_t \rho + u \cdot \nabla \rho = 0, \\
\rho \partial_t u + \rho u \cdot \nabla u - \operatorname{div}\left(\mu\left(\rho\right) \nabla u\right) + \nabla P = -\nabla \cdot \left(\nabla d : \nabla d\right), \\
\partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d, \\
\operatorname{div} u = 0, \\
|d| = 1, \\
(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0),
\end{gathered} \tag{1.1}$$

where ρ and u denote the fluid density and velocity field, respectively, while P is a scalar function representing the fluid pressure. The macroscopic molecular orientation of the nematic liquid crystal is described by the director field $\mathrm{d} \in S^2$, where $S^2 := \{\mathrm{d} = (\mathrm{d}_1, \mathrm{d}_2, \mathrm{d}_3) \in \mathbb{R}^3 || \mathrm{d}| = 1\}$. The nonlinear term $\nabla \mathrm{d} : \nabla \mathrm{d}$ denotes a matrix whose (i, j)-th entry is $\partial_i \mathrm{d} \cdot \partial_j \mathrm{d}$ (where $1 \le i, j \le 3$). The viscosity coefficients μ, λ, θ are positive constants. The initial velocity u_0 satisfies $\operatorname{div} u_0 = 0$, and the initial orientation vector d_0 fulfils $|\mathrm{d}_0| = 1$. The viscosity coefficient $\mu = \mu(\rho)$ is a function of density.

System (1.1) is a simplified version of the original Ericksen-Leslie model (see [11, 15]) that describes the evolutionary behavior of nematic liquid crystal flows. For a comprehensive discussion of the physical foundations of continuum theory of liquid crystals, we refer to the monographs [3, 9].

When the viscosity coefficient is constant, the system reduces to the classical nonhomogeneous incompressible liquid crystal system. Li and Wang [16] established the local well-posedness of this system without requiring initial compatibility conditions. Global well-posedness was later proved under smallness assumptions on the initial data, with the initial density bounded away from vacuum.

Li and Wang [17] studied the initial-boundary value problem for density-dependent incompressible liquid crystal flows in a 3D bounded smooth domain. They obtained local solutions for general initial data (with density away from vacuum) and global small solutions under additional

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smallness conditions. Ding, Huang, and Xia [10] obtained global strong solutions in 3D for small initial data, allowing vacuum.

Liu and Zhang [19] established global well-posedness in 2D with vacuum, assuming a smallness condition on the initial data. They also derived a Serrin-type blowup criterion depending solely on ∇d. De Anna [7] obtained global solutions for small initial data, assuming the initial density is bounded and strictly positive and the initial velocity and ∇d_0 lie in critical Besov spaces. Liu et al. [19] extended these results, proving that 2D nonhomogeneous incompressible nematic liquid crystal flows admit a unique global strong solution, provided that the initial density and ∇d_0 decays sufficiently fast at infinity while the initial orientation satisfies a geometric condition.

Recently, Chen, Liang, Chen and Ye [4] improved the results of De Anna [7] and established the global well-posedness of the three dimensional nonhomogeneous incompressible liquid crystal flow provided that the initial density ρ_0 is bounded, the initial velocity u_0 and the gradient of orientation ∇d_0 are in the critical Besov space $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)$. As for the general case $(\rho_0, u_0, \nabla d_0) \in L^\infty(\mathbb{R}^d) \times \dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d) \times \dot{B}^{-1+\frac{d}{p}}_{p,1}(\mathbb{R}^d)$, the well-posedness of nonhomogeneous incompressible nematic liquid crystal system was established by Wu and Liang [22].

When the director field d is constant, system (1.1) reduces to the nonhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity. Key well-posedness results for this system include: Desjardins [8] proved the existence of global weak solutions with enhanced regularity, assuming the viscosity coefficient $\mu(\rho)$ is a small perturbation of a positive constant in two dimensions. Gui and Zhang [12] established global well-posedness when the initial density ρ_0 is a small perturbation of a positive constant in the Sobolev space $H^s(\mathbb{R}^2)(s \geq 2)$, provided vacuum is absent. Cho and Kim [5] obtained local existence of strong solutions under compatibility conditions on the initial data. Global Strong Solutions (2D) can be found in [14]. Abidi, Gui and Zhang [?] (see also Huang and Wang [14]) established global strong solutions in three dimensions under the smallness condition. He, Li, and Lv [13] extended this result, showing global existence with small initial data in the homogeneous Sobolev space $\dot{H}^\alpha(\mathbb{R}^3)$ with $1/2 < \alpha \leq 1$.

Recently, Ye and Zhu [23] extend the result of He, Li and Lv to three dimensional inhomogeneous nematic liquid crystal equation. They established the following proposition.

Proposition 1.1. *For given numbers $\bar{\rho} \geq 0, p > 3, \frac{1}{2} < \alpha \leq 1$, we assume the initial data $(\rho_0, u_0, \nabla d_0)$ satisfies*

$$\begin{gathered} 0 \leq \rho \leq \bar{\rho}, \quad \rho_0 \in L^{3/2}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3), \quad \nabla \mu(\rho_0) \in L^p(\mathbb{R}^3), \\ u_0 \in H^1_0(\mathbb{R}^3) \cap \dot{H}^\alpha(\mathbb{R}^3), \quad \nabla d_0 \in L^{3/2}(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \cap \dot{H}^\alpha(\mathbb{R}^3), \end{gathered}$$

Then there exists a positive constant ε_0 depending on $\bar{\rho}, p, \mu, \|\rho_0\|_{L^{3/2}}, \|\nabla d_0\|_{L^{3/2}}$ such that if

$$\|u_0\|_{\dot{H}^\alpha} + \|\nabla d_0\|_{\dot{H}^\alpha} \leq \varepsilon_0,$$

then system (1.1) admits a unique global solution (ρ, u, d) satisfying that for any $0 < \tau < T$,

$$\begin{gathered} 0 \leq \rho \in C([0,T], L^{3/2} \cap L^\infty), \quad \nabla \mu(\rho) \in C([0,T], L^p), \\ P \in L^\infty([0,T], L^2 \cap H^1), \quad P_t \in L^2([\tau,T], L^2), \\ (\nabla u, \nabla^2 d) \in L^\infty([0,T], L^2) \cap L^2([0,T], L^2), \\ (\sqrt{\rho} u_t, \nabla d_t) \in L^\infty([\tau,T], L^2) \cap L^2([0,T], L^2), \\ (\nabla u_t, \nabla^2 d_t) \in L^\infty([\tau,T], L^2) \cap L^2([\tau,T], L^2), \\ (\sqrt{\rho} u_{tt}, \nabla d_{tt}) \in L^2([\tau,T], L^2). \end{gathered}$$

For $t > 1$, we have the global decay estimates

$$\begin{gathered} \|\nabla u(\cdot,t)\|^2_{L^2} + \|\nabla^2 d(\cdot,t)\|^2_{L^2} + \|P(\cdot,t)\|^2_{L^2} \leq Ct^{-1} \\ \|\sqrt{\rho} u_t(\cdot,t)\|^2_{L^2} + \|\nabla d_t(\cdot,t)\|^2_{L^2} + \|\nabla P(\cdot,t)\|^2_{L^2} \leq Ct^{-2}, \end{gathered}$$

where C is independent of $\bar{\rho}, p, \mu, \|\rho_0\|_{L^{3/2}}, \|\nabla d_0\|_{L^{3/2}}, \Pi_0$.

Motivated by above references, a natural question arises: How about the global well-posedness of the solution to system (1.1) provided that the initial data $(\rho_0, u_0, \nabla d_0) \in L^\infty \times \dot{H}^{1/2} \times \dot{H}^{1/2}$ satisfies

$$\|u_0\|_{\dot{H}^{1/2}} + \|\nabla d_0\|_{\dot{H}^{1/2}} \le \varepsilon_0,$$

with ε_0 being sufficiently small? To address this challenging problem, we assume that

$$0 < \underline{\mu} \le \mu(\rho_0), \quad \mu(\cdot) \in W^{2,\infty}, \tag{1.2}$$

and formulate our main result as follows.

Theorem 1.2. *Let $0 < m \le \rho_0 \le M$ and $\mu(\rho_0) \ge \underline{\mu} > 0$. Assume that $(u_0, \nabla d_0) \in \dot{H}^{1/2} \times \dot{H}^{1/2}$. Then there exist a sufficient small positive constant ϵ_0, such that if*

$$\|(u_0, \nabla d_0)\|_{\dot{H}^{1/2}} + \|\mu(\rho_0) - 1\|_{L^\infty} \le \epsilon_0, \tag{1.3}$$

then system (1.1) has a global solution $(u, \nabla d, \nabla P)$ satisfying that for any $0 \le t \le T$,

$$\begin{aligned}
&\|(u, \nabla d)\|_{L^\infty_T(\dot{H}^{1/2})} + \|(u, \nabla d)\|_{L^\infty_T(L^3)} + \|(u, \nabla d)\|_{L^2_T(L^\infty)} + \|(\nabla u, \nabla^2 d)\|_{L^4_T(L^2)} \\
&+ \|(\nabla u, \nabla^2 d)\|_{L^2_T(L^3)} + \|t^{1/4}(\nabla u, \nabla^2 d)\|_{L^2_T(L^6)} + \|t^{-1/4}(\nabla u, \nabla^2 d)\|_{L^2_T(L^2)} \\
&+ \|t^{1/4}(\nabla u, \nabla^2 d)\|_{L^\infty_T(L^2)} + \|t^{1/4}(\partial_t u, \partial_t \nabla d)\|_{L^2_T(L^2)} \\
&\le C\|(u_0, \nabla d_0)\|_{\dot{H}^{1/2}}
\end{aligned}$$

If in addition, $(u_0, \nabla d_0) \in \dot{B}^{1/2}_{2,1}, a_0 \in \dot{B}^{3/q}_{q,1}$, then the solution is unique, where $a_0 = \frac{1}{\rho_0} - 1$ and q satisfies some conditions which will be introduced in the proof of uniqueness.

Remark 1.3. Theorem 1.2 extends the existence results in [23] to the initial data $(\rho_0, u_0, \nabla d_0) \in L^\infty \times \dot{H}^{1/2} \times \dot{H}^{1/2}$. When $d = 0$, Theorem 1.2 also improves the global existence of three dimensional nonhomogeneous incompressible Navier-Stokes equations in [13].

The rest of this paper is organized as follows. In the second section, we introduce some useful lemmas and some definition. In section 3, we prove some priori estimates, which are essential to prove the theorem. In sections 4 and 5, we prove the existence and uniqueness for Theorem 1.2.

2. Preliminaries

In this section, we recall key definitions and results that will be used throughout the paper. First we present the Littlewood-Paley theory and the definitions of Besov and Sobolev spaces.

Let $\varphi \in \mathcal{D}^3((\frac{3}{4}, \frac{8}{3}))$, then for $(k, l) \in \mathbb{Z}^2$, we introduce the notation

$$\dot{\Delta}_k a = \mathcal{F}^{-1}\left(\varphi\left(2^{-k}|\xi|\right)\hat{a}\right), \quad \dot{S}_k a = \sum_{k' \le k-1} \dot{\Delta}_{k'} a,$$

where $\mathcal{F}$ denotes the Fourier transform, and φ satisfies

$$\sum_{j \in Z} \varphi(2^{-j}\xi) = 1, \quad \forall \xi > 0.$$

Definition 2.1. Let $\dot{B}^s_{p,r}(\mathbb{R}^3)$ be the completion of $\mathcal{S}_h(\mathbb{R}^3)$ with the norm

$$\|u\|_{\dot{B}^s_{p,r}(\mathbb{R}^3)} = \{2^{js}\|\dot{\Delta}_j u\|_{L^p}\}_{l^r} \tag{2.1}$$

Specifically, when $p = r = 2$, $\dot{B}^s_{2,2}(\mathbb{R}^3) = \dot{H}^s(\mathbb{R}^3)$. For convenience, we introduce the following notation:

$$B^s = \dot{B}^{s-1}_{2,1} \cap \dot{B}^{s-\frac{3}{2}}_{2,2}. \tag{2.2}$$

The following lemmas can be found in [2].

Lemma 2.2 (Bernstein's inequality)**.** *Let $\mathcal{C}$ be an annulus and $\mathcal{B}$ a ball. Then there exists a constant $C > 0$ such that for any non-negative, couple $(p, q) \in [1, \infty]^2$ with $q \ge p \ge 1$ and any $u \in L^p$ we have following results:*

(1) *If* $\operatorname{supp}\hat{u}\subset\lambda\mathcal{B}$, *we have*

$$\sup_{|\alpha|=k}\|\partial^{\alpha}u\|_{L^q}\le C^{k+1}\lambda^{k+N(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$

(2) *If* $\operatorname{supp}\hat{u}\subset\lambda\mathcal{C}$, *we have*

$$C^{-k-1}\lambda^{k}\|u\|_{L^p}\le\|D^{k}u\|_{L^p}\le C^{k+1}\lambda^{k}\|u\|_{L^p}.$$

Lemma 2.3 (Interpolation in Besov space). *If s_1 and s_2 are real numbers such that $s_1<s_2$ and $\theta\in[0,1]$, then we have, for any $(p,r)\in[1,+\infty]^2$ and any $u\in\mathcal{S}_h$,*

$$\|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}}\le C\|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}}\|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}.\tag{2.3}$$

Lemma 2.4 (Law of product in Besov Space). *Let $1\le p,q\le+\infty$, $s_1\le\frac{3}{q}$, $s_2\le\min\{\frac{1}{p},\frac{1}{q}\}$ and $s_1+s_2>3\max\{0,\frac{1}{p}+\frac{1}{q}-1\}$. Then for any $(a,b)\in\dot{B}^{s_1}_{p,1}\times\dot{B}^{s_2}_{p,1}$, we have*

$$\|ab\|_{\dot{B}^{s_1+s_2-\frac{3}{q}}_{p,1}}\lesssim\|a\|_{\dot{B}^{s_1}_{q,1}}\|b\|_{\dot{B}^{s_2}_{p,1}}.\tag{2.4}$$

Lemma 2.5. *Let s be a positive real number and (p,r) be in $[1,\infty]^2$. Then there exists a constant C_s such that if $(u_j)_{j\in\mathbb{Z}}$ is a sequence of smooth functions where $\sum_{j\in\mathbb{Z}}u_j$ converges to u in $\mathcal{S}'_h$ and*

$$N_s((u_j)_{j\in\mathbb{Z}})=\|(\sup_{\{|\alpha|\in[0,s]+1\}}2^{j(s-|\alpha|)}\|\partial^{\alpha}u_j\|_{L^p})_{j\in\mathbb{Z}}\|_{l^r}<\infty,\tag{2.5}$$

then u is in $\dot{B}^{s}_{p,r}$ and $\|u\|_{\dot{B}^{s}_{p,r}}\le C_sN_s((u_j)_{j\in\mathbb{Z}})$.

