

## OSCILLATION CRITERIA FOR FOURTH-ORDER NEUTRAL QUASILINEAR DIFFERENTIAL EQUATIONS IN THE NONCANONICAL CASE

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**ABSTRACT.** This article investigates the monotonic properties and oscillatory behavior of quasi-linear fourth-order neutral differential equations in the noncanonical case. By establishing new monotonic properties for positive solutions, we derive refined relationships between the solutions and their associated auxiliary functions. Based on these results, effective oscillation criteria are obtained to ensure the oscillation of all solutions. The results are supported by illustrative examples and extend and improve several existing works, and provide a useful approach for studying more general classes of higher-order nonlinear neutral differential equations.

### 1. INTRODUCTION

Functional Differential Equations (FDEs) provide a fundamental framework for modeling and predicting the behavior of dynamical systems influenced by both past and present states. They translate phenomena in physics, engineering, life sciences, and economics into quantitative relationships, incorporating time delays and memory effects to simulate real-world processes such as epidemic spread, control system stability, and signal propagation in networks. Among FDEs, neutral differential equations (NDEs) are particularly important, as the system's rate of change depends on both its past state and its past rate of change. This feature captures time-memory effects, introduces significant theoretical and practical complexity, and allows modeling of systems with advanced or lagging delays in mechanical, electrical, and biological contexts. NDEs also provide a basis for developing analytical tools to study oscillatory and convergent solution behavior (See [12, 17, 20]).

Theoretically, oscillation theory is one of the most important avenues of research in neutral differential equations. It aims to determine the necessary and sufficient conditions that ensure the solutions to the equation alternate their signs infinitely around the equilibrium point. The absence of oscillation often indicates a stable system, while its presence suggests instability or a desirable periodicity. Therefore, deriving effective oscillation parameters is not merely a theoretical matter; it is a crucial tool for researchers and engineers to predict system behavior, design control devices to prevent unwanted oscillations, or understand the natural cycles of the phenomena under study (See [9, 16, 21]).

In this research, we analyze the oscillatory behavior of solutions to a fourth-order quasi-linear neutral differential equation

$$(a(t)[z'''(t)]^\gamma)' + \kappa(t)u^\gamma(\delta(t)) = 0, \quad t \geq t_0, \quad (1.1)$$

where  $z(t) = u(t) + p(t)u(\psi(t))$ . We assume the following hypotheses:

- (H1)  $\gamma$  is the ratio of an odd positive integer, with  $0 < \gamma \leq 1$ ;
- (H2)  $p \in C^1([t_0, \infty))$ ,  $\kappa \in C([t_0, \infty))$ , with  $0 < p(t) < p_0 \leq 1$ , and  $\kappa(t) \geq 0$ ;
- (H3)  $\psi, \delta \in C^1([t_0, \infty))$ , with  $\psi(t) \leq t$ ,  $\delta(t) \leq t$ ,  $\delta'(t) \geq 0$ , and  $\lim_{t \rightarrow \infty} \psi(t) = \lim_{t \rightarrow \infty} \delta(t) = \infty$ .

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2020 *Mathematics Subject Classification.* 34C10, 34K11.

*Key words and phrases.* Oscillation criteria; functional differential equations; neutral equations; quasilinear equations; fourth order; noncanonical case.

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Submitted January 10, 2026. Published March 30, 2026.

(H4)  $a \in C^1([t_0, \infty))$ ,  $a(t) > 0$ ,  $a'(t) \geq 0$ , and equation (1.1) in noncanonical form, that is,

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\gamma}(\nu)} d\nu < \infty. \quad (1.2)$$

For convenience, we define

$$\phi_0(t) = \int_t^{\infty} \frac{1}{a^{1/\gamma}(\nu)} d\nu < \infty, \quad \phi_i(t) = \int_t^{\infty} \phi_{i-1}(\nu) d\nu < \infty, \quad i = 1, 2, \text{ for all } t \geq t_0.$$

A function  $u(t) \in C^3([t_u, \infty))$ , with  $t_u \geq t_0$ , is said to be a solution of (1.1) if  $a(t)(z'''(t))^\gamma \in C^1([t_u, \infty))$ , and (1.1) is satisfied for all  $u(t) \in [t_u, \infty)$ . In this study, we consider only those solutions  $u(t)$  of (1.1) that exist on some half-line  $[t_u, \infty)$  and satisfy

$$\sup\{|u(t)| : t \geq T\} > 0, \text{ for all } T \geq t_u.$$

A solution  $u$  is said to be oscillatory if, for every  $b > 0$  the function  $u$  has sign changes on the interval  $[b, \infty)$ ; otherwise, it is said to be non-oscillatory. Equation (1.1) is considered to be in *noncanonical form* if condition (1.2) holds. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory.

The study of oscillations in functional differential equations has seen significant development over the past few decades, with varying levels of interest depending on the order of the equation. Most studies have focused on first and second order equations, aiming to deduce the necessary and sufficient conditions for the oscillations of solutions in equations [14, 13, 10]. This was followed by interest in third-order equations [6, 4, 23]. Fourth-order equations have received less attention compared to lower orders, with studies exploring the derivation of advanced oscillation criteria and analyzing the non-intuitive asymptotic behaviors of solutions [26, 22, 3].

Below is an overview of key research findings published in previous years that have significantly contributed to the study of fourth-order differential equations.

For second-order differential equations, Baculíková [8] derived oscillation parameters for a non-canonical second-order differential equation

$$(a(t)u'(t))' + \kappa(t)u(\delta(t)) = 0,$$

with a delay parameter  $\delta(t)$ , employing iterative techniques to characterize the oscillatory behavior of solutions. Later, Alrashdi et al. [5] extended these results to second-order non-canonical quasi-linear equation

$$(a(t)[z'(t)]^\gamma)' + \kappa(t)u^\gamma(\delta(t)) = 0.$$

Regarding fourth-order equations, pioneering research has emerged, such as that of Jadowska et al. [18], who studied fourth-order linear delay differential equation

$$u''''(t) + \kappa(t)u(\delta(t)) = 0.$$

By refining the asymptotic properties of non-oscillatory solutions, they provided precise oscillation criteria. Grace et al. [15] and Zhang et al. [30] investigated fourth-order quasi-linear delay differential equation

$$(a(t)[u''''(t)]^\gamma)' + \kappa(t)u^\gamma(\delta(t)) = 0,$$

demonstrating how the delay coefficients and degree of nonlinearity influence the oscillatory behavior of solutions.

In the context of neutral differential equations, Parhi and Tripathy [27] and Thandapani and Savitri [29] studied fourth-order neutral differential equation

$$(a(t)(u(t) + p(t)u(\psi(t)))'''' + \kappa(t)u(\delta(t)) = 0.$$

Subsequently, Almutairi et al. [7] and Moaaz et al. [25] derived inequalities to improve the asymptotic and oscillatory properties of solutions, analyzing the more complex equation

$$(a(t)(u(t) + p(t)u(\psi(t)))'''' + \kappa(t)u(\delta(t)) = 0.$$

They established conditions ensuring the nonexistence of positive solutions, thereby proving that all solutions are oscillatory.

Building on this rich research legacy, this study occupies an advanced position, combining two key complexities: neutrality and quasi-linearity. It employs a foundational iterative technique to achieve two main objectives: first, to establish improved and precise relationships between the original solution  $u(t)$  and the associated function  $z(t)$ ; and second, to utilize these relationships in a sophisticated iterative procedure to derive new and efficient oscillation parameters. Notably, the results extend the previous findings and analytical techniques for second-order equations in [5, 8] to fourth-order neutral quasi-linear equations. Finally, the applied examples confirm the validity of the results and support the robustness of the derived criteria.

The remainder of this paper is organized as follows. Section 2 establishes fundamental lemmas forming the basis for the main results. Section 3 investigates the convergence and enhanced monotonicity of positive solutions. In Section 4, new criteria for the oscillation of all solutions of (1.1) are derived and illustrated through theorems. Section 5 provides examples demonstrating the applicability of the theoretical findings. Finally, Section 6 summarizes the main contributions and outlines potential directions for future research.

## 2. AUXILIARY RESULTS

In this section, we review and establish a set of fundamental limits that will form the basis for proving the main results of this study.

$$p_1(t; K) = \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left( \frac{1}{p(\psi^{[2r]}(t))} - 1 \right) \left( \frac{\psi^{[2r]}(t)}{t} \right)^{2/\epsilon}, \tag{2.1}$$

$$p_2(t; K) = \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - \frac{\phi_2(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r]}(t))} \right], \tag{2.2}$$

$$\hat{p}_2(t; K) = \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - \frac{\phi_2(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r]}(t))} \right] \frac{\phi_2^{k_m}(\psi^{[2r]}(t))}{\phi_2^{k_m}(t)}, \tag{2.3}$$

where  $\psi^{[0]}(t) = \psi(t)$  and  $\psi^{[L]}(t) = \psi(\psi^{[L-1]}(t))$ .

A central element in analyzing the asymptotic and oscillatory behavior of solutions to NDEs is the relationship between the solution  $u(t)$  and the associated function  $z(t)$ . In the canonical case for second-order and fourth-order equations, this relationship is typically expressed as

$$u(t) \geq (1 - p(t))z(t), \tag{2.4}$$

whereas in noncanonical cases with decreasing positive solutions, the effect of time delays must be taken into account. Specifically, when  $z'(t) > 0$ , the canonical form (2.4) remains valid. In contrast, if  $z'(t) < 0$ , the relationship is modified. For second-order NDEs, the inequality becomes

$$u(t) \geq \left( 1 - p(t) \frac{\phi_0(\psi(t))}{\phi_0(t)} \right) z(t), \tag{2.5}$$

whereas for fourth-order NDEs, it takes the form [2, 11, 24].

$$u(t) \geq \left( 1 - p(t) \frac{\phi_2(\psi(t))}{\phi_2(t)} \right) z(t). \tag{2.6}$$

**Lemma 2.1** ([19]). *Let  $\omega(t) \in C^m([t_0, \infty), (0, \infty))$ ,  $\omega^i(t) > 0$  for  $i = 1, 2, \dots, m$ , and  $\omega^{(m+1)}(t) \leq 0$ , eventually. Then,*

$$\frac{\omega(t)}{\omega'(t)} \geq \frac{\epsilon}{m}t, \quad \epsilon \in (0, 1).$$

**Lemma 2.2** ([1]). *Let  $f \in C^m([t_0, \infty), \mathbb{R}^+)$ . Suppose that  $f^m$  is of fixed sign and not identically zero, and that there exists  $t_1 \geq t_0$  such that  $f^{(m-1)}(t)f^{(m)}(t) \leq 0$  for all  $t_1 \geq t_0$ . If  $\lim_{t \rightarrow \infty} f(t) \neq 0$ , then, for every  $\epsilon \in (0, 1)$ , there exists  $t_\epsilon \geq t_1$  such that*

$$f(t) \geq \frac{\epsilon}{(m-2)!} t^{m-1} |f^{(m-1)}(t)| \quad \text{for all } t \geq t_\epsilon.$$

**Lemma 2.3** ([30]). *Suppose that  $u(t)$  is a positive solution to Equation (1.1). Then,  $a(t)(z'''(t))^\gamma$  is a decreasing function, and  $z(t)$  is eventually in one of the following cases:*

- (N1)  $z^{(r)}(t) > 0$  for  $r = 0, 1, 3$  and  $z^{(4)}(t) < 0$ ;
- (N2)  $z^{(r)}(t) > 0$  for  $r = 0, 1, 2$  and  $z'''(t) < 0$ ;
- (N3)  $(-1)^r z^{(r)}(t) > 0$  for  $r = 0, 1, 3$ .

