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APPROXIMATIONS OF THE SET OF TRAJECTORIES, ATTAINABLE SETS, AND INTEGRAL FUNNEL OF A CONTROL SYSTEM WITH INTEGRAL AND GEOMETRIC CONSTRAINTS ON THE CONTROL FUNCTIONS

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Abstract. This article studies approximations to the set of trajectories, attainable sets and integral funnel of a control system described by an ordinary differential equation. It is assumed that the equation is nonlinear with respect to the phase state vector and affine with respect to the control vector. The system includes control functions, some of which satisfy the L_p ($p \in (1,\infty)$) norm constraint, while the others satisfy the L_∞ norm constraint. Step by step, the set of admissible control functions is replaced by a set consisting of a finite number of piecewise-constant control functions that generate a finite number of trajectories. Error evaluations are provided for the Hausdorff distances between the set of trajectories, attainable sets, integral funnel, and their approximations, which depend on discretization parameters.

1. Introduction

In mathematical models of control systems different types of constraints on control functions arise depending on the character of the control effort. For example, if the control resource characterizing a given control effort is exhausted by consumption, such as energy, fuel, finance, food, etc., then an integral constraint on control functions becomes necessary. If the control resource characterizing the control effort is not exhausted by consumption, like the steering wheels of a car, ship, plane, etc., then these types of control functions have a geometric constraint. In simple forms, integral and geometric constraints on control functions can be expressed through L_p norms, where $p \in (1,\infty)$, and L_∞ norm type constraints.

One of the most studied areas of control systems theory is control systems, the dynamics of which are described by ordinary differential equations. The concepts of the set of trajectories, attainable sets, and integral funnel, which include comprehensive information about system, are essential tools for studying and forecasting the behavior of these control systems. The set of trajectories consists of trajectories generated by all admissible control functions. The attainable set of system at a given instant of time consists of the points to which the system state can be steered at the given instant of time. The integral funnel of the system is a generalization of the integral curve notion from the theory of ordinary differential equations and consists of the graphs of all trajectories. Different topological properties and approximations of the set of trajectories, attainable sets and integral funnel of control systems with geometric constraints on control functions and described by ordinary differential equations are discussed in a vast number of research papers (see, [7, 15, 16, 21] and references therein). It should be noted that these types of control systems are also studied within the framework of differential and integral inclusions theory (see, e.g. [1, 2, 3, 4, 6]). Similar properties of the set of trajectories, attainable sets and integral funnel of control systems with integral constraints on control functions are considered in [5, 8, 9, 10, 12, 14, 17, 18, 20, 22]. Let us underline that investigating control systems with integral constraints on control functions is generally more difficult task than studying control

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systems with geometric constraints on control functions. These difficulties arise especially in the investigation of nonlinear differential game problems, when the control functions of the players have integral constraints. The methods for constructing attainable sets of control systems with integral constraints on control functions, as outlined in references [9, 10, 18], are based on Pontryagin's maximum principle formulated for motions approaching its boundary. In [8, 12], the constructions of the set of trajectories, attainable sets and integral funnel are based on discretization of the set of admissible control functions. The construction of two-dimensional reachable sets of the Dubins car with both integral and geometric constraints on control functions is discussed in [19]. In [13], approximations of the set of trajectories, attainable sets and integral funnel of the control system are presented, where the same control function satisfies both integral and geometric constraints simultaneously, and the system is affine with respect to the control vector. It also includes error evaluations for the given approximations.

In this article, we consider a control system described by an ordinary differential equation. It is assumed that the system is nonlinear with respect to the phase state vector and affine with respect to the control vector. It is supposed that a part of the control functions have an L_p ($p \in (1,\infty)$) norm constraint, meaning that they are integrally constrained, while another part of the control functions have an L_∞ norm constraint, indicating they are geometrically constrained. The approximations of the set of trajectories, attainable sets and integral funnel are discussed. Step by step, the set of admissible control functions is replaced with a set consisting of a finite number of piecewise-constant control functions that generate a finite number of trajectories. Then, each trajectory generated by a piecewise-constant control function, is replaced by its Euler's broken line and an evaluation for the Hausdorff distance between the set of trajectories and the set, consisting of a finite number of Euler's broken lines, is derived. Using this result, similar evaluations for the Hausdorff distances between the attainable sets, integral funnel and their approximations are provided.

This article is organized as follows. In Section 2, the system's dynamics description, basic conditions for functions describing the behavior of the system and some auxiliary propositions which are used in subsequent arguments, are given. In Section 3, approximations of the set of trajectories (Theorem 3.1) and attainable sets (Corollary 3.2) are presented. An error evaluation depending on the discretization parameters, is given. In Section 4, each trajectory generated by the piecewise-constant control function is replaced with appropriate Euler's broken line and an evaluation between the set of trajectories and the set consisting of a finite number of Euler's broken lines is obtained (Theorem 4.4). The same evaluation is derived for attainable sets (Theorem 4.5). In Section 5, using the obtained results, an approximation of the integral funnel of the system by a set consisting of a finite number of node points of Euler's broken lines is presented (Theorem 5.1). An error evaluation for the obtained approximation is also given.

Let us introduce some notation which will be used in the following arguments. Let $(X, d(\cdot,\cdot))$ be a metric space. The Hausdorff distance between the sets $E \subset X$ and $\Omega \subset X$ is denoted by the symbol $h_X(E,\Omega)$ and is defined as

$$h_X(E,\Omega) = \max\{\sup_{x\in\Omega} d(x,E), \sup_{y\in E} d(y,\Omega)\}$$

where $d(x,E) = \inf\{\|x-y\| : y \in E\}$. By the symbols $h_n(\cdot,\cdot)$ and $h_C(\cdot,\cdot)$ we denote the Hausdorff distance between the subsets of the space $\mathbb{R}^n$ and $C([t_0,\theta];\mathbb{R}^n)$, respectively, where $\mathbb{R}^n$ is an n-dimensional Euclidean space, and $C([t_0,\theta];\mathbb{R}^n)$ is the space of continuous functions $x(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ with norm $\|x(\cdot)\|_C = \max\{\|x(t)\| : t \in [t_0,\theta]\}$. Denote

$$B_C(1) = \{x(\cdot) \in C([t_0,\theta];\mathbb{R}^n) : \|x(\cdot)\|_C \le 1\}. \tag{1.1}$$

Definition 1.1. Let $(X, d(\cdot,\cdot))$ be a metric space, $\sigma > 0$ be a given number. The set $E_\sigma = \{x_1, x_2, \dots, x_{a_*}\} \subset E$ consisting of a finite number of points is called a finite σ-net on E if for each $x \in E$ there exists $x_a \in E_\sigma$ such that the inequality $d(x,x_a) \le \sigma$ is satisfied.

2. Dynamics of the control system

Consider a control system described by the differential equation

$$\dot{x}(t) = f(t, x(t)) + B_1(t, x(t))v(t) + B_2(t, x(t))w(t), \quad x(t_0) = x_0 \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the n-dimensional phase state vector, $v \in \mathbb{R}^{m_1}$, $w \in \mathbb{R}^{m_2}$ are the m_1 and m_2-dimensional control vectors respectively, $t \in [t_0, \theta]$ $(t_0 < \theta < \infty)$ is the time, $f(\cdot, \cdot)$ is an n-dimensional vector function and $B_1(\cdot, \cdot)$ and $B_2(\cdot, \cdot)$ are $n \times m_1$ and $n \times m_2$-dimensional matrix functions respectively.

For given $p \in (1, \infty)$, $\alpha > 0$ and $r > 0$, we denote

$$V_\alpha = \{v(\cdot) \in L_\infty([t_0, \theta]; \mathbb{R}^{m_1}) : \|v(\cdot)\|_\infty \leq \alpha\}, \tag{2.2}$$

$$W_{p,r} = \{w(\cdot) \in L_p([t_0, \theta]; \mathbb{R}^{m_2}) : \|w(\cdot)\|_p \leq r\}, \tag{2.3}$$

$$U_{\alpha,p,r} = V_\alpha \times W_{p,r} \tag{2.4}$$

where $L_\infty([t_0, \theta]; \mathbb{R}^{m_1})$ is the space of Lebesgue measurable functions $v(\cdot) : [t_0, \theta] \to \mathbb{R}^{m_1}$ such that $\|v(\cdot)\|_\infty < \infty$, $\|v(\cdot)\|_\infty = \inf\{c > 0 : \|v(t)\| \leq c \text{ for almost all } t \in [t_0, \theta]\}$, $L_p([t_0, \theta]; \mathbb{R}^{m_2})$ is the space of Lebesgue measurable functions $w(\cdot) : [t_0, \theta] \to \mathbb{R}^{m_2}$ such that $\|w(\cdot)\|_p < \infty$, $\|w(\cdot)\|_p = \big(\int_{t_0}^{\theta} \|w(t)\|^p dt\big)^{1/p}$, $\|\cdot\|$ stands for the Euclidean norm.

The set V_α defined by (2.2) is called the set of geometrically constrained control functions, the set $W_{p,r}$ defined by (2.3) is called the set of integrally constrained control functions, the set $U_{\alpha,p,r}$ defined by (2.4) is called the set of the system's admissible control functions. The control functions with integral constraints characterize the control efforts that are exhausted by consumption, while the control functions with geometric constraints characterize the control efforts that are not exhausted by consumption.

It is also assumed that the functions $f(\cdot, \cdot)$, $B_1(\cdot, \cdot)$ and $B_2(\cdot, \cdot)$ satisfy the following conditions:

(A1) The functions $f(\cdot, \cdot) : [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^n$, $B_1(\cdot, \cdot) : [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^{n \times m_1}$ and $B_2(\cdot, \cdot) : [t_0, \theta] \times \mathbb{R}^n \to \mathbb{R}^{n \times m_2}$ are continuous and for every bounded set $D \subset [t_0, \theta] \times \mathbb{R}^n$ there exist Lipschitz constants $\gamma_i = \gamma_i(D) \in (0, \infty)$ $(i = 0, 1, 2)$ such that

$$\begin{gathered} \|f(t, x_*) - f(t, x^*)\| \leq \gamma_0 \|x_* - x^*\|, \\ \|B_1(t, x_*) - B_1(t, x^*)\| \leq \gamma_1 \|x_* - x^*\|, \\ \|B_2(t, x_*) - B_2(t, x^*)\| \leq \gamma_2 \|x_* - x^*\| \end{gathered}$$

for any $(t, x_*) \in D$, $(t, x^*) \in D$.

(A2) There exist constants $\kappa_i \in (0, \infty)$ $(i = 0, 1, 2)$ such that

$$\|f(t, x)\| \leq \kappa_0(1 + \|x\|), \quad \|B_1(t, x)\| \leq \kappa_1(1 + \|x\|), \quad \|B_2(t, x)\| \leq \kappa_2(1 + \|x\|)$$

for every $(t, x) \in [t_0, \theta] \times \mathbb{R}^n$. Here, $\|B_1(t, x)\|$ and $\|B_2(t, x)\|$ denote the Euclidean norms of the matrices $B_1(t, x)$ and $B_2(t, x)$ respectively.

If the functions $(t, x) \to f(t, x)$, $(t, x) \to B_1(t, x)$, $(t, x) \to B_2(t, x)$, $(t, x) \in [t_0, \theta] \times \mathbb{R}^n$, are continuous with respect to (t, x) and Lipschitz continuous with respect to x, then the conditions (A1) and (A2) hold.

Every pair $(v(\cdot), w(\cdot)) \in U_{\alpha,p,r}$ is said to be an admissible control function. Let $(v_*(\cdot), w_*(\cdot)) \in U_{\alpha,p,r}$. The absolutely continuous function $x_*(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ which satisfies the equation $\dot{x}_*(t) = f(t, x_*(t)) + B_1(t, x_*(t))v_*(t) + B_2(t, x_*(t))w_*(t)$ almost everywhere in $[t_0, \theta]$ and initial condition $x_*(t_0) = x_0$, is said to be a trajectory of system (2.1) generated by the admissible control function $(v_*(\cdot), w_*(\cdot))$. By the symbol $X_{\alpha,p,r}(t_0, x_0)$ we denote the set of trajectories of system (2.1) which are generated by all possible admissible control functions $(v(\cdot), w(\cdot)) \in U_{\alpha,p,r}$. The conditions (A1) and (A2) guarantee that each admissible control function generates a unique trajectory on the interval $[t_0, \theta]$. We denote

$$X_{\alpha,p,r}(t; t_0, x_0) = \{x(t) \in \mathbb{R}^n : x(\cdot) \in X_{\alpha,p,r}(t_0, x_0)\}, \quad t \in [t_0, \theta], \tag{2.5}$$

$$\Phi_{\alpha,p,r}(t_0, x_0) = \{(t, x(t)) \in [t_0, \theta] \times \mathbb{R}^n : x(\cdot) \in X_{\alpha,p,r}(t_0, x_0)\}. \tag{2.6}$$

The set $X_{\alpha,p,r}(t_0,x_0)$ is called the set of trajectories of system (2.1), the set $X_{\alpha,p,r}(t;t_0,x_0)$ is said to be the attainable set of system (2.1) at the instant of time t, the set $\Phi_{\alpha,p,r}(t_0,x_0)$ is referred to as the integral funnel of system (2.1) with constraints (2.4) on the control functions and initial position (t_0,x_0). It is obvious that the integral funnel can also be rewritten as

$$\Phi_{\alpha,p,r}(t_0,x_0)=\{(t,x)\in[t_0,\theta]\times\mathbb{R}^n : x\in X_{\alpha,p,r}(t;t_0,x_0)\}.$$

Approximations of the set of trajectories $X_{\alpha,p,r}(t_0,x_0)$, attainable sets $X_{\alpha,p,r}(t;t_0,x_0)$, $t\in[t_0,\theta]$, and the integral funnel $\Phi_{\alpha,p,r}(t_0,x_0)$ will be presented. To solve the problem, step by step, the set of control functions $U_{\alpha,p,r}$ will be replaced by a set, consisting of a finite number of piecewise-constant control functions that generate a finite number of trajectories. Evaluations of the errors in the presented approximations will also be provided.

For the following arguments, we define the bounded and closed cylinder $D_*\subset[t_0,\theta]\times\mathbb{R}^n$ within which the integral funnel $\Phi_{\alpha,p,r}(t_0,x_0)$ is contained. Let

$$\nu_*=[\|x_0\|+\rho_*]\exp(\rho_*), \tag{2.7}$$

$$D_*=\{(t,x)\in[t_0,\theta]\times\mathbb{R}^n\ :\ \|x\|\le\nu_*+1\} \tag{2.8}$$

where

$$\rho_*=(\kappa_0+\alpha\kappa_1)(\theta-t_0)+r\kappa_2(\theta-t_0)^{\frac{p-1}{p}}.$$

Proposition 2.1. *For every $x(\cdot)\in X_{\alpha,p,r}(t_0,x_0)$ the inequality $\|x(\cdot)\|_C\le\nu_*$ is satisfied.*

The proof of the proposition follows from Condition (A2), Hölder's inequality, and Gronwall-Bellman's inequality. Proposition 2.1 also yields the validity of the following corollary.