3. A PRIORI ESTIMATES

We first establish a priori estimates for smooth solutions of system (1.1). Motivated by [24], we can build the following solutions to (1.1). Let $(\rho,u,d,\nabla P)$ be the smooth enough solutions on $[0,T^*)$. Then we obtain the following equality by the continuity equation of (1.1),

$$\|\mu(\rho)-1\|_{L^\infty}=\|\mu(\rho_0)-1\|_{L^\infty}.\tag{3.1}$$

For $j\in\mathbb{Z}$, let $(u_j,d_j,\nabla P_j)$ be the solutions of the linear system

$$\begin{gathered}
\rho\partial_t u_j+\rho u\cdot\nabla u_j-\operatorname{div}\left(\mu\left(\rho\right)\nabla u_j\right)+\nabla P_j=-\sum_{k=1}^{3}\partial_k\nabla d_j\cdot\partial_k d-\Delta d_j\cdot\nabla d,\\
\partial_t d_j+u\cdot\nabla d_j-\Delta d_j=\nabla d_j\cdot\nabla d\cdot d,\\
\operatorname{div}u_j=0,\\
|d_j|=1,\\
(u_j,d_j)|_{t=0}=(\dot{\Delta}_j u_0,\dot{\Delta}_j d_0).
\end{gathered}\tag{3.2}$$

According to the uniqueness of local smooth solutions, we have

$$u(t)=\sum_{j\in Z}u_j(t),\quad d(t)=\sum_{j\in Z}d_j(t),\quad \nabla P(t)=\sum_{j\in Z}\nabla P_j(t).\tag{3.3}$$

Lemma 3.1. *For a sufficiently small constant c, let*

$$T^1=\sup\{T\in[0,T^*),\ \|(u,\nabla d)\|_{L^\infty_T(L^3)}+\|(\nabla u,\nabla^2 d)\|_{L^4_T(L^2)}\le c\}.\tag{3.4}$$

Then for each $j\in\mathbb{Z}$ and $T\in[0,T^1)$, we have

$$\|(\sqrt{\rho}u_j,\nabla d_j)\|_{L^\infty_T(L^2)}+\|(\sqrt{\mu(\rho)}\nabla u_j,\nabla^2 d_j)\|_{L^2_T(L^2)}\le C\|(\dot{\Delta}_j u_0,\dot{\Delta}_j\nabla d_0)\|_{L^2},\tag{3.5}$$

$$\|(\sqrt{\rho}\nabla u_j,\nabla^2 d_j)\|_{L^\infty_T(L^2)}+\|(\sqrt{\rho}\partial_t u_j,\partial_t\nabla d_j)\|_{L^2_T(L^2)}\le C\|(\nabla\dot{\Delta}_j u_0,\nabla\dot{\Delta}_j\nabla d_0)\|_{L^2},\tag{3.6}$$

$$\|(\nabla u_j,\nabla^2 d_j)\|_{L^2_T(L^6)}\le C\|(\nabla\dot{\Delta}_j u_0,\nabla\dot{\Delta}_j\nabla d_0)\|_{L^2},\tag{3.7}$$

and for time-weighted estimates, we have

$$\|t^{1/2}(\sqrt{\rho}\nabla u_j,\nabla^2 d_j)\|_{L^\infty_T(L^2)}+\|t^{1/2}(\sqrt{\rho}\partial_t u_j,\partial_t\nabla d_j)\|_{L^2_T(L^2)}\le C\|(\dot{\Delta}_j u_0,\dot{\Delta}_j\nabla d_0)\|_{L^2},\tag{3.8}$$

$$\|t^{-\alpha}(\nabla u_j,\nabla^2 d_j)\|_{L^2_T(L^2)}\le C2^{2j\alpha}\|(\dot{\Delta}_j u_0,\dot{\Delta}_j\nabla d_0)\|_{L^2},\quad \alpha\in(0,1/2).\tag{3.9}$$

Proof. Taking the L^2 inner product of $(3.2)_{1,2}$ with $u_j, \Delta d_j$, and using integration by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u_j\|_{L^2}^2 + \int_{\mathbb{R}^3}\mu(\rho)\nabla u_j : \nabla u_j \mathrm{d}x &= -\int_{\mathbb{R}^3}(\partial_k \nabla d_j \cdot \partial_k d)u_j \mathrm{d}x - \int_{\mathbb{R}^3}(\Delta d_j \cdot \nabla d)u_j \mathrm{d}x \\ &= I_{11} + I_{12},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla d_j\|_{L^2}^2 + \int_{\mathbb{R}^3}\nabla^2 d_j : \nabla^2 d_j \mathrm{d}x &= -\int_{\mathbb{R}^3}\nabla(u\cdot\nabla d_j) : \nabla d_j \mathrm{d}x - \int_{\mathbb{R}^3}\nabla(\nabla d_j \cdot \nabla d \cdot d) : \nabla d_j \mathrm{d}x \\ &= J_{11} + J_{12}.\end{aligned} \tag{3.11}$$

Using integration by parts, the Hölder inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}I_{11} &= -\int_{\mathbb{R}^3}(\partial_k \nabla d_j \cdot \partial_k d)u_j \mathrm{d}x \\ &= \int_{\mathbb{R}^3}(\nabla d_j \cdot \partial_k^2 d)u_j \mathrm{d}x + \int_{\mathbb{R}^3}(\nabla d_j \cdot \partial_k d)\partial_k u_j \mathrm{d}x \\ &\le \|\nabla d_j\|_{L^3}\|\nabla^2 d\|_{L^2}\|u_j\|_{L^6} + \|\nabla d_j\|_{L^3}\|\nabla d\|_{L^6}\|\nabla u_j\|_{L^2} \\ &\le C\|\nabla d_j\|_{L^3}\|\nabla^2 d\|_{L^2}\|\nabla u_j\|_{L^2} \\ &\le C\|\nabla d_j\|_{L^2}^{1/2}\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^2 d\|_{L^2}\|\nabla u_j\|_{L^2}.\end{aligned}$$

Similarly for I_{11}, one has

$$\begin{aligned}I_{12} &= -\int_{\mathbb{R}^3}\partial_k^2 d_j \cdot \partial_l d u_j^l \mathrm{d}x \\ &= \int_{\mathbb{R}^3}\partial_k d_j \cdot \partial_{lk} d u_j^l \mathrm{d}x + \int_{\mathbb{R}^3}\partial_k d_j \cdot \partial_l d \partial_k u_j^l \mathrm{d}x \\ &\le C\|\nabla d_j\|_{L^2}^{1/2}\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^2 d\|_{L^2}\|\nabla u_j\|_{L^2}.\end{aligned}$$

Thanks to Young's inequality, we have

$$I_{11} + I_{12} \le \frac{2}{\epsilon}\|\nabla u_j\|_{L^2}^2 + \frac{2}{\eta}\|\nabla^2 d_j\|_{L^2}^2 + C(\epsilon,\eta)\|\nabla^2 d\|_{L^2}^4\|\nabla d_j\|_{L^2}^2. \tag{3.12}$$

Using integration by parts, the Hölder inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}J_{11} &= -\int_{\mathbb{R}^3}\nabla(u\cdot\nabla d_j) : \nabla d_j \mathrm{d}x \\ &= \int_{\mathbb{R}^3}(u\cdot\nabla d_j)\Delta d_j \mathrm{d}x \\ &\le \|u\|_{L^6}\|\nabla d_j\|_{L^3}\|\Delta d_j\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}\|\nabla d_j\|_{L^2}^{1/2}\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^2 d_j\|_{L^2} \\ &\le C\|\nabla u\|_{L^2}\|\nabla d_j\|_{L^2}^{1/2}\|\nabla^2 d_j\|_{L^2}^{3/2},\end{aligned}$$

and

$$\begin{aligned}J_{12} &= -\int_{\mathbb{R}^3}\nabla(\nabla d_j \cdot \nabla d \cdot d) : \nabla d_j \mathrm{d}x \\ &= \int_{\mathbb{R}^3}(\nabla d_j \cdot \nabla d \cdot d)\Delta d_j \mathrm{d}x \\ &\le C\|\nabla d_j\|_{L^3}\|\nabla d\|_{L^6}\|d\|_{L^\infty}\|\Delta d_j\|_{L^2} \\ &\le C\|\nabla^2 d\|_{L^2}\|\nabla d_j\|_{L^2}^{1/2}\|\nabla^2 d_j\|_{L^2}^{3/2}.\end{aligned}$$

Then by Young's inequality,

$$J_{11} + J_{12} \le \frac{2}{\eta}\|\nabla^2 d_j\|_{L^2}^2 + C\left(\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4\right)\|\nabla d_j\|_{L^2}^2. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.10) and (3.11) respectively, selecting appropriate ϵ and η and adding the two inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\rho}u_j,\nabla d_j)\|_{L^2}^2+\|(\sqrt{\mu(\rho)}\nabla u_j,\nabla^2 d_j)\|_{L^2}^2\le C\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\|(u_j,\nabla d_j)\|_{L^2}^2. \tag{3.14}$$

By integrating the above equation over $(0,t)$, with $t\le T^1$, and using the Grönwall inequality, we have

$$\begin{aligned}&\|(\sqrt{\rho}u_j,\nabla d_j)\|_{L_t^\infty(L^2)}^2+\|(\sqrt{\mu(\rho)}\nabla u_j,\nabla^2 d_j)\|_{L_t^2(L^2)}^2\\&\le\exp\{C\int_0^t\left(\|\nabla u\|_{L^2}^4+\|\nabla^2 d\|_{L^2}^4\right)\mathrm{d}t\}\|(\dot{\Delta}_j u_0,\dot{\Delta}_j\nabla d_0)\|_{L^2}^2.\end{aligned}\tag{3.15}$$

Then from the definition of T^1, we establish (3.5).

Now prove the other inequalities in Lemma3.1. Taking the L^2 inner product of $(3.2)_1$ with $\partial_t u_j$, we have

$$\begin{aligned}&\|\sqrt{\rho}\partial_t u_j\|_{L^2}^2-\int_{\mathbb{R}^3}\operatorname{div}(\mu(\rho)\nabla u_j)\partial_t u_j\mathrm{d}x\\&=-\int_{\mathbb{R}^3}(\rho u\cdot\nabla u_j)\partial_t u_j\mathrm{d}x-\int_{\mathbb{R}^3}(\nabla\partial_k d_j\cdot\partial_k d)\partial_t u_j\mathrm{d}x-\int_{\mathbb{R}^3}(\Delta d_j\nabla d)\partial_t u_j\mathrm{d}x,\end{aligned}\tag{3.16}$$

Moreover, since

$$\begin{aligned}-\int_{\mathbb{R}^3}\operatorname{div}(\mu(\rho)\nabla u_j)\partial_t u_j\mathrm{d}x&=-\int_{\mathbb{R}^3}\mu(\rho)\nabla u_j:\partial_t\nabla u_j\mathrm{d}x\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\mu(\rho)}\nabla u_j\|_{L^2}^2-\frac{1}{2}\int_{\mathbb{R}^3}\partial_t\mu(\rho)|\nabla u_j|^2\mathrm{d}x,\end{aligned}\tag{3.17}$$

we deduce the following equation by substituting the above equality into (3.16),

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\mu(\rho)}\nabla u_j\|_{L^2}^2+\|\sqrt{\rho}\partial_t u_j\|_{L^2}^2\\&=-\int_{\mathbb{R}^3}(\rho u\cdot\nabla u_j)\partial_t u_j\mathrm{d}x-\int_{\mathbb{R}^3}(\nabla\partial_k d_j\cdot\partial_k d)\partial_t u_j\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}(\Delta d_j\nabla d_j)\partial_t u_j\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}\partial_t\mu(\rho)|\nabla u_j|^2\mathrm{d}x\\&=I_{21}+I_{22}+I_{23}+I_{24}.\end{aligned}\tag{3.18}$$

Applying the Hölder inequality, we have the following estimates:

$$\begin{aligned}I_{21}&=-\int_{\mathbb{R}^3}(\rho u\cdot\nabla u_j)\partial_t u_j\mathrm{d}x\le C\|\sqrt{\rho}\partial_t u_j\|_{L^2}\|u\|_{L^3}\|\nabla u_j\|_{L^6},\\I_{22}&=-\int_{\mathbb{R}^3}(\nabla\partial_k d_j\cdot\partial_k d)\partial_t u_j\mathrm{d}x\le C\|\sqrt{\rho}\partial_t u_j\|_{L^2}\|\nabla d\|_{L^3}\|\nabla^2 d_j\|_{L^6},\\I_{23}&=-\int_{\mathbb{R}^3}(\Delta d_j\nabla d)\partial_t u_j\mathrm{d}x\le C\|\sqrt{\rho}\partial_t u_j\|_{L^2}\|\nabla d\|_{L^3}\|\nabla^2 d_j\|_{L^6},\end{aligned}$$

which means

$$I_{21}+I_{22}+I_{23}\le C\|(u,\nabla d)\|_{L^3}\|(\nabla u_j,\nabla^2 d_j)\|_{L^6}\|\sqrt{\rho}\partial_t u_j\|_{L^2}. \tag{3.19}$$

Next we handle the last term. According to $(1.1)_1$, we can obtain the equation for $\mu(\rho)$ as follows:

$$\partial_t(\mu(\rho))+u\cdot\nabla(\mu(\rho))=0, \tag{3.20}$$

then thanks to the above equation and integration by parts, we obtain

$$\begin{aligned}K&=-\frac{1}{2}\int_{\mathbb{R}^3}u\cdot\nabla(\mu(\rho))|\nabla u_j|^2\mathrm{d}x\\&=-\sum_{i=1}^3\frac{1}{2}\int_{\mathbb{R}^3}u^i\partial_i\mu(\rho)|\nabla u_j|^2\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{3} \frac{1}{2} \int_{\mathbb{R}^3} u^i \partial_i \Big(\frac{1}{\mu(\rho)}\Big) |\mu(\rho)\nabla u_j|^2 \mathrm{d}x \\
&= -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \nabla u_j : u^i \partial_i (\mu(\rho)\nabla u_j) \mathrm{d}x \\
&= -\sum_{i,l,m=1}^{3} \int_{\mathbb{R}^3} \partial_m u_j^l u^i \partial_i (\mu(\rho)\partial_m u_j^l) \mathrm{d}x \\
&= \sum_{i,l,m=1}^{3} \int_{\mathbb{R}^3} u_j^l u^i \partial_i \partial_m (\mu(\rho)\partial_m u_j^l) \mathrm{d}x + \sum_{i,l,m=1}^{3} \int_{\mathbb{R}^3} u_j^l \partial_m u^i \partial_i (\mu(\rho)\partial_m u_j^l) \mathrm{d}x \\
&= -\sum_{i,l,m=1}^{3} \int_{\mathbb{R}^3} \partial_i u_j^l u^i \partial_m (\mu(\rho)\partial_m u_j^l) \mathrm{d}x - \sum_{i,l,m=1}^{3} \int_{\mathbb{R}^3} \partial_i u_j^l \partial_m u^i \mu(\rho) \partial_m u_j^l \mathrm{d}x \\
&= -\sum_{j=1}^{3} \int_{\mathbb{R}^3} u \cdot \nabla u_j \operatorname{div}(\mu(\rho)\nabla u_j) - \sum_{i,l,m=1}^{3} \int_{\mathbb{R}^3} \partial_i u_j^l \partial_m u^i \mu(\rho) \partial_m u_j^l \mathrm{d}x.
\end{aligned}
$$