**Notation 2.4** ([25]). *Suppose that  $u(t)$  is an eventually positive solution of (1.1). Then, it follows that, eventually,*

$$u(t) > \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{z(\psi^{[2r]}(t))}{p(\psi^{[2r]}(t))} - z(\psi^{[2r+1]}(t)) \right], \quad (2.7)$$

for all integer  $K \geq 0$ .

### 3. ASYMPTOTIC AND MONOTONIC PROPERTIES

This section examines the convergence characteristics and monotonicity of the positive solutions of the equation under study.

**Lemma 3.1.** *Assume that  $u(t)$  is a positive solution of (1.1) with  $z \in N_2$ . Then, eventually,*

- (A1-1)  $z(t) \geq \frac{\epsilon}{2} t z'(t)$ ,
- (A1-2)  $z(t) \geq \frac{\epsilon}{2} t^2 z''(t)$  for all  $\epsilon \in (0, 1)$ ,
- (A1-3)  $z''(t) \geq -a^{1/\gamma}(t) \phi_0(t) z'''(t)$ ,
- (A1-4)  $z''(t)/\phi_0(t)$  is increasing,
- (A1-5)  $u(t) \geq p_1(t; k) z(t)$ ,
- (A1-6)  $(a(t)(z'''(t))^\gamma)' \leq -\kappa(t) p_1^\gamma(\delta(t); K) z^\gamma(\delta(t))$ .

*Proof.* Assume that  $u(t)$  is a positive solution of (1.1) with  $z \in N_2$ . First, applying Lemma 2.1 with  $m = 2$  and  $\omega(t) = z(t)$ , we obtain

$$z(t) \geq \frac{\epsilon}{2} t z'(t). \quad (3.1)$$

Next, by Lemma 2.2 with  $m = 3$  and  $f(t) = z(t)$ , it follows that

$$z(t) \geq \frac{\epsilon}{2} t^2 z''(t), \quad \text{for all } \epsilon_0 \in (0, 1).$$

Because  $a^{1/\gamma}(t) z'''(t)$  is decreasing, we can write

$$z''(t) \geq - \int_t^\infty z'''(\nu) d\nu = - \int_t^\infty \frac{a^{1/\gamma}(\nu)}{a^{1/\gamma}(t)} z'''(\nu) d\nu \geq -a^{1/\gamma}(t) z'''(t) \int_t^\infty \frac{1}{a^{1/\gamma}(\nu)} d\nu,$$

which implies

$$z''(t) \geq -a^{1/\gamma}(t) z'''(t) \phi_0(t).$$

From this, we conclude

$$\left( \frac{z''}{\phi_0} \right)'(t) = \frac{a^{1/\gamma}(t) z'''(t) \phi_0(t) + z''(t)}{a^{1/\gamma}(t) \phi_0^2(t)} \geq 0.$$

Using Lemma 2.4 and the properties of solutions in  $N_2$ , we obtain  $z(\psi^{[2r]}(t)) \geq z(\psi^{[2r+1]}(t))$  for  $r = 1, 2, \dots$  which transforms inequality (2.7) into

$$u(t) > \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - 1 \right] z(\psi^{[2r]}(t)). \quad (3.2)$$

Moreover, from (3.1), we obtain

$$z(\psi^{[2r]}(t)) \geq \left( \frac{\psi^{[2r]}(t)}{t} \right)^{2/\epsilon} z(t),$$

so that

$$u(t) > \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - 1 \right] \left( \frac{\psi^{[2r]}(t)}{t} \right)^{2/\epsilon} z(t) = p_1(t; K)z(t). \tag{3.3}$$

Finally, applying equation (1.1) yields

$$(a(t)(z'''(t))^\gamma)' = -\kappa(t)u^\gamma(\delta(t)) \leq -\kappa(t)p_1^\gamma(\delta(t); K)z^\gamma(\delta(t)).$$

This completes the proof. □

**Remark 3.2.** Setting  $K = 0$  in  $p_1(t; K)$  defined in (3.3) yields  $p_2(t; 0) = 1 - p(t)$ , which coincides with the classical (2.4). Hence,  $p_1(t; K)$  represents an improved extension of the classical inequality, providing sharper bounds for  $K \geq 1$ .

**Lemma 3.3.** Let  $u(t)$  be a positive solution of (1.1) with  $z \in N_2$ , and assume that there exist constants  $\beta > 0$  and  $t_1 \geq t_0$  such that

$$\frac{1}{\gamma} 2^{-\gamma} a^{1/\gamma}(t) \phi_0^{1+\gamma}(t) \delta^{2\gamma}(t) \kappa(t) p_1^\gamma(\delta(t); K) \geq \beta. \tag{3.4}$$

Then, for  $t \geq t_1$ , the following properties hold:

- (A2-1)  $\lim_{t \rightarrow \infty} z''(t) = 0$ ,
- (A2-2)  $z''(t)/\phi_0^{\alpha_0}(t)$  is decreasing,
- (A2-3)  $\lim_{t \rightarrow \infty} z''(t)/\phi_0^{\alpha_0}(t) = 0$ ,
- (A2-4)  $z''(t)/\phi_0^{1-\alpha_0}(t)$  is increasing,

where  $\alpha_0 = \epsilon_0 \beta^{1/\gamma}$ ,  $\epsilon_0 \in (0, 1)$ , and  $\gamma \leq 1$ .

*Proof.* Assume that  $u(t)$  is a positive solution of (1.1) with  $z \in N_2$ . Suppose further that there exist constants  $\beta > 0$  and  $t_1 \geq t_0$  such that condition (3.4) is satisfied.

(A2-1) Since  $z \in N_2$ , all assertions (A1-6)–(A1-6) stated in Lemma 3.1 remain valid for sufficiently large  $t \geq t_1$ . In particular, the function  $z''(t)$  is positive and non-increasing, which implies

$$\lim_{t \rightarrow \infty} z''(t) = \ell_1 \geq 0.$$

We show that  $\ell_1 = 0$ . Assume, on the contrary, that  $\ell_1 > 0$ . Then there exists  $t$  sufficiently large such that  $z''(t) \geq \ell_1$ . Combining this fact with (A1-2), we obtain

$$z(t) \geq \frac{\epsilon}{2} t^2 z''(t) \geq \frac{\epsilon \ell_1}{2} t^2, \quad \epsilon \in (0, 1).$$

Substituting this estimate into (A1-6) yields

$$(a(t)(z'''(t))^\gamma)' \leq -\kappa(t)p_1^\gamma(\delta(t); K)z^\gamma(\delta(t)) \leq -\left(\frac{\epsilon \ell_1}{2} \delta^2(t)\right)^\gamma \kappa(t)p_1^\gamma(\delta(t); K).$$

Using (3.4), we deduce that

$$(a(t)(z'''(t))^\gamma)' \leq -\gamma \epsilon^\gamma \ell_1^\gamma \beta \frac{1}{a^{1/\gamma}(t) \phi_0^{1+\gamma}(t)} \leq -\gamma \ell_1^\gamma \alpha_0^\gamma \frac{1}{a^{1/\gamma}(t) \phi_0^{1+\gamma}(t)}.$$

Integrating this inequality from  $t_2$  to  $t$  leads to

$$\begin{aligned} a(t)(z'''(t))^\gamma &\leq a(t_2)(z'''(t_2))^\gamma - \gamma \ell_1^\gamma \alpha_0^\gamma \int_{t_2}^t \frac{1}{a^{1/\gamma}(\nu) \phi_0^{1+\gamma}(\nu)} d\nu \\ &\leq \ell_1^\gamma \alpha_0^\gamma \left[ \frac{1}{\phi_0^\gamma(t_2)} - \frac{1}{\phi_0^\gamma(t)} \right]. \end{aligned} \tag{3.5}$$

Since  $\phi_0^{-\gamma}(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , there exists  $t_3 \geq t_2$  such that

$$\phi_0^{-\gamma}(t) - \phi_0^{-\gamma}(t_2) \geq \mu_0 \phi_0^{-\gamma}(t), \quad \mu_0 \in (0, 1).$$

Hence, (3.5) becomes

$$z'''(t) \leq -\ell_1 \alpha_0 \mu_0^{1/\gamma} \frac{1}{a^{1/\gamma}(t) \phi_0(t)}, \quad t \geq t_3.$$

Integrating once more, we find that

$$\begin{aligned} z''(t) &\leq z''(t_3) - \ell_1 \alpha_0 \mu_0^{1/\gamma} \int_{t_3}^t \frac{1}{a^{1/\gamma}(\nu) \phi_0(\nu)} d\nu \\ &\leq z''(t_3) - \ell_1 \alpha_0 \mu_0^{1/\gamma} \ln \frac{\phi_0(t_3)}{\phi_0(t)} \rightarrow -\infty \quad \text{as } t \rightarrow \infty, \end{aligned}$$

which contradicts the positivity of  $z''(t)$ . Therefore,  $\ell_1 = 0$ .

