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CODIMENSION IN PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. The topological classification of large families of polynomial differential systems remains an extremely difficult problem. Even for the simplest nonlinear case, the quadratic systems, this problem is still open. In recent years however, significant progress has been achieved on a related challenge: the topological classification modulo limit cycles of the family QS of quadratic differential systems. This progress was enabled by introducing new tools, such as the concepts of the global geometrical and topological configurations of singularities, and by first classifying QS with respect to these notions.

In this work, we extend Sotomayor's notion of codimension, originally based on topological equivalence, so as to allow for other equivalence relations including the geometric one. We provide a rigorous, improved definition of codimension that could be used efficiently for obtaining the classification of the phase portraits of QS modulo limit cycles, and not just for small codimensions as it usually occurs in the present literature. This new definition of codimension can be extended to arbitrary polynomial differential systems, and other differential systems.

In this work, we apply the new concept of codimension by assigning a codimension to each one of the 207 global topological configurations of singularities of systems in QS. This tool is of great help for assigning codimensions even to phase portraits modulo limit cycles. This concept is a potent tool in the topological classification problem modulo limit cycles in QS.

1. Introduction

We consider differential systems of the form

$$\frac{dx}{dt} = p(x,y), \quad \frac{dy}{dt} = q(x,y), \tag{1.1}$$

where $p, q \in \mathbb{R}[x,y]$, i.e. p, q are polynomials with real coefficients in x, y. We call *degree* of a system (1.1) the integer $n = \max(\deg p, \deg q)$. In particular we call *quadratic* a differential system (1.1) with $n = 2$. We denote by QS the whole class of real quadratic differential systems.

Polynomial differential systems occur often in many areas of applied mathematics. They also have theoretical significance and several problems on these systems have been open for over a hundred years even for the lowest degree nonlinear systems, the quadratic ones. These problems are so hard because they are global: they involve systems of any degrees and their solutions in the whole plane.

The topological classification problem for the family QS of quadratic systems is even harder than Hilbert's 16th problem for this family. Although there are over 1000 papers on quadratic differential systems, only around 100 provide systematic classifications of phase portraits for certain subclasses of QS. Studies focus mostly on highly restrictive systems, which reduce the number of parameters and thus simplify the analysis. In recent years several authors have extended these results to systems of codimension up to 2, for example, systems with a weak focus of second order [8], systems with two saddle-nodes [17], or systems featuring a cusp singularity [27].

However, these studies, which produce bifurcation diagrams, face a challenge: the possible existence of "sland" within the parameter space where phase portraits distinct from those described in the classification done in those studies, may appear, either on the boundary or inside the island.

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Usually these islands are mainly related with the existence and disappearance of semistable limit cycles or non-algebraic separatrix connections. Since these islands, if they exist, can be arbitrarily small, they are not in general detectable through numerical methods.

To simplify matters we can first do a topological classification modulo limit cycles. Progress is significant in the topological classification modulo limit cycles for quadratic systems. In QS every limit cycle surrounds a single singularity which is a focus and we always have at most two nests of limit cycles and in one nest we could only have one limit cycle, see [41] in which the incomplete proof given in [42] is repaired.

If we collapse every nest of limit cycles to the focus inside then we obtain a new phase portrait, i.e. the topological quotient space of the original phase portrait after identification of the whole area delimited by the largest one of the limit cycles including the limit cycle with the focus. The resulting topological space with its attached foliation with singularities is *the phase portrait modulo limit cycles*.

The *topological classification problem modulo limit cycles* is the problem of finding all the distinct phase portraits modulo limit cycles for QS. This is still a hard problem but in recent years significant progress was made in this direction. A first step was done in [6] where all structurally stable phase portraits (modulo limit cycles) of QS was obtained. In [7] all structurally unstable codimension one phase portraits (modulo limit cycles) of QS were obtained. The method used does not rely on normal forms nor does it produce bifurcation diagrams. Instead it generates all *potential* phase portraits within a given codimension or subclass. A potential phase portrait is one that is compatible with the number and types of singularities, indices, and other characteristics, although it may not be realizable for other reasons.

For instance, in [6], out of 72 potential phase portraits, only 44 were realizable, with 28 proved impossible. Similarly, in [7], over 500 potential phase portraits only resulted 202 realizable ones, plus 7 conjectured impossible cases lacking formal proofs of impossibility. Current work on codimension 2 continues to close subclasses with realizable phase portraits and a few conjectured impossible cases. The advantage of this approach is that it provides an upper bound on the number of phase portraits (modulo limit cycles) within each codimension.

However, the notion of codimension used for the first few lower codimensions, becomes increasingly difficult to handle as codimension grows. This is mainly due to its limitations in the way it is defined. Thus, a rigorous and more appropriate definition of codimension convenient for being assigned to more complex objects than singularities such as global geometrical or topological configurations of singularities, for graphics and even for phase portraits modulo limit cycles in QS, is necessary.

The concept of codimension in classical mathematics goes back to the end of the 1920's. It began to be used in the 1930's in Functional Analysis. Later in 1940 it was also introduced by Zariski in Algebraic Geometry. All these notions are based on the notion of dimension.

For example the codimension of an n-dimensional submanifold of an m-dimensional topological manifold is $m-n$. In differential equations the definition of codimension is usually based on the notion of structural stability. This notion was first introduced by Andronov and Pontrjiagin in 1937 in their Doklady article [1]. It was later applied by Andronov and collaborators to the theories of oscillators and of bifurcations [2, 3]. It was defined and used for two-dimensional systems. In the late 1950's Peixoto extended the notion of structural stability to n-dimensional systems [32] and proved the so called Peixoto's theorem. Sotomayor wrote a thesis on structural stability under Peixoto's supervision. In [36] he defined the notion of structural stability of order one which in today's language means of codimension 1. From here it is natural to introduce the higher codimension by an induction procedure which results in the Definition 2.1 given in Section 2 and we naturally attribute it to Sotomayor.

Sotomayor's definition is based on the concept of structural stability. This notion is in turn based on the topological equivalence relation as follows: We say that a system S is structurally stable within a family of systems $\mathcal{F}$ if it has a neighborhood in the family such that any other system in this neighborhood has a phase portrait that is *topologically equivalent* to the phase portrait of S. However other equivalence relations need to be considered. To give but one simple example, consider a singularity of a polynomial 2-dimensional differential system which has non-zero complex eigenvalues, a strong focus. It is stable under perturbations. A focus of positive order, a weak focus, is locally topologically equivalent to a strong focus. But a weak focus is unstable as via small perturbations it produces limit cycles as close as we wish to the weak focus. Although the topological equivalence relation does not distinguish a weak from a strong focus, the concept of structural stability for systems does. But there are cases where structural stability of systems does not distinguish. Consider a system endowed with a weak saddle and one endowed with a strong saddle which have no loop. Clearly, they are not distinguished by local homeomorphisms, and nor are they distinguished by structural stability. However in the case the weak saddle forms a loop, it does. Assume the only unstable objects of the system are the weak saddle and the loop. The system with a weak saddle plus a loop (on that saddle) has codimension 2 according to Sotomayor since one can break the weak saddle into a strong one while still maintaining the loop, and later break the loop. However, if one just breaks the loop and maintains the weak saddle, one gets a topologically structurally stable system (codimension 0). With the geometric equivalence, whether one breaks just the weak saddle or the loop, one gets a codimension 1 system.

Sotomayor's definition needs to be made more flexible by allowing for other equivalence relations such as the geometrical one which for example distinguishes between a strong and a weak focus. We need to assign codimension to mathematical objects of diverse types and not just to singularities. For example to global topological or global geometric configurations of singularities, to graphics, to phase portraits. These objects are considerably more complex than just singularities. Allowing other equivalence relations depending on the nature of the specific mathematical objects we consider, allows us to have a finer capacity for distinguishing among various mathematical objects and turns the concept of codimension into a potent tool.

We extend Sotomayor's definition based on the topological equivalence for other equivalence relations and in particular the geometrical one to make it more adaptable for the problems we face.

Using the 1764 distinct global geometrical configurations of singularities occurring in QS (see [10]) we were able to obtain the 207 distinct topological configurations of singularities in this family [9].

In this work we calculate this new codimension for each one of the 207 global topological configurations of singularities of QS and describe how to use this tool in order to further advance in the topological classification problem modulo limit cycles for QS. This new tool enables us to assign codimension not only to singularities or configurations of singularities but also to more complicated objects such as graphics or even phase portraits and it is thus a potent tool in topologically classifying modulo limit cycles the family QS.

In Section 2 we first give a brief survey on how the concept of codimension is defined in the literature. Our definition is more akin with the one defined in algebraic geometry. We next introduced several notions needed in the following sections.

In Section 3 we describe the problems that we found when trying to apply the concept of codimension for different types of objects in differential equations and the distinct possible equivalence relations that can be used.

In Section 4 we describe the problems that we have found when trying to assign a codimension to configurations of singularities with centers or to phase portraits of quadratic systems with centers, and why we do not assign a fixed codimension for such configurations or phase portraits.

In Section 5 we assign a topological codimension to each one of the 207 topologically distinct configurations of singularities obtained in [9] (except for those with a center). We corrected an error in [9] where 208 configurations were found. While in the literature we encounter low codimension (2 or 3) mostly applied to singularities or homoclinic loops, in Section 5 we obtain objects of up to and including codimension 9, more specifically for global topological configurations of singularities.

In Section 6 we introduce a new nomenclature designed to uniquely identify each distinct phase portrait, allowing every phase portrait to be referenced by a single, unambiguous name.

There is also an Appendix presented as Section 8 which contains an example of geometrical codimension applied to a system of degree 4.

In Section 7 we discuss several applications of the codimension concept in the study of phase portraits of the family QS.

2. Definitions and classical results

In lectures of specialists in ordinary differential equations, and in particular on polynomial differential equations, in books or articles, occasionally the term *codimension* is encountered. For example a nilpotent cusp singularity is called a codimension 2 (or greater) singularity [20, Theorem 2], and centers in quadratic differential systems can be classified in four families [35], three of them are said to be of codimension 3 and one of codimension 4 although this holds only generically because inside these families there are subfamilies of systems with centers of higher codimension. As we shall see later, trying to assign a codimension to a phase portrait with centers, is quite a lot more complicated than in this generic case.

It is however rather rare to see a definition of the term appearing in a book or in an article on differential equations. In the books [21, 26, 30, 34] the term does not even appear in the index even though the term may appear in the text of the books. The term does however appear in the index of [24], referring to the page 120 where its meaning only applies to an l-dimensional submanifold of an n-dimensional manifold and in this case the codimension of the submanifold is $n-l$, and on page 123 for a bifurcation where we find the following phrase: "*The codimension of a bifurcation will be the smallest dimension of a parameter space which contains the bifurcation in a persistent way.*" We find a definition on page 9 of [34]: "*We define a singularity Σ of codimension k as a submanifold of codimension $n+k$ (if $n=\dim(S)$) of some space of l-jets of vector fields on S.*" Other more restrictive definitions can also be found in the book.

As mentioned before, Sotomayor [36] gave the definition of structurally stable systems of order one (today called codimension one). He gave this definition for all C^r-vector fields, $r \geq 3$, on a compact 2-dimensional differentiable manifold. This generalizes easily to higher orders (codimensions). We give here an adaptation of this generalization for the context of interest to us here.

The concept of structural stability, which we defined on page 2, is an important one

The concept of structural stability in dynamical systems is an important one. The properties of special interest to us in the study of planar differential systems are the topological ones and we say that two systems have the same topological properties if they are topologically equivalent according to Definition 2.2 given below.

Definition 2.1. Let B be the set of polynomial vector fields of degree n. Let A_0 be the subset of B formed by all the fields which are structurally stable. We will assign codimension 0 to the fields of A_0. Let $B_1 = B \setminus A_0$. This will be called the *bifurcation set.* Let A_1 be the subset of B_1 formed by all the fields which are structurally stable inside B_1. We will assign codimension 1 to the fields of A_1. And the definition continues recursively.

Another issue is that if the polynomial vector fields are real and the stability is considered according to simple real perturbations, the definition is limited and does not allow us to properly deal with situations with two complex singularities with trace or discriminant [1] equal to zero (see Example 3.4 below). We also have another similar issue which will be described in Example 3.5.

Sotomayor's definition uses the concept of "structural stability" and this concept involves an equivalence relation. Even though this is not mentioned in the definition, by default the topological equivalence (see next definition) is assumed in this case.

Definition 2.2. We say that two planar differential systems are topologically equivalent if and only if there exists a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ carrying orbits to orbits and preserving their orientation or reversing the orientation of all the orbits.

Sotomayor's definition involves only one equivalence relation, namely the topological one which is used to define "structural stability". The "structural stability" could also make sense for other

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[1] When we mention the trace, or the determinant or the discriminant of a singularity, we are always referring to the trace (or the determinant) of the Jacobian matrix at that singularity, or the discriminant of the characteristic polynomial of the Jacobian matrix of the differential system at that singularity.

equivalence relations such as for example the geometric one (see [10]). Then an object like a weak saddle (which may already have a topological impact if combined with a loop on the same saddle) would also have an impact geometrically.

We now briefly survey in what way the concept of codimension occurs in mathematics. The simplest case occurs in vector spaces.

Definition 2.3. The codimension of a subspace S of a vector space E over a field K denoted by $\operatorname{codim}_E(S)$, is the dimension of the quotient space E/S and we have $\operatorname{codim}_E(S) = \dim(E) - \dim(S)$.

As already mentioned the concept of codimension is also defined for submanifolds (topological or C^∞) and we have:

Definition 2.4. Let N be a submanifold (topological or C^∞) of a manifold M. The codimension denoted by $\operatorname{codim}(N, M)$ is defined by $\operatorname{codim}(N, M) = \dim(M) - \dim(N)$.

We note that codimension is a relative concept. For example in the above definition we have the codimension of a submanifold. Indeed, we have the codimension of an object endowed with a mathematical structure inherited from the structure of a larger object that contains it. In algebraic geometry we find the following definition of codimension in [18].

Definition 2.5. If V is a closed irreducible subvariety of an irreducible variety X, then we call codimension of V in X, denoted by $\operatorname{codim}(V, X)$, the difference $\dim(X) - \dim(V) \geq 0$, where $\dim(V)$ and $dim(X)$ denote the dimensions of V and X.

In [28] an affine algebraic variety over an algebraically closed field K (for example $\mathbb{C}$) is defined as being an irreducible algebraic subset of the affine space K^n. Irreducible here means that it cannot be decomposed in two non-empty closed algebraic subsets in the Zarisky topology on K^n defined by taking the open sets as being the complements of algebraic sets.

In a real topological manifold of dimension n every point possesses a neighborhood homeomorphic to a ball in $\mathbb{R}^n$. So all neighborhoods are alike. This is not true in an algebraic variety with singularities where neighborhoods of two distinct singularities may look very different.

To a planar differential system we can associate its foliation with singularities and this is an object where also neighborhoods of two distinct singularities may look significantly different. For this reason the objects that we consider for dynamical systems and for which we need to define an adequate concept of codimension are akin to objects in algebraic geometry.

A very general notion of codimension encountered in algebraic geometry is the one given for schemes. These are more general objects than algebraic varieties. An affine scheme is defined starting with a commutative ring and endowing this ring with a topology by using its prime ideals as the points of this topological space called the spectrum of A denoted by $\operatorname{Spec}(A)$. A sheaf of rings is then considered on $\operatorname{Spec}(A)$. Schemes were conceived by Grothendieck with the purpose of unifying number theory with algebraic geometry motivated by the Weil conjectures.

Definition 2.6 ([23])**.** If V is a subvariety of a scheme X, the codimension of V in X denoted by $\operatorname{codim}(V, X)$, is the maximum length of a chain of subvarieties

$$V = V_0 \subset V_1 \subset \cdots \subset V_n = X$$

where $\subset$ denotes the strict inclusion.

We observe that in Definition 2.1 of Sotomayor we construct such a strictly descending sequence, each object B_i in this sequence includes all the structurally stable objects inside B_{i+1}.

Thus the notion of codimension is rigorously defined for very general objects in algebraic geometry. Our definition will also be for more complex mathematical objects than singularities, occurring in differential equations, such as graphics and even for phase portraits.

We could also have infinite codimension. For example the set of local germs of quadratic planar polynomial differential systems inside the local germs of planar analytic differential systems is of infinite codimension.

In this article we shall concentrate on planar polynomial differential systems of arbitrary degree n with emphasis on the quadratic differential systems, i.e. $n = 2$. Since to each such system we can associate a point in $\mathbb{R}^{(n+1)(n+2)}$ by taking the sequence of the coefficients of the first and then of the second equation, we can keep track of all such systems by considering their corresponding points in $\mathbb{R}^{(n+1)(n+2)}$ that is our full parameter space. This provides a simple notion of distance between two polynomial systems of the same degree. When studying a special problem, its associated parameter space may turn out to be in a proper subspace of $\mathbb{R}^{(n+1)(n+2)}$. We will also consider sets of objects (such as singularities, configurations of singularities, graphics, limit periodic sets, phase portraits, ...) for which the concept of "perturbation" must be defined in order to be able to assign a codimension to each one of these objects.

The term unfolding appeared in the theory of singularities of mappings and in catastrophe theory. It appears in the book of René Thom [38] and was used mainly for studying the local behavior around a function in its neighborhood in the space of functions $(C^r, C^\infty, C^\omega)$ and for points around a singularity.

The concept was later considered, first for unfoldings of singularities of vector fields (by Arnold in [4]) and eventually for unfoldings of their graphics. We are mainly concerned with the behavior of systems near a given one and around a compact set in the phase space. By unfolding a graphic or a compact set we expect to eventually see the behavior of all nearby systems and near the compact set. We thus need to specify in what sense "near" is to be considered, in other words we need to specify the topology (C^r, C^∞, C^ω, polynomial) on vector fields X on the plane at $\lambda_0 \in A$ where A is a parameter set. The notion usually used for function spaces is either the *weak* or *strong topology* on the set $C^r(M,N)$ of C^r-mappings from a manifold M into a manifold N. We assume at first that r is finite. The *weak* or *compact-open topology* on $C^r(M,N)$ is generated by sets defined in the following way (see [25]). Let $f \in C^r(M,N)$. Let (ϕ,U), (ψ,V) be charts on the manifolds M,N respectively. Let $K \subset U$ be a compact set such that $f(K) \subset V$; let $0 \le \epsilon \le \infty$. We define a weak sub-basic neighborhood

$$\mathcal{N}^r(f;(\phi,U),(\psi,V),K,\epsilon) \tag{2.1}$$

to be the set of C^r maps $g: M \to N$ such that $g(K) \subset V$ and

$$\|D^k(\psi f \phi^{-1})(x) - D^k(\psi g \phi^{-1})(x)\| < \epsilon$$

for all $x \in \phi(K), k = 1,2,\dots r$. The *weak topology* on $C^r(M,N)$ is generated by these sets.