Thanks to $(3.2)_1$, we obtain

$$
\operatorname{div}(\mu(\rho)\nabla u_j) = \partial_t u_j + u \cdot \nabla u_j + \nabla P_j + \sum_{k=1}^{3} \partial_k \nabla d_j \cdot \partial_k d + \Delta d_j \cdot \nabla d, \tag{3.21}
$$

from which, we have

$$
\begin{aligned}
-I_{24} &= \int_{\mathbb{R}^3} u \cdot \nabla u_j \partial_t u_j \mathrm{d}x + \int_{\mathbb{R}^3} u \cdot \nabla u_j u \cdot \nabla u_j \mathrm{d}x + \int_{\mathbb{R}^3} u \cdot \nabla u_j \nabla P_j \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} u \cdot \nabla u_j \partial_k \nabla d_j \cdot \partial_k d \mathrm{d}x + \int_{\mathbb{R}^3} u \cdot \nabla u_j \Delta d_j \cdot \nabla d \mathrm{d}x \\
&\quad + \sum_{i,l,m=1}^{3} \int_{\mathbb{R}^3} \mu(\rho) \partial_m u^i \partial_i u_j^l \partial_m u_j^l \mathrm{d}x \\
&= I_{241} + I_{242} + I_{243} + I_{244} + I_{245} + I_{246}.
\end{aligned} \tag{3.22}
$$

Thanks to the Hölder inequality, from the above equality, we deduce that

$$
I_{241} = \int_{\mathbb{R}^3} u \cdot \nabla u_j \partial_t u_j \mathrm{d}x \le C\|u\|_{L^3} \|\nabla u_j\|_{L^6} \|\sqrt{\rho}\partial_t u_j\|_{L^2}, \tag{3.23}
$$

$$
I_{242} = \int_{\mathbb{R}^3} u \cdot \nabla u_j u \cdot \nabla u_j \mathrm{d}x \le C\|u \cdot \nabla u_j\|_{L^2}^2, \tag{3.24}
$$

$$
\begin{aligned}
I_{244} + I_{245} &= \int_{\mathbb{R}^3} u \cdot \nabla u_j \partial_k \nabla d_j \cdot \partial_k d \mathrm{d}x + \int_{\mathbb{R}^3} u \cdot \nabla u_j \Delta d_j \cdot \nabla d \mathrm{d}x \\
&\le C\|u \cdot \nabla u_j\|_{L^2} \|\nabla d\|_{L^3} \|\nabla^2 d_j\|_{L^6},
\end{aligned} \tag{3.25}
$$

$$
I_{246} = 2\int_{\mathbb{R}^3} \mu(\rho) \partial_m u^i \partial_i u_j^l \partial_m u_j^l \mathrm{d}x \le \|\nabla u\|_{L^2} \|\nabla u_j\|_{L^4}^2, \tag{3.26}
$$

Now we deal with the term ∇P_j, thanks to the $(3.2)_1$, we obtain

$$
\nabla P_j = \operatorname{div}(\mu(\rho)\nabla u_j) - \rho\partial_t u_j - \rho u \cdot \nabla u_j - \sum_{k=1}^{3} \partial_k \nabla d_j \cdot \partial_k d - \Delta d_j \cdot \nabla d, \tag{3.27}
$$

which means

$$
\begin{aligned}
P_j &= -(-\Delta)^{-1} \operatorname{div}\operatorname{div}(\mu(\rho)\nabla u_j) + (-\Delta)^{-1} \operatorname{div}(\rho\partial_t u_j + \rho u \cdot \nabla u_j \\
&\quad + \sum_{k=1}^{3} \partial_k \nabla d_j \cdot \partial_k d + \Delta d_j \cdot \nabla d),
\end{aligned} \tag{3.28}
$$

from which and the boundedness of the Riesz operator, integration by parts, Hölder inequality and the duality bewteen $\dot H^1$ and $\dot H^{-1}$, we obtain

$$\begin{aligned}
I_{243} &= \int_{\mathbb{R}^3} u\cdot\nabla u_j \nabla P_j \mathrm{d}x\\
&= \sum_{i,l=1}^{3}\int_{\mathbb{R}^3} u^l\partial_l u^i_j \partial_i P_j \mathrm{d}x\\
&= \sum_{i,l=1}^{3}\int_{\mathbb{R}^3} \partial_i u^l\partial_l u^i_j P_j \mathrm{d}x\\
&\le C\|\nabla u_j\|_{L^4}\|\partial_i u^l\partial_l u^i_j\|_{L^{\frac{4}{3}}} + C\|(-\Delta)^{-1}\operatorname{div}(\rho\partial_t u_j + \rho u\cdot\nabla u_j + \partial_k\nabla d_j\cdot\partial_k d\\
&\quad + \Delta d_j\cdot\nabla d)\|_{\dot H^1}\|\partial_i u^l\partial_l u^i_j\|_{\dot H^{-1}}\\
&\le C\|\nabla u_j\|_{L^2}\|\nabla u_j\|_{L^4}^2 + C\|u\cdot\nabla u_j\|_{L^2}\|\sqrt{\rho}\partial_t u_j\|_{L^2} + C\|u\cdot\nabla u_j\|_{L^2}^2\\
&\quad + C\|u\cdot\nabla u_j\|_{L^2}\|\nabla d\|_{L^3}\|\nabla^2 d_j\|_{L^6}.
\end{aligned}\tag{3.29}$$

Combining the estimates from I_{241} to I_{246}, (3.19) and Young's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\mu(\rho)}\nabla u_j\|_{L^2}^2 + \|\sqrt{\rho}\partial_t u_j\|_{L^2}^2 \le C\|(u,\nabla d)\|_{L^3}^2\|(\nabla u_j,\nabla^2 d_j)\|_{L^6}^2 + C\|\nabla u\|_{L^2}\|\nabla u_j\|_{L^4}^2. \tag{3.30}$$

Next we give some estimates for the d_j. Taking the L^2 inner product of $(3.2)_2$ with $-\operatorname{div}\partial_t\nabla d_j$, and integrating by parts, we obtain:

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 d_j\|_{L^2}^2 + \|\partial_t\nabla d_j\|_{L^2}^2 &= -\int_{\mathbb{R}^3}\nabla(u\cdot\nabla d_j)\partial_t\nabla d_j \mathrm{d}x + \int_{\mathbb{R}^3}\nabla(\nabla d_j\cdot\nabla d\cdot d)\partial_t\nabla d_j\mathrm{d}x\\
&= J_{21} + J_{22}.
\end{aligned}\tag{3.31}$$

Applying the Hölder and Gagliardo-Nirenberg inequalities, we deduce the estimates

$$\begin{aligned}
J_{21} &= -\int_{\mathbb{R}^3}\nabla(u\cdot\nabla d_j)\partial_t\nabla d_j\mathrm{d}x\\
&= -\int_{\mathbb{R}^3}\nabla u\cdot\nabla d_j\partial_t\nabla d_j\mathrm{d}x - \int_{\mathbb{R}^3} u\cdot\nabla^2 d_j\partial_t\nabla d_j\mathrm{d}x\\
&\le \|\nabla u\|_{L^2}\|\nabla d_j\|_{L^\infty}\|\partial_t\nabla d_j\|_{L^2} + \|u\|_{L^6}\|\nabla^2 d_j\|_{L^3}\|\partial_t\nabla d_j\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^3 d_j\|_{L^2}^{1/2}\|\partial_t\nabla d_j\|_{L^2}\\
&\le \epsilon_0\|\nabla^3 d\|_{L^2}^2 + C_{\epsilon_0}\|\nabla u\|_{L^2}^4\|\nabla^2 d_j\|_{L^2}^2 + \frac{1}{4\eta}\|\partial_t\nabla d_j\|_{L^2}^2,
\end{aligned}\tag{3.32}$$

and

$$\begin{aligned}
J_{22} &= \int_{\mathbb{R}^3}\nabla^2 d_j\cdot\nabla d\cdot d\partial_t\nabla d_j\mathrm{d}x + \int_{\mathbb{R}^3}\nabla d_j\cdot\nabla^2 d\cdot d\partial_t\nabla d_j\mathrm{d}x + \int_{\mathbb{R}^3}\nabla d_j\cdot\nabla d\cdot\nabla d\partial_t\nabla d_j\mathrm{d}x\\
&\le C\|\nabla^2 d_j\|_{L^3}\|\nabla d\|_{L^6}\|d\|_{L^\infty}\|\partial_t\nabla d_j\|_{L^2} + C\|\nabla d_j\|_{L^\infty}\|\nabla^2 d\|_{L^2}\|d\|_{L^\infty}\|\partial_t\nabla d_j\|_{L^2}\\
&\quad + C\|\nabla d_j\|_{L^6}\|\nabla d\|_{L^6}^2\|d\|_{L^\infty}\|\partial_t\nabla d_j\|_{L^2}\\
&\le C\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^3 d_j\|_{L^2}^{1/2}\|\nabla^2 d\|_{L^2}\|\partial_t\nabla d_j\|_{L^2} + C\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^3 d_j\|_{L^2}^{1/2}\|\nabla^2 d\|_{L^2}\|\partial_t\nabla d_j\|_{L^2}\\
&\quad + C\|\nabla^2 d_j\|_{L^2}\|\nabla^2 d\|_{L^2}^2\|\partial_t\nabla d_j\|_{L^2}\\
&\le C\|\nabla^2 d\|_{L^2}^4\|\nabla^2 d_j\|_{L^2}^2 + \frac{1}{4\eta}\|\partial_t\nabla d_j\|_{L^2}^2 + \epsilon_0\|\nabla^3 d_j\|_{L^2}^2,
\end{aligned}$$

To deal with the above terms, we need to handle $\|\nabla^3 d_j\|_{L^2}^2$. Thanks to $(3.2)_2$, we can conclude that

$$\begin{gathered}
-\Delta d_j = \nabla d_j\cdot\nabla d\cdot d - \partial_t d_j - u\cdot\nabla d_j,\\
d_j|_{t=0} = \dot\Delta_j d_0,
\end{gathered}\tag{3.33}$$

Applying ∇ to both sides of $(3.33)_1$ we obtain

$$\|\nabla^3 d_j\|_{L^2} \le C(\|\partial_t \nabla d_j\|_{L^2} + \|\nabla u \cdot \nabla d_j\|_{L^2} + \|u \cdot \nabla^2 d_j\|_{L^2} + \|\nabla(\nabla d_j \cdot \nabla d \cdot d)\|_{L^2}),$$

By the Hölder inequality and the Gagliardo-Nirenberg inequality, we obtain

$$\|\nabla^3 d_j\|_{L^2} \le C(\|\partial_t \nabla d_j\|_{L^2} + (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\|\nabla^2 d_j\|_{L^2}). \tag{3.34}$$

Selecting the appropriate ϵ_0, we have

$$J_{21} + J_{22} \le C(\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4)\|\nabla^2 d_j\|_{L^2}^2 + \frac{1}{\eta}\|\partial_t \nabla d_j\|_{L^2}^2. \tag{3.35}$$

Substituting the above inequality into (3.31), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 d_j\|_{L^2}^2 + \|\partial_t \nabla d_j\|_{L^2}^2 \le C(\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4)\|\nabla^2 d_j\|_{L^2}^2 + \frac{1}{\eta}\|\partial_t \nabla d_j\|_{L^2}^2,$$

Then by choosing an appropriate η, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^2 d_j\|_{L^2}^2 + \|\partial_t \nabla d_j\|_{L^2}^2 \le C(\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4)\|\nabla^2 d_j\|_{L^2}^2. \tag{3.36}$$

Combining the above equations with (3.30), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\mu(\rho)}\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 + \|(\sqrt{\rho}\partial_t u_j, \partial_t \nabla d_j)\|_{L^2}^2 \\ &\le C\|(u, \nabla d)\|_{L^3}^2 \|(\nabla u_j, \nabla^2 d_j)\|_{L^6}^2 + C\|\nabla u\|_{L^2}\|\nabla u_j\|_{L^4}^2 + C(\|\nabla^2 d\|_{L^2}^4 \\ &\quad + \|\nabla u\|_{L^2}^4)\|\nabla^2 d_j\|_{L^2}^2. \end{aligned} \tag{3.37}$$

Now we estimate $\|\nabla u_j\|_{L^4}$ and $\|(\nabla u_j, \nabla^2 d_j)\|_{L^6}$. By the interpolation inequality

$$\|\nabla u_j\|_{L^p} \le \|\nabla u_j\|_{L^2}^{\frac{3}{p}-\frac{1}{2}}\|\nabla^2 u_j\|_{L^2}^{3(\frac{1}{2}-\frac{1}{p})}, \quad p \in [2,6]$$

and the equation

$$\nabla u_j = \nabla(-\Delta)^{-1}\operatorname{div}\mathbb{P}((\mu(\rho)-1)\nabla u_j) - \nabla(-\Delta)^{-1}\operatorname{div}\mathbb{P}(\mu(\rho)\nabla u_j),$$

the Hölder inequality and the boundedness of the Riesz operator, we have

$$\|\nabla u_j\|_{L^p} \le C\|\mu(\rho_0)-1\|_{L^\infty}\|\nabla u_j\|_{L^p} + C\|\nabla u_j\|_{L^2}^{\frac{3}{p}-\frac{1}{2}}\|\mathbb{P}\operatorname{div}(\mu(\rho)\nabla u_j)\|_{L^2}^{\frac{3}{2}-\frac{3}{p}},$$

Then, thanks to the smallness of $\|\mu(\rho_0)-1\|_{L^\infty}$, we have

$$\|\nabla u_j\|_{L^p} \le C\|\nabla u_j\|_{L^2}^{\frac{3}{p}-\frac{1}{2}}\|\mathbb{P}\operatorname{div}(\mu(\rho)\nabla u_j)\|_{L^2}^{\frac{3}{2}-\frac{3}{p}}.$$

Using the equation

$$\operatorname{div}(\mu(\rho)\nabla u_j) = \rho\partial_t u_j + \rho u \cdot \nabla u_j + \nabla P_j + \partial_k \nabla d_j \cdot \partial_k d + \Delta d_j \cdot \nabla d,$$

we obtain

$$\|\nabla u_j\|_{L^p} \le C\|\nabla u_j\|_{L^2}^{\frac{3}{p}-\frac{1}{2}}\|\rho\partial_t u_j + \rho u \cdot \nabla u_j + \partial_k \nabla d_j \cdot \partial_k d + \Delta d_j \cdot \nabla d\|_{L^2}^{\frac{3}{2}-\frac{3}{p}}. \tag{3.38}$$