(A2-2) Using (3.4), with (A1-2) and (A1-6), we obtain

$$(a(t)(z'''(t))^\gamma)' \leq -\frac{\gamma \alpha_0^\gamma}{a^{1/\gamma}(t) \phi_0^{1+\gamma}(t)} [z''(\delta(t))]^\gamma.$$

Integrating this inequality and using the fact that  $z'''(t) < 0$ , we derive that

$$\begin{aligned} a(t)(z'''(t))^\gamma &\leq a(t_1)(z'''(t_1))^\gamma - \gamma \alpha_0^\gamma (z''(\delta(t)))^\gamma \int_{t_1}^t \frac{1}{a^{1/\gamma}(\nu) \phi_0^{1+\gamma}(\nu)} d\nu \\ &\leq a(t_1)(z'''(t_1))^\gamma + \frac{\alpha_0^\gamma}{\phi_0^\gamma(t_1)} (z''(\delta(t)))^\gamma - \frac{\alpha_0^\gamma}{\phi_0^\gamma(t)} [z''(\delta(t))]^\gamma. \end{aligned}$$

Because  $z''(t) \rightarrow 0$  as  $t \rightarrow \infty$ , there is a  $t_2 \geq t_1$  such that

$$a(t_1)(z'''(t_1))^\gamma + \frac{\alpha_0^\gamma}{\phi_0^\gamma(t_1)} (z''(t))^\gamma \leq 0,$$

for  $t \geq t_2$ . Therefore we have

$$a(t)(z'''(t))^\gamma \leq -\frac{\alpha_0^\gamma}{\phi_0^\gamma(t)} (z''(t))^\gamma,$$

which implies

$$a^{1/\gamma}(t) z'''(t) \leq -\frac{\alpha_0}{\phi_0(t)} z''(t),$$

or equivalently,

$$a^{1/\gamma}(t) z'''(t) \phi_0(t) + \alpha_0 z''(t) \leq 0. \quad (3.6)$$

As a consequence,

$$\left( \frac{z''}{\phi_0^{\alpha_0}} \right)'(t) = \frac{a^{1/\gamma}(t) z'''(t) \phi_0(t) + \alpha_0 z''(t)}{a^{1/\gamma}(t) \phi_0^{1+\alpha_0}(t)} \leq 0.$$

(A2-3) Since the function  $z''(t)/\phi_0^{\alpha_0}(t)$  is positive and decreasing, there exists  $\ell_2 \geq 0$ , such that

$$\lim_{t \rightarrow \infty} \frac{z''(t)}{\phi_0^{\alpha_0}(t)} = \ell_2 \geq 0.$$

Assume, to the contrary, that  $\ell_2 > 0$ . Define the auxiliary function

$$\omega(t) = \frac{z''(t) + a^{1/\gamma}(t) z'''(t) \phi_0(t)}{\phi_0^{\alpha_0}(t)} > 0.$$

Differentiating  $\omega(t)$  yields

$$\begin{aligned} \omega'(t) &= \frac{z'''(t) + (a^{1/\gamma}(t) z'''(t))' \phi_0(t) - z'''(t)}{\phi_0^{\alpha_0}(t)} + \alpha_0 \frac{z''(t) + a^{1/\gamma}(t) z'''(t) \phi_0(t)}{a^{1/\gamma}(t) \phi_0^{1+\alpha_0}(t)} \\ &= \frac{(a^{1/\gamma}(t) z'''(t))'}{\phi_0^{\alpha_0-1}(t)} + \alpha_0 \frac{z''(t)}{a^{1/\gamma}(t) \phi_0^{1+\alpha_0}(t)} + \alpha_0 \frac{z'''(t)}{\phi_0^{\alpha_0}(t)}. \end{aligned}$$

Using the identity

$$(a^{1/\gamma}(t) z'''(t))' = (a(t)(z'''(t))^\gamma)' (a^{1/\gamma}(t) z'''(t))^{1-\gamma},$$

we obtain

$$\omega'(t) = \frac{1}{\gamma} \frac{(a(t)(z'''(t))^\gamma)' (a^{1/\gamma}(t) z'''(t))^{1-\gamma}}{\phi_0^{\alpha_0-1}(t)} + \alpha_0 \frac{z''(t)}{a^{1/\gamma}(t) \phi_0^{1+\alpha_0}(t)} + \alpha_0 \frac{z'''(t)}{\phi_0^{\alpha_0}(t)}.$$

Using (A1-2), (A1-6), and condition (3.4), we obtain

$$\begin{aligned} (a(t)(z'''(t))^\gamma)' &\leq -\kappa(t)p_1^\gamma(\delta(t); K)z^\gamma(\delta(t)) \\ &\leq -\kappa(t)p_1^\gamma(\delta(t); K)\left(\frac{\epsilon}{2}\delta^2(t)\right)^\gamma(z''(\delta(t)))^\gamma \\ &\leq -\gamma\alpha_0^\gamma\frac{1}{a^{1/\gamma}(t)\phi_0^{1+\gamma}(t)}(z''(\delta(t)))^\gamma. \end{aligned} \tag{3.7}$$

Moreover, from (3.6), it follows that

$$a^{1/\gamma}(t)z'''(t) \leq -\alpha_0\frac{z''(t)}{\phi_0(t)},$$

and

$$\left(a^{1/\gamma}(t)z'''(t)\right)^{1-\gamma} \geq \left(\alpha_0\frac{z''(t)}{\phi_0(t)}\right)^{1-\gamma}. \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$\omega'(t) \leq -\frac{\alpha_0^\gamma}{\phi_0^{\alpha_0-1}(t)}\frac{1}{a^{1/\gamma}(t)\phi_0^{1+\gamma}(t)}(z''(\delta(t)))^\gamma\left(\alpha_0\frac{z''(t)}{\phi_0(t)}\right)^{1-\gamma} + \alpha_0\frac{z''(t)}{a^{1/\gamma}(t)\phi_0^{1+\alpha_0}(t)} + \alpha_0\frac{z'''(t)}{\phi_0^{\alpha_0}(t)}.$$

Since  $z'''(t) < 0$  and  $\delta(t) \leq t$ , we have  $z''(\delta(t)) \geq z''(t)$ , and hence

$$\begin{aligned} \omega'(t) &\leq -\frac{\alpha_0^\gamma}{\phi_0^{\alpha_0-1}(t)}\frac{1}{a^{1/\gamma}(t)\phi_0^{1+\gamma}(t)}(z''(t))^\gamma\left(\alpha_0\frac{z''(t)}{\phi_0(t)}\right)^{1-\gamma} + \alpha_0\frac{z''(t)}{a^{1/\gamma}(t)\phi_0^{1+\alpha_0}(t)} + \alpha_0\frac{z'''(t)}{\phi_0^{\alpha_0}(t)} \\ &\leq -\alpha_0\frac{z''(t)}{a^{1/\gamma}(t)\phi_0^{1+\alpha_0}(t)} + \alpha_0\frac{z''(t)}{a^{1/\gamma}(t)\phi_0^{1+\alpha_0}(t)} + \alpha_0\frac{z'''(t)}{\phi_0^{\alpha_0}(t)} \\ &\leq \alpha_0\frac{z'''(t)}{\phi_0^{\alpha_0}(t)}. \end{aligned}$$

Using the fact that  $z''(t)/\phi_0^{\alpha_0}(t) \geq \ell_2$  together with (3.6), we obtain

$$\begin{aligned} \omega'(t) &\leq \alpha_0\frac{1}{\phi_0^{\alpha_0}(t)}z'''(t) \\ &\leq \alpha_0\frac{1}{\phi_0^{\alpha_0}(t)}\cdot\frac{-\alpha_0z''(t)}{a^{1/\gamma}(t)\phi_0(t)} \\ &\leq -\frac{z''(t)}{\phi_0^{\alpha_0}(t)}\frac{\alpha_0^2}{a^{1/\gamma}(t)\phi_0(t)} \\ &\leq \frac{-\alpha_0^2\ell_2}{a^{1/\gamma}(t)\phi_0(t)} < 0. \end{aligned}$$

Thus,  $\omega(t)$  is positive and strictly decreasing, and therefore converges to a finite non-negative limit. Integrating the above inequality from  $t_3$  to  $\infty$  yields

$$-\omega(t_3) \leq -\alpha_0^2\ell_2 \int_{t_3}^\infty \frac{1}{a^{1/\gamma}(\nu)\phi_0(\nu)}d\nu = -\alpha_0^2\ell_2 \lim_{t \rightarrow \infty} \ln \frac{\phi_0(t_3)}{\phi_0(t)},$$

which implies

$$\omega(t_3) \geq \alpha_0^2\ell_2 \lim_{t \rightarrow \infty} \ln \frac{\phi_0(t_3)}{\phi_0(t)} \rightarrow \infty,$$

a contradiction. Hence,  $\ell_2 = 0$ .

(A2-4) Finally, we have

$$\begin{aligned} [a^{1/\gamma}(t)z'''(t)\phi_0(t) + z''(t)]' &= \left(a^{1/\gamma}(t)z'''(t)\right)' \phi_0(t) - z'''(t) + z'''(t) \\ &= \left(a^{1/\gamma}(t)z'''(t)\right)' \phi_0(t) \\ &= \frac{1}{\gamma}(a(t)(z'''(t))^\gamma)'(a^{1/\gamma}(t)z'''(t))^{1-\gamma}\phi_0(t). \end{aligned}$$

Using (3.7) and (3.8), we deduce that

$$\begin{aligned} [a^{1/\gamma}(t)z'''(t)\phi_0(t) + z''(t)]' &\leq -\alpha_0^\gamma \frac{1}{a^{1/\gamma}(t)\phi_0^{1+\gamma}(t)} (z''(\delta(t)))^\gamma \left(\alpha_0 \frac{z''(t)}{\phi_0(t)}\right)^{1-\gamma} \phi_0(t) \\ &\leq -\alpha_0^\gamma \frac{1}{a^{1/\gamma}(t)\phi_0^\gamma(t)} (z''(\delta(t)))^\gamma \left(\alpha_0 \frac{z''(t)}{\phi_0(t)}\right)^{1-\gamma} \\ &\leq \frac{-\alpha_0}{a^{1/\gamma}(t)\phi_0(t)} z''(t). \end{aligned} \quad (3.9)$$

Integrating (3.9) from  $t$  to  $\infty$  gives

$$\begin{aligned} -a^{1/\gamma}(t)z'''(t)\phi_0(t) - z''(t) &\leq -\alpha_0 \int_t^\infty \frac{1}{a^{1/\gamma}(\nu)\phi_0(\nu)} z''(\nu) d\nu \\ &\leq -\alpha_0 \frac{z''(t)}{\phi_0(t)} \int_t^\infty \frac{1}{a^{1/\gamma}(\nu)} d\nu \leq -\alpha_0 z''(t), \end{aligned}$$

or equivalently,

$$a^{1/\gamma}(t)z'''(t)\phi_0(t) + (1 - \alpha_0)z''(t) \geq 0.$$

Consequently,

$$\left(\frac{z''}{\phi_0^{1-\alpha_0}}\right)'(t) = \frac{a^{1/\gamma}(t)z'''(t)\phi_0(t) + (1 - \alpha_0)z''(t)}{a^{1/\gamma}(t)\phi_0^{2-\alpha_0}(t)} \geq 0. \quad (3.10)$$

This completes the proof.  $\square$

When  $\alpha_0 \leq 1/2$ , the results established in Lemma 3.3 can be improved. These enhanced properties are summarized in the next lemma.