A (C^r, C^∞, C^ω, polynomial) vector field X on the plane is a just a (C^r, C^∞, C^ω, polynomial) map $X : \mathbb{R}^2 \to T\mathbb{R}^2$ where $T\mathbb{R}^2$ is the tangent bundle on $\mathbb{R}^2$. A continuous family $\{X_\lambda\}$ of such vector fields defined for λ in a parameter space A is a map $X : A \times \mathbb{R}^2 \to T\mathbb{R}^2$ that is continuous in λ and (C^r, C^∞, C^ω, polynomial) on $\mathbb{R}^2$.

Definition 2.7. We call unfolding of a (C^r,C^∞, C^ω, polynomial) vector field X on the plane at $\lambda_0 \in A$ where $A \subset \mathbb{R}^n$ is a parameter set, a family X_λ depending continuously of λ of vector fields of the same type of differentiability as X on $\mathbb{R}^2$, defined for λ in A such that $X_{\lambda_0} = X$.

Remark 2.8. Even though in our definitions we talk about vector fields on the plane so as to include the cases C^r, C^∞, C^ω, and polynomial, our results will be done just for the polynomial case and we will use the Poincaré compactification on the sphere. We will talk about singularities at infinity, global configurations of singularities (finite and infinite), graphics that may include parts of the infinite line, and phase portraits in the Poincaré disc.

In the theory of singularities of mappings and in catastrophe theory the term *versal unfolding* is used and we consider it here applied to the topic of interest for us here namely to vector fields.

Definition 2.9. [19, 34] Let (X_λ), $\lambda \in A$ (the parameter space A could be a part of $\mathbb{R}^k$) be a C^r-family of vector fields and let $\phi : B \to A$, ($B \subset \mathbb{R}^s$), $\phi(\mu) = \lambda$ be a C^r-map. We say that the family Y_μ, $\mu \in B$ given by $Y_\mu = X_{\phi(\mu)}$ is induced by the map ϕ.

Definition 2.10 ([19])**.** *Two k-parameter families X_μ and Y_μ, both with $\mu \in \mathbb{R}^k$ and with phase space $\mathbb{R}^n$ are called fiber-C^0-equivalent over the identity if there exist homeomorphisms h_μ such that for each $\mu \in \mathbb{R}^k$ the map h_μ is a C^0-equivalence between the vector fields X_μ and Y_μ.*

Roussarie [34] gave a more extended definition.

Definition 2.11. Let X_λ, Y_λ be two C^r, $r = 1, \dots \infty$ or ω families of vector fields with the same parameter space A and the same phase space. Let s be such that $0 \leq s \leq r$. We say that X_λ, Y_λ are fiber-C^s-equivalent if there exists a diffeomorphism ϕ of A of class C^s, such that for each $\lambda \in A$, X_λ, $Y_{\phi(\lambda)}$ are topologically equivalent. If the equivalence is chosen so that it forms a continuous family $h_\lambda(x)$, we say that X_λ, Y_λ are fiber-C^s-equivalent.

We can now define the concept of *versal unfolding* of a vector field X_0. According to Wikipedia the term versal unfolding was initially used by Thom who called it in French "déploiement universel" i.e. universal unfolding. Mather changed it to versal unfolding.

Definition 2.12 ([19])**.** An unfolding X_μ of a vector field X_0 is called a (C^0,C^r)-versal unfolding if all unfoldings of X_0 are fiber-C^0-equivalent over the identity to an induced unfolding that is C^r-induced from X_μ.

Consider a compact set Γ in the plane. We define an equivalence relation on vector fields on the plane along Γ as follows: Two vector fields X and Y on the plane are equivalent with respect to Γ if there is a neighborhood of Γ on which they are topologically equivalent. We call *germ of X along* Γ and denote it by (X, Γ) the equivalence class of X along Γ.

Again Roussarie gave a more extended definition in [34]:

Definition 2.13. Let Γ be a compact non-empty invariant set in the phase space of X_{λ_0}. We say that (X_λ, Γ) is a versal unfolding, $\lambda \in A \subset \mathbb{R}^r$ for the germ (X_{λ_0}, Γ), $\lambda_0 \in A$ for the topological or C^r, C^0 where $r \leq \infty$, or other type of equivalence if

1. Any other unfolding (Y_μ, Γ) of (X_{λ_0}, Γ) with parameter space $B \subset \mathbb{R}^m$ and Γ an invariant set of Y_{μ_0} where $(Y_{\mu_0}, \Gamma) \equiv (X_{\lambda_0}, \Gamma)$ is fiber-C^0-equivalent to an unfolding induced from X_λ by a germ of a C^s-map $(\phi, \mu_0) : (B, \mu_0) \to (A, \lambda_0)$.
2. $\dim(A)$ is minimal for the property 1.

Graphics of planar polynomial vector fields are invariant compact sets. They are part of the characteristics of vector fields and they are important organizing centers in families of vector fields as they produce limit cycles in the neighborhoods of Γ for unfolded vector fields with λ close to λ_0.

Since in this case we are interested in neighborhoods of Γ, we use the term unfolding of X along Γ. We are interested in constructing a notion of codimension which could also be applied to general graphics as well as other non-local concepts.

Singularities of vector fields are the simplest cases of graphics and this explains why the first cases where the concept of codimension was defined was for singularities.

We now describe the unfoldings of objects like singularities, phase portraits, etc.

Definition 2.14. We start with a system S endowed with an object $\mathcal{O}_{\lambda_0}$. We consider an unfolding X_λ such that $X_{\lambda_0} \equiv S$. Then for any fixed value of λ we have an object $\mathcal{O}_\lambda$ or a set of objects $\{(\mathcal{O}_i)_\lambda\}$ such that when $\lambda \to \lambda_0$, then $\{(\mathcal{O}_i)_\lambda\} \to \mathcal{O}_{\lambda_0}$ for all i, as $X_\lambda \to X_{\lambda_0}$. By the perturbation of $\mathcal{O}_{\lambda_0}$ given by X_λ we mean the family $\{(\mathcal{O}_i)_\lambda\}$ within X_λ.

For example, consider a polynomial vector field on the plane possessing an object $\mathcal{O}_{\lambda_0}$ which is a multiple singularity at infinity. Then the perturbation $\{(\mathcal{O}_i)_\lambda\}$ of $\mathcal{O}_{\lambda_0}$ could include finite and infinite singular points of X_λ which tend to $\mathcal{O}_{\lambda_0}$ in the topology of the chart at infinity or the topology of the Poincaré sphere.

The two terms: "unfolding" and "perturbation" have been used as synonymous in the past. In case of polynomial differential systems the definition of the topological equivalence is usually understood on the Poincaré sphere and then it is the following definition.

Definition 2.15. We say that two planar polynomial differential systems X and Y are topologically equivalent on the Poincaré sphere if there is a homeomorphism on $\mathbb{S}^2$ between their analytic extensions $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ on the Poincaré sphere preserving the equator $\mathbb{S}^1$ and carrying oriented orbits of the flow induced by $\mathcal{P}(X)$ into orbits of the flow induced by $\mathcal{P}(Y)$ preserving the orientation or reversing the orientation of all the orbits.

However, in recent works and with the goal of reducing the number of distinct equivalence classes, a weaker definition of topological equivalence allowing also the change in the orientation, is used.

It is often said that a structurally stable object in the full parameter space, or in a subspace of maximal dimension, has codimension 0, see for instance [6, 7]. We underscore that this zero codimension involved this notion of topological equivalence in [6, 7, 37].

The definition of topological equivalence between singularities has already been given in several papers and it is the following.

Definition 2.16. Two singularities p_1 and p_2 are *topologically equivalent* if there exist open neighborhoods N_1 and N_2 of these singularities and a homeomorphism $\Psi : N_1 \to N_2$ carrying orbits to orbits and preserving their orientations. To reduce the number of cases, by *topological equivalence* it is occasionally accepted a variant of this definition in which the homeomorphism Ψ *preserves or reverses* the orientation of all orbits (see [8, 10]).

Apart from the topological equivalence of singularities, in [10] a new equivalence relation was defined which is of geometrical nature.

Definition 2.17. Two finite singularities p_1 and p_2 of two polynomial vector fields are *geometrically equivalent* if they are topologically equivalent, they have the same multiplicity and one of the following conditions is satisfied:

- p_1 and p_2 are order equivalent foci (or saddles),
- p_1 and p_2 are tangent equivalent simple nodes,
- p_1 and p_2 are both elemental centers with the same level of isochronicity,
- p_1 and p_2 are both semi-elemental singularities,
- p_1 and p_2 are blow-up equivalent nilpotent or intricate singularities.

The reader can find the concepts of “order equivalent”, “tangent equivalent”, “level of isochronicity” and “blow-up equivalent” in the book [10]. To give here a hint of their meaning, we recall that they require that the order of weak singularities, the way the orbits arrive at singularities, the centers being isochronous or not, and the desingularization of singularities (respectively) must be the same.

Definition 2.18. We say that two infinite singularities P_1 and P_2 of two polynomial vector fields are *geometrically equivalent* if they are blow-up equivalent finite singularities in the corresponding local charts at infinity and the number, type and ordering of sectors on each side of the line at infinity of P_1 coincide with those of P_2.

A (topological) *bifurcation point* in $\mathbb{R}^{(n+1)(n+2)}$ is an unstable point, i.e. there exist a small continuous perturbation of a system (S_0) which produces systems as close as we wish to (S_0) that are topologically distinct (non-equivalent) from (S_0). The (topological) *bifurcation set* with respect to topological equivalence is the set of all bifurcation points.

The phase portrait of a system involves the behavior of the solutions near all singularities and also near all graphics or near all limit cycles of the system. To construct the bifurcation set of the systems in a parameter space we begin with the study of singularities and their bifurcation set. This set is algebraic. Some singularities may be complex. Since they do not appear in the phase portrait of a system we will not take them into account when considering the topological equivalence. But from a geometrical viewpoint, it is best to take them into account and consider the larger picture that includes them. But the systems we consider are with real coefficients therefore the parameter space is real and hence the bifurcation set of singularities is real and it is an algebraic subset of $R^{(n+1)(n+2)}$.

Thus in the study of polynomial differential systems we encounter parts of the bifurcation set in the parameter space defined by algebraic equations. We may consider the bifurcation points of singularities of a system according to their multiplicities. These algebraic sets in the parameter space $\mathbb{R}^{(n+1)(n+2)}$ are not necessarily smooth. Some of them can be as simple as a hyperplane, or the union of a plane with a straight line. But we can also encounter more complicated bifurcation sets like the cubic surface in the three-dimensional projective space defined by the homogeneous equation $-4m^2h+m^2n-4mg^2-4mgh+4mgn-8mh^2-4g^3-8g^2h+4g^2n=0$, that appeared in [17]. This algebraic equation defines a subset of the real 3-dimensional projective space which has a kind of "swordfish head" formed by the union of a surface and a 1-dimensional "sword" (we will talk a bit more about this surface later on in Example 3.4). As we will see later, these algebraic equations may lead to tricky situations in the real polynomial differential systems.

Algebraic bifurcation subsets could also arise as bifurcation points of breaking connections in case these occur on invariant algebraic curves of the systems.

Studies of these algebraic bifurcation sets involve awareness of real algebraic geometry issues in both contrast and in tandem with the complex ones.

In the case of an algebraic bifurcation subset in the parameter space defined by several algebraic equations, it may happen that some of the equations are not algebraically independent from the others, and thus the codimension cannot be thought to be the number of algebraic equations. Geometric codimension corresponds to the maximum number of equations that are algebraically independent. But detecting the independent equations when one may need to deal with many of them in a 12-parameter space (for quadratic differential systems), is not an easy task. Moreover, not every interesting object in polynomial differential systems to which we may attach a codimension are determined by algebraic equations. The more general separatrix connections than the ones mentioned above and multiple limit cycles are examples of this possibility.

3. Codimension

When in the literature a generic saddle-node is said to be of codimension 1, it is understood that there exists a small perturbation which bifurcates the saddle-node in either a simple saddle and a simple node, or into two complex simple singularities.

When in the literature it is said that a generic cusp is of codimension 2, it is because it can be perturbed into a saddle-node which has codimension 1.

When we say that a generic nilpotent elliptic saddle is of codimension 3, it is because it can be perturbed into a node plus a cusp which has codimension 2.

When we say that an object to which we apply an arbitrary small perturbation leads to objects with *the same properties* we need to give a mathematical meaning to the words *the same properties* and this invokes some equivalence relation, in particular the topological equivalence relation of local or global phase portraits.

When constructing a bifurcation diagram we are forced to take into consideration properties of the systems which are not necessarily of a topological nature. We already mentioned the distinct codimension of a weak from a strong focus. But in global problems like the topological classification of all QS modulo limit cycles we need to consider all possible singularities occurring in this class and necessarily some will have a high codimension due to not only topological properties. Hence there are many other such distinctions that need to be addressed. To these algebraic or geometrical properties correspond other equivalence relations. For example, attached to a quadratic system possessing at least one invariant parabola we have its configuration of parabolas occurring in the system. We have here an equivalence relation defined for such configurations.

To understand differential equations we need to study them inside families. In applications we always deal with approximate measurements and not exact ones. In other words we consider the equations with slightly varying coefficients. In general we need to deal with families of differential systems. So our larger object will be such a family of polynomial vector fields inside of which we may consider a subfamily $\mathcal{F}_1$ defined by specific properties of the vector fields. For example we may take $\mathcal{F}_1$ to be the family of quadratic vector fields having an invariant conic curve, subfamily of QS. We may have a mathematical object $\mathcal{O}$ attached to a vector field within a specific family.

One of the simplest examples is the local phase portrait around a singularity of a vector field. In this case a one direction node n^d or, a star-node n^* (the notation is taken from [10]) have the same codimension 0 as a generic node or a focus under the topological equivalence relation. But if we consider the geometrical properties and the geometrical equivalence relation defined in [10], then n^d is a codimension 1 singularity which can be perturbed into a generic node, and n^* is a codimension 2 singularity since it can be perturbed into an n^d.

Summing up, the concept of codimension applied to differential equations depends on the equivalence relation of the family which interests us and to which the objects are attached.

As in algebraic geometry in differential equations we also have many classification problems. For example we have the open problem of topological classification of the family QS, problem involving the topological equivalence relation. We also have the problem of classifying the global geometrical configurations of singularities occurring in the family of quadratic systems, already solved in [10] where we dealt with the family of global geometric configurations and the geometrical equivalence relation among configurations.

Here we mentioned two equivalence relations: the topological and the geometric one, the first one being weaker than the second as two geometric configurations of singularities are equivalent means that in particular they are topologically equivalent and in addition they both have the same geometric properties (see [10]).

Clearly two local phase portraits around singularities of two local systems may be topologically equivalent, while the singularities may not be geometrically equivalent and do not have the same geometrical codimension. For example, one phase portrait having a finite simple saddle could be topologically equivalent to another with a saddle which is a triple semi-elemental or nilpotent saddle. The local phase portraits are equivalent and the same topological codimension should be assigned to them, but the singularities of the systems may not be geometrically equivalent.

We consider below a similar situation where we have three local systems each one around a singularity such that these singularities are completely different geometrically but their local phase portraits are topologically equivalent. Consider the following three systems:

(a) $x' = y$, $y' = x^2$;
(b) $x' = x^2$, $y' = 2x^2 + y^2$;
(c) $x' = x^2 + y^2$, $y' = 2(x^2 + y^2)$.

System (a) has a finite nilpotent singularity (a cusp at the origin) with two hyperbolic sectors. System (b) has a finite intricate[2] singularity with two hyperbolic sectors (a "flat saddle"), and system (c) which is degenerate; has two complex lines intersecting at a real singular point. The three phase portraits are topologically equivalent. Geometrically these systems are completely different and in principle different geometrical codimensions could be assigned to them.

As we see in the above cases we may need to assign different codimensions according to distinct equivalence relations for objects. For analytic differential systems an infinite number of codimensions could be assigned (for example, as mentioned before, all semi-elemental saddles $\overline{s}_{(i)}$ for i odd are topologically equivalent to elemental saddles s).

One would think that the geometrical equivalence relation of singularities defined in [10] which includes multiplicity as a part of its definition and a number of other things such as for example order of weak foci or saddles, would be strong enough for defining the notion of codimension for all singularities but this turns out not to be true. This equivalence is good enough so as to be able to assign a geometrical codimension to every elemental (different from a center and from an integrable saddle) or semi-elemental singularity. But there are nilpotent and intricate singularities which apart from the geometric properties mentioned in [10] may have additional properties. For example a perturbation of a non-generic cusp may produce a weak focus of order greater than 1 and this could increase its codimension as indicated in [20].

In the classification of phase portraits, in order to simplify matters we can first decide to do a classification modulo limit cycles. Progress in solving Hilbert 16th problem has been incredibly slow even for the simplest non-linear case, the quadratic family. Topological classification problems for special sub-classes of the quadratic family have brought a limited amount of light in connection with Hilbert's 16th problem. For example we saw that in the family QW2 of quadratic system possessing a weak focus of order two, systems with phase portraits possessing the maximum number of two limit cycles which were found for this family, appear in phase portraits that agglutinate around systems with center as close as we wish and in particular agglutinate around a phase portrait having the infinite line filled up with singularities. This suggests that for a certain degree

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[2] The name *intricate* was introduced in [10] as a substitute of the classical *linearly zero* so to use just one word and avoid the use of *degenerate* which is widely used in too many ways leading to confusion.

n although the maximum number of limit cycles appears in the phase portraits of generic systems, actually the neighborhoods of the most degenerate systems with center may have phase portraits possessing this maximum number of limit cycles.

While progress on Hilbert's 16th problem was slow even for quadratic systems, progress is significant in the topological classification modulo limit cycles for quadratic systems. By global topological configuration of singularities we understand a description of the local phase portraits around all the finite and infinite singularities of a polynomial differential system.

Definition 3.1. By the *geometry of a singularity* we understand its local topological phase portrait, together with its multiplicity, its order if it is a weak focus or a weak saddle, its type of tangent equivalence if it is a node, its level of isochronicity if it is a center, and its geometrical sectorial decomposition.

Definition 3.2 ([10])**.** We call a *geometrical configuration of singularities of a polynomial system* an ordered set of symbols encoding the complete geometry of all its singularities. This set encodes the geometry of the isolated singularities (finite and infinite) and occasionally may include a subset encoding the geometry of the whole set of non-isolated singularities if an infinite number of singularities occurs.

The configuration of singularities that includes a center are especially tricky situations for the definitions of codimension even in quadratic systems. We discuss this case in Section 4.

The geometrical equivalence of configurations of singularities was already defined in [10, Definition 3.5] and we do not include it here.

Our first goal is to assign a (topological) codimension to each one of the global topological configurations of singularities in QS. We already mentioned that the geometrical classification of the global geometrical configurations was done in [10] and its bifurcation diagram in terms of invariant polynomials was obtained in the 12-dimensional space of the coefficients of the systems. Using this foundation the classification of global topological configurations in terms of invariant polynomials was obtained in [9] where 208 global topological configurations were obtained. We later realized that we actually have only 207 distinct configurations.