Taking $p = 4, 6$ in (3.38) respectively, and using the Hölder inequality, we obtain

$$\begin{aligned} \|\nabla u_j\|_{L^4} &\le C\|\nabla u_j\|_{L^2}^{1/4}\|\rho\partial_t u_j + \rho u \cdot \nabla u_j + \partial_k \nabla d_j \cdot \partial_k d + \Delta d_j \cdot \nabla d\|_{L^2}^{\frac{3}{4}} \\ &\le C\|\nabla u_j\|_{L^2}^{1/4}(\|\sqrt{\rho}\partial_t u_j\|_{L^2} + \|(u, \nabla d)\|_{L^3}\|(\nabla u_j, \nabla^2 d_j)\|_{L^6})^{\frac{3}{4}}, \end{aligned} \tag{3.39}$$

and

$$\|\nabla u_j\|_{L^6} \le C\left(\|\sqrt{\rho}\partial_t u_j\|_{L^2} + \|(u, \nabla d)\|_{L^3}\|(\nabla u_j, \nabla^2 d_j)\|_{L^6}\right). \tag{3.40}$$

On the other hand, by $(3.33)_1$, we have

$$\begin{aligned} \|\nabla^2 d_j\|_{L^6} &\le \|\partial_t d_j\|_{L^6} + \|\nabla d_j \cdot \nabla d \cdot d\|_{L^6} + \|u \cdot \nabla d_j\|_{L^6} \\ &\le C\|\partial_t \nabla d_j\|_{L^2} + \|\nabla d\|_{L^6}\|\nabla d_j\|_{L^\infty} + \|u\|_{L^6}\|\nabla d_j\|_{L^\infty} \\ &\le C\|\partial_t \nabla d_j\|_{L^2} + \|\nabla^2 d\|_{L^2}\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^3 d_j\|_{L^2}^{1/2} + \|u\|_{L^6}\|\nabla^2 d_j\|_{L^2}^{1/2}\|\nabla^3 d_j\|_{L^2}^{1/2} \\ &\le C\left(\|\partial_t \nabla d_j\|_{L^2} + (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\|\nabla^2 d_j\|_{L^2}\right). \end{aligned} \tag{3.41}$$

where we have used estimate (3.34). This implies

$$\begin{aligned}
\|(\nabla u_j, \nabla d_j)\|_{L^6} &\le C\left(\|\sqrt{\rho}\partial_t u_j\|_{L^2} + \|(u,\nabla d)\|_{L^3}\|(\nabla u_j, \nabla^2 d_j)\|_{L^6}\right)\\
&\quad + C\left(\|\partial_t \nabla d_j\|_{L^2} + (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\|\nabla^2 d_j\|_{L^2}\right)\\
&\le C\|(\sqrt{\rho}\partial_t u_j, \partial_t \nabla d_j)\|_{L^2} + (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)\|\nabla^2 d_j\|_{L^2}
\end{aligned}\tag{3.42}$$

where we have used the smallness of $\|(u,\nabla d)\|_{L^3}$.

From the above estimates, we have

$$\begin{aligned}
\|\nabla u\|_{L^2}\|\nabla u_j\|_{L^4}^2 &\le C\|\nabla u\|_{L^2}\|\nabla u_j\|_{L^2}^{1/2}(\|\sqrt{\rho}\partial_t u_j\|_{L^2} + \|(u,\nabla d)\|_{L^3}\|(\nabla u_j, \nabla^2 d_j)\|_{L^6})^{3/2}\\
&\le C\|\nabla u\|_{L^2}\|\nabla u_j\|_{L^2}^{1/2}\|\sqrt{\rho}\partial_t u_j\|_{L^2}^{3/2}\\
&\quad + \|\nabla u\|_{L^2}\|\nabla u_j\|_{L^2}^{1/2}\|(u,\nabla d)\|_{L^3}^{3/2}\|(\nabla u_j, \nabla^2 d_j)\|_{L^6}^{3/2}\\
&\le C\|\nabla u\|_{L^2}^4\|\nabla u_j\|_{L^2}^2 + \|(u,\nabla d)\|_{L^3}^2\|(\nabla u_j, \nabla^2 d_j)\|_{L^6}^2 + \frac{1}{\epsilon}\|\sqrt{\rho}\partial_t u_j\|_{L^2}^2.
\end{aligned}\tag{3.43}$$

Taking $t \le T^1$, one can deduce from the above inequality

$$\begin{aligned}
&\|\nabla u\|_{L^2}\|\nabla u_j\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}^4\|\nabla u_j\|_{L^2}^2 + \frac{2}{\epsilon}\|(\sqrt{\rho}\partial_t u_j, \partial_t \nabla d_j)\|_{L^2}^2 + C(\|\nabla u\|_{L^2}^4 + \|\nabla^2 d\|_{L^2}^4)\|\nabla^2 d_j\|_{L^2}^2.
\end{aligned}\tag{3.44}$$

Substituting (3.44) into (3.37) and selecting appropriate ϵ, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{\mu(\rho)}\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 + \|(\sqrt{\rho}\partial_t u_j, \partial_t \nabla d_j)\|_{L^2}^2\\
&\le C(\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4)\|(\sqrt{\mu(\rho)}\nabla u_j, \nabla^2 d_j)\|_{L^2}^2.
\end{aligned}\tag{3.45}$$

Thanks to Gronwall's inequality, for $t \le T^1$, we have

$$\begin{aligned}
&\|(\sqrt{\mu(\rho)}\nabla u_j, \nabla^2 d_j)\|_{L_t^\infty(L^2)}^2 + \|(\sqrt{\rho}\partial_t u_j, \partial_t \nabla d_j)\|_{L_t^2(L^2)}^2\\
&\le \exp\{C\int_0^t (\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4)\mathrm{d}\tau\}\|(\nabla \dot{\Delta}_j u_0, \nabla \dot{\Delta}_j \nabla d_0)\|_{L^2}.
\end{aligned}$$

which completes the proof of (3.6).

Multiplying (3.45) by t, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{t}(\sqrt{\mu(\rho)}\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 + \|\sqrt{t}(\sqrt{\rho}\partial_t u_j, \partial_t \nabla d_j)\|_{L^2}^2\\
&\le \|(\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 + C(\|\nabla^2 d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4)\|\sqrt{t}(\sqrt{\mu(\rho)}\nabla u_j, \nabla^2 d_j)\|_{L^2}^2.
\end{aligned}$$

Using the Gronwall inequality, for $t \le T^1$, we have

$$\begin{aligned}
&\|\sqrt{t}(\sqrt{\mu(\rho)}\nabla u_j, \nabla^2 d_j)\|_{L_t^\infty(L^2)}^2 + \|\sqrt{t}(\sqrt{\rho}\partial_t u_j, \partial_t \nabla d_j)\|_{L_t^2(L^2)}^2\\
&\le C\exp\left\{\|(\nabla u, \nabla^2 d)\|_{L_t^4(L^2)}^4\right\}\|(\nabla u_j, \nabla^2 d_j)\|_{L_t^2(L^2)}^2\\
&\le C\|(\nabla u_j, \nabla^2 d_j)\|_{L_t^2(L^2)}^2 \le C\|(\dot{\Delta}_j u_0, \dot{\Delta}_j \nabla d_0)\|_{L^2}^2,
\end{aligned}\tag{3.46}$$

which completes the proof of the (3.8).

Finally, for any $\alpha \in (0, \frac{1}{2})$, if $T \le 2^{-2j}$, then

$$\begin{aligned}
\int_0^T t^{-2\alpha}\|(\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 \mathrm{d}t &\le C\int_0^T t^{-2\alpha}\mathrm{d}t 2^{2j}\|(\dot{\Delta}_j u_0, \dot{\Delta}_j \nabla d_0)\|_{L^2}^2\\
&\le CT^{1-2\alpha}2^{2j}\|(\dot{\Delta}_j u_0, \dot{\Delta}_j \nabla d_0)\|_{L^2}^2\\
&\le C2^{4\alpha j}\|(\dot{\Delta}_j u_0, \dot{\Delta}_j \nabla d_0)\|_{L^2}^2.
\end{aligned}\tag{3.47}$$

And if $2^{-2j} \le T \le T^1$, we have

$$\int_0^T t^{-2\alpha}\|(\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 \mathrm{d}t \le \int_0^{2^{-2j}} t^{-2\alpha}\|(\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 \mathrm{d}t + \int_{2^{-2j}}^{T^1} t^{-2\alpha}\|(\nabla u_j, \nabla^2 d_j)\|_{L^2}^2 \mathrm{d}t$$

$$\leq C2^{4\alpha j}\|(\dot\Delta_j u_0,\dot\Delta_j\nabla d_0)\|_{L^2}^2$$

As a consequence, we obtain the (3.9). □

Proposition 3.2. *For each $T\in[0,T^1]$, we have*

$$\begin{aligned}&\|(u,\nabla d)\|_{L^\infty_T(\dot H^{1/2})}+\|(u,\nabla d)\|_{L^\infty_T(L^3)}+\|(u,\nabla d)\|_{L^2_T(L^\infty)}+\|(\nabla u,\nabla^2 d)\|_{L^4_T(L^2)}\\&+\|(\nabla u,\nabla^2 d)\|_{L^2_T(L^3)}+\|t^{1/4}(\nabla u,\nabla^2 d)\|_{L^2_T(L^6)}+\|t^{-1/4}(\nabla u,\nabla^2 d)\|_{L^2_T(L^2)}\\&+\|t^{1/4}(\nabla u,\nabla^2 d)\|_{L^\infty_T(L^2)}+\|t^{1/4}(\partial_t u,\partial_t\nabla d)\|_{L^2_T(L^2)}\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}.\end{aligned}\tag{3.48}$$

Proof. For $T<T^1$, thanks to the lemma 3.1, we have

$$\begin{gathered}\|(\nabla u_j,\nabla^2 d_j)\|_{L^\infty_T(L^2)}\leq C\|(\nabla\dot\Delta_j u_0,\nabla\dot\Delta_j\nabla d_0)\|_{L^2}\leq Cc_j2^{\frac{j}{2}}\|(u_0,\nabla d_0)\|_{\dot H^{1/2}},\\ \|t^{1/2}(\nabla u_j,\nabla^2 d_j)\|_{L^\infty_T(L^2)}\leq C\|(\dot\Delta_j u_0,\dot\Delta_j\nabla d_0)\|_{L^2}\leq Cc_j2^{-j/2}\|(u_0,\nabla d_0)\|_{\dot H^{1/2}},\end{gathered}$$

where $c_j\in l^2$ and $\|c_j\|_{l^2}=1$. For any $t\leq T<T^1$, we obtain

$$\begin{aligned}\|t^{1/4}\nabla u\|_{L^2}^2&=\sum_{j,k\in Z}\int_{\mathbb R^3}t^{1/2}\nabla u_j\cdot\nabla u_k\mathrm dx\\&\leq C\sum_{j\leq k}\sum_{k\in Z}\|t^{1/2}\nabla u_k\|_{L^\infty_T(L^2)}\|\nabla u_j\|_{L^\infty_T(L^2)}\\&\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}^2\sum_{k\in Z}2^{-\frac{k}{2}}c_k\sum_{j\leq k}2^{\frac{j}{2}}c_j\\&\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}^2.\end{aligned}$$

Along the same lines, we deduce $\|t^{1/4}\nabla^2 d\|_{L^2}\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}$, hence

$$\|(t^{1/4}\nabla u,t^{1/4}\nabla^2 d)\|_{L^\infty_T(L^2)}\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}},\quad T<T^1,\tag{3.49}$$

for any $\epsilon\in(0,\frac14)$. It follows from (3.9) that

$$\|t^{-(\frac12-\epsilon)}(\nabla u_j,\nabla^2 d_j)\|_{L^2_T(L^2)}\leq C2^{j(\frac12-2\epsilon)}c_j\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}$$

and

$$\|t^{-\epsilon}(\nabla u_j,\nabla^2 d_j)\|_{L^2_T(L^2)}\leq C2^{j(2\epsilon-\frac12)}c_j\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}$$

holds for any $T<T^1$. Thanks to the above two equations, we obtain

$$\begin{aligned}\|t^{-1/4}\nabla u\|_{L^2_T(L^2)}^2&\leq 2\sum_{j\leq k}\sum_{k\in Z}\|t^{-(\frac12-\epsilon)}\nabla u_j\|_{L^2_T(L^2)}\|t^{-\epsilon}\nabla u_k\|_{L^2_T(L^2)}\\&\leq\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}^2\sum_{j\leq k}\sum_{k\in Z}2^{j(\frac12-2\epsilon)}c_j2^{k(2\epsilon-\frac12)}c_k\\&\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}^2.\end{aligned}$$

Similarly, we deduce

$$\|t^{-1/4}\nabla^2 d\|_{L^2_T(L^2)}\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}.$$

Therefore, one has

$$\|t^{-1/4}(\nabla u,\nabla^2 d)\|_{L^2_T(L^2)}\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}},\quad T<T^1.\tag{3.50}$$

Using (3.49), (3.50) and the interpolation inequality for any $T<T^1$, we have

$$\begin{aligned}\|(\nabla u,\nabla^2 d)\|_{L^4_T(L^2)}&\leq\|(t^{1/4}\nabla u,t^{1/4}\nabla^2 d)\|_{L^\infty_T(L^2)}^{1/2}\|t^{-1/4}(\nabla u,\nabla^2 d)\|_{L^2_T(L^2)}^{1/2}\\&\leq C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}.\end{aligned}\tag{3.51}$$

By lemma 3.1 and Lemma 2.2 (Bernstein inequality), we obtain

$$\begin{aligned}
&\|(u,\nabla d)\|^2_{L^\infty_T(\dot H^{1/2})}\\
&\le C\sum_{j,k\in Z}2^k\|(\dot\Delta_k u_j,\dot\Delta_k\nabla d_j)\|^2_{L^\infty_T(L^2)}\\
&\le C\sum_{j\in Z,j\le k}2^{-k}\|(\dot\Delta_k\nabla u_j,\dot\Delta_k\nabla^2 d_j)\|^2_{L^\infty_T(L^2)}+\sum_{j\in Z,k\le j}2^k\|(\dot\Delta_k u_j,\dot\Delta_k\nabla d_j)\|^2_{L^\infty_T(L^2)}\\
&\le C\|(u_0,\nabla d_0)\|^2_{\dot H^{1/2}},\quad T<T^1,
\end{aligned}\tag{3.52}$$

which together with L^3 continuous embedding into $\dot H^{1/2}$, ensures that

$$\|(u,\nabla d)\|_{L^\infty_T(L^3)}\le C\|(u,\nabla d)\|_{L^\infty_T(\dot H^{1/2})}\le C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}},\quad T<T^1.\tag{3.53}$$

Using (3.51), (3.53) and using a continuity argument, we can prove that $T^1=T^*$.