**Lemma 3.4.** *Let  $u(t)$  be a positive solution of (1.1) such that  $z \in N_2$ , and assume that condition (3.4) holds. Suppose further that*

$$\lim_{t \rightarrow \infty} \frac{\phi_0(\delta(t))}{\phi_0(t)} = \lambda < \infty. \quad (3.11)$$

Assume that there exists an increasing sequence  $\{\alpha_j\}_{j=1}^m$  defined by

$$\alpha_j = \alpha_0 \frac{\lambda^{\alpha_{j-1}}}{(1 - \alpha_{j-1})^{1/\gamma}},$$

where  $\gamma \leq 1$ ,  $\alpha_0 = \epsilon\beta^{1/\gamma}$ ,  $\alpha_{m-1} \leq 1/2$ , and  $\alpha_m, \epsilon \in (0, 1)$ . Then, eventually, the following properties hold:

(A3-1)  $z''(t)/\phi_0^{\alpha_m}(t)$  is decreasing,

(A3-2)  $\lim_{t \rightarrow \infty} z''(t)/\phi_0^{\alpha_m}(t) = 0$ .

*Proof.* Since  $z \in N_2$ , it follows that (A1-1)–(A1-5) of Lemma 3.1 hold for all  $t \geq t_1$ , with  $t_1$  large enough. Moreover, by Lemma 3.3, the properties (A2-1)–(A2-4) are also satisfied.

Assume now that  $\alpha_0 \leq 1/2$  and define

$$\alpha_1 = \alpha_0 \frac{\lambda^{\alpha_0}}{(1 - \alpha_0)^{1/\gamma}}.$$

We first establish (A3-1) and (A3-2) for the case  $m = 1$ . Proceeding as in the proof of Lemma 3.3, we obtain

$$(a(t)(z'''(t))^\gamma)' \leq -\gamma\alpha_0^\gamma \frac{1}{a^{1/\gamma}(t)\phi_0^{1+\gamma}(t)} (z''(\delta(t)))^\gamma.$$

Integrating this inequality over  $[t_1, t]$  and using the monotonicity property in (A2-2) together with (3.11), yields

$$\begin{aligned} a(t)(z'''(t))^\gamma &\leq a(t_1)(z'''(t_1))^\gamma - \gamma\alpha_0^\gamma \int_{t_1}^t \frac{1}{a^{1/\gamma}(\nu)\phi_0^{1+\gamma}(\nu)} (z''(\delta(\nu)))^\gamma d\nu \\ &\leq a(t_1)(z'''(t_1))^\gamma - \gamma\alpha_0^\gamma \int_{t_1}^t \frac{1}{a^{1/\gamma}(\nu)\phi_0^{1+\gamma}(\nu)} \phi_0^{\gamma\alpha_0}(\delta(\nu)) \left(\frac{z''(\nu)}{\phi_0^{\alpha_0}(\nu)}\right)^\gamma d\nu \end{aligned}$$

$$\begin{aligned} &\leq a(t_1)(z'''(t_1))^\gamma - \gamma\alpha_0^\gamma \left(\frac{z''(t)}{\phi_0^{\alpha_0}(t)}\right)^\gamma \int_{t_1}^t \frac{\phi_0^{-1-\gamma+\gamma\alpha_0}(\nu)}{a^{1/\gamma}(\nu)} d\nu \\ &\leq a(t_1)(z'''(t_1))^\gamma - \frac{\alpha_0^\gamma \lambda^{\gamma\alpha_0}}{1-\alpha_0} \left(\frac{z''(t)}{\phi_0^{\alpha_0}(t)}\right)^\gamma \left(\frac{1}{\phi_0^{\gamma(1-\alpha_0)}(t)} - \frac{1}{\phi_0^{\gamma(1-\alpha_0)}(t_1)}\right) \\ &\leq a(t_1)(z'''(t_1))^\gamma + \alpha_1^\gamma \frac{1}{\phi_0^{\gamma(1-\alpha_0)}(t_1)} \left(\frac{z''(t)}{\phi_0^{\alpha_0}(t)}\right)^\gamma - \alpha_1^\gamma \left(\frac{z''(t)}{\phi_0(t)}\right)^\gamma. \end{aligned}$$

Since  $z''(t)/\phi_0^{\alpha_0}(t) \rightarrow 0$  as  $t \rightarrow \infty$ , it follows that

$$a(t_1)(z'''(t_1))^\gamma + \alpha_1^\gamma \frac{1}{\phi_0^{\gamma(1-\alpha_0)}(t_1)} \left(\frac{z''(t)}{\phi_0^{\alpha_0}(t)}\right)^\gamma \leq 0.$$

Consequently,

$$a(t)(z'''(t))^\gamma \leq -\alpha_1^\gamma \left(\frac{z''(t)}{\phi_0(t)}\right)^\gamma,$$

which is equivalent to

$$a^{1/\gamma}(t)z'''(t)\phi_0(t) + \alpha_1 z''(t) \leq 0.$$

Hence,

$$\left(\frac{z''(t)}{\phi_0^{\alpha_1}(t)}\right)' = \frac{a^{1/\gamma}(t)z'''(t)\phi_0(t) + \alpha_1 z''(t)}{a^{1/\gamma}(t)\phi_0^{1+\alpha_1}(t)} \leq 0.$$

Applying the same reasoning as before, we deduce that

$$\lim_{t \rightarrow \infty} \frac{z''(t)}{\phi_0^{\alpha_1}(t)} = 0,$$

and

$$\left(\frac{z''(t)}{\phi_0^{1-\alpha_1}(t)}\right)' \geq 0.$$

By induction, for any with  $k = 2, 3, \dots, m$  with  $\alpha_{k-1} \leq \alpha_k \leq 1/2$ , one can similarly show that

$$a^{1/\gamma}(t)z'''(t)\phi_0(t) + \alpha_k z''(t) \leq 0, \tag{3.12}$$

and

$$\lim_{t \rightarrow \infty} \frac{z''(t)}{\phi_0^{\alpha_k}(t)} = 0.$$

This completes the proof. □

**Lemma 3.5.** *Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_3$ . Then, eventually,*

(A4-1)  $z(t)/\phi_2(t)$  is increasing,

(A4-2)  $(-1)^{i+1} z^{(2-i)}(t) \leq a^{1/\gamma}(t)z'''(t)\phi_i(t)$ , for  $i = 0, 1, 2$ .

*Proof.* Assume that  $u(t)$  is a positive solution of (1.1) with  $z \in N_3$ .

(A4-1) From (1.1), it follows that  $a^{1/\gamma}(t)z'''(t)$  is decreasing. Consequently,

$$a^{1/\gamma}(t)z'''(t) \int_t^\infty \frac{1}{a^{1/\gamma}(\nu)} d\nu \geq \int_t^\infty \frac{1}{a^{1/\gamma}(\nu)} a^{1/\gamma}(\nu)z'''(\nu) d\nu = \lim_{t \rightarrow \infty} z''(t) - z''(t). \tag{3.13}$$

Since  $z''(t)$  is positive and decreasing, it converges to a finite nonnegative constant as  $t \rightarrow \infty$ . Hence, (3.13) yields

$$a^{1/\gamma}(t)z'''(t)\phi_0(t) \geq -z''(t).$$

Therefore,

$$\left(\frac{z''(t)}{\phi_0(t)}\right)' = \frac{a^{1/\gamma}(t)z'''(t)\phi_0(t) + z''(t)}{a^{1/\gamma}(t)\phi_0^2(t)} \geq 0.$$

Integrating, we obtain

$$-z'(t) \geq \int_t^\infty \frac{z''(\nu)}{\phi_0(\nu)} \phi_0(\nu) d\nu - \geq \frac{z''(t)}{\phi_0(t)} \int_t^\infty \phi_0(\nu) d\nu \geq \frac{z''(t)}{\phi_0(t)} \phi_1(t).$$

Thus,

$$\left(\frac{z'}{\phi_1}\right)'(t) = \frac{\phi_1(t)z''(t) + z'(t)\phi_0(t)}{\phi_1^2(t)} \leq 0.$$

Furthermore,

$$-z'(t) \geq \int_t^\infty \frac{z'(\nu)}{\phi_1(\nu)} \phi_1(\nu) d\nu \geq \frac{z'(t)}{\phi_1(t)} \int_t^\infty \phi_1(\nu) d\nu \geq \frac{z'(t)}{\phi_1(t)} \phi_2(t),$$

which implies

$$\left(\frac{z(t)}{\phi_2(t)}\right)' = \frac{\phi_2(t)z'(t) + z(t)\phi_1(t)}{\phi_2^2(t)} \geq 0.$$

(A4-2) Since  $z(t) \in N_3$ , it follows that

$$a^{1/\gamma}(t)z'''(t)\phi_0(t) \geq \int_t^\infty \frac{a^{1/\gamma}(\nu)z'''(\nu)}{a^{1/\gamma}(\nu)} d\nu \geq -z''(t),$$

or equivalently,

$$z''(t) \geq -a^{1/\gamma}(t)z'''(t)\phi_0(t).$$

Integrating from  $t$  to  $\infty$ , we obtain

$$-z'(t) \geq -a^{1/\gamma}(t)z'''(t)\phi_1(t),$$

which yields

$$z'(t) \leq a^{1/\gamma}(t)z'''(t)\phi_1(t).$$

Integrating once more from  $t$  to  $\infty$ , we have

$$-z(t) \leq a^{1/\gamma}(t)z'''(t)\phi_2(t),$$

or equivalently,

$$z(t) \geq -a^{1/\gamma}(t)z'''(t)\phi_2(t).$$

Therefore, for  $i = 0, 1, 2$ , we conclude that

$$(-1)^{i+1}z^{(2-i)}(t) \leq a^{1/\gamma}(t)z'''(t)\phi_i(t).$$

This completes the proof.  $\square$

**Lemma 3.6.** *Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_3$ . Then, eventually,*

$$(A5-1) \quad u(t) > p_2(t; K)z(t),$$

$$(A5-2) \quad (a(t)(z'''(t))^\gamma)' \leq -\kappa(t)p_2^\gamma(\delta(t); K)z^\gamma(\delta(t)).$$

*Proof.* Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_3$ .

