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SOLUTION STRUCTURE OF NONLOCAL SERRIN-TYPE OVER-DETERMINED PROBLEMS

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Abstract. In this article, we study the solution structures of Serrin-type over-determined problems with Kirchhoff-type nonlocal terms. We prove that the exact number of solutions is the same as those of some transcendental equations defined by the nonlocal terms. We also obtain the explicit form of solutions by using the unique solutions of the over-determined problems without the nonlocal terms.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with C^2 boundary and $k \in \{1, 2, \dots, N\}$. In this article, we consider the fully nonlinear over-determined problem with Kirchhoff-type nonlocal term,

$$\begin{gathered} M\Big(\|u\|_{L^p(\Omega)}, \|\nabla u\|_{L^q(\Omega)}\Big) S_k(D^2 u) = \lambda \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c > 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $0 < p, q \leq \infty$, $\lambda > 0$ are given constants, c is an unknown positive constant, ν is the outer unit normal to $\partial\Omega$, and $M(s,t)$ is a positive function in $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$. We do not assume any continuity of M. For a real symmetric matrix A, let $S_k(A)$ denote the k-th elementary symmetric function of the eigenvalues of A (counted with multiplicity),

$$S_k(A) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_k}.$$

Thus $S_1(A) = \sum_{i=1}^N \lambda_i$ and $S_N(A) = \prod_{i=1}^N \lambda_i$. Finally, let $D^2 u(x) = (\frac{\partial^2 u}{\partial x_i \partial x_j}(x))_{1 \leq i,j \leq N}$ denote the Hessian matrix of a C^2-function u. Then $S_k(D^2 u)$ is called the *k-Hessian operator*. Note that $S_1(D^2 u) = \Delta u$ and $S_N(D^2 u) = \det(D^2 u)$.

In [2], the authors considered the Serrin-type over-determined problem for the k-Hessian operator,

$$\begin{gathered} S_k(D^2 u) = \binom{N}{k} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c > 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $c > 0$ is unknown, $\binom{N}{k} = \frac{N!}{k!(N-k)!}$ for $k \in \{1, 2, \dots, N\}$. They prove that if (1.2) admits a solution $u \in C^2(\overline{\Omega})$ for some $k \in \{1, 2, \dots, N\}$, then, up to a translation, Ω must be a ball, say $\Omega = B_R(x_0)$ for $R > 0$ and $x_0 \in \mathbb{R}^N$, and the solution is of the form

$$u(x) = U_{R,x_0}(x) := \frac{|x - x_0|^2 - R^2}{2} \tag{1.3}$$

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with $c = R$. Note that the k-Hessian operator $S_k(D^2u)$ is fully nonlinear and in general is not elliptic. In spite of these difficulties, the authors in [2] give a shorter alternative proof which does not exploit maximum principles directly and to extend the famous result by Serrin [4] (for $k = 1$) in this setting. Their method uses a Pohozaev-type identity by Tso [5] and reminds us of an alternative proof by Weinberger [6].

Recently, the Serrin-type overdetermined problem with Kirchhof-type nonlocal term,

$$\begin{gathered} M\Big(\|u\|_{L^p(\Omega)}, \|\nabla u\|_{L^q(\Omega)}\Big)\Delta u = \lambda \text{ in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = c \quad \text{on } \partial\Omega, \end{gathered}$$

where $0 < p, q \le \infty$, $\lambda > 0$ are given constants, c is an unknown constant, and $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a positive function, was considered in [3]. There the exact number of solutions according to the values of the bifurcation parameter $\lambda > 0$ is determined by a transcendental equation defined by the nonlocal term for a real unknown variable.

In this article, we extend the main result in [3] to problem (1.1). The original argument comes from [1].