On the other hand, thanks to the (3.45), we deduce for any $t\le T<T^*$, that

$$\frac{\mathrm d}{\mathrm dt}\|(\sqrt{\mu(\rho)}\nabla u_j,\nabla^2 d_j)\|^2_{L^2}+\|(\sqrt\rho\partial_t u_j,\partial_t\nabla d_j)\|^2_{L^2}\le C\|(\nabla u,\nabla^2 d)\|^4_{L^2}\|(\sqrt{\mu(\rho)}\nabla u_j,\nabla^2 d_j)\|^2_{L^2},$$

which implies for the smooth solutions (u,d), that

$$\frac{\mathrm d}{\mathrm dt}\|(\sqrt{\mu(\rho)}\nabla u,\nabla^2 d)\|^2_{L^2}+\|(\sqrt\rho\partial_t u,\partial_t\nabla d)\|^2_{L^2}\le C\|(\nabla u,\nabla^2 d)\|^4_{L^2}\|(\sqrt{\mu(\rho)}\nabla u,\nabla^2 d)\|^2_{L^2}.$$

Multiplying the above equation by $t^{1/2}$, we obtain

$$\begin{aligned}
&\frac{\mathrm d}{\mathrm dt}\|t^{1/4}(\sqrt{\mu(\rho)}\nabla u,\nabla^2 d)\|^2_{L^2}+\|t^{1/4}(\sqrt\rho\partial_t u,\partial_t\nabla d)\|^2_{L^2}\\
&\le\|t^{-1/4}(\sqrt{\mu(\rho)}\nabla u,\nabla^2 d)\|^2_{L^2}+C\|(\nabla u,\nabla^2 d)\|^4_{L^2}\|t^{1/4}(\sqrt{\mu(\rho)}\nabla u,\nabla^2 d)\|^2_{L^2}.
\end{aligned}$$

Using Gronwall's inequality, one obtains

$$\|t^{1/4}(\sqrt{\mu(\rho)}\nabla u,\nabla^2 d)\|_{L^\infty_T(L^2)}+\|t^{1/4}(\sqrt\rho\partial_t u,\partial_t\nabla d)\|_{L^2_T(L^2)}\le C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}.\tag{3.54}$$

According to (3.40) and (3.41), one has

$$\begin{aligned}
\|(\nabla u_j,\nabla^2 d_j)\|_{L^6}&\le C\|(\sqrt\rho\partial_t u_j,\partial_t\nabla d_j)\|_{L^2}+C(\|\nabla u\|^2_{L^2}+\|\nabla^2 d\|^2_{L^2})\|\nabla^2 d_j\|_{L^2}\\
&\quad+C\|(u,\nabla d)\|_{L^3}\|(\nabla u_j,\nabla^2 d_j)\|_{L^6},
\end{aligned}$$

which yields

$$\begin{aligned}
&\|(\nabla u,\nabla^2 d)\|_{L^6}\\
&\le C\|(\partial_t u,\partial_t\nabla d)\|_{L^2}+(\|\nabla u\|^2_{L^2}+C\|\nabla^2 d\|^2_{L^2})\|\nabla^2 d\|_{L^2}+C\|(u,\nabla d)\|_{L^3}\|(\nabla u,\nabla^2 d)\|_{L^6}.
\end{aligned}$$

Multiplying the above equation by $t^{1/2}$, we deduce that

$$\begin{aligned}
\|t^{1/4}(\nabla u,\nabla^2 d)\|^2_{L^6}&\le C\|t^{1/4}(\partial_t u,\partial_t\nabla d)\|^2_{L^2}+(\|\nabla u\|^4_{L^2}+\|\nabla^2 d\|^4_{L^2})\|t^{1/4}\nabla^2 d\|^2_{L^2}\\
&\quad+\|(u,\nabla d)\|^2_{L^3}\|t^{1/4}(\nabla u,\nabla^2 d)\|^2_{L^6},
\end{aligned}$$

and integrating over the interval $(0,T)$, we deduce that

$$\begin{aligned}
&\|t^{1/4}(\nabla u,\nabla^2 d)\|_{L^2_T(L^6)}\\
&\le C\|t^{1/4}(\partial_t u,\partial_t\nabla d)\|_{L^2_T(L^2)}+(\|\nabla u\|^4_{L^4_T(L^2)}+\|\nabla^2 d\|^4_{L^4_T(L^2)})\|t^{1/4}\nabla^2 d\|^2_{L^\infty_T(L^2)}\\
&\quad+\|(u,\nabla d)\|^2_{L^\infty_T(L^3)}\|t^{1/4}(\nabla u,\nabla^2 d)\|^2_{L^2_T(L^6)}.
\end{aligned}$$

Using the inequality

$$\|(u,\nabla d)\|_{L^\infty_T(L^3)}\le C\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}$$

and the smallness of $\|(u_0,\nabla d_0)\|_{\dot H^{1/2}}$, we have

$$\|t^{1/4}(\nabla u,\nabla^2 d)\|_{L^2_T(L^6)}\le C\|t^{1/4}(\partial_t u,\partial_t\nabla d)\|_{L^2_T(L^2)}+(\|\nabla u\|^2_{L^4_T(L^2)}+\|\nabla^2 d\|^2_{L^4_T(L^2)})\|t^{1/4}\nabla^2 d\|_{L^\infty_T(L^2)}.$$

Thanks to (3.54), we obtain

$$\|t^{1/4}(\nabla u, \nabla^2 d)\|_{L^2_T(L^6)} \leq C\|(u_0, \nabla d_0)\|_{\dot{H}^{1/2}}. \tag{3.55}$$

Finally, by the interpolation inequality,

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq C\|\nabla u\|^{1/2}_{L^2(\mathbb{R}^3)}\|\nabla u\|^{1/2}_{L^6(\mathbb{R}^3)},$$

we obtain

$$\begin{aligned} &\|(\nabla u, \nabla^2 d)\|^2_{L^2_T(L^3)} + \|(u, \nabla d)\|^2_{L^2_T(L^\infty)} \\ &\leq \|t^{-1/4}(\nabla u, \nabla^2 d)\|^{1/2}_{L^2_T(L^2)}\|t^{1/4}(\nabla u, \nabla^2 d)\|^{1/2}_{L^2_T(L^6)} \leq C\|(u_0, \nabla d_0)\|^2_{\dot{H}^{1/2}}, \end{aligned} \tag{3.56}$$

which completes the proof. □

Now we reformulate system (1.1) as

$$\begin{gathered} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u - (1+a)(\Delta u + \operatorname{div}(\nu(a)\nabla u) - \nabla P) = -(1+a)\nabla \cdot (\nabla d : \nabla d), \\ \partial_t d + u \cdot \nabla d - \Delta d = |\nabla d|^2 d, \\ \operatorname{div} u = 0, \\ |d| = 1, \\ (a, u, d)|_{t=0} = (a_0, u_0, d_0), \end{gathered} \tag{3.57}$$

where $a = \frac{1}{\rho} - 1$, $\mu(\rho) - 1 = \nu(a)$.

We consider the linear equation

$$\begin{gathered} \partial_t u + w \cdot \nabla u - (1+a)(\Delta u + \operatorname{div}(\nu(a)\nabla u) - \nabla P) = f, \\ \partial_t d + w \cdot \nabla d - \Delta d = g, \\ \operatorname{div} u = \operatorname{div} w = 0, \\ |d| = 1, \\ (u, d)|_{t=0} = (u_0, d_0), \end{gathered} \tag{3.58}$$

In 2022, Qian and Qu [21] obtained the following results.

Proposition 3.3 ([21, Corollary2.8]). *Let $u_0, \omega_0 \in \dot{B}^{1/2}_{2,1}$ and q satisfy*

$$\max\{e\frac{1}{2} - \frac{1}{q}, \frac{1}{q} - \frac{1}{3}\} < \frac{1}{2} < \frac{1}{6} + \frac{1}{q}. \tag{3.59}$$

Let $(u, \nabla\pi, \omega)$ solve system (3.60) on $[0, T]$,

$$\begin{gathered} \partial_t u + v \cdot \nabla u - (1+a)(\Delta u - \nabla\pi + \operatorname{div}(\mu_r(a)Du)) = f, \\ \partial_t \omega + v \cdot \nabla \omega - (1+a)(\Delta\omega + \nabla \operatorname{div} \omega) + 4(1+a)\mu_r(a)\omega = g, \\ \operatorname{div} u = 0, \\ (u, \omega)|_{t=0} = (u_0, \omega_0), \end{gathered} \tag{3.60}$$

with some smooth, positive function μ_r. Further, denote by $b = b(a) = (1+a)\mu_r(a) - \mu_r(0)$ and $\lambda = \lambda(a) = \int_0^a \mu_r(s)\mathrm{d}s$. If for some sufficiently small positive constant c_0 and some integer $m \in \mathbb{Z}$, then it holds

$$\Big(1 + \|a\|_{L^\infty_T(\dot{B}^{3/q}_{q,1})}\Big)^{12} \|(Id - \dot{S}_m)(a, b, \lambda)\|_{L^\infty_T(\dot{B}^{3/q}_{q,1})} \leq \epsilon_0, \tag{3.61}$$

then, for $t \in [o, T]$, we have

$$
\begin{aligned}
&\|u\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|u\|_{L^1_t(\dot{B}^{5/2}_{2,1})} + \|\nabla \pi\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \\
&\lesssim \int_0^t \|v\|_{\dot{B}^{5/2}_{2,1}} \|u\|_{\dot{B}^{1/2}_{2,1}} \mathrm{d}\tau + \|u_0\|_{\dot{B}^{1/2}_{2,1}} \\
&\quad + 2^{4m}(A_m(a)+2)^{12} \int_0^t \|u\|_{\dot{B}^{1/2}_{2,1}} \mathrm{d}\tau + 2^m (A_m(a)+1)^3 \|f\|_{L^1_t(\bar{B}^{3/2})} \\
&\quad + 2^{4m}(A_m(a)+1)^6 \int_0^t \Big(\|V\|^{2/3}_{\dot{B}^{3/2}_{2,1}} \|V\|^{2/3}_{\dot{B}^{5/2}_{2,1}} + \|V\|^{3/2}_{\dot{B}^{3/2}_{2,1}} \|V\|^{1/2}_{\dot{B}^{5/2}_{2,1}} \Big) \|u\|_{B^{1/2}_p} \mathrm{d}\tau \\
&\quad + 2^{2m}(A_m(a)+1)^4 \int_0^t \Big(\|v\|_{\dot{B}^{3/2}_{2,1}} + \|v\|^2_{\dot{B}^{1/2}_{2,1}} \Big) \|u\|_{\dot{B}^{1/2}_{2,1}} \mathrm{d}\tau
\end{aligned}
\tag{3.62}
$$

and

$$
\begin{aligned}
&\|\omega\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|\omega\|_{L^1_t(\dot{B}^{5/2}_{2,1} \bigcap \dot{B}^{1/2}_{2,1})} \\
&\lesssim \int_0^t \|v\|_{\dot{B}^{5/2}_{2,1}} \|\omega\|_{\dot{B}^{1/2}_{2,1}} \mathrm{d}\tau + \|\omega_0\|_{\dot{B}^{1/2}_{2,1}} + \|\omega\|_{L^1_t(\dot{B}^{1/2}_{2,1})} + \|g\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \\
&\quad + 2^{(1+\frac{3}{q})m} \|\dot{S}_m a\|_{L^\infty_t(L^q)} \|\omega\|_{L^1_t(\dot{B}^{3/2}_{2,1})},
\end{aligned}
\tag{3.63}
$$

where $A_m(a) = 2^{3/q} \|\dot{S}_m a\|_{L^\infty_t(L^q)}$.

Comparing (3.58) with (3.60), we obtain the following proposition.

Proposition 3.4. *Let (a, u, w, d, f, g) be smooth enough functions, which satisfy $1 + a \geq c_0 > 0$, then for the smooth solution $(u, d, \nabla P)$ on $[0, T]$, if there exists some m and sufficiently small ϵ_0 such that*

$$
\Big(1 + \|a\|_{L^\infty_T(\dot{B}^{3/q}_{q,1})}\Big)^{12} \|(Id - \dot{S}_m)(a, b, \lambda)\|_{L^\infty_T(\dot{B}^{3/q}_{q,1})} \leq \epsilon_0,
\tag{3.64}
$$

where $b = b(a) = (1+a)\nu(a) - \nu(0)$, $\lambda = \int_0^a \nu(s) \mathrm{d}s$. Then for any $t \in [0, T]$, we have

$$
\begin{aligned}
&\|u\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})} + \|u\|_{L^1_t(\dot{B}^{5/2}_{2,1})} + \|\nabla P\|_{L^1_t(\dot{B}^{1/2}_{2,1})} \\
&\lesssim \int_0^t \|w\|_{\dot{B}^{5/2}_{2,1}} \|u\|_{\dot{B}^{1/2}_{2,1}} \mathrm{d}\tau + \|u_0\|_{\dot{B}^{1/2}_{2,1}} \\
&\quad + 2^{4m}(K_m(a)+2)^{12} \int_0^t \|u\|_{\dot{B}^{1/2}_{2,1}} \mathrm{d}\tau + 2^m (K_m(a)+1)^3 \|f\|_{L^1_t(\bar{B}^{3/2})} \\
&\quad + 2^{4m}(K_m(a)+1)^6 \int_0^t \Big(\|w\|^{2/3}_{\dot{B}^{3/2}_{2,1}} \|w\|^{2/3}_{\dot{B}^{5/2}_{2,1}} + \|w\|^{3/2}_{\dot{B}^{3/2}_{2,1}} \|w\|^{1/2}_{\dot{B}^{5/2}_{2,1}} \Big) \|u\|_{B^{1/2}_p} \mathrm{d}\tau \\
&\quad + 2^{2m}(Km(a)+1)^4 \int_0^t \Big(\|w\|_{\dot{B}^{3/2}_{2,1}} + \|w\|^2_{\dot{B}^{1/2}_{2,1}} \Big) \|u\|_{\dot{B}^{1/2}_{2,1}} \mathrm{d}\tau,
\end{aligned}
\tag{3.65}
$$

and

$$
\begin{aligned}
&\|d\|_{\tilde{L}^\infty_t(\dot{B}^{3/2}_{2,1})} + \|d\|_{L^1_t(\dot{B}^{\frac{7}{2}}_{2,1} \bigcap \dot{B}^{3/2}_{2,1})} \\
&\lesssim \int_0^t \|w\|_{\dot{B}^{5/2}_{2,1}} \|d\|_{\dot{B}^{3/2}_{2,1}} \mathrm{d}\tau + \|d_0\|_{\dot{B}^{3/2}_{2,1}} + \|d\|_{L^1_t(\dot{B}^{3/2}_{2,1})} + \|g\|_{L^1_t(\dot{B}^{3/2}_{2,1})},
\end{aligned}
\tag{3.66}
$$

where $K_m(a) = 2^{3/q} \|\dot{S}_m a\|_{L^\infty_t(L^q)}$.