(A5-1) By Lemma 2.4, condition (2.7) holds. Moreover, from (A4-1), the function  $z(t)/\phi_2(t)$  is non-increasing. Hence,

$$\frac{z(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r+1]}(t))} \leq \frac{z(\psi^{[2r]}(t))}{\phi_2(\psi^{[2r]}(t))},$$

which implies

$$z(\psi^{[2r+1]}(t)) \leq \frac{\phi_2(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r]}(t))} z(\psi^{[2r]}(t)).$$

Combining this with (2.7), we obtain

$$u(t) > \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - \frac{\phi_2(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r]}(t))} \right] z(\psi^{[2r]}(t)). \quad (3.14)$$

Since  $z(t)$  is decreasing, inequality (3.14) yields

$$u(t) > \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - \frac{\phi_2(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r]}(t))} \right] z(t) = p_2(t; K)z(t). \quad (3.15)$$

(A5-2) Substituting the above estimate into (1.1), we obtain

$$(a(t)(z'''(t))^\gamma)' = -\kappa(t)u^\gamma(\delta(t)) \leq -\kappa(t)p_2^\gamma(\delta(t); K)z^\gamma(\delta(t)).$$

This completes the proof. □

**Remark 3.7.** Setting  $K = 0$  in  $p_2(t; K)$  defined in (3.15) yields

$$p_2(t; 0) = 1 - p(t) \frac{\phi_2(\psi(t))}{\phi_2(t)},$$

which coincides with the classical (2.6). Hence,  $p_2(t; K)$  represents an improved extension of the classical inequality, providing sharper bounds for  $K \geq 1$ .

**Lemma 3.8.** *Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_3$ . Assume that*

$$\int_{t_0}^{\infty} \left( \frac{1}{a(\omega)} \int_{t_0}^{\omega} \kappa(\nu) p_2^\gamma(\delta(\nu); K) d\nu \right)^{1/\gamma} d\omega = \infty, \tag{3.16}$$

and that there exists a constant  $k_0 \in (0, 1)$  such that

$$\frac{1}{\gamma} \phi_2^{\gamma+1}(t) \phi_1^{-1}(t) \kappa(t) p_2^\gamma(\delta(t); K) \geq k_0^\gamma. \tag{3.17}$$

Then the following assertions hold:

- (A6-1)  $\lim_{t \rightarrow \infty} z(t) = 0$ ,
- (A6-2)  $z(t)/\phi_2^{k_0}(t)$  is decreasing,
- (A6-3)  $\lim_{t \rightarrow \infty} z(t)/\phi_2^{k_0}(t) = 0$ .

*Proof.* Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_3$ .

(A6-1) Since  $z(t)$  is positive and decreasing, then

$$\lim_{t \rightarrow \infty} z(t) = \ell_3 \geq 0.$$

Assume, to the contrary, that  $\ell_3 > 0$ . Then there exists  $t_2 \geq t_1$  such that  $z(t) \geq \ell_3$ ,  $t \geq t_2$ . From (A5-2), we obtain

$$(a(t)(z'''(t))^\gamma)' \leq -\kappa(t) p_2^\gamma(\delta(t); K) z^\gamma(\delta(t)) \leq -\ell_3^\gamma \kappa(t) p_2^\gamma(\delta(t); K).$$

Integrating from  $t_2$  to  $t$ , we obtain

$$a(t)(z'''(t))^\gamma \leq a(t_2)(z'''(t_2))^\gamma - \ell_3^\gamma \int_{t_2}^t \kappa(\nu) p_2^\gamma(\delta(\nu); K) d\nu.$$

Using case (N3), since  $z'''(t) < 0$  for  $t \geq t_1$ , it follows that  $a(t_2)(z'''(t_2))^\gamma < 0$ , and hence

$$z'''(t) \leq \frac{-\ell_3}{a^{1/\gamma}(t)} \int_{t_2}^t \kappa(\nu) p_2^\gamma(\delta(\nu); K) d\nu.$$

Integrating again from  $t_2$  to  $t$ , we obtain

$$z''(t) \leq z''(t_2) - \ell_3 \int_{t_2}^t \left( \frac{1}{a(\omega)} \int_{t_2}^{\omega} \kappa(\nu) p_2^\gamma(\delta(\nu); K) d\nu \right)^{1/\gamma} d\omega \rightarrow -\infty \text{ as } t \rightarrow \infty.$$

Condition (3.16) implies that the right-hand side tends to  $-\infty$ , as  $t \rightarrow \infty$ , which contradicts the positivity of  $z''(t)$ . Therefore,  $\ell_3 = 0$ , that is  $\lim_{t \rightarrow \infty} z(t) = 0$ .

(A6-2) Integrating (A5-2) from  $t_2$  to  $t$  and using (3.17), we obtain

$$\begin{aligned} a(t)(z'''(t))^\gamma &\leq a(t_2)(z'''(t_2))^\gamma - \int_{t_2}^t \kappa(\nu) p_2^\gamma(\delta(\nu); K) z^\gamma(\delta(\nu)) d\nu \\ &\leq a(t_2)(z'''(t_2))^\gamma - z^\gamma(t) \int_{t_2}^t \gamma k_0^\gamma \frac{\phi_1(\nu)}{\phi_2^{\gamma+1}(\nu)} d\nu \\ &\leq a(t_2)(z'''(t_2))^\gamma - \gamma k_0^\gamma z^\gamma(t) \int_{t_2}^t \frac{\phi_1(\nu)}{\phi_2^{\gamma+1}(\nu)} d\nu \\ &\leq a(t_2)(z'''(t_2))^\gamma - k_0^\gamma \frac{z^\gamma(t)}{\phi_2^\gamma(t)} + k_0^\gamma \frac{z^\gamma(t)}{\phi_2^\gamma(t_2)}, \end{aligned}$$

which, with (A6-1), gives

$$a(t)(z'''(t))^\gamma \leq -k_0^\gamma \frac{z^\gamma(t)}{\phi_2^\gamma(t)},$$

or equivalently,

$$a^{1/\gamma}(t)z'''(t) \leq -k_0 \frac{z(t)}{\phi_2(t)}. \quad (3.18)$$

Using (A4-2) for  $i = 1$ , it follows that

$$\frac{z'(t)}{\phi_1(t)} \leq -k_0 \frac{z(t)}{\phi_2(t)},$$

or equivalently,

$$\phi_2(t)z'(t) + k_0\phi_1(t)z(t) \leq 0. \quad (3.19)$$

Hence,

$$\left(\frac{z(t)}{\phi_2^{k_0}(t)}\right)' = \frac{z'(t)\phi_2(t) + k_0z(t)\phi_1(t)}{\phi_2^{k_0+1}(t)} \leq 0.$$

(A6-3) Since  $z(t)/\phi_2^{k_0}(t)$  is positive and decreasing, there exists  $\ell_4$  such that

$$\lim_{t \rightarrow \infty} z(t)/\phi_2^{k_0}(t) = \ell_4 \geq 0.$$

Assume the contrary, that  $\ell_4 > 0$ . Then there exists  $t_2 \geq t_1$  such that

$$\frac{z(t)}{\phi_2^{k_0}(t)} \geq \ell_4, \quad t \geq t_2.$$

We define

$$\Omega(t) = \frac{z(t) + a^{1/\gamma}(t)z'''(t)\phi_2(t)}{\phi_2^{k_0}(t)}.$$

By (A4-2), we have  $\Omega(t) \geq 0$  for  $t \geq t_2$ . Differentiating  $\Omega(t)$  and using (A4-2) and (A5-2), we obtain

$$\begin{aligned} \Omega'(t) &= \frac{1}{\phi_2^{2k_0}(t)} \left[ \phi_2^{k_0}(t)(z'(t) - a^{1/\gamma}(t)z'''(t)\phi_1(t) + (a^{1/\gamma}(t)z'''(t))'\phi_2(t)) \right. \\ &\quad \left. + k_0\phi_2^{k_0-1}(t)\phi_1(t)(z(t) + a^{1/\gamma}(t)z'''(t)\phi_2(t)) \right] \\ &\leq \frac{1}{\phi_2^{2k_0}(t)} \left[ (a^{1/\gamma}(t)z'''(t))'\phi_2^2(t) + k_0\phi_1(t)(z(t) + a^{1/\gamma}(t)z'''(t)\phi_2(t)) \right] \\ &\leq \frac{1}{\phi_2^{2k_0}(t)} \left[ \frac{1}{\gamma}(a(t)(z'''(t))^\gamma)'(a^{1/\gamma}(t)z'''(t))^{1-\gamma}\phi_2^2(t) \right. \\ &\quad \left. + k_0\phi_1(t)(z(t) + a^{1/\gamma}(t)z'''(t)\phi_2(t)) \right] \\ &\leq \left( \frac{-1}{\gamma} \kappa(t)p_2^\gamma(\delta(t); K)z^\gamma(\delta(t))(a^{1/\gamma}(t)z'''(t))^{1-\gamma}\phi_2^2(t) \right. \\ &\quad \left. + k_0\phi_1(t)z(t) + k_0\phi_1(t)a^{1/\gamma}(t)z'''(t)\phi_2(t) \right) / (\phi_2^{2k_0}(t)). \end{aligned}$$

Since  $\gamma \leq 1$ ,  $z'''(t) \leq 0$ , and from

$$a^{1/\gamma}(t)z'''(t) \leq -k_0 \frac{z(t)}{\phi_2(t)},$$

we obtain

$$\left(a^{1/\gamma}(t)z'''(t)\right)^{1-\gamma} \geq \left(k_0 \frac{z(t)}{\phi_2(t)}\right)^{1-\gamma}.$$

Then,

$$\begin{aligned} \Omega'(t) &\leq \frac{1}{\phi_2^{2k_0}(t)} \left[ \frac{-1}{\gamma} \kappa(t)p_2^\gamma(\delta(t); K)z^\gamma(t) \left(k_0 \frac{z(t)}{\phi_2(t)}\right)^{1-\gamma}\phi_2^2(t) + k_0\phi_1(t)z(t) \right. \\ &\quad \left. + k_0\phi_1(t)a^{1/\gamma}(t)z'''(t)\phi_2(t) \right] \end{aligned}$$

$$\leq \frac{\frac{-k_0^{1-\gamma}}{\gamma} \kappa(t) p_2^\gamma(\delta(t); K) z^\gamma(t) \phi_2^{\gamma+1}(t) + k_0 \phi_1(t) z(t) + k_0 \phi_1(t) a^{1/\gamma}(t) z'''(t) \phi_2(t)}{\phi_2^{2k_0}(t)}.$$

Using (3.17), we have

$$\begin{aligned} \Omega'(t) &\leq \frac{1}{\phi_2^{2k_0}(t)} [-k_0 \phi_1(t) z(t) + k_0 \phi_1(t) z(t) + k_0 \phi_1(t) a^{1/\gamma}(t) z'''(t) \phi_2(t)] \\ &\leq \frac{1}{\phi_2^{k_0}(t)} k_0 \phi_1(t) a^{1/\gamma}(t) z'''(t). \end{aligned} \tag{3.20}$$