Our long term goal is to classify topologically all the different phase portraits of quadratic systems modulo limit cycles. And for this goal we plan to use as a helpful classifying instrument the concept of codimension that we must first define here. The most common equivalence relations that will be used are the topological and the geometrical equivalence defined in [10].

Definition 3.3. Consider a set $\mathcal{F}$ of real differential systems in the plane (C^r, C^∞, analytic or polynomial). Let $\mathcal{F}_1$ be a subset of $\mathcal{F}$. Suppose that for the purpose of a study we consider an equivalence relation E on $\mathcal{F}$ and an equivalence relation E_1 on $\mathcal{F}_1$ compatible with E in the sense that if two objects are E_1-equivalent in $\mathcal{F}_1$ they are also E-equivalent in $\mathcal{F}$. Consider a specific mathematical object (such as for example a configuration of singularities, the configuration of invariant algebraic curves in case they exist, the phase portrait, or the systems themselves, etc.) of a system in the set $\mathcal{F}_1$ with an equivalence relation $\widetilde{E}_1$ on objects compatible with E_1 and E (based on geometrical or topological concepts). Then we say that the object in a system $S \in \mathcal{F}_1$ is of *codimension* 0 *(or structurally stable)* within $\mathcal{F}$ with respect to E (for short E-codimension 0) if any sufficiently small continuous real perturbation $\mathcal{P}_\varepsilon$ within $\mathcal{F}$ for every ϵ, of the system S leaves this object in the same equivalence class. We say that the object is of *E-codimension $m > 0$ (or structurally unstable of codimension $m > 0$)* if any sufficiently small continuous perturbation $\mathcal{P}_\varepsilon$ with systems inside $\mathcal{F}$, for every ε, either leaves this object in the same equivalence class, or perturbs it into objects of lower E-codimension, and moreover, there exists at least one such perturbation which perturbs the object into objects of E-codimension $m-1$.

Definition 3.3 as given here is an extension of the definition given by Sotomayor [37]. There are though several differences: Firstly there is no mention of an equivalence relation in Sotomayor's definition although not directly expressed, the topological equivalence relation was implicitly present. But we do not refer to just the topological equivalence relation and leave the equivalence relation to be specified in each case considered. For example in case we need to consider the codimension of a global geometric configuration of singularities we need to refer to the geometric equivalence of such configurations. We will see that this definition produces some situations requiring much attention because we deal here with real differential systems and their real perturbations producing a finer stratification than in case we would consider their complexification and complex perturbations. We present here below some examples showing that this definition needs to be improved.

The above definition can be applied to global configurations of singularities (either geometrical or topological) considered with their corresponding equivalence relations. It can also be applied to phase portraits of polynomial differential systems with the topological equivalence relation, etc.

Notice that this definition is inductive, that is, one needs to have all the objects of codimension k before starting to study the objects of codimension $k+1$. Moreover one needs to know the versal unfolding of the object which is considered in order to be sure of what codimension it has. For some simple objects of low codimensions, this is very simple, but starting with objects of codimensions higher than or equal to 2, its codimension will only be fixed after a detailed discussion.

We give below three examples that show why Definition 3.3 must be improved.

Example 3.4. In [17] the invariant polynomials $\mathcal{T}_4$ and $\mathcal{T}_3$ appear. These invariants are responsible for the existence of weak singularities (real or complex) for the family of quadratic systems.

For the systems studied in [17] the invariant polynomial $\mathcal{T}_4$ has the value

$$\mathcal{T}_4 = n\big[-4m^2h + m^2n - 4mg^2 - 4mgh + 4mgn - 8mh^2 - 4g^3 - 8g^2h + 4g^2n\big] \equiv n\mathcal{S}_3.$$

The equation $\mathcal{S}_3 = 0$ in four variables with coefficients in $\mathbb{R}$ defines a surface in the 3-dimensional real projective space. We consider the surface $\mathcal{S}_3 = 0$ in the affine chart $g = 1$, in other words in $\mathbb{R}^3$. The surface $\mathcal{S}_3 = 0$ has a singular line (i.e. its gradient is null on this line) and it cuts the planes $n =$const. on an algebraic curve having two components: a point and a curve if $n > 9/4$ and has only one component if $n \leq 9/4$ (see FIGURE 1). On this singular line the invariant $\mathcal{T}_3$ vanishes and this means that the systems defined by points on this line have two weak singularities (or a double point) with trace zero.

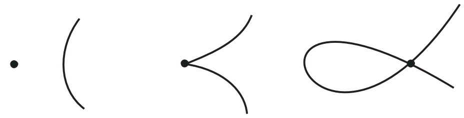
In FIGURE 1 we have the singular cubic curves obtained by intersecting $\mathcal{S}_3$ with the planes $n = constant$, the singular points of these cubics, denoted by s_n lie on the singular line of $\mathcal{S}_3$.

We now describe the types of singularities of the systems corresponding to these points s_n. In picture (c) the parameter value s_n corresponds to a system with a weak saddle and a weak focus. In picture (b) s_n corresponds to a system with a singularity of the cusp type. In picture (a) s_n corresponds to a system with two complex singularities with traces zero.

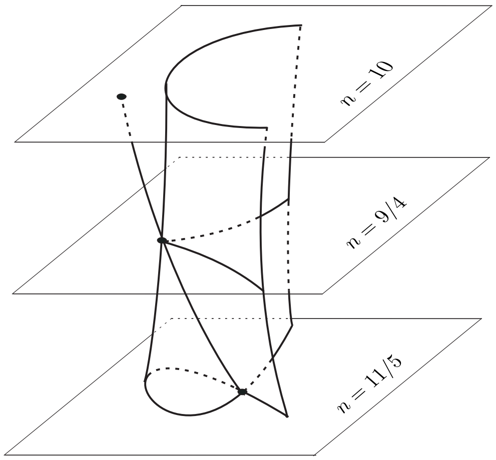
Thus we observe that in the case (c) one can perturb the system corresponding to the singular point of $\mathcal{S}_3$ keeping weak the singularity. In the same way in the case (b) we can perturb the system with the cusp singularity obtaining an elemental weak singularity. But in the case (a) a perturbation that induces one complex singularity to have non-zero trace, simultaneously forces the second singularity to have non-zero trace.

The topological codimension that must be assigned to the systems on the smooth part of the surface $\mathcal{S}_3 = 0$ should be 1. The topological codimension that must be assigned to the systems on the singular part of the surface $\mathcal{S}_3 = 0$ when $n < 9/4$ should be 2. Also clearly the codimension that must be assigned to the system on the singular part of the surface $\mathcal{S}_3 = 0$ when $n = 9/4$ should be 3. But the codimension that must be assigned (according to Definition 3.3) to the systems on the singular part of the surface $\mathcal{S}_3 = 0$ when $n > 9/4$ should be 1. So on this line we would have three different codimensions. This happens because we consider here the real surface. This is similar to what occurs in real algebraic geometry where when trying to understand intersection of curves over real numbers, without Bezout's Theorem, there appear complications due to the fact that the field of the real numbers is not algebraically closed.

A complex perturbation (from the point s_n for $n > 9/4$) could bring us to a system with only one weak complex singularity and the other one strong, and a second perturbation would bring us to a system having both complex singularities strong. Hence for $n > 9/4$ the system corresponding to s_n would have to be of codimension 2 (unlike the codimension 1 occurring in the case of real perturbations) just like for $n < 9/4$.



(a) $n = 10$ (b) $n = 9/4$ (c) $n = 11/5$



(d) 3D image

Figure 1. Global and partial pictures of $\mathcal{S}_3$

Example 3.5. Consider the seven possible finite intricate singularities of a quadratic system: $hh_{(4)}$, $hphp_{(4)}$, $phpphp_{(4)}$, $hhhhhh_{(4)}$, $ee_{(4)}$, $pepe_{(4)}$, and $peppep_{(4)}$ (see [10]). If one considers the geometrical equivalence relation, then it is clear that generically five of them have geometrical codimension at least 4 and two of them ($hphp_{(4)}$ and $pepe_{(4)}$) have codimension one more because they can be perturbed into some of the others of the above list. Moreover, some particular systems having a $phpphp_{(4)}$ singularity may be perturbed into a nilpotent saddle-node of multiplicity 4 ($\widehat{sn}_{(4)}$) which has geometrical codimension 4 (i.e. with respect to the geometrical equivalence of configurations of singularities) and thus, the intricate singularity must be considered of geometrical codimension 5 (see Subsection 3.1).

Consider now just the topological equivalence class of singularities. Then the singularity $hh_{(4)}$ disappears from the list of possible singularities because it coincides with a nilpotent cusp ($\widehat{cp}_{(2)}$) which has topological codimension 2 and is no longer intricate. On the other hand, the singularities $ee_{(4)}$, $pepe_{(4)}$ and $peppep_{(4)}$ can be perturbed into $\widehat{es}_{(3)} + a$ (nilpotent elliptic-saddle plus an anti-saddle, either a focus or a node) which is a configuration of codimension 3 (geometrically and topologically). However the singularity $hhhhhh_{(4)}$ which may be geometrically perturbed into $\widehat{s}_{(3)} + s$ (having topological codimension 0) happens to lack a good topological perturbation into codimension 3 since $\widehat{s}_{(3)}$ is topologically equivalent to an elemental saddle. An extra perturbation from $\widehat{s}_{(3)}$ could produce $\widehat{cp}_{(2)} + s$ which is a topological configuration of codimension 2. The union of the two perturbations is what justifies the topological codimension 4 for $hhhhhh_{(4)}$. Only one perturbation as it was proposed by Definition 3.3 is not enough and we must allow a combination of two perturbations.

Example 3.6. Let us consider again the nilpotent cusp of multiplicity 2. This singularity has been studied in many papers such as [20] which studies its codimension, even though the paper does not include a definition indicating the type of codimension the authors consider.

The most common value for "codimension" assigned to a cusp (of multiplicity 2) is two, and the reason is that it can be transformed by means of a perturbation into a saddle -node which is of codimension 1, because another perturbation splits the saddle-node into a saddle and a node.

The cusp can also bifurcate directly into a saddle and a strong focus, a partial configuration of codimension 0. If the focus coalesces with the saddle, it produces directly the cusp, never a saddle-node.

But there are more possibilities. A cusp is a point with trace zero, so when it bifurcates, it can produce also a weak focus, and this was already considered in [20]. One can obtain a strong saddle and a weak focus of order one, which after a second perturbation may produce a limit cycle. A quadratic system having a cusp of multiplicity 2 can be brought by affine transformations and time rescaling to the normal form

$$\begin{aligned} \dot{x} &= y + gx^2 + 2hxy + ky^2, \\ \dot{y} &= x^2 + 2mxy + ny^2 \end{aligned} \tag{3.1}$$

In [20] it was also mentioned the possibility of having a cusp of codimension 3 if certain conditions on the coefficients occur. This phenomenon can be better studied in quadratic systems using the invariant polynomials introduced in [10]. A cusp in a quadratic system verifies always the equations $\mathcal{T}_4 = \mathcal{T}_3 = 0$ in the invariants $\mathcal{T}_4$, $\mathcal{T}_3$ and in system (3.1) we have $\mathcal{F}_1 = 2(g+m)$. The invariant $\mathcal{F}_1$ is used to determine if an elemental singularity may be weak of order 2 or greater. So, in principle it has nothing to do with a cusp, but in fact it does. If we have a cusp with $\mathcal{T}_4 = \mathcal{T}_3 = 0$ and $\mathcal{T}_2 \neq 0$ this means that the system cannot have any weak elemental singularity. So the value $\mathcal{F}_1$ which somehow represents the first Lyapunov constant, is linked to the cusp. This means that if $g = -m$ then it is possible to bifurcate a weak singularity of order 2 from the cusp. In [20] the authors only considered the possibility of the weak singularity being a focus of order two, but it could also be considered the bifurcation of a weak saddle of order 2.

So, the codimension 2 assigned to generic cusps, or the particular codimension 3 assigned to special cusps fit with the invariants that rule [10, Theorem 6.2].

However there is another option not considered in [20]. As we have already said, the authors only consider the codimension 3 cusp for its possibility of producing a weak focus of order 2, and do not consider the possibility of obtaining a weak saddle of order 2. It is clear that the goal of [20] is more focused on topological features than in geometric ones, and that weak saddles are only of topological interest when concomitantly there is a loop formed by separatrices of the weak saddle. But from the purely geometrical point of view, they are as important as the weak foci as it can be deduced from [10, Theorem 6.2].

Then the fact that a cusp has invariants $\mathcal{T}_4 = \mathcal{T}_3 = 0$ allows that after a perturbation one can obtain a saddle and a focus, both being weak of first order. But this is not enough because [10, Theorem 6.2 item (c)] adds another condition which is $\mathcal{F} = 0$.

If we compute the invariant $\mathcal{F}$ for systems (3.1) one obtains that $\mathcal{F} = -(g+m)(g^2n - 2ghm + h^2 - 2hm^2 + 2hn - m^2n + n^2)$. Then the property of a cusp bifurcating simultaneously into a weak saddle and a weak focus, is possible, but not always. Again, the cusp is generically of codimension 2, but it may have geometrical codimension 3 by the reason just mentioned.

The cusps in quadratic systems may also have higher geometrical codimension under very specific conditions ($\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3\mathcal{F}_4 = 0$) and they may bifurcate into a center, but since we do not assign a codimension to centers, or systems with centers (see Section 4) neither do we nothing for such cusps.

In polynomial systems of higher degree than quadratics, clearly there may be cusps which may bifurcate in weak singularities of higher order, thus having higher codimension.

Summing up, our definition of codimension needs to be enlarged so as to accept the possibility of existence of a set of two simultaneous perturbations which produce an object of codimension 2 less in cases where the 1 less codimension is not achievable. Even though the problems described in the previous examples are of a different kind, they may be addressed in the same manner with the next definitions.

Definition 3.7. Consider a polynomial differential system S of degree m in $\mathbb{R}^k$ with coefficients $\tilde{a} \in \mathbb{R}^n$ (with $n = k(m+2)(m+1)/2$) and two perturbations (real or complex) of the system. Let $\Sigma_1(\tilde{a})$ and $\Sigma_2(\tilde{a})$ be the sequences of the n coefficients of the two perturbations. We define the perturbation sum of the system denoted by $\Sigma_1 + \Sigma_2$ as being the system whose coefficients are given by $(\Sigma_1 + \Sigma_2)(\tilde{a}) = \Sigma_1(\tilde{a}) + \Sigma_2(\tilde{a})$ (in $\mathbb{R}^n$).

Definition 3.8. Given a real polynomial differential system, we define the *conjugate double perturbation* as the perturbation sum of two complex perturbations may be done independently (thus obtaining in each case a system over $\mathbb{C}$), but the perturbation sum leaving the coefficients of the perturbed system in $\mathbb{R}$.

One such conjugate double perturbation may always be easily obtained by using a linear change of the coefficients in complex numbers $\varepsilon_1 + i\delta$ and $\varepsilon_2 - i\delta$ so that they are applied to the same terms with the same coefficients. Then the perturbation sum is a system with real coefficients. Sometimes we will also use the double conjugate perturbation even if $\delta = 0$ as for example to solve the problem exposed in Example 3.5.

Definition 3.9. Consider a set $\mathcal{F}$ of real differential systems in the plane (C^r, C^∞, analytic or polynomial). Let $\mathcal{F}_1$ be a subset of $\mathcal{F}$. Suppose that for the purpose of a study we consider an equivalence relation E on $\mathcal{F}$ and an equivalence relation E_1 on $\mathcal{F}_1$ compatible with E in the sense that if two objects are E_1-equivalent in $\mathcal{F}_1$ they are also E-equivalent in $\mathcal{F}$. Consider a specific mathematical object (such as for example a configuration of singularities, the configuration of invariant algebraic curves in case they exist, the phase portrait, or the systems themselves, etc.) of a system in the set $\mathcal{F}_1$ with an equivalence relation $\widetilde{E}_1$ on objects, compatible with E_1 and E (based on geometrical or topological concepts). Then we say that the object in a system $S \in \mathcal{F}_1$ is of *codimension* 0 *(or structurally stable)* within $\mathcal{F}$ with respect to E (for short E-codimension 0) if any sufficiently small continuous real perturbation $\mathcal{P}_\varepsilon$ within $\mathcal{F}$ for every ϵ, of the system S leaves this object in the same equivalence class. We say that the object is of *E-codimension $m > 0$ (or structurally unstable of codimension $m > 0$)* if any sufficiently small continuous perturbation $\mathcal{P}_\varepsilon$ with systems inside $\mathcal{F}$, for every ε, either leaves this object in the same equivalence class, or perturbs it into objects of lower E-codimension, and moreover, there exists at least: *a*) one such perturbation which perturbs the object into objects of E-codimension $m-1$, or (b) one conjugate double perturbation which produces objects of E-codimension $m-2$.

If we apply this definition to individual singularities one must take into account that a multiple singularity may split into several singularities (real or complex, multiple or elemental) all located in the vicinity of the initial point. The total multiplicity of all the singularities which appear after perturbation must be equal to the multiplicity of the initial singularity. So the resulting set after perturbation is not an individual singularity but it is a set of singularities.

We mentioned already the concept of multiplicity of singularities and we need to recall the definition of this concept that is defined using the notion of intersection number of two algebraic curves at a point in the plane (see [22, 29]) and its relation with the number of singularities that can appear after a perturbation of a multiple one.

The intersection number of two affine algebraic curves $C : f(x, y) = 0$ and $C' : g(x, y) = 0$ over $\mathbb{C}$ at a point a in $\mathbb{C}^2$ is the number $I_a(f, g) = \dim_{\mathbb{C}} \mathbf{0}_a/(f, g)$, where $\mathbf{0}_a$ is the local ring of the affine complex plane $\mathbf{A}^2(\mathbb{C}) = \mathbb{C}^2$ at a; i.e. $\mathbf{0}_a$ is the ring of rational functions $r(x, y)/s(x, y)$ which are defined at a, i.e. with $s(a) \neq 0$.

In our case, since our differential systems are polynomial, the intersection numbers $I_a(p, q)$ for p, q as in (1.1), at the singular points a in $\mathbb{C}^2$ can be computed easily by using the axioms (see [22]). More precisely, for two projective curves in $\mathbb{C}\mathbf{P}^2$, $F(X, Y, Z) = 0$ and $G(X, Y, Z) = 0$, where F and G are homogeneous polynomials in the variables X, Y and Z that are relatively prime over $\mathbb{C}$ we can define $I_W(F, G)$ as follows: Suppose for example that $W = [a : b : c]$ where $c \neq 0$, hence $W = [\frac{a}{c} : \frac{b}{c} : 1]$. Let $f(x, y) = F(x, y, 1)$ and $g(x, y) = G(x, y, 1)$. Then $I_W(F, G) = I_w(f, g)$ where $w = (\frac{a}{c}, \frac{b}{c})$. It is known that $I_W(F, G)$ is independent of the choice of a local chart, and of a projective change of variables, see again [22]. Using all charts and not just the one defined by $c \neq 0$, the multiplicity of intersection is defined everywhere in the projective plane.