Theorem 1.1. *Let $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, $0 < p, q \le \infty$, $\lambda > 0$, and $k \in \{1, 2, \dots, N\}$. Then if (1.1) admits a solution $u \in C^2(\overline{\Omega})$ for an unknown constant $c > 0$, then Ω must be a ball and u must be radially symmetric with respect to the center of the ball.*

Let $\Omega = B_R(x_0)$ with $R > 0$ and $x_0 \in \mathbb{R}^N$. Consider the system of equations with respect to $(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+$:

$$\begin{aligned} s &= \Big\{\frac{M(s,t)}{\lambda}\binom{N}{k}\Big\}^{-1/k}\|U_{R,x_0}\|_{L^p(B_R(x_0))} \\ t &= \Big\{\frac{M(s,t)}{\lambda}\binom{N}{k}\Big\}^{-1/k}\|\nabla U_{R,x_0}\|_{L^q(B_R(x_0))} \end{aligned} \tag{1.4}$$

where U_{R,x_0} defined by (1.3) is the unique solution of (1.2) for $\Omega = B_R(x_0)$. Then for $\lambda > 0$, problem (1.1) for $\Omega = B_R(x_0)$ has the same number of solutions of system (1.4). Also define

$$g(s) = s^k M\Big(s, \frac{\|\nabla U_{R,x_0}\|_{L^q(B_R(x_0))}}{\|U_{R,x_0}\|_{L^p(B_R(x_0)}} s\Big)\binom{N}{k} \tag{1.5}$$

for $s > 0$. Then the number of solutions of (1.4) is the same as the number of solutions to the equation

$$g(s) = \lambda\|U_{R,x_0}\|^k_{L^p(B_R(x_0))} \tag{1.6}$$

with respect to $s > 0$. Moreover, any solution u_λ of (1.1) has the form

$$u_\lambda(x) = s_* \frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^p(B_R(x_0))}}$$

where s_* is any solution of (1.6).

In the second part, we study an exterior over-determined problem with Kirchhoff-type nonlocal term as described below. Let $D \subset \mathbb{R}^N$ $(N \ge 2)$ be a bounded C^2-domain containing the origin as an interior point. We consider

$$\begin{gathered} M\Big(\|u\|_{L^p(\Omega)}, \|\nabla u\|_{L^q(\Omega)}\Big)\Delta u(x) = \lambda|x|^{-N-2}, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \\ |\nabla u(x)| = c|x|^{-N} > 0, \quad x \in \partial\Omega, \\ |u(x)| = o(1) \quad (|x| \to \infty), \quad (\text{if } N \ge 3), \\ |u(x)| = O(1) \quad (|x| \to \infty), \quad (\text{if } N = 2), \end{gathered} \tag{1.7}$$

where $\Omega = \mathbb{R}^N \setminus \overline{D}$ is an exterior domain, $0 < p, q \le \infty$, $\lambda > 0$ are given constants, c is an unknown positive constant, and $M(s, t)$ is a positive function in $(s, t) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Let B be the closed unit ball in $\mathbb{R}^N$ for $N \geq 2$ and define

$$U(x) = \frac{1}{2}\big(|x|^{-N} - |x|^{2-N}\big) \quad \text{for } |x| > 1. \tag{1.8}$$

Then direct computations show that U is a solution of

$$\begin{gathered}\Delta U = N|x|^{-N-2} \text{ in } B^c = \mathbb{R}^N \setminus B,\\ U = 0 \quad \text{on } \partial B^c,\\ |\nabla U| = 1 \quad \text{on } \partial B^c,\\ |U(x)| \to 0 \quad (|x| \to \infty), \quad (\text{if } N \geq 3),\\ |U(x)| \to \frac{1}{2} \quad (|x| \to \infty), \quad (\text{if } N = 2).\end{gathered}$$

Also note that $U \in L^p(B^c)$ if and only if $\frac{N}{N-2} < p \leq \infty$ if $N \geq 3$, $p = \infty$ if $N = 2$; and $|\nabla U| \in L^q(B^c)$ if and only if $\frac{N}{N-1} < q \leq \infty$ if $N \geq 3$, $\frac{2}{3} < q \leq \infty$ if $N = 2$. In this case, we compute