Using that  $z(t)/\phi_2^{k_0}(t) \geq \ell_4$  and (3.18), we have

$$a^{1/\gamma}(t) z'''(t) \leq -k_0 \frac{z(t)}{\phi_2(t)} \leq -k_0 \ell_4 \phi_2^{k_0-1}(t). \tag{3.21}$$

Combining (3.20) and (3.21), it follows that

$$\Omega'(t) \leq \frac{-k_0 \phi_1(t) k_0 \ell_4 \phi_2^{k_0-1}(t)}{\phi_2^{k_0}(t)} \leq -k_0^2 \ell_4 \frac{\phi_1(t)}{\phi_2(t)} < 0.$$

Integrating from  $t_2$  to  $t$ , we have

$$\Omega(t_2) \geq k_0^2 \ell_4 \ln \frac{\phi_2(t_2)}{\phi_2(t)} \rightarrow \infty \quad \text{as } t \rightarrow \infty,$$

which is a contradiction because  $\Omega(t_2)$  is finite. Therefore,  $\ell_4 = 0$ . Hence,

$$\lim_{t \rightarrow \infty} \frac{z(t)}{\phi_2^{k_0}(t)} = 0,$$

and the proof is complete. □

**Lemma 3.9.** *Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_2$ , and suppose that conditions (3.16) and (3.17) hold for some  $k_0 \in (0, 1)$ . Assume further that*

$$k_{i-1} \leq k_i < 1, \quad i = 1, 2, \dots, m - 1.$$

*Then the following assertions are valid:*

(A7-1-m)  $z(t)/\phi_2^{k_m}(t)$  is decreasing,

(A7-2-m)  $\lim_{t \rightarrow \infty} z(t)/\phi_2^{k_m}(t) = 0$ ,

where the sequence  $\{k_j\}_{j=0}^m$  is defined by

$$k_j = k_0 \frac{\lambda_1^{k_{j-1}}}{(1 - k_{j-1})^{1/\gamma}}, \quad j = 1, 2, \dots, m, \tag{3.22}$$

and

$$\frac{\phi_2(\delta(t))}{\phi_2(t)} \geq \lambda_1, \quad \text{for all } t \geq t_1, \tag{3.23}$$

for some  $\lambda \geq 1$ .

*Proof.* Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_2$ . By Lemma 3.8, (A6-1)–(A6-3) are satisfied. Moreover, by an inductive argument combined with Lemma 3.9, we conclude that (A7-1-0)–(A7-2-0) hold.

Assume now that (A7-1-(m-1))–(A7-2-(m-1)) are valid for some  $m \geq 1$ . Integrating inequality (A5-2) from  $t_2$  to  $t$ , we find that

$$a(t)(z'''(t))^\gamma \leq a(t_2)(z'''(t_2))^\gamma - \int_{t_2}^t \kappa(\nu) p_2^\gamma(\delta(\nu); K) z^\gamma(\delta(\nu)) d\nu. \tag{3.24}$$

From (A7-1-(m-1)), we have

$$z(\delta(t)) \geq \phi_2^{k_{m-1}}(\delta(t)) \frac{z(t)}{\phi_2^{k_{m-1}}(t)}.$$

Substituting this into (3.24), we arrive at

$$a(t)(z'''(t))^\gamma \leq a(t_2)(z'''(t_2))^\gamma - \int_{t_2}^t \kappa(\nu)p_2^\gamma(\delta(\nu); K)\phi_2^{\gamma k_{m-1}}(\delta(\nu))\frac{z^\gamma(\nu)}{\phi_2^{\gamma k_{m-1}}(\nu)}d\nu.$$

Since  $z(t)/\phi_2^{k_{m-1}}(t)$  is decreasing, it follows that

$$a(t)(z'''(t))^\gamma \leq a(t_2)(z'''(t_2))^\gamma - \frac{z^\gamma(t)}{\phi_2^{\gamma k_{m-1}}(t)} \int_{t_2}^t \kappa(\nu)p_2^\gamma(\delta(\nu); K)\phi_2^{\gamma k_{m-1}}(\nu)\frac{\phi_2^{\gamma k_{m-1}}(\delta(\nu))}{\phi_2^{\gamma k_{m-1}}(\nu)}d\nu.$$

using conditions (3.17) and (3.23), we deduce that

$$\begin{aligned} a(t)(z'''(t))^\gamma &\leq a(t_2)(z'''(t_2))^\gamma - \lambda_1^{\gamma k_{m-1}} \frac{z^\gamma(t)}{\phi_2^{\gamma k_{m-1}}(t)} \int_{t_2}^t \kappa(\nu)p_2^\gamma(\delta(\nu); K)\phi_2^{\gamma k_{m-1}}(\nu)d\nu \\ &\leq a(t_2)(z'''(t_2))^\gamma - \gamma k_0^\gamma \lambda_1^{\gamma k_{m-1}} \frac{z^\gamma(t)}{\phi_2^{\gamma k_{m-1}}(t)} \int_{t_2}^t \frac{\phi_1(\nu)}{\phi_2^{\gamma(1-k_{m-1})+1}(\nu)}d\nu \\ &\leq a(t_2)(z'''(t_2))^\gamma - k_0^\gamma \frac{\lambda_1^{\gamma k_{m-1}}}{1-k_{m-1}} \frac{z^\gamma(t)}{\phi_2^{\gamma k_{m-1}}(t)} \left[ \frac{1}{\phi_2^{\gamma(1-k_{m-1})}(t)} - \frac{1}{\phi_2^{\gamma(1-k_{m-1})}(t_2)} \right] \\ &\leq a(t_2)(z'''(t_2))^\gamma + k_m^\gamma \frac{z^\gamma(t)}{\phi_2^{\gamma k_{m-1}}(t)} \frac{1}{\phi_2^{\gamma(1-k_{m-1})}(t_2)} - k_m^\gamma \frac{z^\gamma(t)}{\phi_2^\gamma(t)}. \end{aligned}$$

Since  $\lim_{t \rightarrow \infty} z(t)/\phi_2^{k_{m-1}}(t) = 0$ , we finally obtain

$$a(t)(z'''(t))^\gamma \leq -k_m^\gamma \frac{z^\gamma(t)}{\phi_2^\gamma(t)},$$

or equivalently,

$$a^{1/\gamma}(t)z'''(t) \leq -k_m \frac{z(t)}{\phi_2(t)}. \quad (3.25)$$

Applying (A4-2) with  $i = 1$ , we obtain

$$\frac{z'(t)}{\phi_1(t)} \leq -k_m \frac{z(t)}{\phi_2(t)},$$

which is equivalent to

$$z'(t)\phi_2(t) + k_m z(t)\phi_1(t) \leq 0. \quad (3.26)$$

Consequently,

$$\left( \frac{z}{\phi_2^{k_m}} \right)'(t) = \frac{z'(t)\phi_2(t) + k_m z(t)\phi_1(t)}{\phi_2^{k_m+1}(t)} \leq 0.$$

Thus,  $z(t)/\phi_2^{k_m}(t) = 0$  is decreasing. By repeating the argument used to establish (A6-2) in Lemma 3.8, we conclude that

$$\lim_{t \rightarrow \infty} \frac{z(t)}{\phi_2^{k_m}(t)} = 0.$$

This completes the proof.  $\square$

**Lemma 3.10.** *Let  $u(t)$  be an eventually positive solution of (1.1) such that  $z \in N_2$ . Then*

$$u(t) > \widehat{p}_2(t; K)z(t).$$

*Proof.* By applying the same approach as in the proof of Lemma 3.6, we obtain (3.14). Moreover, condition (A7-1-m) implies that

$$\frac{z(\psi^{[2r]}(t))}{\phi_2^{k_m}(\psi^{[2r]}(t))} \leq \frac{z(t)}{\phi_2^{k_m}(t)},$$

which yields

$$z(\psi^{[2r]}(t)) \leq \frac{\phi_2^{k_m}(\psi^{[2r]}(t))}{\phi_2^{k_m}(t)} z(t).$$

Combining this with (3.14), we deduce that

$$\begin{aligned} u(t) &> \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - \frac{\phi_2(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r]}(t))} \right] z(\psi^{[2r]}(t)) \\ &> \sum_{r=0}^K \left( \prod_{L=0}^{2r} p(\psi^{[L]}(t)) \right) \left[ \frac{1}{p(\psi^{[2r]}(t))} - \frac{\phi_2(\psi^{[2r+1]}(t))}{\phi_2(\psi^{[2r]}(t))} \right] \frac{\phi_2^{k_m}(\psi^{[2r]}(t))}{\phi_2^{k_m}(t)} z(t) \\ &= \widehat{p}_2(t; K)z(t). \end{aligned}$$

This completes the proof. □

#### 4. OSCILLATION CRITERIA

In the subsequent theorems, we derive criteria the oscillation of all solutions in (1.1).

**Theorem 4.1.** *Let conditions (3.4), (3.16) and (3.17) hold, and suppose that*

$$\alpha_0 > 1/2. \tag{4.1}$$

*If there exists a positive integer  $m$  such that*

$$\liminf_{t \rightarrow \infty} \int_{\delta(t)}^t \phi_2(\nu) \phi_2^{\gamma-1}(\delta(\nu)) \kappa(\nu) p_2^\gamma(\delta(\nu); K) d\nu > \frac{\gamma k_m^{\gamma-1} (1 - k_m)}{e}, \tag{4.2}$$

$$\liminf_{t \rightarrow \infty} \int_{\delta(t)}^t \kappa(\nu) (1 - p(\delta(\nu)))^\gamma \frac{(\delta^3(\nu))^\gamma}{a(\delta(\nu))} d\nu > \frac{6^\gamma}{\epsilon^\gamma e}, \tag{4.3}$$

*then, (1.1) is oscillatory.*

*Proof.* Suppose, on the contrary, that (1.1) has an eventually positive solution  $u(t)$  satisfying

$$u(t) > 0, \quad u(\psi(t)) > 0, \quad u(\delta(t)) > 0 \quad \text{for } t \geq t_1 \geq t_0.$$

By Lemma 2.2, the associated function  $z(t)$  belongs to one the cases (N1)–(N3).

Case (N2). By Lemma 3.3, the function  $z''(t)/\phi_0^{\alpha_0}(t)$  is decreasing for all  $t \geq t_1$ , while  $z''(t)/\phi_0^{1-\alpha_0}(t)$  is increasing for all  $t \geq t_1$ . Consequently,  $\alpha_0 \leq 1/2$ , which contradicts assumption (4.1).