Corollary 2.2. *The inclusion $\Phi_{\alpha,p,r}(t_0,x_0)\subset D_*$ holds where the cylinder D_* is defined by relation* (2.8).

Here and henceforth, we will consider the cylinder D_* as the set D in Condition (A1). For a given $\Delta>0$, we set

$$\varphi_0(\Delta)=(\kappa_0+\alpha\kappa_1)(1+\nu_*)\Delta+r\kappa_2(1+\nu_*)\Delta^{\frac{p-1}{p}} \tag{2.9}$$

where ν_* is defined by (2.7). It is obvious that $\varphi_0(\Delta)\to0$ as $\Delta\to0^+$.

Proposition 2.3. *For every $t_*\in[t_0,\theta]$, $t^*\in[t_0,\theta]$ and $x(\cdot)\in X_{\alpha,p,r}(t_0,x_0)$, the inequality*

$$\|x(t_*)-x(t^*)\|\le\varphi_0(|t_*-t^*|)$$

is verified.

The proof of this proposition is simple and it follows from Condition (A2) and Proposition 2.1.

Since $\varphi_0(\Delta)\to0$ as $\Delta\to0^+$, then Proposition 2.3 implies that the set of trajectories $X_{\alpha,p,r}(t_0,x_0)$ is a set of equicontinuous functions. Taking into consideration Proposition 2.1 and Proposition 2.3, from Arzela-Ascoli theorem we obtain the validity of following proposition.

Proposition 2.4. *The set of trajectories $X_{\alpha,p,r}(t_0,x_0)$ generated by mixed constrained control functions $U_{\alpha,p,r}$ is a precompact subset of the space $C([t_0,\theta];\mathbb{R}^n)$.*

We denote

$$\varphi_*(\Delta)=\max\{\Delta,\varphi_0(\Delta)\}, \tag{2.10}$$

$$\begin{aligned}\omega_0(\varphi_*(\Delta))=\max\big\{&\|f(t_*,x_*)-f(t^*,x^*)\|:|t_*-t^*|\le\varphi_*(\Delta),\ \|x_*-x^*\|\le\varphi_*(\Delta),\\&(t_*,x_*)\in D_*,\ (t^*,x^*)\in D_*\big\},\end{aligned} \tag{2.11}$$

$$\begin{aligned}\omega_1(\varphi_*(\Delta))=\max\big\{&\|B_1(t_*,x_*)-B_1(t^*,x^*)\|:|t_*-t^*|\le\varphi_*(\Delta),\ \|x_*-x^*\|\le\varphi_*(\Delta),\\&(t_*,x_*)\in D_*,\ (t^*,x^*)\in D_*\big\},\end{aligned} \tag{2.12}$$

$$\begin{aligned}\omega_2(\varphi_*(\Delta))=\max\big\{&\|B_2(t_*,x_*)-B_2(t^*,x^*)\|:|t_*-t^*|\le\varphi_*(\Delta),\ \|x_*-x^*\|\le\varphi_*(\Delta),\\&(t_*,x_*)\in D_*,\ (t^*,x^*)\in D_*\big\}\end{aligned} \tag{2.13}$$

where the cylinder $D_*\subset[t_0,\theta]\times\mathbb{R}^n$ is defined by equality (2.8), $\varphi_0(\Delta)$ is defined by (2.9). It is obvious that $\omega_0(\varphi_*(\Delta))\to0$, $\omega_1(\varphi_*(\Delta))\to0$ and $\omega_2(\varphi_*(\Delta))\to0$ as $\Delta\to0^+$.

If the functions $(t,x) \to f(t,x)$, $(t,x) \to B_1(t,x)$, $(t,x) \to B_2(t,x)$, $(t,x) \in D_*$, are Lipschitz continuous with respect to (t,x) on the cylinder D_*, i.e. if there exist the constants $\rho_i > 0$, $i = 0,1,2$, such that

$$\begin{aligned} \|f(t_*,x_*) - f(t^*,x^*)\| &\le \rho_0[|t_* - t^*| + \|x_* - x^*\|], \\ \|B_1(t_*,x_*) - B_1(t^*,x^*)\| &\le \rho_1[|t_* - t^*| + \|x_* - x^*\|], \\ \|B_2(t_*,x_*) - B_2(t^*,x^*)\| &\le \rho_2[|t_* - t^*| + \|x_* - x^*\|] \end{aligned} \tag{2.14}$$

for any $(t_*,x_*) \in D_*$, $(t^*,x^*) \in D_*$, then the quantities $\omega_0(\varphi_*(\Delta))$, $\omega_1(\varphi_*(\Delta))$ and $\omega_2(\varphi_*(\Delta))$ satisfy the evaluations $\omega_0(\varphi_*(\Delta)) \le 2\rho_0\varphi_*(\Delta)$, $\omega_1(\varphi_*(\Delta)) \le 2\rho_1\varphi_*(\Delta)$ and $\omega_2(\varphi_*(\Delta)) \le 2\rho_2\varphi_*(\Delta)$.

Proposition 2.3 implies the validity of the following corollary.

Corollary 2.5. *For every $t_* \in [t_0,\theta]$ and $t^* \in [t_0,\theta]$ the inequality*

$$h_n\left(X_{\alpha,p,r}(t_*;t_0,x_0), X_{\alpha,p,r}(t^*;t_0,x_0)\right) \le \varphi_0(|t_* - t^*|)$$

holds, where $\varphi_0(\cdot)$ is defined by equality (2.9).

The above corollary implies that $h_n\big(X_{\alpha,p,r}(t;t_0,x_0), X_{\alpha,p,r}(t_*;t_0,x_0)\big) \to 0$ as $t \to t_*$ uniformly on the interval $[t_0,\theta]$.

3. Approximation of the set of trajectories

Let $\beta > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$ be given numbers, $\Gamma = \{t_0, t_1, \dots, t_N = \theta\}$ be a uniform partition of the closed interval $[t_0,\theta]$, $\Lambda_1 = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{c_1} = \alpha\}$ be a uniform partition of the closed interval $[0,\alpha]$, $\Lambda_2 = \{0 = r_0, r_1, \dots, r_{c_2} = \beta\}$ be a uniform partition of the closed interval $[0,\beta]$, $\Delta = \frac{\theta - t_0}{N} = t_{i+1} - t_i$, $i = 0,1,\dots,N-1$, be the diameter of the partition Γ, $\delta_1 = \frac{\alpha}{c_1} = \alpha_{j+1} - \alpha_j$, $j = 0,1,\dots,c_1 - 1$, be the diameter of the partition Λ_1, $\delta_2 = \frac{\beta}{c_2} = r_{l+1} - r_l$, $l = 0,1,\dots,c_2 - 1$, be the diameter of the partition Λ_2, $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$, $S^1_{\sigma_1} = \{b_1, b_2, \dots, b_{c_3}\}$ be a finite σ_1-net on S^1, $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$, $S^2_{\sigma_2} = \{e_1, e_2, \dots, e_{c_4}\}$ be a finite σ_2-net on S^2. Note that $\alpha > 0$ is the number which characterizes the upper bound of the geometric constraint and is given in (2.2). Denote

$$\begin{aligned} V_{\alpha}^{\Gamma,\Lambda_1,S^1_{\sigma_1}} &= \Big\{ v(\cdot) \in L_\infty([t_0,\theta];\mathbb{R}^{m_1}) : v(t) = \alpha_{j_i} b_{k_i} \text{ for every } t \in [t_i, t_{i+1}), \\ &\qquad \alpha_{j_i} \in \Lambda_1,\ b_{k_i} \in S^1_{\sigma_1}, i = 0,1,\dots,N-1 \Big\}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}} &= \Big\{ w(\cdot) \in L_p([t_0,\theta];\mathbb{R}^{m_2}) : w(t) = r_{l_i} e_{g_i} \text{ for every } t \in [t_i, t_{i+1}), \\ &\qquad r_{l_i} \in \Lambda_2,\ e_{g_i} \in S^2_{\sigma_2},\ i = 0,1,\dots,N-1,\ \Delta \cdot \sum_{i=0}^{N-1} r_{l_i}^p \le r^p \Big\}, \end{aligned} \tag{3.2}$$

$$U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}} = V_{\alpha}^{\Gamma,\Lambda_1,S^1_{\sigma_1}} \times W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}}. \tag{3.3}$$

It is obvious that

$$\begin{aligned} &U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}} \\ &= \Big\{ (v(\cdot), w(\cdot)) \in L_\infty([t_0,\theta];\mathbb{R}^{m_1}) \times L_p([t_0,\theta];\mathbb{R}^{m_2}) : v(t) = \alpha_{j_i} b_{k_i}, w(t) = r_{l_i} e_{g_i} \\ &\quad \text{for every } t \in [t_i, t_{i+1}),\ \alpha_{j_i} \in \Lambda_1,\ r_{l_i} \in \Lambda_2,\ b_{k_i} \in S^1_{\sigma_1},\ e_{g_i} \in S^2_{\sigma_2}, \\ &\quad i = 0,1,\dots,N-1,\ \Delta \cdot \sum_{i=0}^{N-1} r_{l_i}^p \le r^p \Big\}. \end{aligned} \tag{3.4}$$

With the symbol $X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ we denote the set of trajectories of system (2.1) generated by all piecewise-constant control functions $(v(\cdot), w(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}$. Since the set

$U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}$ consists of a finite number of control functions, then the set $X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ also consists of a finite number of trajectories and

$$X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0) \subset X_{\alpha,p,r}(t_0,x_0).$$

For a given $t \in [t_0,\theta]$ we also denote

$$X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t;t_0,x_0) = \Big\{x(t) \in \mathbb{R}^n : x(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)\Big\}. \tag{3.5}$$

Let

$$\gamma_* = (\gamma_0+\gamma_1\alpha)(\theta-t_0) + \gamma_2 r(\theta-t_0)^{\frac{p-1}{p}}, \tag{3.6}$$
$$\kappa_* = 2\kappa_2(1+\nu_*)r^p\exp(\gamma_*), \tag{3.7}$$
$$\xi_1(\Delta) = 2\alpha\omega_1(\varphi_*(\Delta))(\theta-t_0) + 2\alpha\kappa_1(1+\nu_*)\Delta, \tag{3.8}$$
$$\xi_2(\Delta) = 2r\omega_2(\varphi_*(\Delta))(\theta-t_0)^{\frac{p-1}{p}} + 2r\kappa_2(1+\nu_*)\Delta^{\frac{p-1}{p}}, \tag{3.9}$$
$$\xi(\Delta) = [\xi_1(\Delta)+\xi_2(\Delta)]\exp(\gamma_*), \tag{3.10}$$
$$\lambda_1(\delta_1) = \kappa_1(1+\nu_*)(\theta-t_0)\delta_1\exp(\gamma_*), \tag{3.11}$$
$$\lambda_2(\delta_2) = \kappa_2(1+\nu_*)(\theta-t_0)\delta_2\exp(\gamma_*), \tag{3.12}$$
$$\chi_1(\sigma_1) = \kappa_1(1+\nu_*)(\theta-t_0)\alpha\sigma_1\exp(\gamma_*), \tag{3.13}$$
$$\chi_2(\beta,\sigma_2) = \kappa_2(1+\nu_*)(\theta-t_0)\beta\sigma_2\exp(\gamma_*), \tag{3.14}$$

where $\varphi_*(\Delta)$, $\omega_1(\varphi_*(\Delta))$ and $\omega_2(\varphi_*(\Delta))$ are defined by relations (2.10), (2.12) and (2.13) respectively, ν_* is defined by equality (2.7), the constants γ_i, κ_i $(i=0,1,2)$ are defined in Conditions (A1) and (A2). We have that $\xi(\Delta)\to 0$ as $\Delta\to 0^+$, $\lambda_1(\delta_1)\to 0$ as $\delta_1\to 0^+$, $\lambda_2(\delta_2)\to 0$ as $\delta_2\to 0^+$, $\chi_1(\sigma_1)\to 0$ as $\sigma_1\to 0^+$ and $\chi_2(\beta,\sigma_2)\to 0$ as $\sigma_2\to 0^+$ for any fixed $\beta>0$.

Theorem 3.1. *The inequality*

$$h_C(X_{\alpha,p,r}(t_0,x_0), X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)) \le \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2)$$

is satisfied for every $\beta>0$, for every uniform partition $\Gamma=\{t_0,t_1,\dots,t_N=\theta\}$ of the closed interval $[t_0,\theta]$, for every uniform partition $\Lambda_1=\{0=\alpha_0,\alpha_1,\dots,\alpha_{c_1}=\alpha\}$ of the closed interval $[0,\alpha]$, for every uniform partition $\Lambda_2=\{0=r_0,r_1,\dots,r_{c_2}=\beta\}$ of the closed interval $[0,\beta]$, for every finite σ_1-net $S^1_{\sigma_1}=\{b_1,b_2,\dots,b_{c_3}\}$ on the unit sphere $S^1=\{b\in\mathbb{R}^{m_1}:\|b\|=1\}$ and for every finite σ_2-net $S^2_{\sigma_2}=\{e_1,e_2,\dots,e_{c_4}\}$ on the unit sphere $S^2=\{e\in\mathbb{R}^{m_2}:\|e\|=1\}$.

Here $\Delta=\frac{\theta-t_0}{N}=t_{i+1}-t_i$, $i=0,1,\dots,N-1$, is the diameter of the partition Γ, $\delta_1=\frac{\alpha}{c_1}=\alpha_{j+1}-\alpha_j$, $j=0,1,\dots,c_1-1$, is the diameter of the partition Λ_1, $\delta_2=\frac{\beta}{c_2}=r_{l+1}-r_l$, $l=0,1,\dots,c_2-1$, is the diameter of the partition Λ_2, $\sigma_1>0$ and $\sigma_2>0$ are given numbers, κ_*, $\xi(\Delta)$, $\lambda_1(\delta_1)$, $\lambda_2(\delta_2)$, $\chi_1(\sigma_1)$ and $\chi_2(\beta,\sigma_2)$ are defined by (3.7), (3.10), (3.11), (3.12), (3.13) *and* (3.14) *respectively.*

Proof. The theorem is proved in 4 steps.

Step 1. Geometric constraint for integrally constrained control functions. We define a new set of control functions, where an additional geometric constraint is introduced for integrally constrained control functions. For a given $\beta>0$ we denote

$$W^\beta_{p,r} = \{w(\cdot)\in L_p([t_0,\theta];\mathbb{R}^{m_2}) : \|w(\cdot)\|_p\le r,\ \|w(\cdot)\|_\infty\le\beta\}, \tag{3.15}$$
$$U^\beta_{\alpha,p,r} = V_\alpha \times W^\beta_{p,r}, \tag{3.16}$$

and denote by $X^\beta_{\alpha,p,r}(t_0,x_0)$ the set of trajectories of system (2.1) generated by all possible admissible control functions $(v(\cdot),w(\cdot))\in U^\beta_{\alpha,p,r}$. It is obvious that

$$X^\beta_{\alpha,p,r}(t_0,x_0) \subset X_{\alpha,p,r}(t_0,x_0). \tag{3.17}$$

Let us prove that

$$X_{\alpha,p,r}(t_0,x_0) \subset X^{\beta}_{\alpha,p,r}(t_0,x_0) + \frac{\kappa_*}{\beta^{p-1}} B_C(1) \tag{3.18}$$

where $B_C(1)$ and κ_* are defined by equalities (1.1) and (3.7) respectively.