Proof. As in [21], we only need to prove the second inequality. Acting the both sides of $(3.58)_2$ with the operator $\dot{\Delta}_j$ and using a standard commutator process, we obtain

$$
\partial_t \dot{\Delta}_j d + \omega \cdot \nabla \dot{\Delta}_j d - \Delta \dot{\Delta}_j d = \dot{\Delta}_j g + [\dot{\Delta}_j, \omega] \nabla d,
\tag{3.67}
$$

By multiplying (3.67) with $\dot{\Delta}_j d$ and integrating on $\mathbb{R}^3$, we have

$$
\frac{1}{2} \|\dot{\Delta}_j d\|_{L^2} + 2^{2j} \|\dot{\Delta}_j d\|_{L^2} \leq \|\dot{\Delta}_j g\|_{L^2} + \|[\dot{\Delta}_j, \omega] \nabla d\|_{L^2},
$$

then thanks to the commutator estimate in [2], we have

$$\|[\dot{\Delta}_j,\omega]\nabla d\|_{L^2}\lesssim d_j 2^{-3j/2}\|\nabla\omega\|_{\dot{B}^{3/2}_{2,1}}\|\nabla d\|_{\dot{B}^{1/2}_{2,1}}\lesssim d_j 2^{-3j/2}\|\nabla\omega\|_{\dot{B}^{3/2}_{2,1}}\|d\|_{\dot{B}^{3/2}_{2,1}},$$

from which, multiplying (3.67) with $2^{\frac{3j}{2}}$, summing up for $j\in\mathbb{Z}$ and integrating on $(0,t)$, we have

$$\|d\|_{L^\infty_t(\dot{B}^{3/2}_{2,1})}+\|d\|_{L^1_t(\dot{B}^{7/2}_{2,1})}\lesssim\|d_0\|_{\dot{B}^{3/2}_{2,1}}+\|g\|_{L^1_t(\dot{B}^{3/2}_{2,1})}+\int_0^t\|\nabla\omega\|_{\dot{B}^{3/2}_{2,1}}\|d\|_{\dot{B}^{3/2}_{2,1}}\mathrm{d}\tau, \tag{3.68}$$

which implies (3.66). □

Proposition 3.5 ([21]). *Under the assumptions of proposition 3.3, with $w=u$, we have*

$$\begin{aligned}&\|u\|_{\tilde{L}^\infty_t(\dot{B}^{1/2}_{2,1})}+\|u\|_{L^1_t(\dot{B}^{5/2}_{2,1})}+\|\nabla P\|_{L^1_t(\dot{B}^{1/2}_{2,1})}\\&\lesssim C\exp(Ct2^{4m}(K_m(a)+2)^{12})\Big(\|u_0\|_{\dot{B}^{1/2}_{2,1}}+2^m(K_m(a)+1)^3\|f\|_{L^1_t(\bar{B}^{3/2})}\\&\quad+2^{16m}(K_m(a)+1)^{24}\int_0^t\Big(\|u\|^3_{\dot{B}^{1/2}_{2,1}}+\|u\|^5_{\dot{B}^{1/2}_{2,1}}+\|u\|_{\dot{B}^{1/2}_{2,1}}\|u\|_{\dot{B}^{5/2}_{2,1}}\Big)\mathrm{d}\tau\Big)\end{aligned}\tag{3.69}$$

where $K_m(a)$ is defined in Proposition 3.4.

4. EXISTENCE

In this section, we will prove the global existence in the $\dot{H}^{1/2}$ and the local existence in the $\dot{B}^{1/2}_{2,1}$.

Local existence in $\dot{B}^{1/2}_{2,1}$.

Theorem 4.1. *Let $a_0\in\dot{B}^{3/q}_{q,1}$ satisfy $1+a_0\geq\epsilon_0$, and assume that $(u_0,\nabla d_0)\in\dot{B}^{1/2}_{2,1}$. Then there exists a positive time T, such that (3.57) admits a local solution satisfying*

$$\|u\|_{\tilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}+\|u\|_{L^1_T(\dot{B}^{5/2}_{2,1})}+\|(\partial_t u,\nabla P)\|_{L^1_T(B^{3/2})}\lesssim\|u_0\|_{\dot{B}^{1/2}_{2,1}}, \tag{4.1}$$

$$\|d\|_{\tilde{L}^\infty_T(\dot{B}^{3/2}_{2,1})}+\|u\|_{L^1_T(\dot{B}^{\frac{7}{2}}_{2,1})}+\|(\partial_t d,d)\|_{L^1_T(\dot{B}^{3/2}_{2,1})}\lesssim\|d_0\|_{\dot{B}^{3/2}_{2,1}}. \tag{4.2}$$

Proof. Following the strategy in [21], we divide the argument into three parts.

Part1: Construct approximate solutions. For any $n\in\mathbb{N}$, let

$$a_0^n=\dot{S}_n a_0-\dot{S}_{-n}a_0,\quad u_0^n=\dot{S}_n u_0-\dot{S}_{-n}u_0,\quad d_0^n=\dot{S}_n d_0-\dot{S}_{-n}d_0.$$

Then using the argument in [3], (3.57) has a unique local solution (a^n,u^n,d^n) with the initial data (a_0^n,u_0^n,d_0^n).

Part2: Uniform estimates of approximate solutions Let $u_L^n=e^{t\Delta}u_0^n$, $u_{NL}^n=u^n-u_L^n$. Then

$$\|u_L^n\|_{\tilde{L}^\infty_T(\dot{B}^{1/2}_{2,1})}\leq C\|u_0\|_{\dot{B}^{1/2}_{2,1}}, \tag{4.3}$$

$$\|u_L^n\|_{\tilde{L}^1_T(\dot{B}^{5/2}_{2,1})}\leq C\sum_{j\in Z}2^{j\frac{3}{2}}(1-e^{-ct2^{2j}})\|\dot{\Delta}_j u_0\|_{L^2}. \tag{4.4}$$

Assume that $(a^n,u_{NL}^n,d^n,\nabla P^n)$ solves the system

$$\begin{gathered}\partial_t a^n+(u_{NL}^n+u_L^n)\cdot\nabla a^n=0\\\partial_t u_{NL}^n+u_L^n\cdot\nabla u_{NL}^n-(1+a)(\Delta u_{NL}^n+\operatorname{div}((\nu(a^n))\nabla u_{NL}^n)-\nabla P^n)=\mathcal{U}_n,\\\partial_t d^n+u_L^n\cdot\nabla d^n-\Delta d^n=\mathcal{D}_n,\\\operatorname{div}u_{NL}^n=0,\\|d^n|=1,\\(a^n,u_{NL}^n,d^n)|_{t=0}=(a_0^n,0,d_0^n),\end{gathered}\tag{4.5}$$

where

$$\begin{aligned}\mathcal{U}_n&=-u_L^n\cdot\nabla u_{NL}^n-u_{NL}^n\cdot\nabla u_L^n-u_{NL}^n\cdot\nabla u_{NL}^n-a^n\Delta u_L^n\\&=(1+a^n)\operatorname{div}(\nu(a^n)\nabla u_L^n)-(1+a^n)\operatorname{div}(\nabla d^n:\nabla d^n),\end{aligned}\tag{4.6}$$

$$\mathcal{D}_n = -u^n_{NL}\cdot\nabla d^n + |\nabla d^n|^2 d^n. \tag{4.7}$$

If we denote

$$b^n = (1+a^n)\nu(a^n) - \nu(0), \quad \lambda^n = \int_0^{a^n}\nu(s)\mathrm{d}s, \tag{4.8}$$

then it is easy to see that $b^n(0) = \lambda^n(0) = 0$, and

$$(1+a^n)\operatorname{div}(\nu(a^n)\nabla u^n_L) = \operatorname{div}((\nu(0)+b^n)\nabla u^n_L) - \nabla u^n_L\cdot\lambda^n.$$

If $a_0\in\dot B^{3/q}_{q,1}$, denoting $\nabla d_0 = (1+a_0)\nu(a_0), \lambda_0 = \int_0^{a_0}\nu(s)\mathrm{d}s$, we deduce that $\nabla d_0, \lambda_0\in\dot B^{3/q}_{q,1}$. Let

$$m = \inf\Big\{k\in\mathbb{N} : \sum_{\ell\geq k} 2^{\frac{3\ell}{q}}\|\dot\Delta_\ell(a_0,b_0,\lambda_0)\|_{L^q} \leq \eta_0\epsilon_0\Big(1+\|a_0\|_{\dot B^{3/q}_{q,1}}\Big)^{-12}\Big\}. \tag{4.9}$$

Then thanks to [21, Prop 2.3] and the equation of a^n, we have

$$\|a^n\|_{L^\infty_t(\dot B^{3/q}_{q,1})} \lesssim \|a_0\|_{\dot B^{3/q}_{q,1}}\exp\Big(C\Big(\|\nabla u^n_L\|_{L^1_t(\dot B^{3/2}_{2,1})} + \|\nabla u^n_{NL}\|_{L^1_t(\dot B^{3/2}_{2,1})}\Big)\Big), \tag{4.10}$$

and

$$\|a^n\|_{L^\infty_t(L^q)} \leq \|a^n_0\|_{L^q} \leq \|a_0\|_{L^q}, \quad \|a^n\|_{L^\infty_t(L^\infty)} \leq \|a^n_0\|_{L^\infty} \leq \|a_0\|_{\dot B^{3/q}_{q,1}}. \tag{4.11}$$

According to the definitions of m, we obtain

$$(1+\|a\|_{L^\infty_t(\dot B^{3/q}_{q,1})})^{12}\|(Id-\dot S_m)(a,b,\lambda)\|_{L^\infty_t(\dot B^{3/q}_{q,1})} \leq 2\eta_0\epsilon_0. \tag{4.12}$$

Then by applying Proposition 3.4 and (4.5), we have

$$\begin{aligned}
&\|u^n_{NL}\|_{\tilde L^\infty_t(\dot B^{1/2}_{2,1})} + \|u^n_{NL}\|_{L^1_t(\dot B^{5/2}_{2,1})} + \|\nabla P^n\|_{L^1_t(\dot B^{1/2}_{2,1})}\\
&\lesssim 2^{4m}(K_m(a^n)+2)^{12}\Big(\int_0^t \|u^n_{NL}\|_{\dot B^{1/2}_{2,1}}\mathrm{d}\tau + \|\mathcal{U}_n\|_{L^1_t(\bar B^{3/2})}\\
&\quad + \int_0^t \Big(\|u^n_L\|^{2/3}_{\dot B^{3/2}_{2,1}}\|u^n_L\|^{2/3}_{\dot B^{5/2}_{2,1}} + \|u^n_L\|^{3/2}_{\dot B^{3/2}_{2,1}}\|u^n_L\|^{1/2}_{\dot B^{5/2}_{2,1}}\Big)\|u^n_{NL}\|_{\dot B^{1/2}_{2,1}}\mathrm{d}\tau\\
&\quad + \int_0^t \Big(\|u^n_L\|_{\dot B^{5/2}_{2,1}} + \|u^n_L\|_{\dot B^{3/2}_{2,1}} + \|u^n_L\|^2_{\dot B^{1/2}_{2,1}}\Big)\|u^n_{NL}\|_{\dot B^{1/2}_{2,1}}\mathrm{d}\tau\Big)
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\|d^n\|_{\tilde L^\infty_t(\dot B^{3/2}_{2,1})} + \|d^n\|_{L^1_t(\dot B^{\frac{7}{2}}_{2,1}\bigcap \dot B^{3/2}_{2,1})}\\
&\lesssim \int_0^t \|u^n_L\|_{\dot B^{5/2}_{2,1}}\|d^n\|_{\dot B^{3/2}_{2,1}}\mathrm{d}\tau + \|d^n_0\|_{\dot B^{3/2}_{2,1}} + \|d^n\|_{L^1_t(\dot B^{3/2}_{2,1})} + \|\mathcal{D}_n\|_{L^1_t(\dot B^{3/2}_{2,1})},
\end{aligned} \tag{4.14}$$

where $K_m(a) = 2^{3/q}\|\dot S_m a\|_{L^\infty_t(L^q)}$.

Now we estimate $\|\mathcal{U}_n\|_{L^1_t(\bar B^{3/2})}$ and $\|\mathcal{D}_n\|_{L^1_t(\dot B^{3/2}_{2,1})}$. According to the product law in the Besov Space and the Lemma 2.4, we obtain

$$\|(1+a^n)\operatorname{div}(\nabla d^n : \nabla d^n)\|_{L^1_t(B^{3/2})} \leq C\int_0^t (1+\|a^n\|_{\dot B^{3/q}_{q,1}})\Big(\|d^n\|_{\dot B^{\frac{7}{2}}_{2,1}} + \|d^n\|_{\dot B^{5/2}_{2,1}}\Big)\|d^n\|_{\dot B^{3/2}_{2,1}}\mathrm{d}\tau,$$

$$\|u^n_{NL}\cdot\nabla d^n\|_{L^1_t(\dot B^{3/2}_{2,1})} \leq C\int_0^t \|u^n_{NL}\|_{\dot B^{3/2}_{2,1}}\|d^n\|_{\dot B^{5/2}_{2,1}}\mathrm{d}\tau,$$

$$\||\nabla d^n|^2 d^n\|_{L^1_t(\dot B^{3/2}_{2,1})} \leq C\int_0^t \|d^n\|_{\dot B^{3/2}_{2,1}}\|d^n\|^2_{\dot B^{5/2}_{2,1}}\mathrm{d}\tau,$$

which implies

$$\begin{aligned}
&\|(1+a^n)\operatorname{div}(\nabla d^n : \nabla d^n)\|_{L^1_t(B^{3/2})} + \|u^n_{NL}\cdot\nabla d^n\|_{L^1_t(\dot B^{3/2}_{2,1})} + \||\nabla d^n|^2 d^n\|_{L^1_t(\dot B^{3/2}_{2,1})}\\
&\leq C\int_0^t \Big(\|u^n_{NL}\|_{\dot B^{1/2}_{2,1}} + \|d^n\|_{\dot B^{3/2}_{2,1}}\Big)\Big(\|u^n_{NL}\|_{\dot B^{5/2}_{2,1}} + \|d^n\|_{\dot B^{\frac{7}{2}}_{2,1}}\Big)\mathrm{d}\tau\\
&\quad + C\int_0^t \|d^n\|^{3/2}_{\dot B^{3/2}_{2,1}}\|d^n\|^{1/2}_{\dot B^{\frac{7}{2}}_{2,1}}\,\mathrm{d}\tau + \int_0^t \|d^n\|^2_{\dot B^{3/2}_{2,1}}\|d^n\|_{\dot B^{\frac{7}{2}}_{2,1}}\,\mathrm{d}\tau
\end{aligned}$$

$$\lesssim Q_n^2(t) + \sqrt{t}Q_n^2(t) + Q_n^3(t),$$

where

$$Q_n(t) = \|u_{NL}^n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{1/2})} + \|u_{NL}^n\|_{L_t^1(\dot{B}_{2,1}^{5/2})} + \|\nabla P^n\|_{L_t^1(\dot{B}_{2,1}^{1/2})} + \|d^n\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{3/2})} + \|d^n\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}} \bigcap \dot{B}_{2,1}^{3/2})}.$$