Intersection multiplicity can also be realized as the maximum number of distinct intersections that exist if the curves are perturbed slightly. More specifically, if p and q define curves which intersect only once at some point a in the closure of an open set U, then for a dense set of

$(\varepsilon(x,y),\delta(x,y)) \in \mathbb{C}[x,y]^2$, $p(x,y)-\varepsilon(x,y)$ and $q(x,y)-\delta(x,y)$ are smooth and intersect transversely (i.e. have different tangent lines) at exactly m points in U. We say then that $I_a(p,q)=m$ (see [22, 28]).

This fact is closely related with the necessity of Definition 3.9 because the geometrical codimension of a singularity of a differential system is related with its multiplicity counted in the complex space.

To show how Definition 3.9 fits with usual definitions of codimension in classical mathematics, we will apply it to the set of quadratic polynomial differential systems under the geometrical equivalence relation.

Let us first observe that while in the classical definition of codimension we start with the dimension n of a larger object and go down to a smaller dimension m (a subspace of a linear space or a submanifold of a manifold) and take the difference $m-n$ for the codimension, in families $\mathcal{F}$ of differential equations we start with a subfamily $\mathcal{F}_1$ of structurally stable systems called of codimension zero and take its complement Σ, called the bifurcation set, and then repeat the process for Σ. The structurally stable ones in Σ are then called the codimension one systems. Therefore the codimension is built by an inductive process going from lower to upper codimensions. For quadratic systems take the larger family to be QS which is of dimension 12. The codimension 1 systems lie in a subspace of dimension 11 and hence a codimension 1 subspace, and the process continues in this way. This is how the notion of codimension for differential equations connects with the classical definitions of codimension.

We also observe that structural stability could be based not only on the equivalence relation of homeomorphism but also on other equivalence relations such as the geometrical one.

We will say that two quadratic systems are geometrically equivalent if their configurations of singularities are geometrically equivalent and their global phase portraits are topologically equivalent. Then the parameter space $\mathbb{R}^{12}$ of quadratic systems is split by hypersurfaces of dimension 11 (codimension 1) corresponding to bifurcations related with the singularities, connections of separatrices, or double limit cycles. Note that some of these hypersurfaces are algebraic and others not even analytic, but to each one of them it could be assigned a generic codimension. The algebraic hypersurfaces may contain singular points in the parameter space which will have a higher codimension. Self intersecting hypersurfaces may present points that can be considered as singularities. Even non algebraic hypersurfaces may present parts that behave as singular inside that hypersurface, and thus will have higher codimension. A quadratic system will either live in the generic part of this bifurcation diagram in $\mathbb{R}^{12}$, or in the generic part of one of the hypersurfaces, or in the singular part of one of the hypersurfaces, or in the intersection of two or more hypersurfaces. According to the number of hypersurfaces where the quadratic system lives (taking into account also the singular parts of these hypersurfaces), the codimension can be calculated. Then the standard codimension of the system will be the sum of the number of hypersurfaces (adding the codimension due to singularities). According to our definition, with the use of perturbations we may move this system towards regions of less codimension up to the generic part of $\mathbb{R}^{12}$ producing then the same codimension as the standard definition gives. Instead of quadratic systems, this works in the same way for polynomial systems of some fixed degree. So our definition of codimension in the space of the polynomial differential systems of a given degree coincides with what the classical definition would do.

Let us propose an example. Consider the quadratic differential system:

$$\dot{x} = x^2 + xy + y^2/5, \quad \dot{y} = y - 2x^2 + 24xy/5 - 3y^2/5. \tag{3.2}$$

This system comes from [12] and corresponds to the part $3.3L_1$ but its phase portrait is topologically equivalent to that given by V_{83}. In [12] a normal form of systems having a finite saddle-node and an infinite $\binom{0}{2}$ saddle-node formed by the coalescence of two infinite singularities, is studied. Moreover this system possesses a weak focus and a weak saddle both of order 1. And it also has another infinite singularity which is a node. There is no other singularity which may be affected by a perturbation and there is no separatrix connection. Now consider the system in $\mathbb{R}^{12}$:

$$\begin{aligned}\dot{x} &= a + cx + dy + (1+g)x^2 + (h+1)xy + (k+1/5)y^2,\\ \dot{y} &= b + ex + (1+f)y + (\ell-2)x^2 + (m+24/5)xy + (n-3/5)y^2.\end{aligned} \tag{3.3}$$

It is clear that systems (3.3) coincide with system (3.2) when all its parameters are zero. Then consider the geometric equivalence class of quadratic systems. The parameter space where systems (3.3) live is $\mathbb{R}^{12}$ where several bifurcation surfaces are present. Among these surfaces we have:

$$\begin{aligned}\mathbf{D} &= 192a/5 - 233664a^2/125 - 3642968064a^3/3125 + <<137163>>\\ &\quad + 6144a^3c^2g\ell^2n^4 - 12288a^4g^2\ell^2n^4 = 0,\\ \eta &= -5324g/125 + 292g^2/25 - 4cg^3/5 + <<84>> + m^2n^2 + 8n^3 - 4\ell n^3 = 0,\\ \mathcal{T}_4 &= 7086244a^2/15625 + 7086244ab/15625 + <<12223>>\\ &\quad + 7086244ab/15625 + 1771561b^2/15625 = 0,\\ \mathcal{T}_3 &= -95832a/625 - 47916b/625 + 5324c/125 + <<5404>>\\ &\quad - 16bch\ell n^3 + 8aeh\ell n^3 - 16ad\ell^2n^3 = 0.\end{aligned}$$

These invariants depend on the 12 parameters of the system and they are independent because we have 12 degrees of freedom and only 4 equalities to hold which control distinct phenomena. Invariant $\mathbf{D}$ controls whether there is a double finite singularity, invariant η controls whether there is a double infinite singularity and invariants $\mathcal{T}_4$ and $\mathcal{T}_3$ control the existence of weak singularities. The four hypersurfaces intersect in a 8-dimensional variety on which $a = b = c = d = e = f = g = h = k = \ell = m = n = 0$ is just a point which is system (3.2). Thus, according to the geometrical equivalence (which also distinguishes weak saddles from strong saddles), system (3.2) stays in the intersection of all four hypersurfaces and it is a codimension 4 system. However, if we consider just a topological equivalence of systems we must get rid of the weak saddle, but we cannot get rid of the weak focus which can produce limit cycles. In this case, we can still work in the space of parameters $\mathbb{R}^{12}$ but considering only three of the hypersurfaces, obtaining only the codimension 3. Finally if we take a topological modulo limit cycles equivalence relation $\sim$, then our parameter space is not anymore $\mathbb{R}^{12}$ but $\mathbb{R}^{12}/\sim$ in which some parts or the previous bifurcation diagram become identified. Under this equivalence class the phase portrait of system (3.2) is just of codimension 2.

Notice that if the phase portrait had a loop formed by the separatrices of the weak saddle, then one cannot use the topological equivalence of systems because then, the geometry of the weak saddle (of order 1) combined with the loop can produce up to 2 limit cycles (or even a double limit cycle) and this phenomenon cannot be properly captured by this equivalence relation. One can use the geometrical equivalence class or the topological modulo limit cycles, but not the topological which still considers the possibility of obtaining limit cycles but forfeits the effects of the weak saddles.

We will also use the concept of index of a singularity, see for example [21].

Lemma 3.10. *If the polynomial differential system of degree n*

$$\frac{dx}{dt} = \sum_{i,j=0}^{i+j=n} a_{i,j}x^iy^j = P(x,y), \quad \frac{dy}{dt} = \sum_{i,j=0}^{i+j=n} b_{i,j}x^iy^j = Q(x,y), \tag{3.4}$$

has a multiple singular point d with $I_d(P,Q) = k$, then the sum of the indices of the singularities which appear from d in the vicinity of d after a sufficiently small perturbation of the differential system inside its family, is equal to the index of the original singularity.

This lemma comes from the conservation of indices in polynomial differential systems. We recall from [21] the two theorems that characterize the local phase portraits of the semi-elemental and nilpotent singularities.

Theorem 3.11 (Semi-elemental Singular Points [21, Theorem 2.19])**.** *Let* $(0,0)$ *be an isolated singular point of the vector field* X *given by*

$$\dot{x} = A(x,y), \quad \dot{y} = \lambda y + B(x,y), \tag{3.5}$$

where A *and* B *are analytic in a neighborhood of the origin with* $A(0,0) = B(0,0) = DA(0,0) = DB(0,0) = 0$ *and* $\lambda > 0$*. Let* $y = f(x)$ *be the solution of the equation* $\lambda y + B(x,y) = 0$ *in a neighborhood of the point* $(0,0)$*, and suppose that the function* $g(x) = A(x, f(x))$ *has the expression* $g(x) = a_m x^m + o(x^m)$*, where* $m \geq 2$ *and* $a_m \neq 0$*. Then there always exists an invariant analytic curve, called the strong unstable manifold, tangent at* 0 *to the* y*–axis, on which* X *is analytically conjugate to*

$$\dot{y} = \lambda y;$$

it represents repelling behavior since $\lambda > 0$*. Moreover the following statements hold.*

(i) *If* m *is odd and* $a_m < 0$*, then* $(0,0)$ *is a topological saddle (see* Figure *2.a). Tangent to the* x*–axis there is a unique invariant* C^∞ *curve, called the center manifold, on which* X *is* C^∞*-conjugate to*

$$\dot{x} = -x^m(1 + a x^{m-1}), \tag{3.6}$$

for some $a \in \mathbb{R}$*. If this invariant curve is analytic, then on it* X *is* C^ω*-conjugate to* (3.6)*.*

System X *is* C^∞*-conjugate to*

$$\dot{x} = -x^m(1 + a x^{m-1}), \quad \dot{y} = \lambda y,$$

and is C^0*-conjugate to*

$$\dot{x} = -x, \quad \dot{y} = y.$$

(ii) *If* m *is odd and* $a_m > 0$*, then* $(0,0)$ *is a unstable topological node (see* Figure *2.b). Every point not belonging to the strong unstable manifold lies on an invariant* C^∞ *curve, called a center manifold, tangent to the* x*–axis at the origin, and on which* X *is* C^∞*-conjugate to*

$$\dot{x} = x^m(1 + a x^{m-1}), \tag{3.7}$$

for some $a \in \mathbb{R}$*. All these center manifolds are mutually infinitely tangent to each other, and hence at most one of them can be analytic, in which case* X *is* C^ω*-conjugate on it to* (3.7)*.*

System X *is* C^∞*-conjugate to*

$$\dot{x} = x^m(1 + a x^{m-1}), \quad \dot{y} = \lambda y,$$

and is C^0*-conjugate to*

$$\dot{x} = x, \quad \dot{y} = y.$$

(iii) *If* m *is even, then* $(0,0)$ *is a saddle-node, that is, a singular point whose neighborhood is the union of one parabolic and two hyperbolic sectors (see* Figure *2.c). Modulo changing* x *into* $-x$*, we suppose that* $a_m > 0$*. Every point to the right of the strong unstable manifold (side* $x > 0$*) lies on an invariant* C^∞ *curve, called a center manifold, tangent to the* x*-axis at the origin, and on which* X *is* C^∞*-conjugate to*

$$\dot{x} = x^m(1 + a x^{m-1}), \tag{3.8}$$

for some $a \in \mathbb{R}$*. All these center manifolds coincide on the side* $x \leq 0$ *and are hence infinitely tangent at the origin. At most one of these center manifolds can be analytic, in which case* X *is* C^ω*-conjugate on it to* (3.8)*.*

System X *is* C^∞*-conjugate to*

$$\dot{x} = x^m(1 + a x^{m-1}), \quad \dot{y} = \lambda y,$$

and is C^0*-conjugate to*

$$\dot{x} = x^2, \quad \dot{y} = y.$$

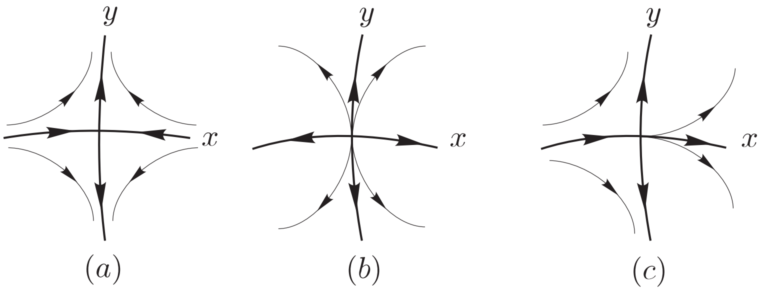


FIGURE 2. Phase portraits of semi hyperbolic singular points.

Theorem 3.12 (Nilpotent Singular Points [21, Theorem 3.5])**.** *Let $(0,0)$ be a singular point of the vector field X given by*

$$\dot{x} = y + A(x,y), \quad \dot{y} = B(x,y), \tag{3.9}$$

where A and B are analytic in a neighborhood of the point $(0,0)$ and $j_1A(0,0) = j_1B(0,0) = 0$. Let $y = f(x)$ be the solution of the equation $y + A(x,y) = 0$ in a neighborhood of the point $(0,0)$, and consider $F(x) = B(x, f(x))$ and $G(x) = (\partial A/\partial x + \partial B/\partial y)(x, f(x))$. Then the following holds:

1. *If $F(x) \equiv G(x) \equiv 0$, then the phase portrait of X is given by* FIGURE *3 (a).*
2. *If $F(x) \equiv 0$ and $G(x) = b_s x^s + o(x^s)$ for $s \in \mathbb{N}$ with $s \geq 1$ and $b_s \neq 0$, then the phase portrait of X is given by* FIGURES *3 (b) or (c).*
3. *If $G(x) \equiv 0$ and $F(x) = a_m x^m + o(x^m)$ for $m \in \mathbb{N}$ with $m \geq 1$ and $a_m \neq 0$, then*
    - (i) *if m is odd and $a_m > 0$, then the origin of X is a saddle (see* FIGURE *3.(d)) and if $a_m < 0$, then it is a center or a focus (see* FIGURES *3.(e), (f) and (g));*
    - (ii) *if m is even then the origin of X is a cusp as in* FIGURE *3.(h).*
4. *If $F(x) = a_m x^m + o(x^m)$ and $G(x) = b_s x^s + o(x^s)$ with $m \in \mathbb{N}$, $m \geq 2$, $s \in \mathbb{N}$, $s \geq 1$, $a_m \neq 0$ and $b_s \neq 0$, then we have:*
    - (i) *if m is even, and*
        - (i1) *$m < 2s + 1$, then the origin of X is a cusp as in* FIGURE *3 (h);*
        - (i2) *$m > 2s + 1$, then the origin of X is a saddle–node as in* FIGURE *3 (i) or (j);*
    - (ii) *if m is odd and $a_m > 0$ then the origin of X is a saddle as in Figure 3 (d);*
    - (iii) *if m is odd, $a_m < 0$ and*
      beginitemize
    - (iii1) *either $m < 2s + 1$, or $m = 2s + 1$ and $b_s^2 + 4a_m(n+1) < 0$, then the origin of X is a center or a focus (see* FIGURES *3 (e), (f) and (g));*
    - (iii2) *s is odd and either $m > 2s + 1$, or $m = 2s + 1$ and $b_s^2 + 4a_m(n+1) \geq 0$, then the phase portrait of the origin of X consists of one hyperbolic and one elliptic sector as in Figure 3 (k);*
    - (iii3) *s is even and either $m > 2s + 1$, or $m = 2s + 1$ and $b_s^2 + 4a_m(n+1) \geq 0$, then the origin of X is a node as in* FIGURES *3 (l) and (m). The node is attracting if $b_s < 0$ and repelling if $b_s > 0$.*

Remark 3.13. If we order the singularities according their degree of degeneracy as elemental, semi-elemental, nilpotent and intricate, it is clear that by means of a perturbation, any of them will remain in the same set or will bifurcate in singularities of less degeneracy.

3.1. **Codimension of semi-elemental and nilpotent singularities.**

In Propositions 3.14 and 3.16 we give a relation between multiplicity and geometrical codimension of the singularities in polynomial differential systems.

Proposition 3.14. *We consider an isolated singularity (finite or infinite) in a polynomial differential system of any degree, and consider the geometrical equivalence relation. Then:*

- (a) *a semi-elemental singularity (finite or infinite) of multiplicity k has geometrical codimension $k - 1$;*

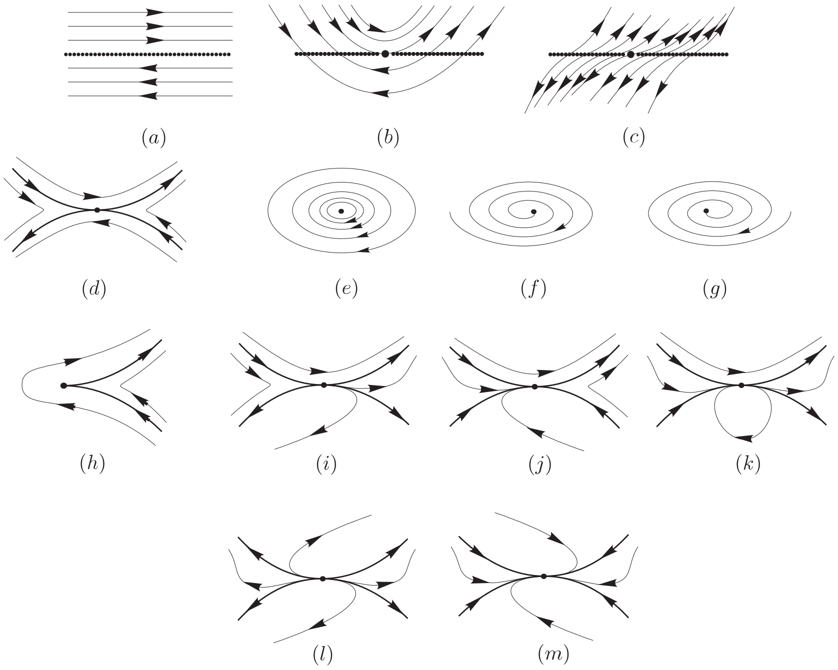


FIGURE 3. Phase portraits of nilpotent singular points.

(b) *a finite nilpotent singularity of multiplicity k has geometrical codimension $\geq k$;*
(c) *an infinite nilpotent singularity of multiplicity k has geometrical codimension $\geq k-1$.*

Proof. Assume we have a polynomial differential system (3.4) of degree n. The polynomial differential system extended to the Poincaré sphere (see [21, Chapter 5]), on the charts at infinity has degree $n+1$ and it may have singularities there. If it has at least one real singularity, a rotation of the original system may make $b_{n,0}=0$, and then the origin of the local chart U_1 is a singularity. The vector field at local chart U_1 is of the form

$$\frac{dw}{dt}=\sum_{i,j=0}^{i+j=n+1} A_{i,j}w^iz^j, \quad \frac{dz}{dt}=\sum_{i,j=0}^{i+j=n+1} B_{i,j}w^iz^j, \tag{3.10}$$

where $A_{i,j}$ and $B_{i,j}$ depend linearly on the coefficients of systems (3.4) and $A_{0,0}=B_{i,0}=0$ for $i=0,\dots,n+1$. That is, the straight line $z=0$ corresponding to the infinite line is invariant.