Choose an arbitrary trajectory $x(\cdot) \in X_{\alpha,p,r}(t_0,x_0)$ generated by the control function $(v(\cdot),w(\cdot)) \in U_{\alpha,p,r} = V_\alpha \times W_{p,r}$. Define a new control function $(v_*(\cdot),w_*(\cdot))$ by setting $v_*(\cdot) = v(\cdot)$ and

$$w_*(t) = \begin{cases} w(t) & \text{if } \|w(t)\| \le \beta, \\ \beta \frac{w(t)}{\|w(t)\|} & \text{if } \|w(t)\| > \beta\,. \end{cases} \tag{3.19}$$

It is not difficult to show that $w_*(\cdot) \in W^{\beta}_{p,r}$, and hence $(v_*(\cdot),w_*(\cdot)) \in U^{\beta}_{\alpha,p,r} = V_\alpha \times W^{\beta}_{p,r}$. Let $x_*(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ be the trajectory of system (2.1) generated by the control function $(v_*(\cdot),w_*(\cdot))$. Then $x_*(\cdot) \in X^{\beta}_{\alpha,p,r}(t_0,x_0)$.

Denote $T(\beta) = \{t \in [t_0,\theta] : \|w(t)\| > \beta\}$. Since $w_*(\cdot) \in W^{\beta}_{p,r}$, then by virtue of Tchebyshev's inequality (see, e.g., [23, p. 82]) we have that

$$\text{meas}(T(\beta)) \le \frac{r^p}{\beta^p} \tag{3.20}$$

where meas$(T(\beta))$ stands for Lebesgue measure of the set $T(\beta)$.

Taking into consideration that $v_*(\cdot) = v(\cdot)$ and $v(\cdot) \in V_\alpha$, $w(\cdot) \in W_{p,r}$, $w_*(\cdot) \in W^{\beta}_{p,r} \subset W_{p,r}$, from Conditions (A1), (A2), Proposition 2.1, Hölder's inequality, Minkowski's inequality, (3.19) and (3.20) we obtain that

$$\begin{aligned}
&\|x(t) - x_*(t)\| \\
&\le \int_{t_0}^{t} \|f(\tau,x(\tau)) - f(\tau,x_*(\tau))\| d\tau + \int_{t_0}^{t} \|B_1(\tau,x(\tau))v(\tau) - B_1(\tau,x_*(\tau))v_*(\tau)\| d\tau \\
&\quad + \int_{t_0}^{t} \|B_2(\tau,x(\tau))w(\tau) - B_2(\tau,x_*(\tau))w_*(\tau)\| d\tau \\
&\le \int_{t_0}^{t} \gamma_0 \|x(\tau)) - x_*(\tau)\| d\tau + \int_{t_0}^{t} \gamma_1 \|x(\tau) - x_*(\tau)\| \|v(\tau)\| d\tau \\
&\quad + \int_{t_0}^{t} \|B_2(\tau,x(\tau)) - B_2(\tau,x_*(\tau))\| \|w(\tau)\| d\tau + \int_{t_0}^{t} \|B_2(\tau,x_*(\tau))\| \|w(\tau) - w_*(\tau)\| d\tau \\
&\le \int_{t_0}^{t} [\gamma_0 + \gamma_1\alpha + \gamma_2\|w(\tau)\|]\, \|x(\tau) - x_*(\tau)\| d\tau + \int_{T(\beta)} \kappa_2(1+\nu_*) \|w(\tau) - w_*(\tau)\| d\tau \\
&\le \int_{t_0}^{t} [\gamma_0 + \gamma_1\alpha + \gamma_2\|w(\tau)\|]\, \|x(\tau) - x_*(\tau)\| d\tau \\
&\quad + \kappa_2(1+\nu_*)[meas\,(T(\beta))]^{\frac{p-1}{p}} \Big(\int_{T(\beta)} \|w(\tau) - w_*(\tau)\|^p d\tau \Big)^{1/p} \\
&\le \int_{t_0}^{t} [\gamma_0 + \gamma_1\alpha + \gamma_2\|w(\tau)\|]\, \|x(\tau) - x_*(\tau)\| d\tau + 2r\kappa_2(1+\nu_*) \frac{r^{p-1}}{\beta^{p-1}} \\
&= \int_{t_0}^{t} [\gamma_0 + \gamma_1\alpha + \gamma_2\|w(\tau)\|]\, \|x(\tau) - x_*(\tau)\| d\tau + 2\kappa_2(1+\nu_*) \frac{r^p}{\beta^{p-1}}
\end{aligned}$$

for every $t \in [t_0,\theta]$. Now the last inequality, the Gronwall-Bellman inequality, (3.6) and (3.7) yield that

$$\begin{aligned}
\|x(t) - x_*(t)\| &\le 2\kappa_2(1+\nu_*) \frac{r^p}{\beta^{p-1}} \exp\Big[\int_{t_0}^{t} (\gamma_0 + \gamma_1\alpha + \gamma_2\|w(\tau)\|)\, d\tau \Big] \\
&\le 2\kappa_2(1+\nu_*) \frac{r^p}{\beta^{p-1}} \exp\Big[(\gamma_0 + \gamma_1\alpha)(\theta - t_0) + \gamma_2 r(\theta - t_0)^{\frac{p-1}{p}} \Big] = \frac{\kappa_*}{\beta^{p-1}}
\end{aligned}$$

for every $t \in [t_0, \theta]$, and consequently

$$\|x(\cdot) - x_*(\cdot)\|_C \le \frac{\kappa_*}{\beta^{p-1}}\,. \tag{3.21}$$

Since $x(\cdot) \in X_{\alpha,p,r}(t_0, x_0)$ is an arbitrarily chosen trajectory, $x_*(\cdot) \in X^{\beta}_{\alpha,p,r}(t_0, x_0)$, then (3.21) yields the validity of the inclusion (3.18). From (3.17) and (3.18) it follows that

$$h_C\left(X_{\alpha,p,r}(t_0, x_0), X^{\beta}_{\alpha,p,r}(t_0, x_0)\right) \le \frac{\kappa_*}{\beta^{p-1}}\,. \tag{3.22}$$

Step 2. Piecewise-constant control functions. Now we define the new class of control functions which consists of piecewise-constant functions on a uniform partition Γ of the interval $[t_0, \theta]$. We set

$$V^{\Gamma}_{\alpha} = \{v(\cdot) \in V_\alpha : v(t) = v_i \text{ for every } t \in [t_i, t_{i+1}),\ i = 0, 1, \dots, N-1\}, \tag{3.23}$$

$$W^{\beta,\Gamma}_{p,r} = \{w(\cdot) \in W^{\beta}_{p,r} : w(t) = w_i \text{ for every } t \in [t_i, t_{i+1}),\ i = 0, 1, \dots, N-1\}, \tag{3.24}$$

$$U^{\beta,\Gamma}_{\alpha,p,r} = V^{\Gamma}_{\alpha} \times W^{\beta,\Gamma}_{p,r}, \tag{3.25}$$

where the set $W^{\beta}_{p,r}$ is defined by equality (3.15). It is obvious that

$$\begin{aligned} U^{\beta,\Gamma}_{\alpha,p,r} = \big\{&(v(\cdot), w(\cdot)) \in U^{\beta}_{\alpha,p,r} : v(t) = v_i, w(t) = w_i \\ &\text{for every } t \in [t_i, t_{i+1}),\ i = 0, 1, \dots, N-1\big\} \end{aligned} \tag{3.26}$$

where the set of control functions $U^{\beta}_{\alpha,p,r}$ is defined by equality (3.16). Let $X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0)$ be the set of trajectories of system (2.1) generated by all control functions $(v(\cdot), w(\cdot)) \in U^{\beta,\Gamma}_{\alpha,p,r}$. It is obvious that

$$X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0) \subset X^{\beta}_{\alpha,p,r}(t_0, x_0). \tag{3.27}$$

Now we prove that

$$X^{\beta}_{\alpha,p,r}(t_0, x_0) \subset X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0) + \xi(\Delta) B_C(1) \tag{3.28}$$

where $\xi(\Delta)$ is defined by (3.10), $B_C(1)$ is defined by (1.1).

Let us choose an arbitrary $\tilde{x}(\cdot) \in X^{\beta}_{\alpha,p,r}(t_0, x_0)$ generated by the control function $(\tilde{v}(\cdot), \tilde{w}(\cdot)) \in U^{\beta}_{\alpha,p,r}$ and define a new piecewise-constant control function $(\tilde{v}_*(\cdot), \tilde{w}_*(\cdot)) : [t_0, \theta] \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, setting

$$\tilde{v}_*(t) = \frac{1}{\Delta} \int_{t_i}^{t_{i+1}} \tilde{v}(\tau) d\tau, \quad t \in [t_i, t_{i+1}),\ i = 0, 1, \dots, N-1, \tag{3.29}$$

$$\tilde{w}_*(t) = \frac{1}{\Delta} \int_{t_i}^{t_{i+1}} \tilde{w}(\tau) d\tau, \quad t \in [t_i, t_{i+1}),\ i = 0, 1, \dots, N-1. \tag{3.30}$$

It is not difficult to prove that $\|\tilde{v}_*(t)\| \le \alpha$, $\|\tilde{w}_*(t)\| \le \beta$ for every $t \in [t_0, \theta]$, $\|\tilde{w}_*(\cdot)\|_p \le r$. Hence, by (3.26) we have $(\tilde{v}_*(\cdot), \tilde{w}_*(\cdot)) \in U^{\beta,\Gamma}_{\alpha,p,r}$. Let $\tilde{x}_*(\cdot)$ be the trajectory of system (2.1) generated by the control function $(\tilde{v}_*(\cdot), \tilde{w}_*(\cdot))$. Then, $\tilde{x}_*(\cdot) \in X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0)$ and taking into account condition (A1), we obtain

$$\begin{aligned} &\|\tilde{x}(t) - \tilde{x}_*(t)\| \\ &\le \int_{t_0}^{t} \|f(\tau, \tilde{x}(\tau)) - f(\tau, \tilde{x}_*(\tau))\| d\tau + \int_{t_0}^{t} \|B_1(\tau, \tilde{x}(\tau)) - B_1(\tau, \tilde{x}_*(\tau))\| \|\tilde{v}(\tau)\| d\tau \\ &\quad + \Big\| \int_{t_0}^{t} B_1(\tau, \tilde{x}_*(\tau))(\tilde{v}(\tau) - \tilde{v}_*(\tau)) d\tau \Big\| + \int_{t_0}^{t} \|B_2(\tau, \tilde{x}(\tau)) - B_2(\tau, \tilde{x}_*(\tau))\| \|\tilde{w}(\tau)\| d\tau \\ &\quad + \Big\| \int_{t_0}^{t} B_2(\tau, \tilde{x}_*(\tau))(\tilde{w}(\tau) - \tilde{w}_*(\tau)) d\tau \Big\| \\ &\le \int_{t_0}^{t} (\gamma_0 + \gamma_1 \|\tilde{v}(\tau)\| + \gamma_2 \|\tilde{w}(\tau)\|) \|\tilde{x}(\tau) - \tilde{x}_*(\tau)\| d\tau \end{aligned}$$

$$+\Big\|\int_{t_0}^{t}B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau\Big\|+\Big\|\int_{t_0}^{t}B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau\Big\| \tag{3.31}$$

for all $t\in[t_0,\theta]$. Let us derive evaluations for $\big\|\int_{t_0}^{t}B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau\big\|$ and $\big\|\int_{t_0}^{t}B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau\big\|$, taking into consideration equalities (3.29), (3.30) and the inclusions $\tilde{v}(\cdot)\in V_\alpha$, $\tilde{v}_*(\cdot)\in V_\alpha^{\Gamma}\subset V_\alpha$, $\tilde{w}(\cdot)\in W_{p,r}^{\beta}$ and $\tilde{w}_*(\cdot)\in W_{p,r}^{\beta,\Gamma}$.

Now choose an arbitrary $t\in[t_0,\theta]$ and fix it. Let $t\in[t_0,t_1]$. From Condition (A2), Proposition 2.1, inclusions $\tilde{v}(\cdot)\in V_\alpha$, $\tilde{v}_*(\cdot)\in V_\alpha^{\Gamma}\subset V_\alpha$, $\tilde{w}(\cdot)\in W_{p,r}^{\beta}\subset W_{p,r}$, $\tilde{w}_*(\cdot)\in W_{p,r}^{\beta,\Gamma}\subset W_{p,r}$, Hölder's inequality and inequality $t-t_0\le\Delta$ it follows that

$$\int_{t_0}^{t}\|B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))\|d\tau\le 2\alpha\kappa_1(1+\nu_*)\Delta, \tag{3.32}$$

$$\int_{t_0}^{t}\|B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))\|d\tau\le 2r\kappa_2(1+\nu_*)\Delta^{\frac{p-1}{p}}. \tag{3.33}$$

Now let $t>t_1$. Then there exists $k=1,2,\dots,N-1$ such that $t\in(t_k,t_{k+1}]$. Then

$$\begin{aligned}&\int_{t_0}^{t}B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau\\&=\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau+\int_{t_k}^{t}B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau\,,\end{aligned} \tag{3.34}$$

$$\begin{aligned}&\int_{t_0}^{t}B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau\\&=\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau+\int_{t_k}^{t}B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau\,.\end{aligned} \tag{3.35}$$

Choose an arbitrary $i=0,1,\dots,k-1$. From (3.29) and (3.30) we have

$$\int_{t_i}^{t_{i+1}}(\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau=0,\quad \int_{t_i}^{t_{i+1}}(\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau=0$$

and consequently

$$\begin{aligned}&\int_{t_i}^{t_{i+1}}B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau\\&=\int_{t_i}^{t_{i+1}}[B_1(\tau,\tilde{x}_*(\tau))-B_1(t_i,\tilde{x}_*(t_i))](\tilde{v}(\tau)-\tilde{v}_*(\tau))d\tau\,,\end{aligned} \tag{3.36}$$

$$\begin{aligned}&\int_{t_i}^{t_{i+1}}B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau\\&=\int_{t_i}^{t_{i+1}}[B_2(\tau,\tilde{x}_*(\tau))-B_2(t_i,\tilde{x}_*(t_i))](\tilde{w}(\tau)-\tilde{w}_*(\tau))d\tau\,.\end{aligned} \tag{3.37}$$