Moreover, thanks to the product law in Besov space and Lemma 2.4, we have

$$\begin{aligned}\|\mathcal{U}_n\|_{L_t^1(\bar{B}^{3/2})} \lesssim \int_0^t &(\|u_L^n\|_{\dot{B}_{2,1}^{5/2}}(\|u_L^n\|_{\dot{B}_{2,1}^{1/2}} + \|u_{NL}^n\|_{\dot{B}_{2,1}^{1/2}}) + \|u_L^n\|_{\dot{B}_{2,1}^{1/2}}(\|u_L^n\|_{\dot{B}_{2,1}^{3/2}} + \|u_{NL}^n\|_{\dot{B}_{2,1}^{3/2}}) \\ &+ \|u_{NL}^n\|_{\dot{B}_{2,1}^{1/2}}(\|u_{NL}^n\|_{\dot{B}_{2,1}^{3/2}} + \|u_{NL}^n\|_{\dot{B}_{2,1}^{5/2}}) + \|a^n\|_{\dot{B}_{2,1}^{3/2}}(\|u_L^n\|_{\dot{B}_{2,1}^{5/2}} + \|u_L^n\|_{\dot{B}_{2,1}^{3/2}}) \\ &+ (1 + \|a^n\|_{\dot{B}_{q,1}^{3/q}})\Big(\|d^n\|_{\dot{B}_{2,1}^{\frac{7}{2}}} + \|d^n\|_{\dot{B}_{2,1}^{5/2}}\Big)\|d^n\|_{\dot{B}_{2,1}^{3/2}})\mathrm{d}\tau,\end{aligned}$$

and

$$\|\mathcal{D}_n\|_{L_t^1(\dot{B}_{2,1}^{3/2})} \lesssim \int_0^t (\|u_{NL}^n\|_{\dot{B}_{2,1}^{3/2}}\|d^n\|_{\dot{B}_{2,1}^{5/2}} + \|d^n\|_{\dot{B}_{2,1}^{3/2}}\|d^n\|_{\dot{B}_{2,1}^{5/2}}^2)\mathrm{d}\tau.$$

If we denote

$$D(t) = 1 + \|u_n^L\|_{\dot{B}_{2,1}^{5/2}} + \|u_n^L\|_{\dot{B}_{2,1}^{1/2}}^2 + \|u_n^L\|_{\dot{B}_{2,1}^{1/2}}^{3/2}\|u_n^L\|_{\dot{B}_{2,1}^{5/2}}^{3/2} + \|u_n^L\|_{\dot{B}_{2,1}^{1/2}}^{3/2}\|u_n^L\|_{\dot{B}_{2,1}^{5/2}}^{1/2} + \|u_n^L\|_{\dot{B}_{2,1}^{1/2}}^{1/2}\|u_n^L\|_{\dot{B}_{2,1}^{5/2}}^{1/2},$$

then according to an estimates in [21], for a sufficient large number N_0, and $t \leq 1$, we have

$$\begin{aligned}Q_n(t) &\leq C_m(Q_n^2(t) + Q_n^3(t) + ct + \sum_j 2^{1/2}(1 - e^{-ct2^{2j}})\|\dot{\Delta}_j u_0\|_{L^2}) \\ &\leq C_m(Q_n^2(t) + Q_n^3(t) + ct + Ct2^{2N_0}\|u_0\|_{\dot{B}_{2,1}^{1/2}}).\end{aligned} \tag{4.15}$$

Let $T_1 = \min(1, (C_m c(1 + 2^{2N_0}))^{-1}\epsilon))$. Thanks to [21], there exists n_1, such that for any $n > n_1$, we obtain

$$Q_n(t) \leq \frac{4}{3}\epsilon. \tag{4.16}$$

According to the proof in [21, theorem3.1], there exists a positive time T_1, such that for any n,

$$Q_n(T_1) \leq C, \quad \|a^n\|_{L_T^\infty(\dot{B}_{q,1}^{3/q})} \leq C. \tag{4.17}$$

Part3: Convergence of the approximate solutions Thanks to (4.17), we can use the method in [21, Theorem 3.1], which means there existence (a, u, d, P) such that

$$\begin{gathered}\|u\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{1/2})} + \|u\|_{L_t^1(\dot{B}_{2,1}^{5/2})} + \|\nabla P\|_{L_t^1(\dot{B}_{2,1}^{1/2})} + \|d\|_{\tilde{L}_t^\infty(\dot{B}_{2,1}^{3/2})} + \|d\|_{L_t^1(\dot{B}_{2,1}^{\frac{7}{2}} \bigcap \dot{B}_{2,1}^{3/2})} \leq C, \\ \|a^n\|_{L_T^\infty(\dot{B}_{q,1}^{3/q})} \leq C, \\ (1 + \|a\|_{L_t^\infty(\dot{B}_{q,1}^{3/q})})^{12}\|(Id - \dot{S}_m)(a, b, \lambda)\|_{L_t^\infty(\dot{B}_{q,1}^{3/q})} \leq 2\eta_0\epsilon_0.\end{gathered}$$

This completes the proof. □

Global existence in $\dot{H}^{1/2}$. By mollifying the initial data $(\rho_0, u_0, \nabla d_0)$, we obtain the smooth initial data $(\rho_{0\epsilon}, u_{0\epsilon}, \nabla d_{0\epsilon})$. By using modifications of the classical well-posedness theory of inhomogeneous incompressible Navier-Stokes system, (1.1) has a unique local solution $(\rho_\epsilon, u_\epsilon, \nabla d_\epsilon)$ on $[0, T_\epsilon^*)$. If the ε_0 is sufficiently small, we deduce from Proposition 3.3 that $(\rho_\epsilon, u_\epsilon, B_\epsilon)$ satisfy the estimates (3.48) for any $T < T_\epsilon^*$. By a standard continuous argument, it has been proved in the Proposition 3.3 that $T_\epsilon^* = +\infty$. In particular, we have $(u_\epsilon, \nabla d_\epsilon) \in (C([0, +\infty); \dot{H}^{1/2}) \cap L^4(\mathbb{R}^+; \dot{H}^1))^2$, and for any $T \in [0, +\infty]$, $(u_\epsilon, \nabla d_\epsilon)$ satisfy (3.48). By using a compactness argument similar to that in [18], there exists $\rho \in C_{\mathrm{w}}([0, \infty); L^\infty)$, such that, for any $r < \infty$,

$$\begin{aligned}\rho_\epsilon &\rightharpoonup \rho \quad \text{weak} * \text{ in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^3), \\ \rho_\epsilon &\to \rho \quad \text{strongly in } L_{\mathrm{loc}}^r(\mathbb{R}^+ \times \mathbb{R}^3).\end{aligned} \tag{4.18}$$

According to the interpolation inequality in Lorentz space, for any $T < \infty$,

$$\|(\partial_t u_\epsilon, \partial_t \nabla d_\epsilon)\|_{L_T^{\frac{4}{3},\infty}(L^2)} \lesssim \|t^{-1/4}\|_{L_T^{4,\infty}}\|(\partial_t u_\epsilon, \partial_t \nabla d_\epsilon)\|_{L_T^2(L^2)}, \tag{4.19}$$

which together with (3.48) yields

$$\|(\partial_t u_\epsilon, \partial_t \nabla d_\epsilon)\|_{L_T^{\frac{4}{3},\infty}(L^2)} + \|(\nabla u_\epsilon, \nabla^2 d_\epsilon)\|_{L_T^2(L^3)} \lesssim \|(u_0, \nabla d_0)\|_{\dot{H}^{1/2}}. \tag{4.20}$$

Then according to the Arzela-Ascoli Theorem, there exist

$$(u, \nabla d) \in \big(L^\infty([0,+\infty); \dot{H}^{1/2}) \cap L^4(\mathbb{R}^+; \dot{H}^1) \cap L^2(\mathbb{R}^+; \dot{W}^{1,3})\big)^2$$

such that, for any $r < \infty$

$$\begin{gathered}(u_\epsilon, \nabla d_\epsilon) \rightharpoonup (u, \nabla d) \quad \text{weakly in } L^4(\mathbb{R}^+; \dot{H}^1)\\ (u_\epsilon, \nabla d_\epsilon) \to (u, \nabla d) \quad \text{strongly in } L^2_{\rm loc}(\mathbb{R}^+;\ L^r_{\rm loc}(\mathbb{R}^3)).\end{gathered} \tag{4.21}$$

Thanks to (4.18) and (4.21), we conclude that $(\rho, u, \nabla d)$ is a global weak solution of (1.1). Moreover, it follows from (3.48) and Fatou's Lemma that $(\rho, u, \nabla\Pi, \nabla d)$ satisfies the estimates (3.48) for any $T \in [0, +\infty]$.

Finally, let prove that $(u, \nabla d) \in (C([0,\infty); \dot{H}^{1/2}))^2$. It follows from (3.48) that

$$\|u\|_{\widetilde{L}^\infty(\mathbb{R}^+; \dot{H}^{1/2})} \le C\|(u_0, \nabla d_0)\|_{\dot{H}^{1/2}}.$$

Then for any $\varepsilon > 0$, there exists a $m_0 \in \mathbb{N}$ such that

$$4 \sum_{|j| \ge m_0} 2^j \|\dot{\Delta}_j u\|^2_{L^\infty(\mathbb{R}^+; L^2)} < \varepsilon.$$

Then for any $t \in [0, +\infty)$, $h > 0$, we have

$$\begin{aligned}\|u(t+h) - u(t)\|^2_{\dot{H}^{1/2}} &= \sum_{j\in\mathbb{Z}} 2^j \|\dot{\Delta}_j (u(t+h) - u(t))\|^2_{L^2}\\ &\le \sum_{|j| \le m_0 - 1} 2^j \|\dot{\Delta}_j (u(t+h) - u(t))\|^2_{L^2} + 4 \sum_{|j| \ge m_0} 2^j \|\dot{\Delta}_j u\|^2_{L^\infty_T(L^2)}\\ &\le 2^{m_0} \|u(t+h) - u(t)\|^2_{L^2} + \varepsilon,\end{aligned}$$

from which, we infer that

$$\begin{aligned}\|u(t+h) - u(t)\|^2_{\dot{H}^{1/2}} &\le 2^{m_0} \Big\| \int_t^{t+h} \tau^{-\frac14}\, \tau^{1/4} u_\tau(\tau)\, d\tau \Big\|^2_{L^2} + \varepsilon\\ &\le 2^{m_0} \|\tau^{-\frac14}\|^2_{L^2(t,t+h)} \|\tau^{1/4} u_\tau\|^2_{L^2(t,t+h;L^2)} + \varepsilon\\ &\le C\, 2^{m_0+2} \|(u_0, \nabla d_0)\|_{\dot{H}^{1/2}}\, h^{1/2} + \varepsilon,\end{aligned} \tag{4.22}$$

where we used $\|t^{1/4} u_t\|_{L^2(\mathbb{R}^+; L^2)} \le C\|(u_0, \nabla d_0)\|_{\dot{H}^{1/2}}$ in (3.48) and the inequality

$$(t+h)^{1/2} - t^{1/2} = \frac{h}{(t+h)^{1/2} + t^{1/2}} \le h^{1/2}, \quad t > 0.$$

Then (4.22) shows that $u \in C([0,\infty); \dot{H}^{1/2})$.

Along the same lines, we can prove that $\nabla d \in C([0,\infty); \dot{H}^{1/2})$. Thus we completed the proof of global existence in $\dot{H}^{1/2}$ and the local existence in the $\dot{B}^{1/2}_{2,1}$.

5. UNIQUENESS

To prove uniqueness, we first consider the linear system

$$\begin{gathered}\partial_t u - (1+a)(\Delta u + \operatorname{div}(\nu(a)\nabla u) - \nabla P) = f,\\ \operatorname{div} u = g,\\ \partial_t g = \operatorname{div} R,\\ u|_{t=0} = 0.\end{gathered} \tag{5.1}$$

Proposition 5.1 ([21]). *For the smooth solutions* $(u,\nabla P)$ *of* (5.1)*, there exists some* m*, such that if*

$$(1+\|a\|_{L^\infty_t(\dot B^{3/q}_{q,1})})^{12}\|(Id-\dot S_m)(a,b,\lambda)\|_{L^\infty_t(\dot B^{3/q}_{q,1})}\le \epsilon_0,$$

then we have the following estimates

$$\|u\|_{\tilde L^\infty_T(B^{3/2})}+\|u\|_{L^1_T(B^{7/2})}+\|(\partial_t u,\nabla P)\|_{L^1_T(B^{3/2})}\le C\int_0^T\|u\|_{B^{3/2}}+\|(f,\nabla g,R)\|_{L^1_T(B^{3/2})}. \tag{5.2}$$

For the linear system of d:

$$\begin{gathered}\partial_t d-\Delta d=h,\\ |d|=1,\\ d|_{t=0}=0,\end{gathered} \tag{5.3}$$

Proposition 5.2. *For the smooth solutions* d *of* (5.3)*, we have the estimate*

$$\|d\|_{\tilde L^\infty_T(B^{5/2})}+\|d\|_{L^1_T(B^{\frac92})}+\|\partial_t d\|_{L^1_T(B^{5/2})}\le C\int_0^T\|d\|_{B^{5/2}}+\|h\|_{L^1_T(B^{5/2})}. \tag{5.4}$$

Proof. Thanks to the proof in Proposition 3.3 and (5.3), we have

$$\begin{gathered}\|d\|_{L^\infty_T(\dot B^{5/2}_{2,1})}+\|d\|_{L^1_T(\dot B^{\frac92}_{2,1})}\lesssim\|h\|_{L^1_T(\dot B^{5/2}_{2,1})},\\ \|d\|_{L^\infty_T(\dot B^{1}_{2,2})}+\|d\|_{L^1_T(\dot B^{3}_{2,2})}\lesssim\|h\|_{L^1_T(\dot B^{1}_{2,2})},\end{gathered}$$

from which and (5.3), the proposition is proved. □

We now establish the uniqueness of solutions to the system via the Lagrangian coordinate method. By the existence result, we can choose a sufficiently small $T>0$ such that

$$\int_0^T\|u\|_{\dot B^{3/2}_{3,1}}\mathrm{d}t\le\frac14. \tag{5.5}$$

Hence, for any $y\in\mathbb{R}^3$, the ordinary differential equation

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}X(t,y)=u(t,X(t,y)),\\ X(t,y)|_{t=0}=y.\end{gathered} \tag{5.6}$$

has a unique solution on $[0,T]$. It is easy to see that the relation between Euler coordinates x and Lagrangian coordinates y:

$$x=X(t,y)=y+\int_0^t u(\tau,X(t,y))\mathrm{d}\tau. \tag{5.7}$$

Let $Y(t,\cdot)$ be the inverse mapping of $X(t,\cdot)$, then $\nabla_x Y(t,x)=(\nabla_y X(t,y))^{-1}$.