Case (N3). By Lemma 3.9, both conditions (A7-1-m) and (A7-2-m) hold. We define the auxiliary function

$$\omega(t) = a^{1/\gamma}(t)z'''(t)\phi_2(t) + z(t).$$

From condition (A4-2) with  $i = 0$ , we have  $\omega(t) \geq 0$  for  $t \geq t_2$ . Moreover, (3.25) yields

$$a^{1/\gamma}(t)z'''(t)\phi_2(t) \leq -k_m z(t).$$

Hence,

$$\begin{aligned} \omega(t) &= a^{1/\gamma}(t)z'''(t)\phi_2(t) + k_m z(t) - k_m z(t) + z(t) \\ &\leq -k_m z(t) + z(t) \leq (1 - k_m)z(t). \end{aligned} \tag{4.4}$$

Then

$$\omega'(t) = (a^{1/\gamma}(t)z'''(t))'\phi_2(t) - a^{1/\gamma}(t)z'''(t)\phi_1(t) + z'(t).$$

Using condition (A4-2), with  $i = 1$ , it follows that

$$\omega'(t) \leq (a^{1/\gamma}(t)z'''(t))'\phi_2(t) = \frac{1}{\gamma} (a(t)(z'''(t))^\gamma)' \left( a^{1/\gamma}(t)z'''(t) \right)^{1-\gamma} \phi_2(t).$$

Applying conditions (A5-2) and (3.25), we arrive at

$$\begin{aligned} \omega'(t) &\leq \frac{-1}{\gamma} \kappa(t) p_2^\gamma(\delta(t); K) z^\gamma(\delta(t)) (a^{1/\gamma}(t)z'''(t))^{1-\gamma} \phi_2(t) \\ &\leq \frac{-1}{\gamma} \kappa(t) p_2^\gamma(\delta(t); K) z^\gamma(\delta(t)) \left( -k_m \frac{z(t)}{\phi_2(t)} \right)^{1-\gamma} \phi_2(t) \\ &\leq \frac{-k_m^{1-\gamma}}{\gamma} \kappa(t) p_2^\gamma(\delta(t); K) z^\gamma(\delta(t)) \left( \frac{z(t)}{\phi_2(t)} \right)^{1-\gamma} \phi_2(t). \end{aligned}$$

By condition (A4-1), the function  $z(t)/\phi_2(t)$  is increasing, and hence

$$\frac{z(\delta(t))}{\phi_2(\delta(t))} \leq \frac{z(t)}{\phi_2(t)}.$$

Therefore,

$$\begin{aligned} \omega'(t) &\leq \frac{-k_m^{1-\gamma}}{\gamma} \kappa(t) p_2^\gamma(\delta(t); K) z^\gamma(\delta(t)) \left(\frac{z(\delta(t))}{\phi_2(\delta(t))}\right)^{1-\gamma} \phi_2(t) \\ &\leq \frac{-k_m^{1-\gamma}}{\gamma} \phi_2^{\gamma-1}(\delta(t)) \phi_2(t) \kappa(t) p_2^\gamma(\delta(t); K) z(\delta(t)). \end{aligned}$$

Combining this inequality with (4.4), we conclude that

$$\omega'(t) + \frac{1}{\gamma} \frac{k_m^{1-\gamma}}{1 - k_m} \phi_2^{\gamma-1}(\delta(t)) \phi_2(t) \kappa(t) p_2^\gamma(\delta(t); K) \omega(\delta(t)) \leq 0. \tag{4.5}$$

Thus,  $\omega(t)$  is a positive solution of the differential inequality (4.5) and is positive. However, by [21, Theorem 2.1.1], condition (4.2) guarantees that inequality (4.5) is oscillatory, which contradicts the positivity of  $\omega$ .

Case (N1). Since  $z(t) = u(t) + p(t)u(\psi(t))$ , we have

$$u(t) = z(t) - p(t)u(\psi(t)) \geq z(t) - p(t)z(\psi(t)).$$

Because  $z'(t) > 0$ , and  $\psi(t) \leq t$ , it follows that  $z(t) \geq z(\psi(t))$ , so  $u(t) \geq (1 - p(t))z(t)$ . Using (1.1), we obtain

$$(a(t)(z'''(t))^\gamma)' = -\kappa(t)u^\gamma(\delta(t)) \leq -\kappa(t)(1 - p(\delta(t)))^\gamma z^\gamma(\delta(t)). \tag{4.6}$$

By (N1), we see that  $\lim_{t \rightarrow \infty} z(t) \neq 0$ . By Lemma 2.2, for every  $\epsilon \in (0, 1)$ , we have that

$$z(t) \geq \frac{\epsilon}{3!} t^3 z'''(t).$$

Consequently,

$$z(\delta(t)) \geq \frac{\epsilon}{6} \delta^3(t) [a(\delta(t)) z'''(\delta(t))], \tag{4.7}$$

eventually. Substituting (4.7) into (4.6), we obtain

$$(a(t)(z'''(t))^\gamma)' \leq -\kappa(t)(1 - p(\delta(t)))^\gamma \left(\frac{\epsilon}{6} \delta^3(t)\right)^\gamma \frac{1}{a(\delta(t))} (a(\delta(t)) [z'''(\delta(t))]^\gamma).$$

We define

$$\vartheta(t) = a(t)(z'''(t))^\gamma.$$

Then,  $\vartheta(t) > 0$  for all sufficiently large  $t$ , and the above inequality implies

$$\vartheta'(t) + \frac{\epsilon^\gamma}{3^\gamma} \kappa(t)(1 - p(\delta(t)))^\gamma \frac{\delta^{3\gamma}(t)}{a(\delta(t))} \vartheta(\delta(t)) \leq 0. \tag{4.8}$$

Thus,  $\vartheta(t)$  is a positive solution of the delay differential inequality (4.8). However, by [21, Theorem 2.1.1], condition (4.3) guarantees that inequality (4.8) is oscillatory, which contradicts the positivity of  $\vartheta(t)$ . This contradiction completes the proof.  $\square$

**Theorem 4.2.** *Let conditions (3.4), (3.11), (3.16) and (3.17) hold. Assume that (4.2) and (4.3) are satisfied. If there exists a positive integer  $m$  such that*

$$\liminf_{t \rightarrow \infty} \int_{\delta(t)}^t \phi_0(\nu) \phi_0^{\gamma-1}(\delta(\nu)) \delta^{2\gamma}(\nu) \kappa(\nu) p_1^\gamma(\delta(\nu); K) d\nu > \frac{\gamma^{2\gamma} \alpha_m^{\gamma-1} (1 - \alpha_m)}{e}, \tag{4.9}$$

then, (1.1) is oscillatory.

*Proof.* Suppose, on the contrary, that (1.1) has an eventually positive solution  $u(t)$  satisfying

$$u(t) > 0, \quad u(\psi(t)) > 0, \quad u(\delta(t)) > 0 \quad \text{for } t \geq t_1 \geq t_0.$$

By Lemma 2.2, the associated function  $z(t)$  belongs to one of cases (N1)–(N3).

Case (N2). By Lemma 3.4, conditions (A3-1) and (A3-2) are satisfied. We define

$$\omega_1(t) = a^{1/\gamma}(t)z'''(t)\phi_0(t) + z''(t).$$

From condition (A1-3), it follows that  $\omega_1(t) > 0$  for all  $t \geq t_1$ . Moreover, by (A3-1), we obtain

$$a^{1/\gamma}(t)z'''(t)\phi_0(t) \leq \alpha_m z''(t).$$

Hence, using the definition of  $\omega_1(t)$ , we deduce

$$\begin{aligned} \omega_1(t) &= a^{1/\gamma}(t)z'''(t)\phi_0(t) + \alpha_m z''(t) - \alpha_m z''(t) + z''(t) \\ &\leq (1 - \alpha_m)z''(t). \end{aligned} \tag{4.10}$$

Applying Lemma 3.1, conditions (A1-1)–(A1-6) hold. From (A1-2) and (A1-6), we obtain

$$\begin{aligned} \omega_1'(t) &= \left( a^{1/\gamma}(t)z'''(t) \right)' \phi_0(t) \\ &\leq \frac{1}{\gamma} \left( a(t)(z'''(t))^\gamma \right)' \left( a^{1/\gamma}(t)z'''(t) \right)^{1-\gamma} \phi_0(t) \\ &\leq -\frac{1}{\gamma} \kappa(t)p_1^\gamma(\delta(t); K)z^\gamma(\delta(t)) \left( \alpha_m \frac{z''(t)}{\phi_0(t)} \right)^{1-\gamma} \phi_0(t) \\ &\leq -\frac{1}{\gamma} \alpha_m^{1-\gamma} \kappa(t)p_1^\gamma(\delta(t); K)\phi_0(t) \left( \frac{\epsilon}{2} \beta^2(t) \right)^\gamma (z''(\delta(t)))^\gamma \left( \frac{z''(t)}{\phi_0(t)} \right)^{1-\gamma}. \end{aligned}$$

Applying (A1-4) and the monotonicity of  $z''(t)/\phi_0(t)$ ,  $w$  is increasing. We further obtain

$$\begin{aligned} \omega'(t) &\leq -\frac{1}{\gamma} \alpha_m^{1-\gamma} \kappa(t)p_1^\gamma(\delta(t); K)\phi_0(t) \left( \frac{\epsilon}{2} \beta^2(t) \right)^\gamma (z''(\delta(t)))^\gamma \left( \frac{z''(\delta(t))}{\phi_0(\delta(t))} \right)^{1-\gamma} \\ &\leq -\frac{1}{\gamma} \frac{\alpha_m^{1-\gamma} \epsilon^\gamma}{2^\gamma} \kappa(t)p_1^\gamma(\delta(t); K) \frac{\phi_0(t)}{\phi_0^{1-\gamma}(\delta(t))} (\beta^2(t))^\gamma z''(\delta(t)). \end{aligned}$$

Consequently, from (4.10), it follows that

$$\omega_1'(t) + \frac{1}{\gamma} \frac{\alpha_m^{1-\gamma} \epsilon^\gamma}{(2!)^\gamma (1 - \alpha_m)} \frac{\phi_0(t)}{\phi_0^{1-\gamma}(\delta(t))} (\beta^2(t))^\gamma \kappa(t)p_1^\gamma(\delta(t); K)\omega_1(\delta(t)) \leq 0. \tag{4.11}$$

Thus,  $\omega(t)$  is a positive solution of the differential inequality (4.11). However, by [21, Theorem 2.1.1], condition (4.9) guarantees that inequality (4.11) oscillatory, which yields a contradiction.