$$\begin{gathered}\|U\|^p_{L^p(B^c)} = (\frac{1}{2})^{p+1}|\mathbb{S}^{N-1}|B\Big(\frac{p(N-2)-N}{2}, p+1\Big), \quad (\frac{N}{N-2} < p < \infty,\ N \geq 3),\\ \|U\|_{L^\infty(B^c)} = \Big(\frac{1}{N-2}\Big)\Big(\frac{N}{N-1}\Big)^{-N/2}, \quad (N \geq 3),\\ \|U\|_{L^\infty(B^c)} = \frac{1}{2}, \quad (N = 2),\\ \|\nabla U\|^q_{L^q(B^c)} = (\frac{1}{2})^{q+1}|\mathbb{S}^{N-1}|N^q\Big(\frac{N-2}{N}\Big)^{\frac{q(N+1)-N}{2}}\Big(B\big(\frac{q(N-1)-N}{2}, q+1\big)\\ + B_{2/N}\big(q+1, \frac{-(N+1)q+N}{2}\big)\Big), \quad (\frac{N}{N-1} < q < \infty,\ N \geq 3),\\ \|\nabla U\|^q_{L^q(B^c)} = \frac{2\pi}{3q-2} \quad (\frac{2}{3} < q < \infty,\ N = 2),\\ \|\nabla U\|_{L^\infty(B^c)} = \big(\frac{N}{N-1}\big)\big(\frac{N(N+1)}{(N-1)(N-2)}\big)^{-(N+1)/2}, \quad (N \geq 3),\\ \|\nabla U\|_{L^\infty(B^c)} = 1, \quad (N = 2),\end{gathered}$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ denotes the Beta function, and $B_z(x,y) = \int_0^z t^{x-1}(1-t)^{y-1}dt$ denotes the incomplete Beta function for $z \in (0,1)$.

Assume $\frac{N}{N-2} < p \leq \infty$ if $N \geq 3$, $p = \infty$ if $N = 2$, and $\frac{N}{N-1} < q \leq \infty$ if $N \geq 3$, $2/3 < q \leq \infty$ if $N = 2$. Let us consider the system of equations

$$\begin{aligned}s &= \big\{\frac{M(s,t)}{\lambda}N\big\}^{-1}\|U\|_{L^p(B^c)}\\ t &= \big\{\frac{M(s,t)}{\lambda}N\big\}^{-1}\|\nabla U\|_{L^q(B^c)}\end{aligned} \tag{1.9}$$

with respect to $(s,t) \in \mathbb{R}_+ \times \mathbb{R}_+$, where U is in (1.8). Then we have the following:

Theorem 1.2. *Let $N \geq 2$, $M : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, and*

$$\begin{gathered}\frac{N}{N-2} < p \leq \infty \quad \textit{and} \quad \frac{N}{N-1} < q \leq \infty, \quad \textit{if } N \geq 3,\\ p = \infty \quad \textit{and} \quad \frac{2}{3} < q \leq \infty, \quad \textit{if } N = 2.\end{gathered}$$

Assume that (1.7) admits a solution $u \in C^2(\overline{\Omega})$ for an unknown constant $c > 0$, where $\Omega = \mathbb{R}^N \setminus \overline{D}$ is an exterior domain. Then D must be a ball and u must be radially symmetric with respect to the center of the ball.

Assume, without loss of generality, that $\overline{D} = B$. *Then for* $\lambda > 0$, *the problem* (1.7) *for* $\Omega = B^c$ *has the same number of solutions of the system* (1.9). *Also define*

$$g(s) = sM\Big(s, \frac{\|\nabla U\|_{L^q(B^c)}}{\|U\|_{L^p(B^c)}} s\Big) N$$

for $s > 0$. *Then the number of solutions of* (1.9) *is the same as the number of solutions to the equation*

$$g(s) = \lambda \|U\|_{L^p(B^c)} \tag{1.10}$$

with respect to $s > 0$. *Moreover, any solution* u_λ *of* (1.7) *has the form*

$$u_\lambda(x) = s_* \frac{U(x)}{\|U\|_{L^p(B^c)}}$$

where s_* *is any solution of* (1.10).

Concerning the strategy of the proof of Theorem 1.1, we cannot use the maximum principle directly for problem (1.1) because of the nonlocal terms. Also it seems difficult to carry out the integral approach in [2], again because of the nonlocal terms. The key point of the proof is using the homogeneity of the k-Hessian operator

$$S_k(\gamma D^2 u) = \gamma^k S_k(D^2 u), \quad (\gamma > 0)$$

to eliminate the nonlocal terms in the equation. Then we can appeal to the uniqueness of the solution for the over-determined problem for the k-Hessian operators (1.2) in [2].