Since $\tilde{x}_*(\cdot)\in X_{\alpha,p,r}^{\beta,\Gamma}(t_0,x_0)\subset X_{\alpha,p,r}(t_0,x_0)$, by Proposition 2.3 we have that

$$\|\tilde{x}_*(t)-\tilde{x}_*(t_i)\|\le\varphi_0(\Delta)\le\varphi_*(\Delta) \tag{3.38}$$

for every $t\in[t_i,t_{i+1}]$, where $\varphi_0(\Delta)$ and $\varphi_*(\Delta)$ are defined by (2.9) and (2.10) respectively. Since $\Delta\le\varphi_*(\Delta)$, from (2.12), (2.13) and (3.38) we obtain

$$\begin{aligned}&\int_{t_i}^{t_{i+1}}\|B_1(\tau,\tilde{x}_*(\tau))-B_1(t_i,\tilde{x}_*(t_i))\|\|\tilde{v}(\tau)-\tilde{v}_*(\tau)\|d\tau\\&\le\omega_1(\varphi_*(\Delta))\int_{t_i}^{t_{i+1}}\|\tilde{v}(\tau)-\tilde{v}_*(\tau)\|d\tau\,,\end{aligned} \tag{3.39}$$

$$
\begin{aligned}
&\int_{t_i}^{t_{i+1}} \|B_2(\tau,\tilde{x}_*(\tau)) - B_2(t_i,\tilde{x}_*(t_i))\| \|\tilde{w}(\tau)-\tilde{w}_*(\tau)\| d\tau \\
&\le \omega_2(\varphi_*(\Delta)) \int_{t_i}^{t_{i+1}} \|\tilde{w}(\tau)-\tilde{w}_*(\tau)\| d\tau\,.
\end{aligned} \tag{3.40}
$$

Taking into consideration that $i = 0,1,\dots,k-1$ is arbitrarily chosen, the inclusions $\tilde{v}(\cdot) \in V_\alpha$, $\tilde{v}_*(\cdot) \in V_\alpha^\Gamma \subset V_\alpha$, $\tilde{w}(\cdot) \in W_{p,r}^\beta \subset W_{p,r}$, $\tilde{w}_*(\cdot) \in W_{p,r}^{\beta,\Gamma} \subset W_{p,r}$, the inequalities (3.39), (3.40) and applying Hölder's and Minkowski's inequalities we obtain that

$$
\begin{aligned}
&\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))\| d\tau \\
&\le \sum_{i=0}^{k-1} \omega_1(\varphi_*(\Delta)) \int_{t_i}^{t_{i+1}} \|\tilde{v}(\tau)-\tilde{v}_*(\tau)\| d\tau \\
&\le \omega_1(\varphi_*(\Delta)) \int_{t_0}^{t_k} \|\tilde{v}(\tau)-\tilde{v}_*(\tau)\| d\tau \le 2\alpha\omega_1(\varphi_*(\Delta))(t_k-t_0) \\
&\le 2\alpha\omega_1(\varphi_*(\Delta))(\theta-t_0),
\end{aligned} \tag{3.41}
$$

$$
\begin{aligned}
&\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \|B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))\| d\tau \\
&\le \sum_{i=0}^{k-1} \omega_2(\varphi_*(\Delta)) \int_{t_i}^{t_{i+1}} \|\tilde{w}(\tau)-\tilde{w}_*(\tau)\| d\tau \\
&\le \omega_2(\varphi_*(\Delta)) \int_{t_0}^{t_k} \|\tilde{w}(\tau)-\tilde{w}_*(\tau)\| d\tau \le 2r\omega_2(\varphi_*(\Delta))(t_k-t_0)^{\frac{p-1}{p}} \\
&\le 2r\omega_2(\varphi_*(\Delta))(\theta-t_0)^{\frac{p-1}{p}}.
\end{aligned} \tag{3.42}
$$

Analogously to inequalities (3.32) and (3.33), from Condition (A2), Proposition 2.1, inclusions $\tilde{v}(\cdot) \in V_\alpha$, $\tilde{v}_*(\cdot) \in V_\alpha^\Gamma \subset V_\alpha$, $\tilde{w}(\cdot) \in W_{p,r}^\beta \subset W_{p,r}$, $\tilde{w}_*(\cdot) \in W_{p,r}^{\beta,\Gamma} \subset W_{p,r}$, and $t \in (t_k, t_{k+1}]$ it follows that

$$
\int_{t_k}^{t} \|B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau))\| d\tau \le 2\alpha\kappa_1(1+\nu_*)\Delta, \tag{3.43}
$$

$$
\int_{t_k}^{t} \|B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau))\| d\tau \le 2r\kappa_2(1+\nu_*)\Delta^{\frac{p-1}{p}} \tag{3.44}
$$

where $k \ge 1$.

The relations (3.34), (3.36), (3.41), and (3.43) imply that

$$
\Big\| \int_{t_0}^{t} B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau)) d\tau \Big\| \le 2\alpha\omega_1(\varphi_*(\Delta))(\theta-t_0) + 2\alpha\kappa_1(1+\nu_*)\Delta, \tag{3.45}
$$

and from (3.35), (3.37), (3.42) ,and (3.44) we have that

$$
\Big\| \int_{t_0}^{t} B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau)) d\tau \Big\| \le 2r\omega_2(\varphi_*(\Delta))(\theta-t_0)^{\frac{p-1}{p}} + 2r\kappa_2(1+\nu_*)\Delta^{\frac{p-1}{p}} \tag{3.46}
$$

where $t \in (t_k, t_{k+1}]$, $k = 1,2,\dots,N-1$.

Concluding, from (3.8), (3.32), (3.45) we obtain that

$$
\Big\| \int_{t_0}^{t} B_1(\tau,\tilde{x}_*(\tau))(\tilde{v}(\tau)-\tilde{v}_*(\tau)) d\tau \Big\| \le 2\alpha\omega_1(\varphi_*(\Delta))(\theta-t_0) + 2\alpha\kappa_1(1+\nu_*)\Delta = \xi_1(\Delta), \tag{3.47}
$$

and from (3.9), (3.33) and (3.46) it follows that

$$
\Big\| \int_{t_0}^{t} B_2(\tau,\tilde{x}_*(\tau))(\tilde{w}(\tau)-\tilde{w}_*(\tau)) d\tau \Big\| \le 2r\omega_2(\varphi_*(\Delta))(\theta-t_0)^{\frac{p-1}{p}} + 2r\kappa_2(1+\nu_*)\Delta^{\frac{p-1}{p}} = \xi_2(\Delta), \tag{3.48}
$$

where t is an arbitrarily fixed instant of time in the interval $[t_0, \theta]$.

Inequalities (3.31), (3.47), and (3.48) yield

$$\|\tilde{x}(t) - \tilde{x}_*(t)\| \le \int_{t_0}^{t} (\gamma_0 + \gamma_1 \|\tilde{v}(\tau)\| + \gamma_2 \|\tilde{w}(\tau)\|) \|\tilde{x}(\tau) - \tilde{x}_*(\tau)\| d\tau + \xi_1(\Delta) + \xi_2(\Delta)$$

for every $t \in [t_0, \theta]$. Finally, the last inequality, the Gronwall-Bellman inequality, inclusions $\tilde{v}(\cdot) \in V_\alpha$, $\tilde{w}(\cdot) \in W^{\beta}_{p,r} \subset W_{p,r}$, (3.6) and (3.10) yield

$$\begin{aligned} \|\tilde{x}(t) - \tilde{x}_*(t)\| &\le [\xi_1(\Delta) + \xi_2(\Delta)] \exp \Big[\int_{t_0}^{t} (\gamma_0 + \gamma_1 \|\tilde{v}(\tau)\| + \gamma_2 \|\tilde{w}(\tau)\|) d\tau \Big] \\ &\le [\xi_1(\Delta) + \xi_2(\Delta)] \exp \Big[(\gamma_0 + \gamma_1 \alpha)(\theta - t_0) + r\gamma_2 (\theta - t_0)^{\frac{p-1}{p}} \Big] = \xi(\Delta) \end{aligned}$$

for every $t \in [t_0, \theta]$, and hence

$$\|\tilde{x}(\cdot) - \tilde{x}_*(\cdot)\|_C \le \xi(\Delta).$$

Since $\tilde{x}(\cdot) \in X^{\beta}_{\alpha,p,r}(t_0, x_0)$ is an arbitrarily chosen trajectory, $\tilde{x}_*(\cdot) \in X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0)$, then the last inequality implies that the inclusion (3.28) is valid. From inclusions (3.27) and (3.28) it follows that

$$h_C \left(X^{\beta}_{\alpha,p,r}(t_0, x_0), X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0) \right) \le \xi(\Delta) \tag{3.49}$$

where $\Delta > 0$ is the diameter of the uniform partition Γ.

Step 3. Piecewise-constant control functions with norms from uniform partitions. Using uniform partitions $\Lambda_1 = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{c_1} = \alpha\}$ of the closed interval $[0, \alpha]$ and $\Lambda_2 = \{0 = r_0, r_1, \dots, r_{c_2} = \beta\}$ of the closed interval $[0, \beta]$, where $\delta_1 = \frac{\alpha}{c_1} = \alpha_{j+1} - \alpha_j$, $j = 0, 1, \dots, c_1 - 1$, is the diameter of the partition Λ_1, and $\delta_2 = \frac{\beta}{c_2} = r_{l+1} - r_l$, $l = 0, 1, \dots, c_2 - 1$, is the diameter of the partition Λ_2, define a new set of control functions, by setting

$$V^{\Gamma,\Lambda_1}_{\alpha} = \{v(\cdot) \in V^{\Gamma}_{\alpha} : v(t) = v_i, \|v_i\| \in \Lambda_1 \text{ for every } t \in [t_i, t_{i+1}),\ i = 0, 1, \dots, N-1\}, \tag{3.50}$$

$$\begin{aligned} W^{\beta,\Gamma,\Lambda_2}_{p,r} = \big\{ &w(\cdot) \in W^{\beta,\Gamma}_{p,r} : w(t) = w_i, \|w_i\| \in \Lambda_2 \text{ for every} \\ &t \in [t_i, t_{i+1}),\ i = 0, 1, \dots, N-1 \big\}, \end{aligned} \tag{3.51}$$

$$U^{\beta,\Gamma,\Lambda_1,\Lambda_2}_{\alpha,p,r} = V^{\Gamma,\Lambda_1}_{\alpha} \times W^{\beta,\Gamma,\Lambda_2}_{p,r} \tag{3.52}$$

where the sets V^{Γ}_{α} and $W^{\beta,\Gamma}_{p,r}$ are defined by (3.23) and (3.24), respectively. It is obvious that

$$\begin{aligned} U^{\beta,\Gamma,\Lambda_1,\Lambda_2}_{\alpha,p,r} = \{ &(v(\cdot), w(\cdot)) \in U^{\beta,\Gamma}_{\alpha,p,r} : v(t) = v_i, w(t) = w_i \text{ for every } t \in [t_i, t_{i+1}), \\ &\|v_i\| \in \Lambda_1,\ \|w_i\| \in \Lambda_2,\ i = 0, 1, \dots, N-1 \} \end{aligned} \tag{3.53}$$

where the set of control functions $U^{\beta,\Gamma}_{\alpha,p,r}$ is defined by equality (3.25). Let $X^{\beta,\Gamma,\Lambda_1,\Lambda_2}_{\alpha,p,r}(t_0, x_0)$ be the set of trajectories of system (2.1) generated by all control functions $(v(\cdot), w(\cdot)) \in U^{\beta,\Gamma,\Lambda_1,\Lambda_2}_{\alpha,p,r}$. It is obvious that

$$X^{\beta,\Gamma,\Lambda_1,\Lambda_2}_{\alpha,p,r}(t_0, x_0) \subset X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0). \tag{3.54}$$

Now we prove that

$$X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0) \subset X^{\beta,\Gamma,\Lambda_1,\Lambda_2}_{\alpha,p,r}(t_0, x_0) + [\lambda_1(\delta_1) + \lambda_2(\delta_2)] B_C(1) \tag{3.55}$$

where $\lambda_1(\delta_1)$ and $\lambda_2(\delta_2)$ are defined by equalities (3.11) and (3.12) respectively, $B_C(1)$ is defined by (1.1).

Choose an arbitrary trajectory $z(\cdot) \in X^{\beta,\Gamma}_{\alpha,p,r}(t_0, x_0)$ generated by the control function $(\hat{v}(\cdot), \hat{w}(\cdot)) \in U^{\beta,\Gamma}_{\alpha,p,r}$. According to (3.25) we have that $\hat{v}(\cdot) \in V^{\Gamma}_{\alpha}$ and $\hat{w}(\cdot) \in W^{\beta,\Gamma}_{\alpha,p,r}$. Then from (3.23) and (3.24) we obtain that

$$\hat{v}(t) = \hat{v}_i \text{ for every } t \in [t_i, t_{i+1}),\ \|\hat{v}_i\| \le \alpha,\ i = 0, 1, \dots, N-1, \tag{3.56}$$

$$\hat{w}(t) = \hat{w}_i \quad \text{for every } t \in [t_i, t_{i+1}),\ \|\hat{w}_i\| \le \beta,\ i = 0, 1, \dots, N-1,\ \Delta \cdot \sum_{i=0}^{N-1} \|\hat{w}_i\|^p \le r^p. \tag{3.57}$$

From (3.56) it follows that for each $\hat{v}_i$ there exists $\alpha_{j_i} \in \Lambda_1$ such that

$$\|\hat{v}_i\| \in [\alpha_{j_i}, \alpha_{j_i+1}) \text{ or } \|\hat{v}_i\| = \alpha,\ i = 0, 1, \dots, N-1. \tag{3.58}$$

Relation (3.57) implies that for each $\hat{w}_i$ there exists $r_{l_i} \in \Lambda_2$ such that

$$\|\hat{w}_i\| \in [r_{l_i}, r_{l_i+1}) \text{ or } \|\hat{w}_i\| = \beta,\ i = 0, 1, \dots, N-1. \tag{3.59}$$

Let us define a new control function $(\hat{v}_*(\cdot), \hat{w}_*(\cdot)) : [t_0, \theta] \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, setting

$$\hat{v}_*(t) = \begin{cases} \alpha_{j_i} \cdot \frac{\hat{v}_i}{\|\hat{v}_i\|} & \text{if } 0 < \|\hat{v}_i\| < \alpha, \\ \hat{v}_i & \text{if } \|\hat{v}_i\| = 0 \text{ or } \|\hat{v}_i\| = \alpha\,, \end{cases} \tag{3.60}$$

$$\hat{w}_*(t) = \begin{cases} r_{l_i} \cdot \frac{\hat{w}_i}{\|\hat{w}_i\|} & \text{if } 0 < \|\hat{w}_i\| < \beta, \\ \hat{w}_i & \text{if } \|\hat{w}_i\| = 0 \text{ or } \|\hat{w}_i\| = \beta \end{cases} \tag{3.61}$$

where $t \in [t_i, t_{i+1})$, $i = 0, 1, \dots N-1$. It is not difficult to verify that (3.58), (3.59), (3.60), and (3.61) imply that $\hat{v}_*(\cdot) \in V_\alpha^{\Gamma,\Lambda_1}$, $\hat{w}_*(\cdot) \in W_{p,r}^{\beta,\Gamma,\Lambda_2}$ and the inequalities

$$\|\hat{v}(t) - \hat{v}_*(t)\| \le \delta_1\,, \quad \|\hat{w}(t) - \hat{w}_*(t)\| \le \delta_2 \tag{3.62}$$

are satisfied for every $t \in [t_0, \theta]$. Suppose $z_*(\cdot) : [t_0, \theta] \to \mathbb{R}^n$ is the trajectory of system (2.1) generated by the control function $(\hat{v}_*(\cdot), \hat{w}_*(\cdot)) \in V_\alpha^{\Gamma,\Lambda_1} \times W_{p,r}^{\beta,\Gamma,\Lambda_2} = U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}$. Then, $z_*(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0, x_0)$ and by Conditions (A1), (A2), Proposition 2.1, and (3.62) we obtain

$$\begin{aligned} \|z(t) - z_*(t)\| &\le \int_{t_0}^t (\gamma_0 + \gamma_1 \|\hat{v}(\tau)\| + \gamma_2 \|\hat{w}(\tau)\|) \|z(\tau) - z_*(\tau)\| d\tau \\ &\quad + \kappa_1(1+\nu_*) \int_{t_0}^t \|\hat{v}(\tau) - \hat{v}_*(\tau)\| d\tau + \kappa_2(1+\nu_*) \int_{t_0}^t \|\hat{w}(\tau) - \hat{w}_*(\tau)\| d\tau \\ &\le \int_{t_0}^t (\gamma_0 + \gamma_1 \|\hat{v}(\tau)\| + \gamma_2 \|\hat{w}(\tau)\|) \|z(\tau) - z_*(\tau)\| d\tau \\ &\quad + \kappa_1(1+\nu_*)(\theta - t_0)\delta_1 + \kappa_2(1+\nu_*)(\theta - t_0)\delta_2 \end{aligned} \tag{3.63}$$

for every $t \in [t_0, \theta]$, where ν_* is defined by (2.7).