The codimension 0 singularities according to the geometrical equivalence are the elemental saddle s with trace different from zero, the node n with discriminant different from zero and the strong focus f with trace different from zero. All these are singularities of multiplicity 1. Thus $I_u(P,Q)=1$ where u is any of these singular points. There are also elemental singularities with geometrical codimension greater than one like the one-direction node (n^d), the star node (n^*), the weak saddles or foci ($s^{(i)}$ or $f^{(i)}$), the integrable saddles and the centers. However, all these imply that either the trace of the Jacobian matrix or the discriminant of the characteristic polynomial associated to the Jacobian matrix, must be zero. Since the trace and the discriminant of the Jacobian matrix of a semi-elemental singularity are always different from zero, the elemental singularities with geometrical codimension greater than one cannot be obtained by perturbation of a semi-elemental singularity.

(a) Assume that system (3.4) has a semi-elemental finite singularity. Then by means of an affine change of variables and time rescaling, the system can be transformed to one having $a_{0,0}=a_{1,0}=a_{0,1}=b_{0,0}=b_{1,0}=0$ and $b_{0,1}=1$. We can also consider the case of the semi-elemental infinite singularity changing the coefficient $a's$ and $b's$ by $A's$ and $B's$ respectively. Then we obtain the expression for $g(x)=a_mx^m+o(x^m)$ as described in [21, Theorem 2.19].

Assume that we are in the most generic situation of a semi-elemental singularity. This will happen when $a_2 \neq 0$, then $m = 2$ and this implies $a_{2,0} \neq 0$. So the singularity is a saddle-node. It is well known that a generic saddle-node is a singularity of multiplicity 2 and geometrical codimension 1 and that its index is 0. This can be easily checked because a small perturbation of the linear part of the system can transform the saddle-node into a saddle and a node.

Assume now that $a_2 = 0$ but we are in the most generic possibility in this case, that is $a_3 \neq 0$. Then the singularity will be a saddle or a node, so the index is -1 or $+1$. How has the index changed from 0 to ± 1? The only possibility is that the origin has coalesced with some singularities (totaling an odd number counting the multiplicity of the singularities) to produce the new semi-elemental singularity. Of course the simplest possibility is that the number of singularities that coalesced with the origin is just one and the singularity is elemental. But it could also occur that three or five or more singularities had coalesced simultaneously. We claim that this is not possible, and not just for the step from a_2 to a_3 but for any other consecutive step from a_m to a_{m+1}.

Indeed assume that $a_i = 0$ from $i = 2, \dots, m$ and $a_{m+1} \neq 0$. We obtain the functions $f(x)$ and $g(x)$ of Theorem 3.11 [21, Theorem 2.19]. Then $x = 0$ is a zero of multiplicity $m + 1$ of the equation $g(x) = 0$ and the point $(0, 0)$ is a multiple singularity of the system (3.5). This system may of course have other singularities, simple or multiple different from the origin. If we make a convenient perturbation depending on ε in one or several parameters so that $a_m \neq 0$ (maintaining all previous $a_i = 0$) and we recompute the functions $f_\varepsilon(x)$ and $g_\varepsilon(x)$, then the equation $g_\varepsilon(x) = 0$ will have $x = 0$ as a zero of multiplicity m and another solution $x = \varepsilon$ which must be simple. Then another singularity $(\varepsilon, f_\varepsilon(\varepsilon))$ appears close to the origin, and this singularity is simple. In case that there were other multiple singularities, they may have also split, but they will not be close to the origin. So the claim is proved.

Then in the case $a_2 = 0 \neq a_3$ we have proved that the singularity at the origin has multiplicity 3 and codimension 2, and by induction we have proved statement *a*) of the Lemma (for finite singularities).

Consider now an infinite singularity of system (3.4). If a singular point of system (3.10) is semi-elemental, then either the first or the second eigenvalue vanishes. In both cases the system can be transformed by means of an affine change of variables into a system with the same linear part as proposed for the finite case. The same arguments as we have applied for finite singularities can be applied to infinite singularities and so statement (a) is proved.

(b) Assume that a polynomial differential system (3.4) of degree n has a nilpotent finite singularity. Then by means of an affine change of variables, this system can be converted to one having $a_{0,0} = a_{1,0} = b_{0,0} = b_{1,0} = b_{0,1} = 0$ and $a_{0,1} = 1$. That is, we obtain the system

$$\frac{dx}{dt} = y + \sum_{i=2}^{n} p_i(x, y) = y + A(x, y), \quad \frac{dy}{dt} = \sum_{i=2}^{n} q_i(x, y) = B(x, y), \tag{3.11}$$

where p_i and q_i are homogeneous polynomials in x and y.

We obtain the expressions for $y = f(x)$ (solution of $y + A(x, y) = 0$), $F(x) = a_m x^m + o(x^m)$ and $G(x) = b_s x^s + o(x^s)$ as described in Theorem 3.12. We remark that for any k, the coefficient a_k is a polynomial in the coefficients $a_{i,j}$ and $b_{i,j}$ of the system but always $k < i + j \leq n$ except for the coefficient $b_{k,0}$ which is always linearly present in a_k (if $n \geq k$).

Theorem 3.12 offers two distinct possibilities for nilpotent singularities, whether $G(x) \equiv 0$ or not. We consider first the case that we have a nilpotent singularity and $G(x)$ does not vanish. In this case the proof follows a very similar argument as in the semi-elemental case after a first argument that will compute the codimension of the most generic case.

Assume that we are in the most generic situation of a nilpotent singularity. This will happen when $a_2 \neq 0$, then $m = 2$ and this implies $b_{2,0} \neq 0$. So the singularity is a cusp (in fact, for this case, it does not matter if $G(x) \equiv 0$ or not). It is well known that a generic cusp is a singularity of multiplicity 2 and geometrical codimension ≥ 2 (see [20]). This can be easily checked because a small perturbation of the linear part of the system can transform the cusp into a saddle-node without ejecting any singularity and this saddle-node is of multiplicity 2 and codimension 1. Another possibility is that after perturbation the cusp splits into an elemental saddle and a weak focus of order one, that under the geometrical equivalence relation is also a codimension 1 singularity. In the case of quadratic systems, if the system with the cusp has some vanishing invariants $\mathcal{F}_i$ from [10] (equivalent conditions can be seen in [20]), the cusp can split into an elemental saddle and a weak focus of order two, that is a geometrical codimension 2 singularity. Finally if more conditions hold then the cusp can split into an elemental saddle and a center which yields an even higher codimension for the cusp.

For polynomial systems that are not quadratic, the analogs of these invariants $\mathcal{F}_i$ are still unknown but they must exist and be more numerous to reflect the greater number of relevant Lyapunov constants. Thus a cusp of multiplicity 2 has geometrical codimension ≥ 2.

Assume that $a_2 = 0$ and the next generic situation is $a_3 \neq 0$. Then $m = 3$ and with m odd, all the different options of Theorem 3.12 lead of a singularity with index $+1$ or -1. So an odd number of singularities (counting multiplicity) have coalesced with the origin. If we move to more degenerated situations with $m > 3$ we see that always m even implies that the singularity has index 0, and m odd implies index ± 1. We claim that only one singularity coalesces for every level.

Indeed assume that all $a_i = 0$ from $i = 2, \dots, m$ and $a_{m+1} \neq 0$. We obtain the functions $f(x)$, $F(x)$ and $G(x)$ of Theorem 3.12. Then $x = 0$ is a zero of multiplicity $m + 1$ of the equation $F(x) = 0$ and the point $(0, 0)$ is a multiple singularity of system (3.9). This system may of course have other singularities, simple or multiple different from the origin. If we make a perturbation depending on ε on one or more parameters so that $a_m \neq 0$ (maintaining all previous $a_i = 0$) and we recompute the functions $f_\varepsilon(x)$, $F_\varepsilon(x)$ and $G_\varepsilon(x)$, then the equation $F_\varepsilon(x) = 0$ will have $x = 0$ as a zero of multiplicity m and another solution $x = \varepsilon$, that must be simple. Then another singularity $(\varepsilon, f_\varepsilon(\varepsilon))$ appears close to the origin, and this singularity is simple. In case that there were other multiple singularities, they may have also split, but they will not be close to the origin. So the claim is proved.

The addition of one in the codimension that nilpotent singularities have with respect to semi-elemental singularities when both have the same multiplicity, comes from the fact that in the last step the cusp can be turned into a semi-elemental saddle-node. It changes then from a singularity with one characteristic direction to a singularity with two characteristic directions. The increased geometrical codimension that a nilpotent singularity may have above its multiplicity comes always from weak singularities, centers, double limit cycles, one-direction nodes or star nodes that may split from them in some special combinations of parameters. This fact will not have any consequences when we consider topological codimension of singularities modulo limit cycles.

Repeating this argument inductively, the statement *b*) is proved in the case $G(x) \not\equiv 0$.

Assume we have $G(x) \equiv 0$ and $F(x) = a_m x^m + o(x^m)$ for $m \in \mathbb{N}$ with $m \geq 1$ and $a_m \neq 0$. Then one may do perturbations maintaining $G(x) \equiv 0$, or breaking it. Assume first that we maintain $G(x) \equiv 0$. Then one must perturb $a_i \neq 0$ from $i = m - 1, m - 2, \dots, 2$ as we have done in the previous case. In this way the index of the singularity will be changing from $+1$ (or -1) to 0 and back to $+1$ (or -1). So the perturbations will eject elemental singularities of indices $+1$ or -1 from the nilpotent singularity, one at every step. These perturbations on a_i from $i = m-1, m-2, \dots, 2$ may need perturbations in several coefficients of the system in order to maintain $G(x) \equiv 0$.

So we have already proved that the multiplicity of the origin of system (3.9) is directly related with the number of coefficients a_i that vanish in the function $F(x)$. If we make a perturbation on the system so that $G(x) \not\equiv 0$, but maintaining the same m, the singularity may turn from cusp (case 3.ii) to saddle-node (case 4.i2), or from center-focus (case 3.i) to elliptic-saddle, implying a change in codimension, but this cannot change the multiplicity of the singularity. Summing up, statement (b) has been proved.

(c) Why infinite nilpotent singularities have one codimension less than finite nilpotent singularities? This is because the polynomial differential systems cannot have cusps at infinity. This can easily be checked looking at (3.10) and considering that all the coefficients $B_{i,0}$ are zero for $i = 0, \dots, n + 1$. A nilpotent singularity at infinity comes always from the coalescence of at least one finite with at least two infinite singularities, and when we start perturbing a multiple nilpotent singularity at infinity, we either eject singularities along the line of infinity, or into the affine plane until this nilpotent point becomes a semi-elemental singularity and the possibility of having a cusp does not exist.

But one could think also about the possibility that an infinite nilpotent (or intricate) singularity could eject a cusp into the affine region. We claim that this is not possible and this statement deserves to be a lemma that we give just below. So, statement (c) is proved and this completes the proof of Proposition 3.14. □

Lemma 3.15. *An infinite nilpotent or intricate singularity cannot eject just one generic cusp of multiplicity* 2 *into the affine region by means of a perturbation.*

Proof. The proof is quite simple. The system at infinity has always the coefficient $B_{2,0} = 0$. And in order to have a generic cusp of multiplicity 2 we need it to be different from zero. And there is no way to modify this coefficient by means of a perturbation of the original system. The coefficient $B_{2,0}$ is always equal to zero by construction of the Poincaré compactification. □

One may eject a more degenerate cusp (of multiplicity 4 or greater) or other degenerate singularities which may immediately split in the affine region into a generic cusp plus other singularities (real or complex), but not just a single generic cusp. In other words, a finite generic cusp cannot escape to infinity alone. It has to move along with some other finite singularities (real or complex) and coalesce with the same infinite singularity at the same time. It is true that all of them coalesce with the infinite singularity in the projective space but it is also possible that they coalesce to opposite points at infinity in the Poincaré disk.

For example, the system $x' = 1 + x$, $y' = -x^2$ has the geometric configuration of singularities $\emptyset$, $\binom{4}{3}\, \overset{\curvearrowright}{P}_{\curlywedge} \overset{\curvearrowleft}{P} \overset{\curvearrowright}{P} \overset{\curvearrowleft}{P}_{\curlywedge} - \overset{\curvearrowright}{P}\, \overset{\curvearrowleft}{P}$, that is, it has an infinite intricate singularity of multiplicity 7 (3 infinite and 4 finite) which behaves topologically like an elemental node. This system has geometrical codimension 6 but topological codimension 0. Now take the perturbed system

$$
\begin{gathered}
x' = 1 + x + \frac{1}{3}\varepsilon(9\varepsilon - 2)y + 2\varepsilon x^2 - \varepsilon^2 xy, \\
y' = -x^2 + 2\varepsilon xy - \varepsilon^2 y^2.
\end{gathered}
$$

This system has the geometrical configuration of singularities $\widehat{cp}_{(2)}, \overline{sn}_{(2)}; \overline{\binom{0}{3}} N$, that is, the intricate point of multiplicity 7 has split into three singularities, a semi-elemental infinite one of multiplicity 3 which is a node, and two finite ones of multiplicity 2 which are a semi-elemental saddle-node and a nilpotent cusp. And this configuration has geometrical codimension 5 (2 from the cusp, plus 1 from the finite saddle-node, plus 2 from the infinite semi-elemental node). The topological codimension is just 3 (coming from the cusp and the saddle-node) because the triple infinite node behaves like a normal node.

The perturbed system has singularities $(-3, -\frac{3}{\varepsilon})$ and $(-\frac{1}{3\varepsilon}, -\frac{1}{3\varepsilon^2})$, so if $\varepsilon > 0$ both singularities appear in the neighborhood of the same infinite singular point in the Poincaré disk, but if $\varepsilon < 0$ one of them appears close to the opposite infinite singularity in the Poincaré disk.

3.2. **Codimension of intricate singularities.** We consider now the case of intricate singularities and aim to obtain a lemma for the codimension of these singularities in terms of their multiplicity. We first give some motivation.

We may think that an intricate singularity of multiplicity k would have a higher (by one) codimension than a nilpotent singularity of the same multiplicity. One may also think that given the most generic case of intricate singularities, one could make a perturbation by adding some εy to the first equation and obtain a nilpotent singularity of the same multiplicity. But we claim that this is not always possible.

If the intricate singularity has index greater than +1 or lower than −1, then it cannot turn by means of a perturbation in a single nilpotent singularity (whose index may be +1, 0 or −1) without ejecting some singular points, otherwise, it could not change its index.

So only intricate singularities with index −1, 0 or +1 are suitable to be turned into a nilpotent singularity with the same multiplicity (and index) by means of a perturbation. We have checked that this is possible for finite intricate singularities of quadratic differential systems, but the fact that this may be produced it is not just a local property related only to the intricate singularity, but it is a global fact which involves the corresponding complete configuration of singularities (finite and infinite). For example having an intricate singularity of a quadratic system with geometrical description $hh_{(4)}$, one cannot be sure if it will have geometrical codimension at least 4, that is, if its perturbation may produce at most a semi-elemental saddle-node of multiplicity 4 ($\overline{sn}_{(4)}$ of codimension 3) or may produce a nilpotent saddle-node of multiplicity 4 ($\widehat{sn}_{(4)}$ of codimension 4). The really surprising fact is that if the global configuration is $hh_{(4)}; N^r, \copyright, \copyright$ (with $r \in \{f, \infty\}$) the system cannot be perturbed into a more degenerate configuration than $\overline{sn}_{(4)}; N^r, \copyright, \copyright$. But if the global configuration is $hh_{(4)}; N^*, \copyright, \copyright$ the system can be perturbed into $\widehat{sn}_{(4)}; N^*, \copyright, \copyright$. That is, since the finite nilpotent saddle-node implies the existence of the infinite star node (which corresponds to the case with the invariant $\theta = 0$), then only from a configuration having $hh_{(4)}$ and an infinite star node, we can obtain the configuration with a nilpotent saddle-node by means of a perturbation.

This brings us into an even trickier situation which needs to be carefully explained. Consider the next diagram in quadratic systems of unfoldings indicated by arrows: These are all the geometrical

$$\begin{tikzcd}
\overline{sn}_{(4)}; N^f, \copyright, \copyright \Rightarrow cod = 3 & hh_{(4)}; N^f, \copyright, \copyright \Rightarrow cod = 4 \arrow[l, "-1"'] \\
\overline{sn}_{(4)}; N^d, \copyright, \copyright \Rightarrow cod = 4 \arrow[u, "-1"'] & hh_{(4)}; N^*, \copyright, \copyright \Rightarrow cod??? \arrow[l, "-1"'] \arrow[u, "-1"] \arrow[dl, "-1"] \\
\widehat{sn}_{(4)}; N^*, \copyright, \copyright \Rightarrow cod = 5 \arrow[u, "-1"'] &
\end{tikzcd}$$

FIGURE 4. First approach to the geometric bifurcation diagram of $hh_{(4)}; N^*$ in quadratic systems.

global configurations of singularities in quadratic differential systems having a finite singular point of multiplicity 4 and of index 0 and a single real elemental infinite singularity (as proved in [10, Diagram 8.14]). All these configurations hold only if the comitants (from [10]) **D**, **T**, **P** and **R** vanish. This implies that all the coefficients of these comitants must be zero yielding 16 algebraic equations in terms of the coefficients of the system. More exactly these equations are provided by the coefficients of the T-comitants (1 for **D**, 7 for **T**, 5 for **P** and 3 for **R**). However it can be checked directly that these 16 algebraic equations can be reduced to just 3 independent equations, i.e. the vanishing of these three equations implies the annihilation of all 16 equations.

We now make a deeper analysis of FIGURE 4. Since all the configurations in FIGURE 4 have two complex singularities at infinity, in this discussion we remove the couple of infinite complex singularities. We start from the configuration $\overline{sn}_{(4)}; N^r$. This configuration has clearly geometric codimension 3 and this codimension is determined just by the saddle-node. Below this one, we have $\overline{sn}_{(4)}; N^d$, the finite singularity remains the same, but the infinite singularity has turned into an N^d. This is clear by [10, Diagram 6.5] where the invariant θ applies to this case and this adds one codimension to the configuration. In this case we can assign the codimension 4 to $\overline{sn}_{(4)}; N^d$ and this codimension can logically split as 3 coming from the finite $\overline{sn}_{(4)}$ and one from the infinite N^d. But if we go one step further down, we see that when trying to move from a semi-elemental saddle-node of multiplicity 4 to a nilpotent one with the same multiplicity, this forces that the infinite N^d must be transformed into an N^*. There does not exist the possibility of producing these two phenomena independently, and the reason is that the invariant condition $\mathcal{T}_4 = 0$, that normally controls the change from a semi-elemental singularity to a nilpotent one, is equivalent (in this case) to the condition $\theta_2 = 0$, that controls the change from N^d to N^*. For sure the complete unfolding covering all possibilities will be possible in higher degree systems, with plenty more coefficients which allow the required invariants to be independent, but not for quadratic systems. In the quadratic case from the geometrical definition of codimension as well as from Definition 3.3, we must assign codimension 5 to this configuration. In this case the codimension cannot be split among the non-stable objects. We cannot simply say that the nilpotent point has geometrical codimension 4 and the infinite N^* one, because by itself a N^* has geometrical codimension 2. So when the codimension is high, we cannot expect that the geometrical codimension of a global configuration of singularities is the sum of the codimension of each singularity.