The proof of Theorem 1.2 goes also along the the same lines. In this case, we use a classical Kelvin transform to rewrite the problem from the exterior to the interior of the domain. Then we use the Serrin's characterization of the over-determined problem without nonlocal terms. Since the Kelvin transform works well only for the linear operator, we have to restrict ourselves in the linear problem. Finally, nonlocal terms are treated by the use of the homogeneity of the Laplacian, as in the proof of Theorem 1.1.

The structure of this article is as follows: In §2, we prove Theorem 1.1. In §3, we obtain a characterization of (the outside of) a ball for some exterior over-determined problem, Proposition 3.1, and prove Theorem 1.2.

2. Proof of Theorem 1.1

Proof. First assume that there exists a solution $u \in C^2(\overline{\Omega})$ of (1.1) and put

$$v = \gamma u$$

where $\gamma > 0$ is chosen so that

$$\gamma = \Big(\frac{M(\|u\|_{L^p(\Omega)}, \|\nabla u\|_{L^q(\Omega)})}{\lambda}\binom{N}{k}\Big)^{1/k}.$$

Note that the k-Hessian operator is homogeneous of degree k. Then

$$\begin{aligned} S_k(D^2 v) &= S_k(\gamma D^2 u) \\ &= \gamma^k S_k(D^2 u) \\ &\overset{(1.1)}{=} \gamma^k \frac{\lambda}{M(\|u\|_{L^p(\Omega)}, \|\nabla u\|_{L^q(\Omega)})} \\ &= \binom{N}{k} \end{aligned}$$

in Ω. Also we see that $v = 0$ on $\partial\Omega$ and $\frac{\partial v}{\partial \nu} = \gamma c = \text{const.}$ on $\partial\Omega$. Thus by the result of [2], we see that Ω must be a ball, say $\Omega = B_R(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^N$ and $v \equiv U_{R,x_0}(x)$. This implies that

$$u(x) = \gamma^{-1} U_{R,x_0}(x), \quad \nabla u(x) = \gamma^{-1} \nabla U_{R,x_0}(x). \tag{2.1}$$

We define

$$s = \|u\|_{L^p(B_R(x_0))} \quad \text{and} \quad t = \|\nabla u\|_{L^q(B_R(x_0))}. \tag{2.2}$$

Then by (2.1), we have

$$s = \gamma^{-1}\|U_{R,x_0}\|_{L^p(B_R(x_0))},$$
$$t = \gamma^{-1}\|\nabla U_{R,x_0}\|_{L^q(B_R(x_0))},$$

which is equivalent to (1.4). This shows that

$$(s,t) = (\|u\|_{L^p(B_R(x_0))}, \|\nabla u\|_{L^q(B_R(x_0))})$$

is a solution to (1.4) and thus

$$\#\{u : \text{solutions of (1.1) for } \Omega = B_R(x_0)\} \le \#\{(s,t) \in (\mathbb{R}_+)^2 : \text{solutions of (1.4)}\},$$

where $\#A$ denotes the cardinality of the set A.

On the other hand, let $\Omega = B_R(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^N$ and let $(s,t) \in (\mathbb{R}_+)^2$ be a solution to (1.4). Note that by (1.4), we have

$$\frac{s}{\|U_{R,x_0}\|_{L^p(B_R(x_0))}} = \frac{t}{\|\nabla U_{R,x_0}\|_{L^q(B_R(x_0))}} = \Big(\frac{M(s,t)}{\lambda}\binom{N}{k}\Big)^{-1/k}. \tag{2.3}$$