Cases (N1) and (N3). These cases follow the same procedure as in Theorem 4.1 to establish conditions (4.2) and (4.3). Therefore, no eventually positive solution exists in these cases. This completes the proof.  $\square$

**Theorem 4.3.** *Let conditions (3.4), (3.11), (3.16) and (3.17) hold. Suppose that (4.1) and (4.3) are satisfied. If there exists a positive integer  $m$  such that*

$$\liminf_{t \rightarrow \infty} \int_{\delta(t)}^t \phi_2(\nu)\phi_2^{\gamma-1}(\delta(\nu))\kappa(\nu)\widehat{p}_2^\gamma(\delta(\nu); K)d\nu > \frac{\gamma k_m^{\gamma-1}(1 - k_m)}{e}, \tag{4.12}$$

then, (1.1) is oscillatory.

*Proof.* Suppose, on the contrary, that (1.1) has an eventually positive solution  $u(t)$  satisfying

$$u(t) > 0, \quad u(\psi(t)) > 0, \quad u(\delta(t)) > 0 \quad \text{for } t \geq t_1 \geq t_0.$$

By Lemma 2.2, the associated function  $z(t)$  belongs to one of cases (N1)–(N3).

Case (N3). To proof (4.12) we applying the inequality

$$u(t) \geq \widehat{p}_2(t; K)z(t),$$

to equation (1.1), and follow the same line of reasoning as in the proof of (4.9) in theorem 4.1 to obtain the required estimate.

Cases (N1) and (N2). These cases follow the same procedure as in Theorem 4.1 to establish conditions (4.2) and (4.3), showing that no eventually positive solution exists. This completes the proof.  $\square$

**Theorem 4.4.** *Let conditions (3.4), (3.11), (3.16) and (3.17) hold. Suppose that (4.9), (4.12) and (4.3) hold. Then, (1.1) is oscillatory.*

*Proof.* This theorem follows immediately from the preceding three theorems by applying the standard method found in the literature. The proof is straightforward and, therefore, is omitted.  $\square$

## 5. EXAMPLES

**Example 5.1.** Consider the fourth-order neutral differential equation

$$\left[ t^{4\gamma} [(u(t) + p_0 u(\psi_0 t))''']^\gamma \right]' + \kappa_0 t^{\gamma-1} u^\gamma(\delta_0 t) = 0, \quad (5.1)$$

where  $\gamma \leq 1$ ,  $t > 0$ ,  $p_0 \in (0, 1)$ ,  $\psi_0, \delta_0 \in (0, 1)$ , and  $\kappa_0 > 0$ . By comparing (1.1) with (5.1), we identify

$$a(t) = t^{4\gamma}, \quad p(t) = p_0, \quad \psi(t) = \psi_0 t, \quad \kappa(t) = \kappa_0 t^{\gamma-1}, \quad \delta(t) = \delta_0 t.$$

Straightforward calculations yield

$$\phi_0(t) = \frac{1}{3t^3}, \quad \phi_1(t) = \frac{1}{6t^2}, \quad \phi_2(t) = \frac{1}{6t}, \quad \lambda = \frac{1}{\delta_0^3}, \quad \lambda_1 = \frac{1}{\delta_0}.$$

For each  $K \in \mathbb{N}$ , the auxiliary functions reduce to

$$p_1(t; K) = (1 - p_0) \sum_{r=0}^K (p_0^2 \psi^{4/\epsilon})^r, \quad \epsilon \in (0, 1), \quad (5.2)$$

$$p_2(t; K) = \frac{\psi_0 - p_0}{\psi_0} \sum_{r=0}^K p^{2r} = \frac{\psi_0 - p_0}{\psi_0} \cdot \frac{1 - p_0^{2(K+1)}}{1 - p_0}. \quad (5.3)$$

From condition (3.4), we have

$$\beta = \frac{\delta_0^{2\gamma} \kappa_0}{3 \cdot 6^{\gamma} \gamma} \left[ (1 - p_0) \sum_{r=0}^K (p_0 \psi^{\frac{2}{\epsilon}})^{2r} \right]^\gamma. \quad (5.4)$$

Furthermore, from equation (3.17), we find that

$$k_0 = \frac{1}{6} \left( \frac{\kappa_0}{\gamma} \right)^{\frac{1}{\gamma}} \frac{\psi_0 - p_0}{\psi_0} \cdot \frac{1 - p_0^{2(K+1)}}{1 - p_0}. \quad (5.5)$$

Condition (4.1) is satisfied whenever

$$\frac{\kappa_0}{3\gamma} > \left( \frac{3}{\epsilon \delta_0^2 (1 - p_0) \sum_{r=0}^K (p_0^2 \psi^{4/\epsilon})^r} \right)^\gamma. \quad (5.6)$$

Condition (4.9) leads to

$$\kappa_0 > \frac{6^\gamma \gamma \alpha_m^{\gamma-1} (1 - \alpha_m)}{e \delta_0^{3-\gamma} [(1 - p_0) \sum_{r=0}^K (p_0 \psi^{2/\epsilon})^{2r}]^\gamma \ln \frac{1}{\delta_0}}. \quad (5.7)$$

Similarly, condition (4.2) implies

$$\kappa_0 > \frac{6^\gamma \gamma k_m^{\gamma-1} (1 - k_m)}{e \delta_0^{1-\gamma} \ln \frac{1}{\delta_0}} \left( \frac{\psi_0}{\psi_0 - p_0} \cdot \frac{1 - p_0^2}{1 - p_0^{2(K+1)}} \right)^\gamma. \quad (5.8)$$

Condition (4.12) yields

$$\kappa_0 > \frac{6^\gamma \gamma k_m^{\gamma-1} (1 - k_m)}{[e \delta_0^{1-\gamma}]^\gamma \ln \frac{1}{\delta_0}} \left( \frac{\psi_0}{\psi_0 - p_0} \frac{1}{\sum_{r=0}^K (p_0^2 / \psi_0^{2k_m})^r} \right)^\gamma. \quad (5.9)$$

Finally, condition (4.3) reduces to

$$\kappa_0 > \left(\frac{6\delta_0}{1-p_0}\right)^\gamma \frac{1}{e \ln \frac{1}{\delta_0}}. \tag{5.10}$$

Consequently, the oscillatory behavior of (5.1) can be characterized using different criteria. Specifically, Theorem 4.1 guarantees oscillation if conditions (5.6), (5.8), and (5.10) hold.

Theorem 4.2 ensures oscillation under conditions (5.7), (5.8), and (5.10).

Moreover, Theorem 4.3 applies when (5.6), (5.9), and (5.10) are satisfied, whereas Theorem 4.4 asserts oscillation provided that (5.7), (5.9), and (5.10) hold.

By assigning appropriate values to the parameters, the previously established criteria produce explicit oscillatory results for equation (5.1), thereby illustrating the applicability and effectiveness of the derived conditions.

In the following example, specific values will be assigned to the coefficients, and the derived criteria will be applied to illustrate the resulting oscillatory behavior.

**Example 5.2.** Consider the special case of (5.1), with  $p_0 = 1/2$ ,  $\psi_0 = 0.7$ , and  $\delta_0 = 1/3$ . Then, equation (5.1) reduces to

$$(t^{4/3}[(u(t) + 0.5u(0.7t))''' ]^{1/3})' + \kappa_0 t^{-2/3} u^{1/3}(0.6t) = 0, \quad t \geq 1. \tag{5.11}$$

In this case, we clearly have

$$\begin{aligned} \gamma = 1/3, \quad a(t) = t^{4/3}, \quad p(t) = 0.5, \quad \psi(t) = 0.9t, \quad \delta(t) = 0.6t, \quad \kappa(t) = \kappa_0 t^{-2/3}, \\ \phi_0(t) = \frac{1}{3t^3}, \quad \phi_1(t) = \frac{1}{6t^2}, \quad \phi_2(t) = \frac{1}{6t}. \end{aligned}$$

We then obtain  $\lambda = 4.630$  and  $\lambda_1 = 1.667$ . From condition (5.2), it follows that

$$p_1(t; 10) = (1 - 0.5) \sum_{r=0}^{10} ((0.5)^2 (0.9)^{4/0.8})^r \approx 0.59, \quad \epsilon = 0.8.$$

From condition (5.3), we obtain

$$p_2(t; 10) = \left(\frac{0.9 - 0.5}{0.9}\right) \sum_{r=0}^{10} (0.25)^r = \frac{0.9 - 0.5}{0.9} \cdot \frac{1 - (0.25)^{11}}{1 - 0.25} \approx 0.59.$$

From (5.4), we have

$$\begin{aligned} \beta = 0.328\kappa_0, \quad \alpha_0 = 0.035\epsilon_0\kappa_0^3, \\ \alpha_j = 0.566\epsilon_0\kappa_0^3 \frac{(4.63)^{\alpha_{j-1}}}{(1 - \alpha_{j-1})^3}, \quad j = 1, 2, \dots, m. \end{aligned}$$

From (5.5), we have

$$k_0 = 4\kappa_0^3, \quad k_j = 4\kappa_0^3 \frac{(1.667)^{k_{j-1}}}{(1 - k_{j-1})^3}, \quad j = 1, 2, \dots, m.$$

The oscillation conditions (5.6)-(5.10) yield

$$\kappa_0 > 3.05, \quad \kappa_0 > 2.03\alpha_m^{-2/3}(1 - \alpha_m), \quad \kappa_0 > 0.934k_m^{-2/3}(1 - k_m), \quad \kappa_0 > 1.391.$$

Therefore, by Theorems 4.1 and 4.4, we conclude that (5.11) is oscillatory.

It is worth noting that the studies cited in [7, 15, 25, 27, 29, 30] do not apply to (5.11), since here  $\gamma = 1/3 < 1$  and  $p = 0.5$ , and these works do not cover the nonlinear case. Consequently, the results obtained in this study constitute an extension and refinement of the existing literature.

## 6. CONCLUSION

This research investigated the oscillatory behavior and monotonic properties of a class of fourth-order quasi-linear neutral differential equations in the noncanonical case, a class less explored compared to lower-order equations. The results expand the existing literature by overcoming limitations of previous studies, particularly regarding strict oscillation conditions and reliance on traditional analytical tools. New precise monotonic properties of positive solutions were derived, improving analytical connections between solutions and their derivatives, and enabling the development of more general oscillation criteria. Examples confirmed that these criteria enhance known results and extend their applicability to cases not previously addressed. This work enriches the oscillation theory of higher-order differential equations and provides a foundation for future research, including extensions to fourth-order nonlinear equations of the form

$$(a(t)(z'''(t))^\gamma)' + \kappa(t)u^\alpha(\delta(t)) = 0,$$

as well as higher-order equations of the form

$$\left(a(t)\left(z^{(n-1)}(t)\right)^\gamma\right)' + \kappa(t)u^\alpha(\delta(t)) = 0,$$

and equations involving advanced time arguments, variable coefficients, or more general nonlinear structures.

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