We continue to study FIGURE 4. There are more surprises awaiting for us. Now we move to the right of FIGURE 4. On the topmost spot we have $hh_{(4)}; N^r$ and geometrically this means the vanishing of the invariant polynomial $\widetilde{D}$, which is just one more condition. This is consistent with the given codimension 4. Moreover, the possibility of perturbing the intricate singularity $hh_{(4)}$ into a nilpotent $\widehat{sn}_{(4)}$ and afterwards into a semi-elemental $\overline{sn}_{(4)}$ which would force codimension 5 for the first one, does not exist as this cannot happen as we have an N^f (or N^∞). And now we go one step down and move the infinite N^r for obtaining an N^*. The fact is that we do not have the intermediate possibility of N^d in quadratics. Just the fact that the invariant θ is zero forces the singularity to be N^* (because θ_2 is already zero in this case). So it seems that we should have to assign codimension 5 (or more) to this configuration.

But if we just assign codimension 5 then we arrive to a contradiction because there exists a perturbation that moves a system with configuration $hh_{(4)}; N^*$ to a system with $\widehat{sn}_{(4)}; N^*$, and since the last one has already codimension 5, the first one must have at least codimension 6. How can this be explained?

Now we take a look to FIGURE 5.

$$\begin{tikzcd}
 & & \underline{\overline{sn}_{(4)}; N^f} \Rightarrow \mathit{cod} = 3 & & \\
 & \underline{\overline{sn}_{(4)}; N^d} \Rightarrow \mathit{cod} = 4 \arrow[ur, "-1"] & & \widehat{sn}_{(4)}; N^f \Rightarrow \mathit{cod} = 4 \arrow[ul, "-1"'] & \\
\overline{sn}_{(4)}; N^* \Rightarrow \mathit{cod} = 5 \arrow[ur, "-1"] & & \widehat{sn}_{(4)}; N^d \Rightarrow \mathit{cod} = 5 \arrow[ul, "-1"'] \arrow[ur, "-1"] & & \underline{hh_{(4)}; N^f} \Rightarrow \mathit{cod} = 5 \arrow[ul, "-1"'] \\
 & \underline{\widehat{sn}_{(4)}; N^*} \Rightarrow \mathit{cod} = 6 \arrow[ul, "-1"'] \arrow[ur, "-1"] & & hh_{(4)}; N^d \Rightarrow \mathit{cod} = 6 \arrow[ul, "-1"'] \arrow[ur, "-1"] & \\
 & & \underline{hh_{(4)}; N^*} \Rightarrow \mathit{cod} = 7 \arrow[ul, "-1"'] \arrow[ur, "-1"] & &
\end{tikzcd}$$

FIGURE 5. Geometric bifurcation diagram of $hh_{(4)}; N^*$ in polynomial systems.

The diagram in FIGURE 5 represents the complete tree of perturbations that the sub-configuration $hh_{(4)}; N^*$ (where all other singularities are complex) could have in any polynomial differential system of even degree (up to semi-elemental singularities) and surely it has, but in a polynomial system of degree higher than 2, starting with degree 4.

In FIGURE 5 we have underlined the ones which appear in quadratic differential systems. In a higher degree polynomial system, with many more parameters, the independence of algebraic conditions must persist some degrees further up, so as to allow the geometrical codimension of a configuration of singularities to be equal to the sum of codimensions of each one, at least for this level of degeneracy. The lack of independence among algebraic conditions (of invariants and comitants that control singularities) may start again to appear in higher codimensions.

We will prove that this really happens for quartic systems. But for the sake of the continuity of the arguments on codimension, we move this proof to the Section 8.

Finally, what is the geometrical codimension of $hh_{(4)}; N^*, \copyright, \copyright$ inside the family of quadratic differential systems? The only logical explanation is given in the FIGURE 6.

$$\begin{tikzcd}
\overline{sn}_{(4)}; N^r, \copyright, \copyright \Rightarrow cod = 3 & hh_{(4)}; N^r, \copyright, \copyright \Rightarrow cod = 4 \arrow[l, "-1"'] \\
\overline{sn}_{(4)}; N^d, \copyright, \copyright \Rightarrow cod = 4 \arrow[u, "-1"'] & \\
\widehat{sn}_{(4)}; N^*, \copyright, \copyright \Rightarrow cod = 5 \arrow[u, "-1"'] & hh_{(4)}; N^*, \copyright, \copyright \Rightarrow cod = 6 \arrow[l, "1"'] \arrow[uu, "-2"]
\end{tikzcd}$$

FIGURE 6. Final geometric bifurcation diagram of $hh_{(4)}; N^*$ in quadratic systems.

According to the definition of geometric codimension, and also according to Definition 3.3, the geometrical codimension of $hh_{(4)}; N^*, \copyright, \copyright$ must be 6 in quadratic systems because there exists a path of perturbations from $hh_{(4)}; N^*, \copyright, \copyright$ to $\overline{sn}_{(4)}; N^r$ reducing the codimension one by one, even though the path through $hh_{(4)}; N^r$ (which exists) implies a jump of two codimensions.

Proposition 3.16. *A finite intricate singularity of multiplicity k has geometrical codimension greater than or equal to k. An infinite intricate singularity of multiplicity k has geometrical codimension greater than or equal to $k-1$.*

Proof. It is clear that intricate singularities will not have lower codimension than nilpotent ones with the same multiplicity. Apart from the possibility of splitting singularities with higher geometrical codimension like weak foci, the intricate singularities may also produce in perturbations other singularities like n^d, n^* which also imply higher codimension.

The example is quite simple and appears already with the different geometrical phase portraits that finite intricate singularities of quadratic systems can have. There are seven different possibilities, but we will concentrate on $phpphp_{(4)}$, $hphp_{(4)}$ and $hh_{(4)}$ (an identical argument could be done for $peppep_{(4)}$, $pepe_{(4)}$ and $ee_{(4)}$); see [10] and the phase portraits in [11]. The first has three characteristic directions, but the second only two, and the third has one. If we study their blow-ups (or the invariants that define them [10]) we see that the second implies the existence of a double singularity in the blow-up, equivalent to the vanishing of a new invariant (in this case η). In fact, in the second case one of the characteristic directions is double. So, under the geometrical point of view, the second phase portrait can either bifurcate (inside the class of polynomial phase portraits) into the first or the third type. So, geometrically, it has one codimension more than the other two. Moreover, from the topological point of view, $phpphp_{(4)}$ is equivalent to $hphp_{(4)}$ but different from $hh_{(4)}$.

This is not limited to just one more codimension. An intricate singularity of a high degree system could have as many characteristic directions as the degree permits, and so these directions could coalesce, hence it will be unstable inside that family. So the geometric codimension could be greater than the multiplicity up to some value depending on the degree of the system.

If we concentrate on quadratic systems, can a generic finite intricate singularity have codimension 5? The answer is negative. As we have already explained in Example 3.5, there is the possibility that a finite intricate singularity of a quadratic system with index 0 may geometrically bifurcate into a nilpotent saddle-node, but this only happens if some other conditions on the parameters occur. So this is not generic.

In the case of infinite intricate singularities the reduction of one codimension from the finite singularities with the same multiplicity comes from the same reduction that infinite nilpotent singularities have compared to the finite nilpotent ones. This completes the proof of Proposition 3.16. □

3.3. **Relation between geometrical and topological codimensions.**

Theorem 3.17. *Consider a mathematical object defined for polynomial differential systems (for example, a singularity, a global configuration of singularities, a phase portrait, etc.) and the topological equivalence relation. The topological codimension of this object is the minimum of the geometrical codimensions of all topologically equivalent objects.*

Proof. Clearly the geometrical equivalence relation is finer than the topological one. When moving from the geometric classification of singularities (of configurations of singularities) to the topological one, all the singularities which are multiple but with the same topological properties become identified.

Also the weak foci, strong foci and all types of nodes are identified and become just anti-saddles. So the perturbations that one may obtain considering only the topological equivalence are exactly the same as the ones one may obtain from the most generic representation of such an object in the geometrical equivalence. ☐

Studying the global phase portraits of polynomial systems, another important fact is that codimension does not come only from the codimension of the configuration of singularities, but it can also come from other phenomena such as separatrix connections or multiple limit cycles. If the study is done modulo limit cycles, the multiple limit cycles disappear and only separatrix connections count. On lower codimension cases, the codimension of the configuration of singularities and the number of separatrix connections can be added in order to form the final codimension of the phase portraits. But there are some high codimension cases in which the configuration of singularities forces the existence of some connections, and then these cases must require additional study.

4. Centers

In this section we describe the problems that we have found when trying to assign a codimension to configurations of singularities with centers or to phase portraits of quadratic systems with centers, and why we have decided not to assign a fixed codimension for such configurations or phase portraits.

The centers (points surrounded by a continuum of periodic orbits) are a very important subset of the set of singularities. The problem of Poincaré of distinguishing a focus from a center has been solved only for quadratic systems, for Hamiltonian systems (of any degree), for some symmetric cubic systems, for some cubic Kukles' systems and for some other especial classes of polynomial differential systems. It is worthwhile to stress that apart from elemental singularities which can be centers, also nilpotent and intricate singularities can be centers starting in cubic polynomial systems. The topological classification of phase portraits of polynomial differential systems with centers, is only given in the quadratic case. There are several papers which have determined that there are exactly 31 topologically distinct phase portraits with centers in quadratic systems (see [31, 35, 39, 43]). We use the notation of Vulpe and will call them from Vul_2 to Vul_{32} (Vul_1 corresponds to the unique linear system with center modulo the group action).

The geometrical codimension of a center depends on its multiplicity (in case of nilpotent and intricate singularities) but also on the number of independent Lyapunov polynomial constants that polynomial systems may have according to its degree. For example, if for a center of a quadratic system the first three Lyapunov constants L_1, L_2, L_3 are zero one can find a set of four perturbations such that the first one makes $L_3 \neq 0$, the second one makes $L_2 \neq 0$, the third one makes $L_1 \neq 0$ and finally the last one makes the trace non zero, then this center will have geometrical codimension 4. But if any perturbation which breaks $L_3 = 0$ unavoidably makes also $L_2 \neq 0$, the geometrical codimension of this center will be 3.

The first classification of quadratic systems with centers was done in [39]. In [35, 43, 31] the topological bifurcation diagrams of quadratic systems with a center where done for all four classes in the real four dimensional projective parameter space. In addition in [35] the key role of the invariant algebraic curves in the global geometry of the systems and the Darboux integrability, was shown. Quadratic systems with centers split in four classes with some non-empty intersections:

Hamiltonian, Lotka-Volterra, symmetrical and class IV. In [35] one can see two 2-dimensional bifurcation diagrams for the second and third families and one 1-dimensional bifurcation diagram for the class IV and in [31] the 2-dimensional bifurcation diagram for Hamiltonian systems with centers. Bifurcation diagrams for Hamiltonian, Lotka-Volterra and symmetrical centers can also be found in [8] with respect to different normal forms. Using the Kapteyn's normal form for quadratic systems having a weak focus of second order, each of the three main families occupies a plane in a 3-dimensional real projective space. The three planes of course intersect among them but not all of them on a single line and they even have a common point. We can fit all four classes in a four dimensional projective space. A phase portrait may belong to several classes but this is not just because it lies at the intersection of the planes; they may also exist in generic regions. For example, Vul_{10} which is the phase portrait with the center inside a triangle formed by three straight lines, always belongs to the Lotka-Volterra class in which it is even generic. In some cases, it may be Hamiltonian non-symmetrical, in others may be symmetrical but non-Hamiltonian, and there is the possibility that it belongs to all three classes.

The three main classes of quadratic centers have traditionally been called of codimension 3 and the class IV of codimension 4 inside quadratic systems. The last is clear because it is always possible to make a perturbation and obtain a weak focus of order three. There is just one phase portrait which lives exclusively in class IV which is Vul_{32}. In the case of the three main classes, the possibility to obtain a weak focus of third order from the center is not guaranteed. There are some cases in which this is possible, and others in which it is not. We incorporate here a piece of the bifurcation diagram given in [8, Fig. 51] and explain a bit of it (see FIGURE 7). The article [8] provides a complete study of phase portraits of the family QW2 of quadratic systems having a weak focus of order 2, and also studies the border of this family which is also included in the chosen normal form. This border contains systems with weak foci of order three and centers. There is a slice of the parameter space in which there are always symmetric centers. In parts $8S_1$, $8S_2$ and $3.8L_4$ (see FIGURE 7 of the bifurcation diagram for systems with symmetric centers in [8], the corresponding phase portrait is Vul_8. However, the part $3.8L_4$ corresponds to the intersection of a plane whose generic points have a weak focus of third order. Topologically the part $3.8L_4$ plays no role in this slice, and so, the phase portrait does not change when moving from $8S_1$ to $8S_2$. But points of $3.8L_4$ posses a hidden geometrical property their neighboring points do not possess. Indeed. if we start with a system in $3.8L_4$, we will be able to perturb it and produce a weak focus of order 3, thus such a system (and the phase portrait) should be assigned geometrical codimension 4. But if we start from a system in $8S_1$ or $8S_2$, the most we can obtain is a weak focus of order two, and the geometric codimension cannot be other than 3. The relevant topological consequence of this is that one can perturb the systems in the whole $\mathbb{R}^{12}$ and obtain 3 limit cycles surrounding the same point from $3.8L_4$ but only two from $8S_1$ or $8S_2$.

A phase portrait of a quadratic system with a center could appear in distinct classes of centers with distinct codimensions as we shall now see. Consider now the phase portrait Vul_{10} which is generic inside the Lotka-Volterra class but not in the symmetrical or Hamiltonian families. Phase portraits Vul_8, Vul_9, Vul_{10} and Vul_{11} have all the same configuration of singularities (2) $s, s, s, c; N, N, N$ and clearly Vul_{11} has just one separatrix connection, Vul_8 and Vul_9 have two and Vul_{10} has three. The four exist only in the Hamiltonian class where one can bifurcate Vul_{10} into Vul_8 or Vul_9 and these last two into Vul_{11}. We include FIGURES 52 and 53 from [8] to show the distribution of Hamiltonian and Lotka-Volterra phase portraits inside the closure of the family QW2 in the normal form for QW2 (see FIGURES 8 and 9).

The same problem appears in other cases. We show two examples: (1) Vul_{31} only appears as a Lotka-Volterra center and there it can bifurcate into Vul_{27} which is not a generic phase portrait within systems with center that could be obtained from the configuration of singularities $s, a, a, c; S, S, N$ because as a symmetrical system Vul_{27} may bifurcate into Vul_{25} or Vul_{26}; (2) Phase portrait Vul_{20} may be just symmetrical (and there it is generic), or may be both symmetrical and Lotka-Volterra (in which case it is not generic). Phase portrait Vul_{22} appears only as a symmetrical center and there it can bifurcate into Vul_{20}. But if we perturb Vul_{20} inside the Lotka-Volterra, we can get Vul_{30} which cannot be symmetrical. So the codimension to be given to Vul_{20} depends on the family that the system belongs, and this affects the codimension of Vul_{22}.

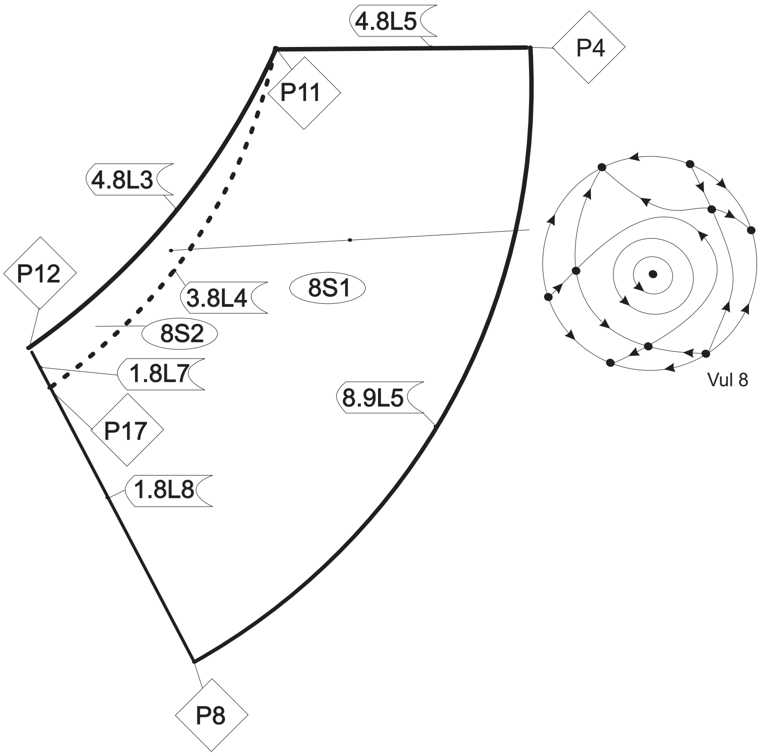


Figure 7. Quadratic symmetrical centers. Extract from [8, Fig. 51].

From the discussion above we see that to assign a codimension to a phase portrait with a center is more complicated. We may assign a codimension to a system with center and not just to the phase portrait. The codimension of a system with center depends on its particular position in the bifurcation set. Two systems with center with identical phase portraits may lead to distinct codimensions depending on how they are placed in the bifurcation set.

Moreover, since assigning a specific codimension to every phase portrait with a center occurring in the quadratic class is not actually necessary for our main goal that is the classification of all phase portraits of quadratic systems modulo limit cycles, we have decided not to assign a codimension to phase portraits or to configurations of singularities (either topological or geometrical) that include a center.

5. Topological codimension of the configuration of singularities in QS

We could give the geometrical codimension of each one of the 1764 distinct geometrical configurations of singularities obtained in [10]. But it is more useful for giving the topological codimension of the 207 topological configurations of singularities in [9] given in terms of algebraic invariants because this is one of the tools that we need in order to continue the classification of topologically distinct phase portraits of quadratic systems modulo limit cycles.

As we have already said the determination of the codimension must be done in an inductive way starting from the cases of codimension 0, 1, ….

We consider first the non-degenerated quadratic systems and after, the degenerated ones. That is, those with a finite number of finite singularities, and those with an infinite number of finite singularities, respectively.