Thus if we define

$$u(x) = s\frac{U_{R,x_0}(x)}{\|U_{R,x_0}\|_{L^p(B_R(x_0))}} \quad \Big(= t\frac{U_{R,x_0}(x)}{\|\nabla U_{R,x_0}\|_{L^q(B_R(x_0))}}\Big), \tag{2.4}$$

then we have $u = 0$, $\frac{\partial u}{\partial \nu}$ =const. on $\partial B_R(x_0)$ and (2.2) holds by (2.4). Moreover, we have

$$\begin{aligned}
&M\Big(\|u\|_{L^p(B_R(x_0))}, \|\nabla u\|_{L^q(B_R(x_0))}\Big)S_k(D^2u)\\
&\overset{(2.2)}{=} M(s,t)S_k(D^2u)\\
&\overset{(2.4)}{=} M(s,t)\Big(\frac{s}{\|U_{R,x_0}\|_{L^p(B_R(x_0))}}\Big)^k \underbrace{S_k(D^2U_{R,x_0})}_{=\binom{N}{k}}\\
&\overset{(2.3)}{=} M(s,t)\Big(\frac{M(s,t)}{\lambda}\binom{N}{k}\Big)^{-1}\binom{N}{k} = \lambda
\end{aligned}$$

on $B_R(x_0)$. This shows that

$$\#\{u : \text{solutions of (1.1) for } \Omega = B_R(x_0)\} \ge \#\{(s,t) \in (\mathbb{R}_+)^2 : \text{solutions of (1.4)}\}.$$

Thus the number of solutions of (1.1) for $\Omega = B_R(x_0)$ and that of (1.4) are the same.

Also by (2.3), we can rewrite the system of equations (1.4) into a single equation for s,

$$s = \Big(\frac{M\big(s, \frac{\|\nabla U_{R,x_0}\|_{L^q(B_R(x_0))}}{\|U_{R,x_0}\|_{L^p(B_R(x_0))}}\big)s}{\lambda}\binom{N}{k}\Big)^{-1/k}\|U_{R,x_0}\|_{L^p(B_R(x_0))},$$

which is equivalent to (1.6) with $g(s)$ in (1.5). Thus the number of solutions of (1.4) for $\Omega = B_R(x_0)$ and that of (1.6) are also the same. □

3. Exterior over-determined problem

Let $D \subset \mathbb{R}^N$, $N \ge 2$ be a bounded C^2-domain containing the origin as an interior point, and put $\Omega = \mathbb{R}^N \setminus \overline{D}$. First, we consider an exterior over-determined problem without Kirchhoff term:

$$\begin{gathered}
\Delta u = N|x|^{-N-2} \text{ in } \Omega,\\
u = 0 \quad \text{on } \partial\Omega,\\
|\nabla u| = c|x|^{-N} \quad \text{on } \partial\Omega,\\
|u(x)| = o(1) \quad (|x| \to \infty), \quad (\text{if } N \ge 3),\\
|u(x)| = O(1) \quad (|x| \to \infty), \quad (\text{if } N = 2),
\end{gathered} \tag{3.1}$$

where c is an unknown positive constant. Direct computation shows that $U(x)$ in (1.8) is the exact solution of (3.1) for $\Omega = B^c$ with $c = 1$. On the uniqueness, we have the following proposition.

Proposition 3.1. *Let $N \geq 2$ and $\Omega = \mathbb{R}^N \setminus \overline{D}$ be an exterior domain. If problem* (3.1) *admits a solution $u \in C^2(\overline{\Omega})$, then D must be a ball and u must be radially symmetric with respect to the center of the ball. Let $\Omega = B^c = \mathbb{R}^N \setminus B$. Then $U(x)$ in* (1.8) *is the unique solution of* (3.1) *and c must satisfy $c = 1$.*

Proof. First we consider $N \geq 3$ and assume that $u \in C^2(\overline{\Omega})$ is a solution of (3.1). Define a function $w : \Omega^* \setminus \{0\} \to \mathbb{R}$ such that

$$w(y) = |y|^{2-N} u\Big(\frac{y}{|y|^2}\Big), \quad (y \in \Omega^* \setminus \{0\}), \tag{3.2}$$

where

$$\Omega^* = \{y = \frac{x}{|x|^2} : x \in \Omega\} \cup \{0\},$$

that is, w is the Kelvin transform of u. Note that $x = \frac{y}{|y|^2} \in \Omega$ is equivalent to $y = \frac{x}{|x|^2} \in \Omega^* \setminus \{0\}$. By a well-known formula for the Kelvin transform and the equation in (3.1), we have

$$\Delta_y w(y) = |y|^{-2-N} (\Delta_x u)\big(\frac{y}{|y|^2}\big) = |x|^{2+N} \Delta_x u(x) = N$$