From inclusions $\hat{v}_*(\cdot) \in V_\alpha^{\Gamma,\Lambda_1}$, $\hat{w}_*(\cdot) \in W_{p,r}^{\beta,\Gamma,\Lambda_2}$, the Gronwall-Bellman inequality, (3.11), (3.12) and (3.63) we conclude that

$$\begin{aligned} &\|z(t) - z_*(t)\| \\ &\le [\kappa_1(1+\nu_*)(\theta - t_0)\delta_1 + \kappa_2(1+\nu_*)(\theta - t_0)\delta_2] \exp\Big(\int_{t_0}^t (\gamma_0 + \gamma_1 \|\hat{v}(\tau)\| + \gamma_2 \|\hat{w}(\tau)\|) d\tau\Big) \\ &\le [\kappa_1(1+\nu_*)(\theta - t_0)\delta_1 + \kappa_2(1+\nu_*)(\theta - t_0)\delta_2] \exp\Big((\gamma_0 + \gamma_1\alpha)(\theta - t_0) + \gamma_2 r (\theta - t_0)^{\frac{p-1}{p}}\Big) \\ &= \lambda_1(\delta_1) + \lambda_2(\delta_2) \end{aligned}$$

for every $t \in [t_0, \theta]$, and hence

$$\|z(\cdot) - z_*(\cdot)\|_C \le \lambda_1(\delta) + \lambda_2(\delta_2).$$

Since $z(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma}(t_0, x_0)$ is an arbitrarily chosen trajectory, $z_*(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0, x_0)$, the last inequality yields the validity of inclusion (3.55). Inclusions (3.54) and (3.55) imply that

$$h_C\left(X_{\alpha,p,r}^{\beta,\Gamma}(t_0, x_0), X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0, x_0)\right) \le \lambda_1(\delta_1) + \lambda_2(\delta_2)\,. \tag{3.64}$$

Step 4. Finite number of control functions. In this step we prove the inequality

$$h_C(X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0, x_0), X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0, x_0)) \le \chi_1(\sigma_1) + \chi_2(\beta, \sigma_2), \tag{3.65}$$

where $X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0, x_0)$ is the set of trajectories of system (2.1) generated by the set of a finite number of control functions $U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}$ defined by equality (3.4), $\chi_1(\sigma)$ and $\chi_2(\beta, \sigma)$ are defined by (3.13) and (3.14), respectively.

It is obvious that

$$X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0, x_0) \subset X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0, x_0). \tag{3.66}$$

Now let us prove that

$$X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0,x_0) \subset X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0) + [\chi_1(\sigma_1)+\chi_2(\beta,\sigma_2)]B_C(1)\,. \tag{3.67}$$

Choose an arbitrary $y(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0,x_0)$ generated by the control function $(\overline{v}(\cdot),\overline{w}(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2} = V_{\alpha}^{\Gamma,\Lambda_1} \times W_{p,r}^{\beta,\Gamma,\Lambda_2}$. Then according to (3.50), (3.51), (3.52), and (3.53) we have

$$\begin{aligned} \overline{v}(t) &= \alpha_{j_i}\overline{b}_i, \quad \alpha_{j_i} \in \Lambda_1, \quad \overline{b}_i \in S^1, \\ \overline{w}(t) &= r_{l_i}\overline{e}_i, \quad r_{j_i} \in \Lambda_2, \quad \overline{e}_i \in S^2 \end{aligned} \tag{3.68}$$

for all $t \in [t_i,t_{i+1})$ $(i=0,1,\dots,N-1)$, where $\Delta \cdot \sum_{i=0}^{N-1} r_{l_i}^p \le r^p$.

From the definitions of $S^1_{\sigma_1}$ and $S^2_{\sigma_2}$, for each $\overline{b}_i \in S^1$ and $\overline{e}_i \in S^2$ there exist $b_{k_i} \in S^1_{\sigma_1}$ and $e_{g_i} \in S^2_{\sigma_2}$ such that

$$\|\overline{b}_i - b_{k_i}\| \le \sigma_1, \quad \|\overline{e}_i - e_{g_i}\| \le \sigma_2, \quad i=0,1,\dots,N-1\,. \tag{3.69}$$

We define new control functions, setting

$$\overline{v}_*(t) = \alpha_{j_i} b_{k_i}, \quad \overline{w}_*(t) = r_{l_i} e_{g_i}, \quad t \in [t_i,t_{i+1}),\ i=0,1,\dots,N-1 \tag{3.70}$$

where $\alpha_{j_i} \in \Lambda_1$, $r_{l_i} \in \Lambda_2$ are defined by (3.68).

One can verify that

$$(\overline{v}_*(\cdot),\overline{w}_*(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}} = V_{\alpha}^{\Gamma,\Lambda_1,S^1_{\sigma_1}} \times W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}},$$

where the set of piecewise-constant control functions $V_{\alpha}^{\Gamma,\Lambda_1,S^1_{\sigma_1}}$ and $W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}}$ are defined by equalities (3.1) and (3.2) respectively. The relations (3.68), (3.69) and (3.70) imply that

$$\|\overline{v}(t) - \overline{v}_*(t)\| \le \alpha\sigma_1, \quad \|\overline{w}(t) - \overline{w}_*(t)\| \le \beta\sigma_2 \tag{3.71}$$

for every $t \in [t_0,\theta]$.

Assume $y_*(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ is the trajectory of system (2.1) generated by the control function $(\overline{v}_*(\cdot),\overline{w}_*(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}$. Then, $y_*(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ and Conditions (A1), (A2), Proposition 2.1 and inequality (3.71) we have that

$$\begin{aligned} \|y(t) - y_*(t)\| \le &\int_{t_0}^{t} (\gamma_0 + \gamma_1\|\overline{v}(\tau)\| + \gamma_2\|\overline{w}(\tau)\|)\|y(\tau) - y_*(\tau)\|d\tau \\ &+ \kappa_1(1+\nu_*)(\theta - t_0)\alpha\sigma_1 + \kappa_2(1+\nu_*)(\theta-t_0)\beta\sigma_2 \end{aligned} \tag{3.72}$$

for every $t \in [t_0,\theta]$. Now, taking into consideration that $\overline{v}(\cdot) \in V_{\alpha}^{\Gamma,\Lambda_1,S^1_{\sigma_1}} \subset V_\alpha$ and $\overline{w}(\cdot) \in W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}} \subset W_{p,r}$, where the sets of control functions V_α and $W_{p,r}$ are defined by equalities (2.2) and (2.3) respectively, from (3.13), (3.14), (3.72) and the Gronwall-Bellman inequality we obtain

$$\begin{aligned} &\|y(t) - y_*(t)\| \\ &\le [\kappa_1(1+\nu_*)(\theta-t_0)\alpha\sigma_1 + \kappa_2(1+\nu_*)(\theta-t_0)\beta\sigma_2] \exp\Big[\int_{t_0}^{t} (\gamma_0 + \gamma_1\|\overline{v}(\tau)\| + \gamma_2\|\overline{w}(\tau)\|)d\tau\Big] \\ &\le [\kappa_1(1+\nu_*)(\theta-t_0)\alpha\sigma_1 + \kappa_2(1+\nu_*)(\theta-t_0)\beta\sigma_2] \exp\Big[(\gamma_0+\gamma_1\alpha)(\theta-t_0) + \gamma_2 r(\theta-t_0)^{\frac{p-1}{p}}\Big] \\ &= \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2) \end{aligned}$$

for every $t \in [t_0,\theta]$, and consequently

$$\|y(\cdot) - y_*(\cdot)\|_C \le \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2). \tag{3.73}$$

Since $y(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2}(t_0,x_0)$ is an arbitrarily chosen trajectory and

$$y_*(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0),$$

inequality (3.73) implies the validity of the inclusion (3.67). Finally, the inclusions (3.66) and (3.67) yield the validity of the inequality (3.65).

Now, inequalities (3.22), (3.49), (3.64), and (3.65) complete the proof of the theorem. $\square$

Theorem 3.1 implies the validity of the following corollary characterizing the approximation of the attainable sets of the control system (2.1) with mixed constraints on the control functions.

Corollary 3.2. *The inequality*

$$h_n(X_{\alpha,p,r}(t;t_0,x_0), X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t;t_0,x_0)) \le \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2)$$

holds for every $\beta > 0$, for every uniform partition $\Gamma = \{t_0, t_1, \dots, t_N = \theta\}$ of the closed interval $[t_0,\theta]$, for every uniform partition $\Lambda_1 = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{c_1} = \alpha\}$ of the closed interval $[0,\alpha]$, for every uniform partition $\Lambda_2 = \{0 = r_0, r_1, \dots, r_{c_2} = \beta\}$ of the closed interval $[0,\beta]$, for every finite σ_1-net $S^1_{\sigma_1} = \{b_1, b_2, \dots, b_{c_3}\}$ on the unit sphere $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$ and for every finite σ_2-net $S^2_{\sigma_2} = \{e_1, e_2, \dots, e_{c_4}\}$ on the unit sphere $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$ and for every $t \in [t_0,\theta]$.

Here, the discretization parameters Δ, δ_1, δ_2, σ_1, σ_2 and the quantities κ_*, $\xi(\Delta)$, $\lambda_1(\delta_1)$, $\lambda_2(\delta_2)$, $\chi_1(\sigma_1)$, $\chi_2(\beta,\sigma_2)$ are defined as in Theorem 3.1. The attainable set $X_{\alpha,p,r}(t;t_0,x_0)$ $(t \in [t_0,\theta])$ of system (2.1) is defined by equality (2.5), the set $X^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_\sigma}_{\alpha,p,r}(t;t_0,x_0)$ $(t \in [t_0,\theta])$ which consists of a finite number of points, is defined by relation (3.5).

Remark 3.3. Since $X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t;t_0,x_0) \subset X_{\alpha,p,r}(t;t_0,x_0)$ for every $t \in [t_0,\theta]$, we conclude that the presented approximation is an inner one.

Remark 3.4. According to Theorem 2 from [11] we have $h_{L_1}(W_{p,r}, W_r^*) \to 0$ as $p \to \infty$ where the set $W_{p,r}$ is defined by (2.3), $W_r^* = \{w(\cdot) \in L_\infty([t_0,\theta];\mathbb{R}^{m_2}) : \|w(\cdot)\|_\infty \le r\}$, $h_{L_1}(\cdot,\cdot)$ stands for the Hausdorff distance between subsets of the space $L_1([t_0,\theta];\mathbb{R}^{m_2})$. Denote $U^*_{\alpha,r} = V_\alpha \times W_r^*$, and let $X^*_{\alpha,r}(t_0,x_0)$ be the set of trajectories of system (2.1) generated by all control functions $u(\cdot) = (v(\cdot), w(\cdot)) \in U^*_{\alpha,r}$. It is obvious that the control functions chosen from the set $U^*_{\alpha,r}$ have only geometric constraints. The conditions (A1) and (A2) imply that

$$h_C(X_{\alpha,p,r}(t_0,x_0), X^*_{\alpha,r}(t_0,x_0)) \to 0 \quad \text{as } p \to \infty\,.$$

This implies that for sufficiently large p, the integral constraint on the control functions $w(\cdot)$ can be replaced by geometric constraint, and thus, it is possible to consider an approximation of the set of trajectories of the control system (2.1) where the control functions have only geometric constraints.

Remark 3.5. Similarly to Remark 3.4, it is possible to replace geometric constraint on the control functions for system (2.1) with appropriate integral constraint. Let the set $W_{p,r}$ be defined by (2.3), $V^0_{p',\alpha} = \{v(\cdot) \in L_{p'}([t_0,\theta];\mathbb{R}^{m_1}) : \|v(\cdot)\|_{p'} \le \alpha\}$. Denote $U^0_{p',\alpha,p,r} = V^0_{p',\alpha} \times W_{p,r}$, and let $X^0_{p',\alpha,p,r}(t_0,x_0)$ be the set of trajectories of system (2.1) generated by all control functions $u(\cdot) = (v(\cdot), w(\cdot)) \in U^0_{p',\alpha,p,r}$. Now, the control functions from the set $U^0_{p',\alpha,p,r}$ have only integral norm constraints. According to Theorem 2 from [11] we have that $h_{L_1}(V_\alpha, V^0_{p',\alpha}) \to 0$ as $p' \to \infty$. Again, applying the conditions (A1) and (A2) one can show that

$$h_C(X^0_{p',\alpha,p,r}(t_0,x_0), X_{\alpha,p,r}(t_0,x_0)) \to 0 \text{ as } p' \to \infty\,.$$

This yields that by choosing a sufficiently large p', the geometric constraint on the control functions $v(\cdot)$ can be replaced by appropriate integral constraint, and thereby, it is possible to consider an approximation of the set of trajectories where the control functions have only the integral constraints.

4. Euler's broken lines

In this section we present an approximation of the set of trajectories of system (2.1) with finite number of Euler's broken lines.