5.1. **Codimensions of configurations for non-degenerate quadratic systems.** In this subsection we will assign the topological codimension to each topological configuration of singularities of non-degenerate quadratic systems. We indicate the number of every topological configuration given in [9] and also use the notation from [9, 10].

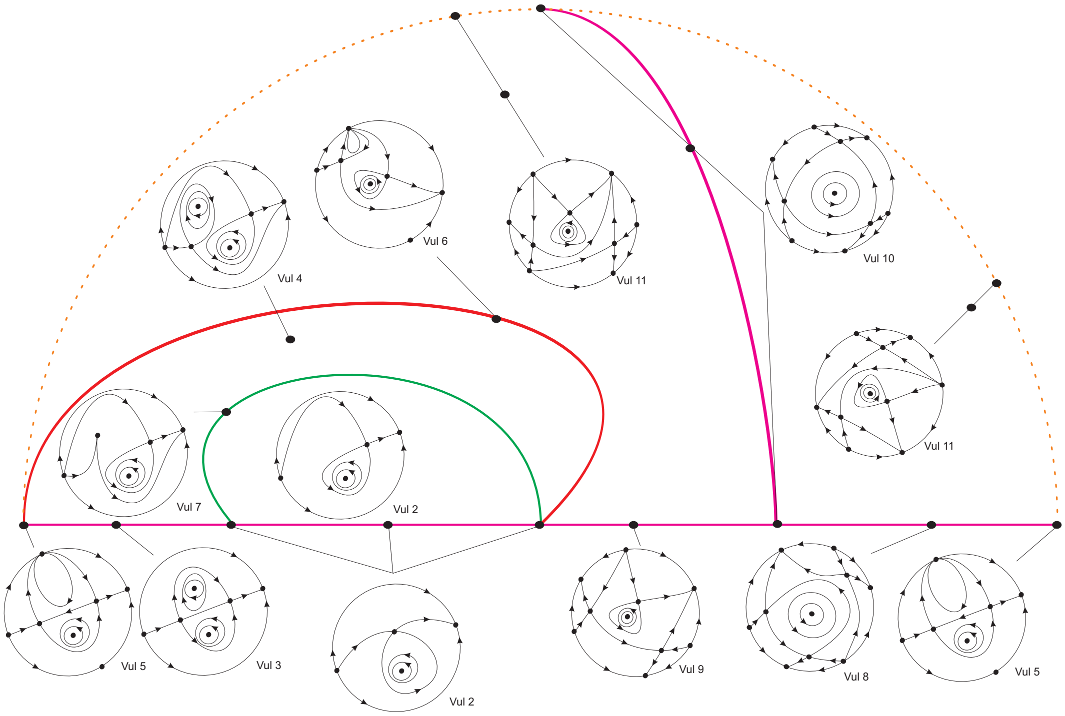


FIGURE 8. Quadratic Hamiltonian centers from [8, Fig. 52].

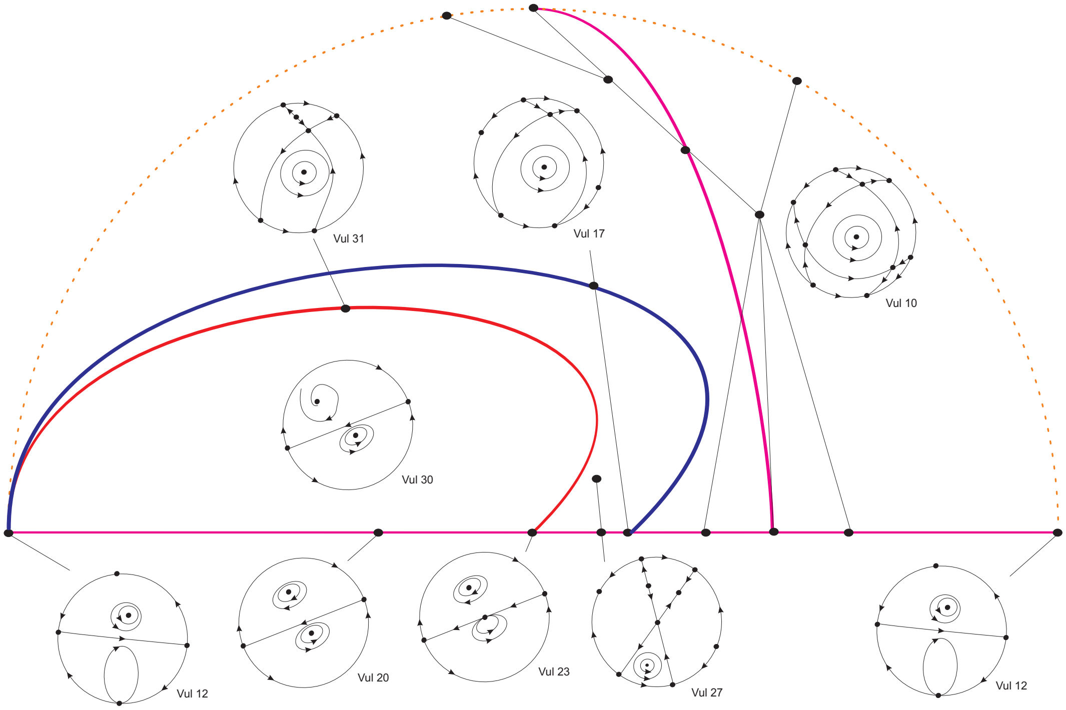


FIGURE 9. Quadratic Lotka-Volterra centers from [8, Fig. 53].

1: Codimension 0 configurations. We must start with the topological configurations of singularities which are structurally stable (topological codimension 0). It is easy to find that they are

(1) $s, s, s, a; N, N, N$;
(3) $s, a, a, a; S$;
(5) $s, a, a, a; S, S, N$;
(8) $s, s, a, a; N$;
(10) $s, s, a, a; S, N, N$;
(12) $\emptyset; N$;
(13) $\emptyset; S, N, N$;
(15) $s, s; N, N, N$;
(16) $a, a; S$;
(19) $a, a; S, S, N$;
(23) $s, a; N$;
(25) $s, a; S, N, N$.

These 12 configurations correspond to the 12 classes in [6, Table 5.1] which produced the 44 structurally stable quadratic systems.

2: Codimension 1 configurations. Each one of the topological configurations of singularities has a unique semi-elemental saddle-node (finite or infinite). We give beside each case the number of the two codimension 0 configurations in which they may bifurcate (whether the saddle-node bifurcates in two real or complex singularities). Those bifurcations are very simple and they are all described in [7].

(7) $s, a, a, a; \binom{0}{2}SN, S \ \to$ (3) or (5);
(11) $s, s, a, a; \binom{0}{2}SN, N \ \to$ (8) or (10);
(14) $\emptyset; \binom{0}{2}SN, N \ \to$ (12) or (13);
(22) $a, a; \binom{0}{2}SN, S \ \to$ (16) or (19);
(27) $s, a; \binom{0}{2}SN, N \ \to$ (23) or (25);
(28) $s, s, sn; N, N, N \ \to$ (15) or (1);
(30) $a, a, sn; S \ \to$ (16) or (3);
(31) $a, a, sn; S, S, N \ \to$ (19) or (5);
(37) $s, a, sn; N \ \to$ (23) or (8);
(38) $s, a, sn; S, N, N \ \to$ (25) or (10);
(44) $sn; N \ \to$ (12) or (23);
(45) $sn; S, N, N \ \to$ (13) or (25);
(73) $s, a, a; \binom{1}{1}SN \ \to$ (3) or (8);
(74) $s, a, a; \binom{1}{1}SN, S, N \ \to$ (5) or (10);
(76) $s, s, a; \binom{1}{1}SN, N, N \ \to$ (1) or (10);
(84) $a; \binom{1}{1}SN \ \to$ (23) or (16);
(85) $a; \binom{1}{1}SN, S, N \ \to$ (19) or (25);
(87) $s; \binom{1}{1}SN, N, N \ \to$ (25) or (15).

All these configurations are used in [7] (even though the numbers appeared later) in order to produce the codimension 1 phase portraits of the quadratic systems. In [7] the authors denote by (A) the class of phase portraits with a finite saddle-node, i.e. the cases (28), (30), (31), (37), (38), (44) and (45). In [7] the authors denote by (B) the class of phase portraits with an infinite saddle-node which is obtained by coalescing two infinite singularities, i.e. the cases (7), (11), (14), (22) and (27). In [7] the authors denote by (C) the class of phase portraits with an infinite saddle-node which is obtained by coalescing a finite and an infinite singularity, i.e. the cases (73), (74), (76), (84), (85) and (87). In [7] we also find class (D) of those phase portraits having one separatrix connection. These come from the configurations of codimension 0.

3: Codimension 2 configurations. These topological configurations will either have one cusp, two saddle-nodes (finite or infinite), or an infinite nilpotent singularity of multiplicity 3. Of course they all may bifurcate in several codimension 0 configurations. But we are more interested to know which are the codimension 1 configurations into which they bifurcate. We will see that the cases with a cusp have only one codimension 1 configuration into which they bifurcate. Others with an infinite nilpotent singularity, or with two finite saddle-nodes will have two options, and the rest will have four options. We will detail all of them. The needed bifurcations are of the same type as those already used to bifurcate configurations of codimension 1 into codimension 0 except that one must take care not to break the second unstable object. In addition we also need the perturbation to turn a cusp into a finite saddle-node which has already been described in this paper.

(29) $s, s, cp; N, N, N \rightarrow$ (28);
(32) $a, a, sn; \binom{0}{2}SN, S \rightarrow$ (22) or (7) or (30) or (31);
(34) $a, a, cp; S \rightarrow$ (30);
(35) $a, a, cp; S, S, N \rightarrow$ (31);
(39) $s, a, sn; \binom{0}{2}SN, N \rightarrow$ (27) or (11) or (37) or (38);
(40) $s, a, cp; N \rightarrow$ (37);
(42) $s, a, cp; S, N, N \rightarrow$ (38);
(46) $sn; \binom{0}{2}SN, N \rightarrow$ (14) or (27) or (44) or (45);
(47) $cp; N \rightarrow$ (44);
(48) $cp; S, N, N \rightarrow$ (45);
(50) $sn, sn; N \rightarrow$ (44) or (37);
(51) $sn, sn; S, N, N \rightarrow$ (45) or (38);
(75) $s, a, a; \binom{1}{1}SN, \binom{0}{2}SN \rightarrow$ (7) or (11) or (73) or (74);
(77) $s, s, a; \binom{1}{2}E{-}H, N \rightarrow$ (11) or (76);
(79) $s, a, a; \binom{1}{2}E{-}H, S \rightarrow$ (7) or (74);
(80) $s, a, a; \binom{1}{2}PHP{-}E, S \rightarrow$ (7) or (74);
(81) $s, a, a; \binom{1}{2}HHH{-}H, N \rightarrow$ (11) or (74);
(86) $a; \binom{1}{1}SN, \binom{0}{2}SN \rightarrow$ (22) or (27) or (84) or (85);
(88) $s; \binom{1}{2}E{-}H, N \rightarrow$ (27) or (87);
(89) $a; \binom{1}{2}E{-}H, S \rightarrow$ (22) or (85);
(91) $a; \binom{1}{2}PHP{-}E, S \rightarrow$ (22) or (85);
(93) $a; \binom{1}{2}HHH{-}H, N \rightarrow$ (27) or (85);
(98) $a, sn; \binom{1}{1}SN \rightarrow$ (84) or (73) or (37) or (30);
(99) $a, sn; \binom{1}{1}SN, S, N \rightarrow$ (85) or (74) or (38) or (31);
(104) $s, sn; \binom{1}{1}SN, N, N \rightarrow$ (87) or (76) or (28) or (38);
(132) $s, a; \binom{1}{1}SN, \binom{1}{1}SN, N \rightarrow$ (76) or (74);
(134) $s, a; \binom{1}{1}SN, \binom{1}{1}NS, N \rightarrow$ (76) or (74);
(140) $\emptyset; \binom{1}{1}SN, \binom{1}{1}SN, N \rightarrow$ (87) or (85);
(141) $\emptyset; \binom{1}{1}SN, \binom{1}{1}NS, N \rightarrow$ (87) or (85).

Configurations (50) and (51) with two finite saddle-nodes as well as configurations (29), (34), (35), (40), (42) and (47) were used in [16] to find all the phase portraits of codimension 2 in the class (AA) with two finite saddle-nodes or a cusp.

Configurations (32), (39) and (46) with a finite saddle-node and a $\binom{0}{2}SN$ were used in [15] to find all the phase portraits of codimension 2 in the class (AB). Also in [15] configurations (98), (99) and (104) with a finite saddle-node and a $\binom{1}{1}SN$ were used to find all the phase portraits of codimension 2 in the class (AC). There is also [5] which classifies the phase portraits of codimension 2 of the class (AD) with a finite saddle-node and a separatrix connection but this starts from the configurations of codimension 1 having a finite saddle-node plus a separatrix connection. The remaining configurations of codimension 2 are the classes (BC) and (CC) (the class (BB) does not need to be studied since the singularities $\overline{\binom{0}{3}}S$ and $\overline{\binom{0}{3}}N$ are topologically equivalent to elemental saddles and nodes and do not produce new phase portraits). Class (CC) corresponds to phase portraits where two finite singularities has escaped to infinity. The remaining configurations of codimension 1 plus a separatrix connection will produce all the codimension 2 phase portraits of the classes (BD) and (CD). Finally all the codimension 0 configurations which can have two separatrix connections will complete the class (DD) and thus, all the codimension 2 topologically distinct phase portraits will be classified.

This previous discussion was needed in order to make clear all the facts involved in the codimension 2 phase portraits. This will help us to obtain the codimension 3 (and higher) phase portraits. It will even be the instrument to do it.

4: Codimension 3 configurations. Note that a configuration in this class having a finite saddle-node and two infinite saddle-nodes (of different types), may have up to 6 non-equivalent perturbations inside the class of codimension 2. For codimension 3 we will still give here all the "potential unfoldings" of codimension 2. By *potential unfolding* we mean an unfolding which is coherent with the initial configuration, but we need to prove its existence. We will only describe some of them. In fact, according to Definition 3.9, we only need to prove the existence of one unfolding of one less codimension. The specific perturbations of at least one of the examples for each configuration (which produces the proof of the given codimension) can be checked on a Mathematica file and a pdf file that it is at https://mat.uab.cat/∼artes/articles/codimension/codimension.html. In these perturbations the bifurcation diagram (done in terms of polynomial invariants) of the geometrical configurations of singularities plays a major role.

(36) $a, a, cp; \binom{0}{2}SN, S \rightarrow$ (32) or (34) or (35);
(43) $s, a, cp; \binom{0}{2}SN, N \rightarrow$ (39) or (40) or (42);
(49) $cp; \binom{0}{2}SN, N \rightarrow$ (46) or (47) or (48);
(52) $sn, sn; \binom{0}{2}SN, N \rightarrow$ (46) or (39) or (50);
(53) $sn, cp; N \rightarrow$ (47) or (40) or (50);
(54) $sn, cp; S, N, N \rightarrow$ (48) or (42) or (51);
(59) $a, es; S \rightarrow$ (34);
(61) $a, es; S, S, N \rightarrow$ (35);
(64) $s, es; N \rightarrow$ (40);
(65) $s, es; S, N, N \rightarrow$ (42);
(82) $s, a, a; \binom{1}{3}HHP{-}H \rightarrow$ (75) or (79) or (80) or (81);
(95) $a; \binom{1}{3}HHP{-}H \rightarrow$ (86) or (89) or (91) or (93);
(100) $a, sn; \binom{1}{1}SN, \binom{0}{2}SN \rightarrow$ (86) or (75) or (39) or (32) or (98) or (99);
(101) $a, cp; \binom{1}{1}SN \rightarrow$ (98) or (34) or (40);
(102) $a, cp; \binom{1}{1}SN, S, N \rightarrow$ (99) or (35) or (42);
(105) $s, cp; \binom{1}{1}SN, N, N \rightarrow$ (104) or (29) or (42);
(106) $s, sn; \binom{1}{2}E{-}H, N \rightarrow$ (88) or (77) or (104) or (98);
(108) $a, sn; \binom{1}{2}E{-}H, S \rightarrow$ (89) or (79) or (32) or (99);
(109) $a, sn; \binom{1}{2}PHP{-}E, S \rightarrow$ (91) or (80) or (32) or (99);
(110) $a, sn; \binom{1}{2}HHH{-}H, N \rightarrow$ (93) or (81) or (98) or (99);
(123) $s, s; \binom{2}{2}E{-}E, N \rightarrow$ (77);
(124) $s, a; \binom{2}{2}PH{-}PH, N \rightarrow$ (77) or (132);
(125) $a, a; \binom{2}{2}PHP{-}PHP, S \rightarrow$ (80);
(126) $a, a; \binom{2}{2}HHH{-}HHH, N \rightarrow$ (81) or (88);
(129) $s, a; \binom{2}{2}E{-}E, S \rightarrow$ (79);
(135) $s, a; \binom{2}{2}PH{-}H, N \rightarrow$ (134) or (77) or (81);
(136) $s, a; \binom{1}{2}E{-}H, \binom{1}{1}SN \rightarrow$ (134) or (79) or (77) or (75);
(137) $\emptyset; \binom{2}{2}H{-}H, N \rightarrow$ (140) or (88);
(138) $\emptyset; \binom{2}{2}E{-}E, S \rightarrow$ (89) or (91);
(142) $\emptyset; \binom{2}{2}PH{-}H, N \rightarrow$ (140) or (88) or (93);
(143) $\emptyset; \binom{1}{2}E{-}H, \binom{1}{1}SN \rightarrow$ (140) or (86) or (89) or (88);
(148) $sn; \binom{1}{1}SN, \binom{1}{1}SN, N \rightarrow$ (140) or (132) or (104) or (99);
(149) $sn; \binom{1}{1}SN, \binom{1}{1}NS, N \rightarrow$ (141) or (134) or (104) or (99).

It is now worthwhile to mention why some of the configurations here have just one potential unfolding in codimension 2. We explain this. We take for example configuration (123) $s, s; \binom{2}{2}E{-}E, N$. There is just one multiple singularity, an intricate infinite one formed by the coalescence of two finite singularities and two infinite ones. Having just two elliptic sectors, it has index 2. So any split of this singularity in a simple one plus a triple one, or two double ones forces the indices of these new singularities to be both $+1$ or one 0 and another one 2. But there are no triple singularities of index 2, and besides, double singularities have index 0. So the only possible unfolding is into a simple and a triple singularity, both of index 1. The case presented in the table is when we split a finite anti-saddle from infinity and a nilpotent elliptic saddle still remains. If instead we try to split an infinite node, the triple point that remains is a semi-elemental node $\overline{\binom{2}{1}}N$ which is topologically equivalent to an elemental node. From a geometric point of view the configuration $s, s; \overline{\binom{2}{1}}N, N, N$ is of codimension 2 and according to our definition of codimension it could be used to form the first step of the couple of perturbations needed to prove the codimension 3 of configuration (123). The next perturbation would lead to configuration (76) $s, s, a; \binom{1}{1}SN, N, N$. But since we already have an unfolding into codimension 2, we do not need this path.