Denoting $A(t,y)=(D_yX(t,y))^{-1}$, where $(D_{y_j}X)_{ij}=D_{y_j}X^i$. if $\nabla_y=D_y^\top$, then

$$\begin{gathered}\partial_t u(t,X(t,y))=(\partial_t u+u\cdot\nabla u)(t,x),\\ \nabla_x u(t,x)=A(t,y)^\top\nabla_y u(t,X(t,y)),\\ \operatorname{div}_x u(t,x)=\operatorname{div}_y(A(t,y)u(t,X(t,y))),\\ \Delta_x u(t,x)=\operatorname{div}_y(A(t,y)A(t,y)^\top\nabla_y u(t,X(t,y))).\end{gathered}$$

We introduce the following notation:

$$\begin{gathered}\nabla_u=A^\top\cdot\nabla_y,\quad \operatorname{div}_u=\operatorname{div}_y(A\cdot),\quad \Delta_u=\operatorname{div}_u\nabla_u,\\ b(t,y)=a(t,X(t,y)),\quad v(t,y)=u(t,X(t,y)),\\ \pi(t,y)=P(t,X(t,y)),\quad D(t,y)=d(t,X(t,y)).\end{gathered} \tag{5.8}$$

Then $(b, v, \nabla\pi, D)$ solves the system

$$\begin{gathered}
\partial_t b = 0, \\
\partial_t v - (1+b)(\Delta_u v - \nabla_u \pi + \operatorname{div}_u(\nu(b)\nabla_u v)) = \operatorname{div}_u(\nabla_u D : \nabla_u D), \\
\partial_t D - \Delta_u D = |\nabla_u D|^2 D, \\
\operatorname{div}_u v = 0, \\
|D| = 1, \\
(b, v, D)|_{t=0} = (a_0, u_0, d_0).
\end{gathered} \tag{5.9}$$

It is obvious that $b = a_0$. Let (a_i, u_i, P_i, d_i) be the two solutions with the same initial data which obtained by the first part of section 3 and satisfy (4.1) and (4.2). Let (v_i, π_i, D_i) be given by the (5.8). Then $(\delta v, \delta\pi, \delta D) = (v_2 - v_1, \pi_2 - \pi_1, D_2 - D_1)$ solves the system

$$\begin{gathered}
\partial_t \delta v - (1+a_0)(\Delta\delta v - \nabla\delta\pi + \operatorname{div}(\nu(a_0)\nabla\delta v)) = \sum_{j=1}^{5} \delta F_j + \sum_{j=1}^{6} \delta k_i, \\
\operatorname{div} \delta v = \delta g, \\
\partial_t \delta g = \operatorname{div}(\delta R), \\
\partial_t \delta D - \Delta\delta D = \sum_{i=1}^{6} \delta h_i, \\
(\delta v, \delta D)|_{t=0} = (0, 0),
\end{gathered} \tag{5.10}$$

where

$$\begin{gathered}
\delta F_1 = (1+a_0)\operatorname{div}[\delta A \cdot A_2^\top \cdot \nabla D_2 : A_2^\top \cdot \nabla D_2], \\
\delta F_2 = (1+a_0)\operatorname{div}[A_1 \cdot \delta A^\top \cdot \nabla D_2 : A_2^\top \cdot \nabla D_2], \\
\delta F_3 = (1+a_0)\operatorname{div}[A_1 \cdot A_1^\top \cdot \nabla \delta D : A_2^\top \cdot \nabla D_2], \\
\delta F_4 = (1+a_0)\operatorname{div}[A_1 \cdot A_1^\top \cdot \nabla D_1 : \delta A^\top \cdot \nabla D_2], \\
\delta F_5 = (1+a_0)\operatorname{div}[A_1 \cdot A_1^\top \cdot \nabla D_1 : A_1^\top \cdot \nabla \delta D], \\
\delta k_1 = (1+a_0)[(Id - A_2^\top)\nabla\delta\pi - \delta A^\top \nabla\pi_1], \\
\delta k_2 = (1+a_0)\operatorname{div}[(A_2 A_2^\top - Id)\nabla\delta v + (A_2 A_2^\top - A_1 A_1^\top)\nabla v_1], \\
\delta k_3 = (1+a_0)\operatorname{div}[\nu(a_0)(A_2 - Id)\nabla\delta v], \\
\delta k_4 = (1+a_0)\operatorname{div}[\nu(a_0)A_2(A2 - Id)^\top \cdot \nabla\delta D], \\
\delta k_5 = (1+a_0)\operatorname{div}[\nu(a_0)A_2 \delta A^\top v_1], \\
\delta k_6 = (1+a_0)\operatorname{div}[\nu(a_0)\delta A A_1 \cdot \nabla v_1], \\
\delta g = (Id - A_2) \cdot \nabla\delta v - \delta A \cdot \nabla v_1, \quad \delta R = \partial_t[(Id - A_2)\delta v] - \partial_t[\delta A \cdot \nabla v_1], \\
\delta h_1 = \delta A^\top \cdot \nabla D_2 A_2^\top \cdot \nabla D_2 D_2, \\
\delta h_2 = A_1^\top \cdot \nabla\delta D A_2^\top \cdot \nabla D_2 D_2, \\
\delta h_3 = A_1^\top \cdot \nabla D_1 \delta A^\top \cdot \nabla D_2 D_2, \quad \delta h_4 = A_1^\top \cdot \nabla D_1 A_1^\top \cdot \nabla\delta D D_2, \\
\delta h_5 = A_1^\top \cdot \nabla D_1 A_1^\top \cdot \nabla D_1 \delta D, \\
\delta h_6 = \operatorname{div}[(A_2 A_2^\top - Id)\nabla\delta D + (A_2 A_2^\top - A_1 A_1^\top)\nabla D_1].
\end{gathered}$$

Thanks to the estimates in [6], we have

$$\begin{gathered}
\|\partial_t \delta A\|_{\dot B^{3/2}_{2,1}} \lesssim \|v_i\|_{\dot B^{5/2}_{2,1}}, \quad \|\delta A\|_{L^\infty_t(\dot B^{3/2}_{2,1})} \lesssim \|\delta v\|_{L^1_t(\dot B^{5/2}_{2,1})}, \\
\|A_i - Id\|_{L^\infty_t(\dot B^{3/2}_{2,1})} \lesssim \|\delta v\|_{L^1_t(\dot B^{5/2}_{2,1})}, \\
\|\partial_t \delta A\|_{L^2_t(\dot B^{1/2}_{2,1})} \lesssim \|(v_1, v_2)\|_{L^1_t(\dot B^{3/2}_{2,1})} \|\delta v\|_{L^1_t(\dot B^{5/2}_{2,1})} + \|\delta v\|_{L^2_t(\dot B^{3/2}_{2,1})}.
\end{gathered} \tag{5.11}$$

Let

$$\delta G(t) = \|(\delta v, \nabla \delta D)\|_{L^\infty_t(B^{3/2})} + \|(\delta v, \nabla \delta D)\|_{L^1_t(B^{7/2})} + \|(\partial_t \delta v, \partial_t \nabla \delta D, \nabla \delta \pi)\|_{L^1_t(B^{3/2})}. \tag{5.12}$$

Then we obtain the following estimates by the Propositions 5.1 and 5.2,

$$\begin{aligned} \delta G(t) &\lesssim \int_0^t \|(\delta v, \nabla \delta D)\|_{B^{3/2}} \mathrm{d}\tau + \|(\delta F_i, \delta k_i, \nabla \delta h_i, \delta g, \delta R)\|_{L^1_t(B^{3/2})} \\ &\lesssim t\delta G(t) + \|(\delta F_i, \delta k_i, \nabla \delta h_i, \delta g, \delta R)\|_{L^1_t(B^{3/2})}. \end{aligned} \tag{5.13}$$

Now we handle $\delta F_1 - \delta F_5$ and $\delta h_1 - \delta h_5$. According to the product law in Besov Space, Lemma 2.4 and (5.11), we have

$$\begin{aligned} \|\delta F_1\|_{L^1_t(B^{3/2})} &\le C \int_0^t (1 + \|a_0\|_{\dot B^{3/q}_{q,1}}) \|\delta A\|_{B^{5/2}} \|A_2^\top \cdot \nabla D_2 : A_2^\top \cdot \nabla D_2\|_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\le C \|A_2^\top \cdot \nabla D_2 : A_2^\top \cdot \nabla D_2\|_{L^1_t(\dot B^{3/2}_{2,1})} \|\delta A\|_{L^\infty_t(B^{5/2})} \\ &\le C \|\delta v\|_{L^1_t(B^{7/2})} \int_0^t \|\nabla D_2\|^2_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\le C \delta G(t) \int_0^t \|\nabla D_2\|_{\dot B^{1/2}_{2,1}} \|\nabla D_2\|_{\dot B^{5/2}_{2,1}} \mathrm{d}\tau \\ &\le C (\|\nabla D_2\|_{L^\infty_t(\dot B^{1/2}_{2,1})} \|\nabla D_2\|_{L^1_t(\dot B^{5/2}_{2,1})}) \delta G(t), \end{aligned} \tag{5.14}$$

$$\begin{aligned} \|\delta F_2\|_{L^1_t(B^{3/2})} &\le C \int_0^t (1 + \|a_0\|_{\dot B^{3/q}_{q,1}}) \|A_1 \cdot \delta A^\top \cdot \nabla D_2 : A_2^\top \cdot \nabla D_2\|_{B^{5/2}} \mathrm{d}\tau \\ &\le C \|\delta A^\top\|_{L^\infty_t(B^{5/2})} \int_0^t \|\nabla D_2\|^2_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\le C (\|\nabla D_2\|_{L^\infty_t(\dot B^{1/2}_{2,1})} \|\nabla D_2\|_{L^1_t(\dot B^{5/2}_{2,1})}) \delta G(t) \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} \|\delta F_3\|_{L^1_t(B^{3/2})} &\le C \int_0^t (1 + \|a_0\|_{\dot B^{3/q}_{q,1}}) \|A_1 \cdot A_2^\top \cdot \nabla \delta D : A_2^\top \cdot \nabla D_2\|_{B^{5/2}} \mathrm{d}\tau \\ &\le C \int_0^t \|\delta D\|_{B^{5/2}} \|\nabla D_2\|_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\le C \int_0^t \|\delta D\|^{1/2}_{B^{3/2}} \|\delta D\|^{1/2}_{B^{7/2}} \|\nabla D_2\|_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\le C \int_0^t \|\delta D\|^{1/2}_{B^{3/2}} \|\delta D\|^{1/2}_{B^{7/2}} \|\nabla D_2\|^{1/2}_{\dot B^{1/2}_{2,1}} \|\nabla D_2\|^{1/2}_{\dot B^{5/2}_{2,1}} \mathrm{d}\tau \\ &\le C (\|\nabla D_2\|_{L^\infty_t(\dot B^{1/2}_{2,1})} \|\nabla D_2\|_{L^1_t(\dot B^{5/2}_{2,1})})^{1/2} \delta G(t). \end{aligned} \tag{5.16}$$

Similarity to the above estimates, we have

$$\|(\delta F_4, \delta F_5)\|_{L^1_t(B^{3/2})} \lesssim \delta G(t). \tag{5.17}$$

For $\delta h_1 - \delta h_5$, according to the product law in Besov Space, Lemma 2.4 and (5.11), we have

$$\begin{aligned} \|\delta h_1\|_{L^1_t(B^{5/2})} &\le C \int_0^t \|\delta A^\top \cdot \nabla D_2 A_2^\top \cdot \nabla D_2 D_2\|_{B^{5/2}} \mathrm{d}\tau \\ &\le C \|\delta A^\top\|_{L^\infty_t(B^{5/2})} \int_0^t \|\nabla D_2 A_2^\top \cdot \nabla D_2 D_2\|_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\le C \delta G(t) \int_0^t \|D_2\|^2_{\dot B^{5/2}_{2,1}} \|D_2\|_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\le C \delta G(t) \int_0^t \|D_2\|_{\dot B^{7/2}_{2,1}} \|D_2\|^2_{\dot B^{3/2}_{2,1}} \mathrm{d}\tau \\ &\lesssim \delta G(t), \end{aligned} \tag{5.18}$$

$$
\begin{aligned}
\|\delta h_2\|_{L^1_t(B^{5/2})} &\le C\int_0^t \|A_1^\top\cdot\nabla\delta D A_2^\top\cdot\nabla D_2 D_2\|_{B^{5/2}}\mathrm{d}\tau\\
&\le C\int_0^t \|\delta D\|_{B^{7/2}}\|\nabla D_2\|_{\dot B^{3/2}_{2,1}}\|D_2\|_{\dot B^{3/2}_{2,1}}\mathrm{d}\tau\\
&\le C\int_0^t \|\delta D\|^{1/2}_{B^{5/2}}\|\delta D\|^{1/2}_{B^{\frac{9}{2}}}\|D_2\|^{3/2}_{\dot B^{3/2}_{2,1}}\|D_2\|^{1/2}_{\dot B^{7/2}_{2,1}}\mathrm{d}\tau\\
&\lesssim \delta G(t)
\end{aligned}
\tag{5.19}
$$

and

$$
\begin{aligned}
\|\delta h_5\|_{L^1_t(B^{5/2})} &\le C\int_0^t \|A_1^\top\cdot\nabla D_1A_1^\top\cdot\nabla D_1\delta D\|_{B^{5/2}}\mathrm{d}\tau\\
&\le C\int_0^t \|A_1^\top\cdot\nabla D_1A_1^\top\cdot\nabla D_1\|_{\dot B^{3/2}_{2,1}}\|\delta D\|_{B^{5/2}}\mathrm{d}\tau\\
&\le C\int_0^t \|\nabla D_1\|^2_{\dot B^{3/2}_{2,1}}\|\delta D\|_{B^{5/2}}\mathrm{d}\tau\\
&\le C\delta G(t)\int_0^t \|D_1\|_{\dot B^{3/2}_{2,1}}\|D_1\|_{\dot B^{7/2}_{2,1}}\mathrm{d}\tau\\
&\lesssim \delta G(t).
\end{aligned}
\tag{5.20}
$$

Similarly, we obtain

$$
\|(\delta h_3,\delta h_4)\|_{L^1_t(B^{5/2})}\lesssim \delta G(t),
\tag{5.21}
$$

where we have used the estimates of (4.1), (4.2), $\|A_i\|_{L^\infty_t(\dot B^{3/2}_{2,1})}<+\infty$ and the fact that (v_i,D_i) share the same regularity with (u,d). Then thanks to the uniqueness part in [21], we obtain the rest estimates on the right side of (5.10) that

$$
\|(\delta k_i,\delta g,\delta R,\nabla\delta h_6)\|_{B^{3/2}}\lesssim (t+t^{1/2})\delta G(t).
$$

From which and the above estimates, we deduce that

$$
\delta G(t)\lesssim \phi(t)\delta G(t),
\tag{5.22}
$$

where $\phi(t)$ is a positive continuous function with $\phi(t)\to 0$, as $t\to 0$, which leads to the uniqueness.

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