Let $x(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)$ be an arbitrary trajectory of system (2.1) generated by the control function $(v(\cdot),w(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}$. Then according to (3.4) we have that there exist $\alpha_{j_i} \in \Lambda_1$, $r_{l_i} \in \Lambda_2$, $b_{k_i} \in S_{\sigma_1}^1$, $e_{g_i} \in S_{\sigma_2}^2$, $i = 0,1,\dots,N-1$, such that

$$v(t) = \alpha_{j_i} b_{k_i},\ w(t) = r_{l_i} e_{g_i} \text{ for every } t \in [t_i,t_{i+1}),\ i = 0,1,\dots,N-1,\ \Delta \cdot \sum_{i=0}^{N-1} r_{l_i}^p \le r^p\,. \quad (4.1)$$

In this case the trajectory $x(\cdot)$ can be defined by the equality

$$x(t) = x(t_i) + \int_{t_i}^{t} [f(\tau,x(\tau)) + B_1(\tau,x(\tau))\alpha_{j_i} b_{k_i} + B_2(\tau,x(\tau)) r_{l_i} e_{g_i}]\, d\tau, \quad x(t_0) = x_0 \quad (4.2)$$

for every $t \in (t_i,t_{i+1}]$, $i = 0,1,\dots,N-1$.

Now let us give the definition of the Euler's broken line corresponding to the trajectory $x(\cdot)$ defined by (4.2).

Definition 4.1. Let $x(\cdot)$ be a trajectory of system (2.1) generated by the control function $(v(\cdot),w(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}$ which is defined by equality (4.1). The function $z(\cdot) : [t_0,\theta] \to \mathbb{R}^n$ defined by the relation

$$z(t) = z(t_i) + (t-t_i)[f(t_i,z(t_i)) + B_1(t_i,z(t_i))\alpha_{j_i} b_{k_i} + B_2(t_i,z(t_i)) r_{l_i} e_{g_i}], \quad z(t_0) = x_0 \quad (4.3)$$

for every $t \in (t_i,t_{i+1}]$, $i = 0,1,\dots,N-1$, is called Euler's broken line of system (2.1) corresponding to the trajectory $x(\cdot)$.

For a given uniform partition $\Gamma = \{t_0,t_1,\dots,t_N = \theta\}$ of the closed interval $[t_0,\theta]$, a uniform partition $\Lambda_1 = \{0 = \alpha_0,\alpha_1,\dots,\alpha_{c_1} = \alpha\}$ of the closed interval $[0,\alpha]$, a uniform partition $\Lambda_2 = \{0 = r_0,r_1,\dots,r_{c_2} = \beta\}$ of the closed interval $[0,\beta]$, a finite σ_1-net $S_{\sigma_1}^1 = \{b_1,b_2,\dots,b_{c_3}\}$ on the unit sphere $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$ and a finite σ_2-net $S_{\sigma_2}^2 = \{e_1,e_2,\dots,e_{c_4}\}$ on the unit sphere $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$ we set

$$\begin{aligned} Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0) = \Big\{ & z(\cdot) \in C([t_0,\theta];\mathbb{R}^n) : z(\cdot) \quad \text{is Euler's broken line of a} \\ & x(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0) \Big\}\,. \end{aligned} \quad (4.4)$$

For a given $t \in [t_0,\theta]$ we also set

$$Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t;t_0,x_0) = \{z(t) \in \mathbb{R}^n : z(\cdot) \in Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)\}. \quad (4.5)$$

We denote

$$\psi(\beta,\Delta) = \omega_0(\varphi_*(\Delta)) + \alpha\omega_1(\varphi_*(\Delta)) + \beta\omega_2(\varphi_*(\Delta))\,, \quad (4.6)$$

$$\zeta(\beta,\Delta) = \psi(\beta,\Delta) \cdot \Delta \quad (4.7)$$

where $\varphi_*(\Delta)$, $\omega_0(\varphi_*(\Delta))$, $\omega_1(\varphi_*(\Delta))$ and $\omega_2(\varphi_*(\Delta))$ are defined by relations (2.10), (2.11), (2.12) and (2.13), respectively. It is obvious that $\psi(\beta,\Delta) \to 0$ as $\Delta \to 0^+$ for every fixed $\beta > 0$.

Proposition 4.2. *The inequality*

$$h_C\Big(X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0), Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)\Big) \le \psi(\beta,\Delta)(\theta-t_0)\exp(\gamma_*)$$

holds for every $\beta > 0$, for every uniform partition $\Gamma = \{t_0,t_1,\dots,t_N = \theta\}$ of the closed interval $[t_0,\theta]$, for every uniform partition $\Lambda_1 = \{0 = \alpha_0,\alpha_1,\dots,\alpha_{c_1} = \alpha\}$ of the closed interval $[0,\alpha]$, for every uniform partition $\Lambda_2 = \{0 = r_0,r_1,\dots,r_{c_2} = \beta\}$ of the closed interval $[0,\beta]$, for every finite σ_1-net $S_{\sigma_1}^1 = \{b_1,b_2,\dots,b_{c_3}\}$ on the unit sphere $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$ and for every finite σ_2-net $S_{\sigma_2}^2 = \{e_1,e_2,\dots,e_{c_4}\}$ on the unit sphere $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$, where $\Delta = \frac{\theta-t_0}{N} = t_{i+1}-t_i$, $i = 0,1,\dots,N-1$, is the diameter of the partition Γ, $\gamma_* > 0$ and $\psi(\beta,\Delta)$ are defined by relations (3.6) and (4.6), respectively, $X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)$ is the set of trajectories *generated by all piecewise-constant control functions* $(v(\cdot),w(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}$ *and the set* $Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ *is defined by equality* (4.4).

Proof. Let $x(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ be a trajectory of system(2.1) generated by control function $(v(\cdot),w(\cdot)) \in U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}} = V_\alpha^{\Gamma,\Lambda_1,S^1_{\sigma_1}} \times W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}}$ defined by relation (4.2) where the sets of control functions $V_\alpha^{\Gamma,\Lambda_1,S^1_{\sigma_1}}$ and $W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}}$ are defined by (3.1) and (3.2) respectively, and let $z(\cdot) \in Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ be the Euler's broken line corresponding to the trajectory $x(\cdot)$.

Let us evaluate $\|x(\cdot)-z(\cdot)\|_C$. Let $t \in (t_0,t_1]$. Taking into consideration that $\alpha_{j_0} \le \alpha$, $r_{l_0} \le \beta$, $\|b_{k_0}\| = 1$, $\|e_{g_0}\| = 1$, from Condition (A1), Corollary 2.2, Proposition 2.3, (2.10), (2.11), (2.12), (2.13), (4.2), (4.3), (4.6), and (4.7) it follows that

$$
\begin{aligned}
\|x(t)-z(t)\| &\le \int_{t_0}^{t} \big[\|f(\tau,x(\tau))-f(t_0,x_0)\| + \|B_1(\tau,x(\tau))-B_1(t_0,x_0)\|\|\alpha_{j_0}b_{k_0}\| \\
&\quad + \|B_2(\tau,x(\tau))-B_2(t_0,x_0)\|\|r_{l_0}e_{g_0}\|\big]\,d\tau \\
&\le [\omega_0(\varphi_*(\Delta)) + \alpha\omega_1(\varphi_*(\Delta)) + \beta\omega_2(\varphi_*(\Delta))]\,(t-t_0) \\
&\le \psi(\beta,\Delta)\Delta = \zeta(\beta,\Delta)
\end{aligned} \tag{4.8}
$$

for every $t \in (t_0,t_1]$.

Let $t \in (t_1,t_2]$. Taking into consideration that $\alpha_{j_1} \le \alpha$, $r_{l_1} \le \beta$, $\|b_{k_1}\| = 1$, $\|e_{g_1}\| = 1$, from Corollary 2.2, Proposition 2.3, (2.10), (2.11), (2.12), (2.13), (4.2), (4.3), (4.6), (4.7), and (4.8) we obtain that

$$
\begin{aligned}
&\|x(t)-z(t)\| \\
&\le \|x(t_1)-z(t_1)\| + \int_{t_1}^{t} \big[\|f(\tau,x(\tau))-f(t_1,z(t_1))\| \\
&\quad + \|B_1(\tau,x(\tau))-B_1(t_1,z(t_1))\|\|\alpha_{j_1}b_{k_1}\| + \|B_2(\tau,x(\tau))-B_2(t_1,z(t_1))\|\|r_{l_1}e_{g_1}\|\big]\,d\tau \\
&\le \|x(t_1)-z(t_1)\| + \int_{t_1}^{t} [\|f(\tau,x(\tau))-f(t_1,x(t_1))\| + \|f(t_1,x(t_1))-f(t_1,z(t_1))\|]\,d\tau \\
&\quad + \int_{t_1}^{t} [\|B_1(\tau,x(\tau))-B_1(t_1,x(t_1))\| + \|B_1(t_1,x(t_1))-B_1(t_1,z(t_1))\|]\,\alpha\,d\tau \\
&\quad + \int_{t_1}^{t} [\|B_2(\tau,x(\tau))-B_2(t_1,x(t_1))\| + \|B_2(t_1,x(t_1))-B_2(t_1,z(t_1))\|]\,\|r_{l_1}\|\,d\tau \\
&\le \|x(t_1)-z(t_1)\| + \omega_0(\varphi_*(\Delta))\Delta + \gamma_0\|x(t_1)-z(t_1)\|\Delta \\
&\quad + \alpha\omega_1(\varphi_*(\Delta))\Delta + \alpha\gamma_1\|x(t_1)-z(t_1)\|\Delta + r_{l_1}\omega_2(\varphi_*(\Delta))\Delta + r_{l_1}\gamma_2\|x(t_1)-z(t_1)\|\Delta \\
&\le \|x(t_1)-z(t_1)\| + \big[\omega_0(\varphi_*(\Delta)) + \alpha\omega_1(\varphi_*(\Delta)) + \beta\omega_2(\varphi_*(\Delta))\big]\Delta r \\
&\quad + [\gamma_0+\gamma_1\alpha+\gamma_2 r_{l_1}]\Delta\|x(t_1)-z(t_1)\| \\
&= \psi(\beta,\Delta)\Delta + \big[1+(\gamma_0+\gamma_1\alpha+\gamma_2 r_{l_1})\Delta\big]\|x(t_1)-z(t_1)\|] \\
&\le \zeta(\beta,\Delta) + \zeta(\beta,\Delta)\exp[(\gamma_0+\gamma_1\alpha+\gamma_2 r_{l_1})\Delta] \\
&= \zeta(\beta,\Delta)\big[1+\exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_1})\big]
\end{aligned} \tag{4.9}
$$

for every $t \in (t_1,t_2]$.

Now, assuming that

$$
\begin{aligned}
\|x(t)-z(t)\| \le \zeta(\beta,\Delta)\big[&1+\exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_{i-1}}) \\
&+ \exp(2\Delta\gamma_0+2\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_{i-1}}+r_{l_{i-2}})) + \dots \\
&+ \exp((i-1)\Delta\gamma_0+(i-1)\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_{i-1}}+r_{l_{i-2}}+\dots+r_{l_1}))\big]
\end{aligned}
$$

for every $t \in (t_{i-1}, t_i]$, we will prove that

$$\begin{aligned} \|x(t)-z(t)\| \le \zeta(\beta,\Delta)\big[&1+\exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_i}) \\ &+\exp\big(2\Delta\gamma_0+2\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_i}+r_{l_{i-1}})\big)+\dots \\ &+\exp(i\Delta\gamma_0+i\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_i}+r_{l_{i-1}}+\dots+r_{l_1}))\big] \end{aligned} \tag{4.10}$$

for every $t \in (t_i, t_{i+1}]$ where $2 \le i \le N-1$.

Taking into account that $0 \le \alpha_{j_i} \le \alpha$, $0 \le r_{l_i} \le \beta$, $\|b_{k_i}\| = 1$, $\|e_{g_i}\| = 1$, from Condition (A1), Corollary 2.2, Proposition 2.3, (2.10), (2.11), (2.12), (2.13), (4.2), (4.3), (4.6), (4.7), and (4.10), we have that

$$\begin{aligned} &\|x(t)-z(t)\| \\ &\le \|x(t_i)-z(t_i)\| + \int_{t_i}^t \big[\|f(\tau,x(\tau))-f(t_i,z(t_i))\| \\ &\quad + \|B_1(\tau,x(\tau))-B_1(t_i,z(t_i))\|\cdot\|\alpha_{j_i}b_{k_i}\| + \|B_2(\tau,x(\tau))-B_2(t_i,z(t_i))\|\cdot\|r_{l_i}e_{g_i}\|\big]d\tau \\ &\le \|x(t_i)-z(t_i)\| + \int_{t_i}^t [\|f(\tau,x(\tau))-f(t_i,x(t_i))\| + \|f(t_i,x(t_i))-f(t_i,z(t_i))\|]\,d\tau \\ &\quad + \int_{t_i}^t [\|B_1(\tau,x(\tau))-B_1(t_i,x(t_i))\| + \|B_1(t_i,x(t_i))-B_1(t_i,z(t_i))\|]\,\alpha\,d\tau \\ &\quad + \int_{t_i}^t [\|B_2(\tau,x(\tau))-B_2(t_i,x(t_i))\| + \|B_2(t_i,x(t_i))-B_2(t_i,z(t_i))\|]\,r_{l_i}\,d\tau \\ &\le \|x(t_i)-z(t_i)\| + \int_{t_i}^t [\omega_0(\varphi_*(\Delta))+\gamma_0\|x(t_i)-z(t_i)\|]\,d\tau \\ &\quad + \int_{t_i}^t [\omega_1(\varphi_*(\Delta))+\gamma_1\|x(t_i)-z(t_i)\|]\,\alpha\,d\tau + \int_{t_i}^t [\omega_2(\varphi_*(\Delta)\beta)+\gamma_2\|x(t_i)-z(t_i)\|]\,r_{l_i}\,d\tau \\ &\le [\omega_0(\varphi_*(\Delta))+\omega_1(\varphi_*(\Delta))\alpha+\omega_2(\varphi_*(\Delta))\beta]\cdot\Delta + [1+(\gamma_0+\gamma_1\alpha+\gamma_2 r_{l_i})\Delta]\|x(t_i)-z(t_i)\| \\ &\le \psi(\beta,\Delta)\Delta + \exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_i})\cdot\|x(t_i)-z(t_i)\| \\ &\le \zeta(\beta,\Delta) + \exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_i}) \\ &\quad \times \Big\{\zeta(\beta,\Delta)\big[1+\exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_{i-1}})+\exp(2\Delta\gamma_0+2\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_{i-1}}+r_{l_{i-2}})) \\ &\quad + \dots + \exp((i-1)\Delta\gamma_0+(i-1)\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_{i-1}}+r_{l_{i-2}}+\dots+r_{l_1}))\big]\Big\} \\ &= \zeta(\beta,\Delta)\big[1+\exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_i})+\exp(2\Delta\gamma_0+2\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_i}+r_{l_{i-1}})) \\ &\quad + \exp(3\Delta\gamma_0+3\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_i}+r_{l_{i-1}}+r_{l_{i-2}}))+\dots \\ &\quad + \exp(i\Delta\gamma_0+i\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_i}+r_{l_{i-1}}+\dots+r_{l_1}))\big] \end{aligned} \tag{4.11}$$

for every $t \in (t_i, t_{i+1}]$. So, the validity of the inequality (4.10) is proved. Finally, it follows from (4.8), (4.9), (4.10) and (4.11) that the inequality

$$\begin{aligned} &\|x(t)-z(t)\| \\ &\le \zeta(\beta,\Delta)\big[1+\exp(\Delta\gamma_0+\Delta\gamma_1\alpha+\Delta\gamma_2 r_{l_i})+\exp(2\Delta\gamma_0+2\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_i}+r_{l_{i-1}})) \\ &\quad + \exp(3\Delta\gamma_0+3\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_i}+r_{l_{i-1}}+r_{l_{i-2}}))+\dots \\ &\quad + \exp((N-1)\Delta\gamma_0+(N-1)\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_{N-1}}+r_{l_{N-2}}+\dots+r_{l_1}))\big] \end{aligned} \tag{4.12}$$

is verified for every $t \in (t_i, t_{i+1}]$, $i = 0, 1, \dots, N-1$. Taking into account that $x(t_0) = z(t_0) = x_0$, $N \cdot \Delta = \theta - t_0$, from (4.6), (4.7) and (4.12) we have

$$\begin{aligned} &\|x(t)-z(t)\| \\ &\le \zeta(\beta,\Delta)N\exp(N\Delta\gamma_0+N\Delta\gamma_1\alpha+\Delta\gamma_2(r_{l_{N-1}}+r_{l_{N-2}}+\dots+r_{l_1}+r_{l_0})) \\ &= \psi(\beta,\Delta)(\theta-t_0)\exp(\gamma_0(\theta-t_0)+\gamma_1\alpha(\theta-t_0)+\Delta\gamma_2(r_{l_{N-1}}+r_{l_{N-2}}+\dots+r_{l_1}+r_{l_0})) \end{aligned} \tag{4.13}$$

for every $t \in [t_0, \theta]$. Now, from (4.1), inclusion $w(\cdot) \in W_{p,r}^{\beta,\Gamma,\Lambda_2,S^1_{\sigma_2}} \subset W_{p,r}$ and Hölder's inequality we obtain

$$\Delta \cdot \sum_{i=0}^{N-1} r_{l_i} = \int_{t_0}^{\theta} \|w(t)\| dt \le (\theta - t_0)^{\frac{p-1}{p}} \Big(\int_{t_0}^{\theta} \|w(t)\|^p dt \Big)^{1/p} \le r(\theta - t_0)^{\frac{p-1}{p}}. \tag{4.14}$$