for $y \in \Omega^* \setminus \{0\}$. Since $y \in \partial\Omega^*$ is equivalent to $x \in \partial\Omega$, $w(y)\big|_{y \in \partial\Omega^*} = u(x)\big|_{x \in \partial\Omega} = 0$. Furthermore, direct computations give that

$$\nabla_y w(y) = |y|^{-N} \left\{(2-N)u(x)y - 2(\nabla_x u(x) \cdot x)y + \nabla_x u(x)\right\}$$

where $x = \frac{y}{|y|^2}$, $y \in \Omega^* \setminus \{0\}$. Thus

$$\begin{aligned}
\nabla_y w(y)\big|_{y \in \partial\Omega^*} &= \Big[|x|^N \Big\{(2-N) \underbrace{u(x)}_{=0} \frac{x}{|x|^2} - 2(\nabla_x u(x) \cdot x) \frac{x}{|x|^2} + \nabla_x u(x)\Big\}\Big]_{x \in \partial\Omega} \\
&= |x|^N \Big(-2(\nabla_x u(x) \cdot \frac{x}{|x|}) \frac{x}{|x|} + \nabla_x u(x)\Big)_{x \in \partial\Omega} \\
&= |x|^N \Big(\vec{a}(x) - \vec{b}(x)\Big),
\end{aligned}$$

where

$$\vec{a}(x) = \nabla_x u(x) - (\nabla_x u(x) \cdot \frac{x}{|x|}) \frac{x}{|x|}, \quad \vec{b}(x) = (\nabla_x u(x) \cdot \frac{x}{|x|}) \frac{x}{|x|}.$$

Note that

$$\vec{a}(x) + \vec{b}(x) = \nabla_x u(x), \quad \vec{a}(x) \perp \vec{b}(x) \quad \text{for } x \in \partial\Omega,$$

thus $|\vec{a}(x)|^2 + |\vec{b}(x)|^2 = |\nabla_x u(x)|^2$. Therefore,

$$|\nabla_y w(y)|^2\big|_{y \in \partial\Omega^*} = |x|^{2N} |\nabla_x u(x)|^2\big|_{x \in \partial\Omega}.$$

Since u is a solution of (3.1), we see that w in (3.2) satisfies

$$\begin{gathered}
\Delta w = N \quad \text{in } \Omega^* \setminus \{0\}, \\
w = 0 \quad \text{on } \partial\Omega^*, \\
|\nabla w| = c > 0 \quad \text{on } \partial\Omega^*.
\end{gathered} \tag{3.3}$$

Furthermore, since u satisfies $\lim_{|x| \to \infty} u(x) = 0$, its Kelven transform w satisfies

$$\lim_{|y| \to 0} \frac{|w(y)|}{|y|^{2-N}} = 0.$$

Thus $y = 0$ is a removable singularity and w satisfies the equation on the whole Ω^*. Then Serrin's result assures that Ω^* must be a ball and w must be radially symmetric with respect to the center of the ball. This implies that Ω must be the compliment of the ball and u is radially symmetric with respect to the same point. If $\Omega = B^c$, then $\Omega^* = \mathrm{int}(B)$ and

$$w(y) = \frac{|y|^2 - 1}{2} \quad (y \in \mathrm{int}(B))$$

is the unique solution of (3.3) with $c = 1$. Then by (3.2), we have

$$u(x) = |y|^{N-2} w(y) = \frac{|y|^N - |y|^{N-2}}{2} = \frac{|x|^{-N} - |x|^{2-N}}{2}.$$

When $N = 2$, the same computation also holds for the Kelvin transform $w(y) = u(\frac{y}{|y|^2})$. Moreover, we can claim that $y = 0$ is again removable since in this case

$$\lim_{|y|\to 0} \frac{|w(y)|}{\log(1/|y|)} = 0$$

follows from the weaker condition at ∞: $u(x) = O(1)$ as $|x| \to \infty$. We have finished the proof of Proposition 3.1. □

Proof of Theorem 1.2. Once the uniqueness of the explicit solution U in (1.8) to problem (3.1) is proven in Proposition 3.1, the proof of Theorem 1.2 can be done in exactly the same way as the proof of Theorem 1.1. □

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