The inequalities (4.13) and (4.14) yield that

$$\|x(t) - z(t)\| \le \psi(\beta, \Delta)(\theta - t_0) \exp\Big(\gamma_0(\theta - t_0) + \gamma_1 \alpha(\theta - t_0) + \gamma_2 r(\theta - t_0)^{\frac{p-1}{p}}\Big) \tag{4.15}$$

for every $t \in [t_0, \theta]$. Taking into account (3.6) and (4.15) we have

$$\|x(\cdot) - z(\cdot)\|_C \le \psi(\beta, \Delta)(\theta - t_0) \exp(\gamma_*).$$

Since $x(\cdot) \in X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0, x_0)$ is an arbitrarily chosen trajectory and $z(\cdot)$ belongs to $Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0, x_0)$ is its Euler's broken line, then the last inequality completes the proof. □

Remark 4.3. In the proof of Proposition 4.2 (in the inequality (4.13)) by applying the estimations $r_{l_i} \le \beta$ $(i = 0, 1, \dots, N-1)$, we can obtain the evaluation

$$\begin{aligned} &h_C\Big(X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0, x_0), Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0, x_0)\Big) \\ &\le \psi(\beta, \Delta)(\theta - t_0) \exp\big[\gamma_0(\theta - t_0) + \gamma_1 \alpha(\theta - t_0) + \gamma_2 \beta(\theta - t_0)\big]. \end{aligned} \tag{4.16}$$

According to Theorem 3.1, in order to obtain a sufficiently small evaluation for the Hausdorff distance between the set of trajectories $X_{\alpha,p,r}(t_0, x_0)$ and its approximation $X_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0, x_0)$, the number β must be chosen to be sufficiently large. If the inequality $\beta > \frac{r}{(\theta - t_0)^{1/p}}$ is satisfied, then the evaluation presented in Proposition 4.2 is better than the evaluation given by (4.16).

From Theorem 3.1 and Proposition 4.2 we obtain the validity of the following theorem.

Theorem 4.4. *The inequality*

$$\begin{aligned} &h_C(X_{\alpha,p,r}(t_0, x_0), Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0, x_0)) \\ &\le \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) + \chi_1(\sigma_1) + \chi_2(\beta, \sigma_2) + \psi(\beta, \Delta)(\theta - t_0) \exp(\gamma_*) \end{aligned}$$

is verified for every $\beta > 0$, for every uniform partition $\Gamma = \{t_0, t_1, \dots, t_N = \theta\}$ of the closed interval $[t_0, \theta]$, for every uniform partition $\Lambda_1 = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{c_1} = \alpha\}$ of the closed interval $[0, \alpha]$, for every uniform partition $\Lambda_2 = \{0 = r_0, r_1, \dots, r_{c_2} = \beta\}$ of the closed interval $[0, \beta]$, for every finite σ_1-net $S^1_{\sigma_1} = \{b_1, b_2, \dots, b_{c_3}\}$ on the unit sphere $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$ and for every finite σ_2-net $S^2_{\sigma_2} = \{e_1, e_2, \dots, e_{c_4}\}$ on the unit sphere $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$.

Here, $\Delta = \frac{\theta - t_0}{N} = t_{i+1} - t_i$, $i = 0, 1, \dots, N-1$, is the diameter of the partition Γ, $\delta_1 = \frac{\alpha}{c_1} = \alpha_{j+1} - \alpha_j$, $j = 0, 1, \dots, c_1 - 1$, is the diameter of the partition Λ_1, $\delta_2 = \frac{\beta}{c_2} = r_{l+1} - r_l$, $l = 0, 1, \dots, c_2 - 1$, is the diameter of the partition Λ_2, $\sigma_1 > 0$ and $\sigma_2 > 0$ are given numbers, γ_*, κ_*, $\xi(\Delta)$, $\lambda_1(\delta_1)$, $\lambda_2(\delta_2)$, $\chi_1(\sigma_1)$, $\chi_2(\beta, \sigma_2)$ and $\psi(\beta, \Delta)$ are defined by (3.6), (3.7), (3.10), (3.11), (3.12), (3.13), (3.14) and (4.6) respectively, $X_{\alpha,p,r}(t_0, x_0)$ is the set of trajectories of system (2.1) generated by the set of control functions $U_{\alpha,p,r}$ defined by (2.4), and the set $Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0, x_0)$ is defined by equality (4.4).

Theorem 4.4 implies the following assertion which is approximation of the attainable sets of system (2.1) with mixed constraints on the control functions.

Theorem 4.5. *The inequality*

$$\begin{aligned} &h_n(X_{\alpha,p,r}(t; t_0, x_0), Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t; t_0, x_0)) \\ &\le \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) + \chi_1(\sigma_1) + \chi_2(\beta, \sigma_2) + \psi(\beta, \Delta)(\theta - t_0) \exp(\gamma_*) \end{aligned}$$

is satisfied for every $\beta > 0$, *for every uniform partition* $\Gamma = \{t_0, t_1, \dots, t_N = \theta\}$ *of the closed interval* $[t_0, \theta]$, *for every uniform partition* $\Lambda_1 = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{c_1} = \alpha\}$ *of the closed interval* $[0, \alpha]$, *for every uniform partition* $\Lambda_2 = \{0 = r_0, r_1, \dots, r_{c_2} = \beta\}$ *of the closed interval* $[0, \beta]$, *for every finite* σ_1*-net* $S^1_{\sigma_1} = \{b_1, b_2, \dots, b_{c_3}\}$ *on the unit sphere* $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$, *for every finite* σ_2*-net* $S^2_{\sigma_2} = \{e_1, e_2, \dots, e_{c_4}\}$ *on the unit sphere* $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$ *and for every* $t \in [t_0, \theta]$.

Here, the discretization parameters Δ, δ_1, δ_2, σ_1, σ_2 *and the quantities* κ_*, $\xi(\Delta)$, $\lambda_1(\delta_1)$, $\lambda_2(\delta_2)$, $\chi_1(\sigma_1)$, $\chi_2(\beta, \sigma_2)$, $\psi(\beta, \Delta)$ *are defined in the same way as in Theorem 4.2, the set* $X_{\alpha,p,r}(t; t_0, x_0)$ *defined by equality* (2.5) *is the attainable set of system* (2.1) *at the instant of time* t, *the set* $Z^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t; t_0, x_0)$ *is defined by* (4.5).

5. Integral funnel's approximation

In this section we present an approximation of the integral funnel $\Phi_{\alpha,p,r}(t_0, x_0)$ of system (2.1) defined by equality (2.6).

Given a uniform partition $\Gamma = \{t_0, t_1, \dots, t_N = \theta\}$ of the closed interval $[t_0, \theta]$, a uniform partition $\Lambda_1 = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{c_1} = \alpha\}$ of the closed interval $[0, \alpha]$, a uniform partition $\Lambda_2 = \{0 = r_0, r_1, \dots, r_{c_2} = \beta\}$ of the closed interval $[0, \beta]$, a finite σ_1-net $S^1_{\sigma_1} = \{b_1, b_2, \dots, b_{c_3}\}$ on the unit sphere $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$ and a finite σ_2-net $S^2_{\sigma_2} = \{e_1, e_2, \dots, e_{c_4}\}$ on the unit sphere $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$, we define

$$\Phi^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t_0, x_0) = \cup_{i=0}^{N} \Big(t_i, Z^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t_i; t_0, x_0) \Big) \tag{5.1}$$

where $Z^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t_i; t_0, x_0)$ is defined by (4.5). The set $Z^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t_i; t_0, x_0)$ consists of the values of a finite number of Euler's broken lines at $t = t_i$.

Theorem 5.1. *The inequality*

$$\begin{aligned} &h_{n+1}\big(\Phi_{\alpha,p,r}(t_0, x_0), \Phi^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t_0, x_0)\big) \\ &\le \Delta + \varphi_0(\Delta) + \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) + \chi_1(\sigma_1) + \chi_2(\beta, \sigma_2) + \psi(\beta, \Delta)(\theta - t_0)\exp(\gamma_*) \end{aligned}$$

holds for every $\beta > 0$, *for every uniform partition* $\Gamma = \{t_0, t_1, \dots, t_N = \theta\}$ *of the closed interval* $[t_0, \theta]$, *for every uniform partition* $\Lambda_1 = \{0 = \alpha_0, \alpha_1, \dots, \alpha_{c_1} = \alpha\}$ *of the closed interval* $[0, \alpha]$, *for every uniform partition* $\Lambda_2 = \{0 = r_0, r_1, \dots, r_{c_2} = \beta\}$ *of the closed interval* $[0, \beta]$, *for every finite* σ_1*-net* $S^1_{\sigma_1} = \{b_1, b_2, \dots, b_{c_3}\}$ *on the unit sphere* $S^1 = \{b \in \mathbb{R}^{m_1} : \|b\| = 1\}$ *and finite* σ_2*-net* $S^2_{\sigma_2} = \{e_1, e_2, \dots, e_{c_4}\}$ *on the unit sphere* $S^2 = \{e \in \mathbb{R}^{m_2} : \|e\| = 1\}$.

Here $\Delta = \frac{\theta - t_0}{N} = t_{i+1} - t_i$, $i = 0, 1, \dots, N-1$, *is the diameter of the partition* Γ, $\delta_1 = \frac{\alpha}{c_1} = \alpha_{j+1} - \alpha_j$, $j = 0, 1, \dots, c_1 - 1$, *is the diameter of the partition* Λ_1, $\delta_2 = \frac{\beta}{c_2} = r_{l+1} - r_l$, $l = 0, 1, \dots, c_2 - 1$, *is the diameter of the partition* Λ_2, $\sigma_1 > 0$ *and* $\sigma_2 > 0$ *are given numbers,* $\varphi_0(\Delta)$, γ_*, κ_*, $\xi(\Delta)$, $\lambda_1(\delta_1)$, $\lambda_2(\delta_2)$, $\chi_1(\sigma_1)$, $\chi_2(\beta, \sigma_2)$, *and* $\psi(\beta, \Delta)$ *are defined by* (2.9), (3.6), (3.7), (3.10), (3.11), (3.12), (3.13), (3.14), *and* (4.6), *respectively. The set* $\Phi_{\alpha,p,r}(t_0, x_0)$ *defined by equality* (2.6) *is the integral funnel of system* (2.1), *the set* $\Phi^{\beta,\Gamma,\Lambda_1,\Gamma_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t_0, x_0)$ *consists of a finite number of points and is defined by* (5.1).

Proof. Choose an arbitrary $(t_*, x_*) \in \Phi_{\alpha,p,r}(t_0, x_0)$. According to the definition of the integral funnel $\Phi_{\alpha,p,r}(t_0, x_0)$, there exists a trajectory $x_*(\cdot) \in X_{\alpha,p,r}(t_0, x_0)$ such that $x_*(t_*) = x_* \in X_{\alpha,p,r}(t_*; t_0, x_0)$ where $t_* \in [t_0, \theta]$. Suppose that $t_* \notin \Gamma = \{t_0, t_1, \dots, t_N = \theta\}$. Then there exists i_* such that $t_* \in (t_{i_*}, t_{i_*+1})$ where $0 \le i_* \le N-1$. Denote $y_* = x_*(t_{i_*})$. Then $y_* \in X_{\alpha,p,r}(t_{i_*}; t_0, x_0)$ and from Proposition 2.5 we have that

$$\|x_* - y_*\| \le \varphi_0(\Delta) \,. \tag{5.2}$$

If $t_* \in \Gamma$, then we choose $y_* = x_*$. Since $y_* \in X_{\alpha,p,r}(t_{i_*}; t_0, x_0)$ and $Z^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}_{\alpha,p,r}(t_{i_*}; t_0, x_0)$ consists of a finite number of points, then Theorem 4.5 implies that there exists

$z_* \in Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_{i_*};t_0,x_0)$ such that

$$\begin{aligned}\|y_* - z_*\| &\le \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) \\ &\quad + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2) + \psi(\beta,\Delta)(\theta - t_0)\exp(\gamma_*).\end{aligned} \tag{5.3}$$

It is obvious that $(t_{i_*}, z_*) \in \Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)$. The inequalities (5.2) and (5.3) yield that

$$\begin{aligned}\|(t_*,x_*) - (t_{i_*} z_*)\| &\le |t_* - t_{i_*}| + \|x_* - z_*\| \le |t_* - t_{i_*}| + \|x_* - y_*\| + \|y_* - z_*\| \\ &\le \Delta + \varphi_0(\Delta) + \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) \\ &\quad + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2) + \psi(\beta,\Delta)(\theta - t_0)\exp(\gamma_*)\,.\end{aligned} \tag{5.4}$$

Since $(t_*,x_*) \in \Phi_{\alpha,p,r}(t_0,x_0)$ is an arbitrarily chosen point, $(t_{i_*}, z_*) \in \Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)$, then from (5.4) we conclude that

$$\begin{aligned}\Phi_{\alpha,p,r}(t_0,x_0) &\subset \Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0) + \Big[\Delta + \varphi_0(\Delta) + \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) \\ &\quad + \lambda_2(\delta_2) + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2) + \psi(\beta,\Delta)(\theta - t_0)\exp(\gamma_*)\Big] \cdot B_{n+1}(1)\end{aligned} \tag{5.5}$$

where $B_{n+1}(1) = \{s \in \mathbb{R}^{n+1} : \|s\| \le 1\}$.

Choose an arbitrary $\eta > 0$ and fix it. Now, let $(t_i, \tilde{z}_*) \in \Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)$ ($i = 0,1,\dots,N$) be an arbitrarily chosen point. Then $\tilde{z}_* \in Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_i;t_0,x_0)$. By applying Theorem 4.5 we conclude that there exists $\tilde{x}_* \in X_{\alpha,p,r}(t_i;t_0,x_0)$ such that

$$\begin{aligned}\|\tilde{z}_* - \tilde{x}_*\| &\le \eta + \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) \\ &\quad + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2) + \psi(\beta,\Delta)(\theta - t_0)\exp(\gamma_*)\,,\end{aligned}$$

and consequently

$$\begin{aligned}\|(t_i,\tilde{z}_*) - (t_i,\tilde{x}_*)\| &\le \eta + \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) \\ &\quad + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2) + \psi(\beta,\Delta)(\theta - t_0)\exp(\gamma_*)\end{aligned} \tag{5.6}$$

where $\eta > 0$ is an arbitrarily chosen number. From the inclusion $\tilde{x}_* \in X_{\alpha,p,r}(t_i;t_0,x_0)$ it follows that $(t_i,\tilde{x}_*) \in \Phi_{\alpha,p,r}(t_0,x_0)$. Since $(t_i,\tilde{z}_*) \in \Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)$ is an arbitrarily chosen point, $(t_i,\tilde{x}_*) \in \Phi_{\alpha,p,r}(t_0,x_0)$, then (5.6) yields the validity of the inclusion

$$\begin{aligned}\Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0) &\subset \Phi_{\alpha,p,r}(t_0,x_0) + \Big[\eta + \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) \\ &\quad + \chi_1(\sigma_1) + \chi_2(\beta,\sigma_2) + \psi(\beta,\Delta)(\theta - t_0)\exp(\gamma_*)\Big] \cdot B_{n+1}(1)\,.\end{aligned} \tag{5.7}$$

From (5.5) and (5.7) we have that

$$\begin{aligned}&h_{n+1}\Big(\Phi_{\alpha,p,r}(t_0,x_0), \Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S_{\sigma_1}^1,S_{\sigma_2}^2}(t_0,x_0)\Big) \\ &\le \eta + \Delta + \varphi_0(\Delta) + \frac{\kappa_*}{\beta^{p-1}} + \xi(\Delta) + \lambda_1(\delta_1) + \lambda_2(\delta_2) + \chi_1(\sigma) \\ &\quad + \chi_2(\beta,\sigma) + \psi(\beta,\Delta)(\theta - t_0)\exp(\gamma_*)\,.\end{aligned}$$

Since $\eta > 0$ is an arbitrarily chosen number, then last inequality completes the proof of the theorem. □

6. Conclusion

In the presented approximations of the set of trajectories, attainable sets and integral funnel, the quantities β, Δ, δ_1, δ_2, σ_1, and σ_2 are discretization parameters. From (2.9), (3.10), (3.11), (3.12), (3.13), (3.14), and (4.6) it follows that $\varphi_0(\Delta) \to 0$ as $\Delta \to 0^+$, $\xi(\Delta) \to 0$ as $\Delta \to 0^+$, $\lambda_1(\delta_1) \to 0^+$ as $\delta_1 \to 0^+$, $\lambda_2(\delta_2) \to 0^+$ as $\delta_2 \to 0^+$, $\chi_1(\sigma_1) \to 0$ as $\sigma_1 \to 0^+$ and for each fixed $\beta > 0$ we have that $\chi_2(\beta,\sigma_2) \to 0^+$ as $\sigma_2 \to 0^+$, and $\psi(\beta,\Delta) \to 0$ as $\Delta \to 0^+$. Therefore, for a given $\varepsilon > 0$ it is possible to choose $\beta_*(\varepsilon) > 0$, $\Delta_*(\varepsilon,\beta_*(\varepsilon)) > 0$, $\delta_1^*(\varepsilon) > 0$, $\delta_2^*(\varepsilon) > 0$, $\sigma_1^*(\varepsilon) > 0$ and $\sigma_2^*(\varepsilon,\beta_*(\varepsilon)) > 0$ such that for every $\Delta \in (0,\Delta_*(\varepsilon,\beta_*(\varepsilon)))$, $\delta_1 \in (0,\delta_1^*(\varepsilon))$, $\delta_2 \in (0,\delta_2^*(\varepsilon))$, $\sigma_1 \in (0,\sigma_1^*(\varepsilon))$ and $\sigma_2 \in (0,\sigma_2^*(\varepsilon,\beta_*(\varepsilon))$, the inequalities

$$h_C\Big(X_{\alpha,p,r}(t_0,x_0), Z_{\alpha,p,r}^{\beta_*(\varepsilon),\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)\Big) \le \varepsilon\,,$$
$$h_n\Big(X_{\alpha,p,r}(t;t_0,x_0), Z_{\alpha,p,r}^{\beta_*(\varepsilon),\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t;t_0,x_0)\Big) \le \varepsilon\,, \quad t\in[t_0,\theta]\,,$$
$$h_{n+1}\Big(\Phi_{\alpha,p,r}(t_0,x_0), \Phi_{\alpha,p,r}^{\beta_*(\varepsilon),\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)\Big) \le \varepsilon$$

are satisfied where $\Delta = \frac{\theta - t_0}{N} = t_{i+1} - t_i$, $i = 0,1,\dots,N-1$ is the diameter of the partition Γ, $\delta_1 = \frac{\alpha}{c_1} = \alpha_{j+1} - \alpha_j$, $j = 0,1,\dots,c_1-1$ is the diameter of the partition Λ_1, $\delta_2 = \frac{\beta}{c_2} = r_{l+1} - r_l$, $l = 0,1,\dots,c_2-1$ is the diameter of the partition Λ_2, $\sigma_1 > 0$ and $\sigma_2 > 0$ are given numbers which specify finite σ_1 and σ_2 nets on the unit spheres $S^1 = \{v \in \mathbb{R}^{m_1} : \|v\| = 1\}$ and $S^2 = \{w \in \mathbb{R}^{m_2} : \|w\| = 1\}$, respectively.

The quantities $\beta_*(\varepsilon) > 0$, $\Delta_*(\varepsilon,\beta_*(\varepsilon)) > 0$, $\delta_1^*(\varepsilon) > 0$, $\delta_2^*(\varepsilon) > 0$, $\sigma_1^*(\varepsilon) > 0$, and $\sigma_2^*(\varepsilon,\beta_*(\varepsilon)) > 0$ can be determined from the definitions of $\varphi_0(\Delta)$, $\frac{\kappa_*}{\beta^{p-1}}$, $\xi(\Delta)$, $\lambda_1(\delta_1)$, $\lambda_2(\delta_2)$, $\chi_1(\sigma_1)$, $\chi_2(\beta,\sigma_2)$ and $\psi(\beta,\Delta)$. If the functions $(t,x) \to f(t,x)$, $(t,x) \to B_1(t,x)$, $(t,x) \to B_2(t,x)$, $(t,x) \in D_*$, are Lipschitz continuous with respect to (t,x), i.e., if these functions satisfy the inequalities (2.14), then all quantities $\beta_*(\varepsilon) > 0$, $\Delta_*(\varepsilon,\beta_*(\varepsilon)) > 0$, $\delta_1^*(\varepsilon) > 0$, $\delta_2^*(\varepsilon) > 0$, $\sigma_1^*(\varepsilon) > 0$, and $\sigma_2^*(\varepsilon,\beta_*(\varepsilon)) > 0$ can be defined explicitly. Note that to obtain the appropriate evaluations, the choice of the quantities $\Delta_*(\varepsilon,\beta_*(\varepsilon)) > 0$ and $\sigma_2^*(\varepsilon,\beta_*(\varepsilon)) > 0$ should be coordinated with the quantity $\beta_*(\varepsilon) > 0$, i.e., the choice of $\Delta_*(\varepsilon,\beta_*(\varepsilon)) > 0$ and $\sigma_2^*(\varepsilon,\beta_*(\varepsilon)) > 0$ must be made after the choice of the quantity $\beta_*(\varepsilon) > 0$.

By applying the algorithm outlined in [8] we can align a finite number of control functions from the sets $V_\alpha^{\Gamma,\Lambda_1,S^1_{\sigma_1}}$ and $W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}}$ defined by relations (3.1) and (3.2), respectively, where $V_\alpha^{\Gamma,\Lambda_1,S^1_{\sigma_1}} \times W_{p,r}^{\beta,\Gamma,\Lambda_2,S^2_{\sigma_2}} = U_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}$. Then, by calculating Euler's broken lines as given in (4.4), it is possible to construct an approximation $Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ of the set of trajectories $X_{\alpha,p,r}(t_0,x_0)$. Moreover, by computing approximations $Z_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_i;t_0,x_0)$ of the attainable sets $X_{\alpha,p,r}(t_i;t_0,x_0)$ at the instants of time t_i, $i = 1,2,\dots,N$, it is possible to provide an approximation $\Phi_{\alpha,p,r}^{\beta,\Gamma,\Lambda_1,\Lambda_2,S^1_{\sigma_1},S^2_{\sigma_2}}(t_0,x_0)$ of the integral funnel $\Phi_{\alpha,p,r}(t_0,x_0)$ of the control system (2.1) with integral and geometric constraints on control functions.

Finally, let us note that in the discretization procedure, it is possible to apply an arbitrary partition $\Gamma = \{t_0,t_1,\dots,t_N = \theta\}$ of the closed interval $[t_0,\theta]$, an arbitrary partition $\Lambda_1 = \{0 = \alpha_0,\alpha_1,\dots,\alpha_{c_1} = \alpha\}$ of the closed interval $[0,\alpha]$ and an arbitrary partition $\Lambda_2 = \{0 = r_0,r_1,\dots,r_{c_2} = \beta\}$ of the closed interval $[0,\beta]$ instead of the uniform partitions. However, in this case, to obtain the best evaluations, we have to operate with diameters of the presented partitions defined as $\operatorname{diam}(\Gamma) = \max\{t_{i+1} - t_i : i = 0,1,\dots,N-1\}$, $\operatorname{diam}(\Lambda_1) = \max\{\alpha_{j+1} - \alpha_j : j = 0,1,\dots,c_1-1\}$, $\operatorname{diam}(\Lambda_2) = \max\{r_{l+1} - r_l : l = 0,1,\dots,c_2-1\}$.

References

[1] T. S. Angell, R. K. George, J. P. Sharma; Controllability of Urysohn integral inclusions of Volterra type, *Electron. J. Diff. Equat.*, **2010**(79) (2010), 1-12.
[2] J.-P. Aubin, A. Cellina; *Differential Inclusions. Set-Valued Maps and Viability Theory*, Berlin: Springer-Verlag, 1984.
[3] V. I. Blagodatskikh, A. F. Filippov; Differential inclusions and optimal control, *Proc. Steklov Inst. Math.*, **169** (1986), 199-256.
[4] A. Bressan, G. Facchi; Trajectories of differential inclusions with state constraints, *J. Differ. Equat.* **250**(4) (2011), 2267-2281.
[5] R. Conti; *Problemi di Controllo e di Controllo Ottimale*, Torino: UTET, 1974.
[6] K. Deimling; *Multivalued Differential Equations*, Berlin: D. Gruyter, 1992.
[7] A. A. Ershov, A. V. Ushakov, V. N. Ushakov; An approach problem for a control system with a compactum in the phase space in the presence of phase constraints, *Sb. Mat.*, **210**(8) (2019), 1092-1128.

[8] Kh. G. Guseinov, A. S. Nazlipinar; An algorithm for approximate calculation of the attainable sets of the nonlinear control systems with integral constraint on controls, *Comput. Math. Appl.,* **62**(4) (2011), 1887-1895.
[9] M. I. Gusev; On some properties of reachable sets for nonlinear systems with control constraints in L_p, *Tr. Inst. Mat. Mekh. UrO RAN,* **30**(3) (2024), 99-112.
[10] M. I. Gusev, I. V. Zykov; On extremal properties of the boundary points of reachable sets for control systems with integral constraints, *Tr. Inst. Mat. Mekh. UrO RAN*, **23**(1) (2017), 103-115.
[11] N. Huseyin, A. Huseyin; On the continuity properties of the L_p balls, *J. Appl. Anal.*, **29**(1) (2023), 151-159.
[12] N. Huseyin, A. Huseyin, Kh. G. Guseinov; Approximations of the set of trajectories and integral funnel of the non-linear control systems with L_p norm constraints on the control functions, *IMA J. Math. Control Inform.*, **39**(4) (2022), 1213-1231.
[13] N. Huseyin, A. Huseyin, Kh. G. Guseinov, V. N. Ushakov; Approximations of the set of trajectories, attainable sets and integral funnel of the control system with mixed constraints on the control functions, *European J. Contr.*, **87**(Paper no. 101422) (2026), 1-11.
[14] N. N. Krasovskii; *Theory of Control of Motion: Linear Systems*, Moscow: Nauka, 1968.
[15] N. N. Krasovskii, A. I. Subbotin; *Game-Theoretical Control Problems*, New York: Springer-Verlag, 1988.
[16] A. B. Kurzhanskii, P. Varaiya; *Dynamics and Control of Trajectory Tubes. Theory and Computation*, Cham: Birkhäuser, 2014.
[17] M. Motta, C. Sartori; Minimum time with bounded energy, minimum energy with bounded time, *SIAM J. Control Optim.*, **42** (2003), 789-809.
[18] V. Patsko, G. Trubnikov, A. Fedotov; Numerical study of a three-dimensional reachable set for a Dubins car under an integral control constraint, *Commun. Optim. Theory,* **2025**(Article ID 24) (2025), 1-33.
[19] V. S. Patsko, G. I. Trubnikov, A. A. Fedotov; Two-dimensional reachable set of Dubins car with both geometric and integral constraints on control, *Tr. Inst. Mat. Mekh. UrO RAN,* **31**(2) (2025), 162-180.
[20] B. T. Polyak; Convexity of the reachable set of nonlinear systems under L_2 bounded controls, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **11**(2-3) (2004), 255-267.
[21] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, E. F. Mishchenko; *The Mathematical Theory of Optimal Processes*, New York: John Wiley & Sons, 1962.
[22] P. Rousse, P.-L. Garoche, D. Henrion; Parabolic set simulation for reachability analysis of linear time-invariant systems with integral quadratic constraint, *European J. Contr.*, **28** (2021), 152-167.
[23] R. L. Wheeden, A. Zygmund; *Measure and Integral. An Introduction to Real Analysis*, New York: M. Dekker Inc., 1977.

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