At this point of codimension 3 it is worthwhile to start asking if any of the configurations described here could in fact be of codimension 4 by unfolding into another one of this group. But we can check that each configuration cannot bifurcate into any other from the list of codimension 3 configurations of singularities. In particular it is interesting to see that configuration (125) $a, a; \binom{2}{2}PHP\text{–}PHP, S$ cannot bifurcate into (36) $a, a, cp; \binom{0}{2}SN, S$ because we have already proved Lemma 3.15 which states that an intricate infinite singularity cannot eject a single generic cusp into the affine region. Since configuration (125) already has two finite singularities, the cusp that appears in (36) cannot come from the infinite intricate singularity.

5: Codimension 4 configurations. In this group there will appear the first configuration that needs a double conjugate perturbation in order to compute its codimension. More specifically this is configuration (69) $hhhhhh; N, N, N$, that contains precisely the singularity described in Example 3.6. Thus configuration (69) unfolds into $s, \widehat{s}_{(3)}; N, N, N$ which is topologically equivalent to configuration (15), but which can unfold into configuration (20) $s, s, \widehat{cp}_{(2)}; N, N, N$, that has codimension 2, thus proving the codimension 4 of (69). The other codimension 4 configurations are (we just show one of the unfoldings that we have proved to exist for each case):

(55) $sn, cp; \binom{0}{2}SN, N \ \rightarrow$ (52);
(56) $cp, cp; N \ \rightarrow$ (53);
(57) $cp, cp; S, N, N \ \rightarrow$ (54);
(63) $a, es; \binom{0}{2}SN, S \ \rightarrow$ (36);
(66) $s, es; \binom{0}{2}SN, N \ \rightarrow$ (43);
(67) $ee; S \ \rightarrow$ (59);
(68) $ee; S, S, N \ \rightarrow$ (61);
(71) $phph; S, N, N \ \rightarrow$ (65);
(83) $s, a, a; [\infty; \emptyset] \ \rightarrow$ (82);
(96) $a; [\infty; \emptyset] \ \rightarrow$ (95);
(103) $a, cp; \binom{1}{1}SN, \binom{0}{2}SN \ \rightarrow$ (100);
(107) $s, cp; \binom{1}{2}E{-}H, N \ \rightarrow$ (106);
(111) $a, sn; \binom{1}{3}HHP{-}P \ \rightarrow$ (95);
(113) $a, cp; \binom{1}{2}E{-}H, S \ \rightarrow$ (108);
(114) $a, cp; \binom{1}{2}PHP{-}E, S \ \rightarrow$ (109);
(115) $a, cp; \binom{1}{2}HHH{-}H, N \ \rightarrow$ (110);
(117) $es; \binom{1}{1}SN, S, N \ \rightarrow$ (102);
(127) $a, a; \binom{2}{3}HHP{-}PHH \ \rightarrow$ (82);
(130) $s, a; \binom{2}{3}HE{-}P \ \rightarrow$ (82);
(144) $sn; \binom{2}{2}PH{-}PH, N \ \rightarrow$ (137);
(145) $sn; \binom{2}{2}E{-}E, S \ \rightarrow$ (138);

(150) $cp; \binom{1}{1}SN, \binom{1}{1}SN, N \rightarrow$ (148);
(151) $cp; \binom{1}{1}SN, \binom{1}{1}NS, N \rightarrow$ (149);
(152) $sn; \binom{2}{2}PH{-}H, N \rightarrow$ (142);
(154) $sn; \binom{1}{2}E{-}H\ \binom{1}{1}SN \rightarrow$ (143);
(156) $s; \binom{3}{2}E{-}PH, N \rightarrow$ (123);
(157) $a; \binom{3}{2}HP{-}HHH, N \rightarrow$ (126);
(160) $s; \binom{2}{2}E{-}E, \binom{1}{1}SN \rightarrow$ (136);
(161) $a; \binom{2}{2}PH{-}PH, \binom{1}{1}SN \rightarrow$ (136).

In this set we have found the first configuration with the infinite line filled up with singularities. The bifurcation of (83) into (82) is quite simple because as it can be seen in [9, Diagram 2], both configurations can be distinguished by the comitant C_2 being zero or not. We must mention that $C_2 = 0$ also implies $\widetilde{L} = 0$. So we simply need to perturb a parameter in C_2 for making it non zero, while maintaining $\mu_0 = \kappa = \widetilde{L} = 0$, that is possible as it is shown in the examples given in https://mat.uab.cat/∼artes/articles/codimension/codimension.html.

As before to prove that none of these configurations can have codimension 5, one needs to check (and we have checked) that none of them can unfold in any of the others of this list. This will be applied for all the upcoming codimensions, so we will skip mentioning it.

6: Codimension 5 configurations. The configurations of singularities with codimension 5 are:

(58) $cp, cp; \binom{0}{2}SN, N \rightarrow$ (55);
(70) $ee; \binom{0}{2}SN, S \rightarrow$ (68);
(72) $phph; \binom{0}{2}SN, N \rightarrow$ (71);
(112) $a, sn; [\infty; \emptyset] \rightarrow$ (83);
(116) $a, cp; \binom{1}{3}HHP{-}P \rightarrow$ (111);
(118) $es; \binom{1}{1}SN, \binom{0}{2}SN \rightarrow$ (103);
(119) $es; \binom{1}{2}E{-}H, S \rightarrow$ (113);
(120) $es; \binom{1}{2}PHP{-}E, S \rightarrow$ (114);
(121) $es; \binom{1}{2}HHH{-}H, N \rightarrow$ (115);
(128) $a, a; [\infty; S] \rightarrow$ (127);
(131) $s, a; [\infty; N] \rightarrow$ (130);
(146) $sn; \binom{2}{3}HE{-}P \rightarrow$ (130);
(153) $cp; \binom{2}{2}PH{-}H, N \rightarrow$ (144);
(155) $cp; \binom{1}{2}E{-}H,\ \binom{1}{1}SN \rightarrow$ (154);
(158) $a; \binom{3}{3}HE{-}PHH \rightarrow$ (157);
(162) $s; \binom{3}{3}EE{-}P \rightarrow$ (130);
(163) $a; \binom{3}{3}HPH{-}P \rightarrow$ (130);
(164) $\emptyset; \binom{4}{2}PHP{-}PHP, N \rightarrow$ (156);
(165) $\emptyset; \binom{4}{2}E{-}HHH, N \rightarrow$ (157);
(166) $\emptyset; \binom{3}{2}E{-}PH, \binom{1}{1}SN \rightarrow$ (161).

Now we start having configurations with the infinite line filled up with singularities that are not generic inside the class $C_2 = 0$ (like (112), (128) or (131)). They easily bifurcate into the most generic elements of this class, or into other configurations such as (127) or (130) as we have checked.

In this codimension we see some configurations with no finite singularities. We already saw some before, but those were due to the existence of some finite complex singularities. The ones we see here are due to the fact that all four finite singularities have escaped to infinity. Their bifurcations are simple and obtained by returning one singularity to the affine plane yielding an already known configuration of codimension 4. Note that if we send all four finite singularities to the same infinite singularity to obtain a geometrical codimension 4 configuration, the infinite singularity will be semi-elemental of type $\overline{\binom{4}{1}}N$ or $\overline{\binom{4}{1}}S$. This will be topologically equivalent to an elemental node or saddle. That is, this will have geometrical but not topological codimension 4.

7: Codimension 6 configurations. The codimension 6 configurations are:

(122) $es; [\infty; \emptyset] \ \rightarrow\ (112)$;
(139) $\emptyset; [\infty; C] \ \rightarrow\ (97)\ c; [\infty; \emptyset] \ \rightarrow\ (96)$;
(147) $sn; [\infty; N] \ \rightarrow\ (131)$;
(159) $a; [\infty; \binom{2}{0}SN] \ \rightarrow\ (131)$;
(167) $\emptyset; \binom{4}{3}EH\!-\!HE \ \rightarrow\ (158)$;
(168) $\emptyset; \binom{4}{3}EE\!-\!HH \ \rightarrow\ (162)$;
(169) $\emptyset; \binom{4}{3}EH\!-\!P \ \rightarrow\ (163)$.

In this group we find three configurations ((167), (168) and (169)) where a singularity of multiplicity 7 occurs. The phase portraits with singularities of multiplicity 7 were studied in [33]. In fact, there are up to six geometrically different configurations of singularities with a singularity of multiplicity 7 (see [10]), but only the three we have here are not topologically equivalent with other configurations of lower topological codimension.

Regarding configuration (139), even though it does not have a finite isolated center, its simplest perturbation is the one which takes out the center from the infinity and moves it into the affine plane, obtaining configuration (97). We initially wanted to assign a codimension to every configuration and phase portrait with a center, but as we have already explained in Section 4, we have decided not to do this. We can justify the codimension 6 given to configuration (139) by the double conjugate perturbation (which does not even need to be done in the complex field) which after passing through (97) arrives at (96), i.e. a configuration of codimension 4.

8: Codimension 7 configuration. Finally there is only one configuration of codimension 7 which is:

(170) $a; [\infty; \binom{3}{0}ES] \ \rightarrow\ (159)$.

5.2. **Codimensions of configurations for degenerate quadratic systems.** We consider now the configurations of singularities for degenerate quadratic systems. We call degenerate such configurations. We start by taking the most generic cases which are degenerate and determine their codimension by perturbing them into some of the non degenerate configurations.

1: Codimension 4 configurations. Among all the configurations of singularities of degenerate quadratic systems, the most generic ones are the following:

(171) $a, (\ominus\, {\scriptstyle [|];\emptyset}); \big(\ominus\, [|]; \emptyset\big) \ \rightarrow\ (59)$;
(176) $a, (\ominus\, {\scriptstyle [|];\emptyset}); N, S, \big(\ominus\, [|]; \emptyset\big) \ \rightarrow\ (61)$;
(178) $s, (\ominus\, {\scriptstyle [|];\emptyset}); N, N, \big(\ominus\, [|]; \emptyset\big) \ \rightarrow\ (65)$.

We claim that these configurations have topological (and geometrical) codimension 4 and not only because the indicated unfoldings are of codimension 3. These configurations have the invariant $\eta \neq 0$ (see [10]) and this implies that we cannot get any infinite singularity with $\eta = 0$ from their unfolding. So all the codimensions of an unfolding must come from the finite singularities. The only way to obtain a codimension 4 finite singularity would be to have a multiplicity 4 singularity (or two cusps). But we already have a simple finite singularity outside the line of degeneracy, so the most that we can obtain from there will be a triple nilpotent singularity, which corresponds to the indicated configurations on the right. Note that we have the topological codimension of the most generic configurations of degenerate systems, the others may simply be derived from them (except for two cases that we will see in the next block).

2: Codimension 5 configurations. The codimension 5 degenerate configurations are:

(173) $(\ominus\, {\scriptstyle [|];f}); \big(\ominus\, [|]; \emptyset\big) \ \rightarrow\ (171)$;
(175) $(\ominus {\scriptstyle [\circ];\emptyset}); N, \big(\ominus\, [\circ]; \emptyset, \emptyset\big) \ \rightarrow\ (56)$;
(177) $(\ominus\, {\scriptstyle [|];n}); S, N, \big(\ominus\, [|]; \emptyset\big) \ \rightarrow\ (176)$;
(179) $(\ominus\, {\scriptstyle [|];s}); N, N, \big(\ominus\, [|]; \emptyset\big) \ \rightarrow\ (178)$;

(180) $(\ominus_{[|];\emptyset})$; $\binom{1}{1}SN, N, \big(\ominus\,[|];\emptyset\big) \ \to\ (176)$;
(181) $(\ominus_{[\,)(\,];\emptyset}); N, \big(\ominus\,[\,)(\,];\emptyset,\emptyset\big) \ \to\ (57)$;
(183) $a, (\ominus_{[|];\emptyset})$; $\binom{0}{2}SN, \big(\ominus\,[|];\emptyset\big) \ \to\ (176)$;
(185) $s, (\ominus_{[|];\emptyset})$; $N, \big(\ominus\,[|];N_3^f\big) \ \to\ (178)$;
(187) $a, (\ominus_{[|];\emptyset})$; $S, \big(\ominus\,[|];N_3^\infty\big) \ \to\ (176)$;
(189) $a, (\ominus_{[|];\emptyset})$; $N, \big(\ominus\,[|];S_3\big) \ \to\ (176)$.

Configurations (175) and (181) deserve a comment. There is no way to perturb an irreducible conic formed by singularities into a line of singularities. Hence if we perturb a system with an ellipse formed by singularities (so as to obtain something topologically different), we must move into a non-degenerate system. What is the greatest codimension that such a system may have? The original system with the conic formed by singularities has already non-zero invariants like η and $\widetilde{D}$ (see [10]). Thus the perturbed system will also have them different from zero. This implies that we cannot have an intricate singularity either finite (because this would imply $\widetilde{D} = 0$) or infinite (because this would imply $\eta = 0$). We can neither have a nilpotent infinite singularity (because this would also imply $\eta = 0$). So the largest geometrical codimension we may have is 4 with finite configurations like $\widehat{sn}_{(4)}$ or $\widehat{cp}_{(2)}, \widehat{cp}_{(2)}$. Both are realizable as it can be checked in the file https://mat.uab.cat/∼artes/articles/codimension/codimension.html. The first option does not have topological codimension 4 but a second perturbation could produce $\widehat{es}_{(3)}, s$ and thus would also be acceptable. We have presented here the configuration (56) which is the one with two cusps. Other possible bifurcations into codimension 4 could be a triple nilpotent finite singularity plus a $\binom{1}{1}SN$.

We present an example related to configuration (181) (which shows the necessity of improving the Definition 3.3 to Definition 3.9) where the hyperbolas are dotted.

Example 5.1. Consider the systems

(a) $x' = -1 + xy$, $y' = 1 - xy$;
(b) $x' = 0$, $y' = 1 - xy$;
(c) $x' = 1 - xy$, $y' = 1 - xy$.

All three systems are degenerate having a hyperbola filled up with singular points (see the phase portraits in Figure 10).

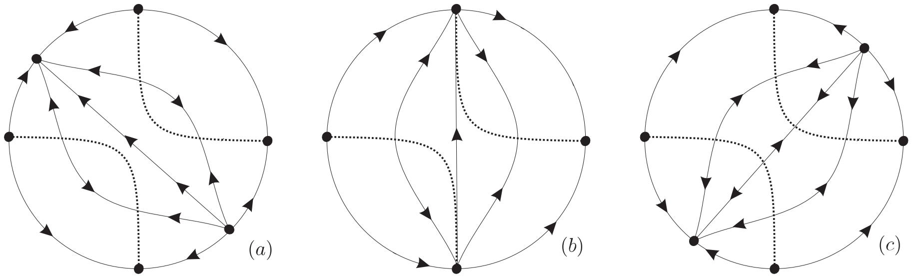


Figure 10. Phase portraits with degenerated hyperbolas

The first and the third have the topological configuration of singularities $(\ominus_{[\,)(\,];\emptyset})$; $N^*, \big(\ominus\,[\,)(\,];\emptyset,\emptyset\big)$ while the second has $(\ominus_{[\,)(\,];\emptyset})$; $\big(\ominus\,[\,)(\,];N^*,\emptyset\big)$, respectively cases (181) and (199) from [9]. Notice that all three phase portraits are topologically different, and thus, there is no affine change of variables which can transform one system into another.

Clearly case (b) is one codimension higher than (a) and (c) which must have the same codimension. Case (a) can easily be perturbed into a system which has a nilpotent saddle-node of multiplicity 4. Even though this singularity is not topologically of codimension 4 by being topologically equivalent to a semi-elemental saddle-node, using the double conjugate perturbation we obtain $\widehat{es}_{(3)} + s$ which has codimension 3. Thus we obtain codimension 5 for case (181). However, we claim that case (c) can never be perturbed into a system having a nilpotent saddle-node of multiplicity 4, nor any other configuration of codimension 4.

To prove the claim we first show how case (a) can be perturbed into an $\widehat{sn}_{(4)}$. The singularities of system (a) are of the form $(x_0, 1/x_0)$. We move one such point to the origin so that when perturbed, this will remain as an isolated singularity with determinant and trace of the Jacobian equal to zero. To obtain this, we need that the linear part of the degenerate system has determinant and trace zero. This can be obtained only if the chosen singularity is $(1,1)$ or $(-1,-1)$. Then the system becomes

(a1) $x' = x + y + xy$, $y' = -x - y - xy$,

which after the change $(\bar{x}, \bar{y}, \bar{t}) \to (x + y, -x + y, t/2)$ becomes

(a2) $\bar{x}' = \bar{y} - \bar{x}^2/2 + \bar{y}^2/2$, $y' = 0$,

and then a simple perturbation on y^2 in the second equation produces the nilpotent saddle-node. Subsequent perturbations on xy, x^2, y and x all in the second equation produce the chain $\widehat{sn}_{(4)} \to \widehat{es}_{(3)} + s \to \widehat{cp}_{(2)} + s + a \to \overline{sn}_{(2)} + s + a \to s + s + a + a$.

However, if we try to do the same with system (c), after translating any singularity $(x_0, 1/x_0)$ of the hyperbola to the origin, the condition to be hold so that the linear part of the system has trace zero, is that $x_0 + 1/x_0 = 0$ and this implies $x_0 = \pm i$. So according to Definition 3.3 which does not accept the double conjugate perturbation, this system cannot be considered of codimension 5, even though logic says that the configuration of singularities of systems (a) and (c) are the same and they should have the same codimension. So our final Definition 3.9 solves this problem too.

3: Codimension 6 configurations. The codimension 6 degenerate configurations are:

(174) $(\ominus\,{\scriptstyle [|];c});\ \big(\ominus\,[|];\emptyset\big) \ \to\ (173)$;
(182) $(\ominus\,{\scriptstyle [\times];\emptyset});\ \big(\ominus\,[\times];\emptyset,\emptyset\big), N \ \to\ (181)$;
(184) $(\ominus\,{\scriptstyle [|];n^d});\ \big(\ominus\,[|];\emptyset\big), \binom{0}{2}SN \ \to\ (183)$;
(186) $(\ominus\,{\scriptstyle [|];s});\ \big(\ominus\,[|];N_2^f\big), N \ \to\ (185)$;
(188) $(\ominus\,{\scriptstyle [|];n});\ \big(\ominus\,[|];N_2^\infty\big), S \ \to\ (187)$;
(190) $(\ominus\,{\scriptstyle [|];n});\ \big(\ominus\,[|];S_2\big), N \ \to\ (189)$;
(191) $(\ominus\,{\scriptstyle [\cup];\emptyset});\ \big(\ominus\,[\cup];\emptyset\big), N \ \to\ (175)$;
(193) $(\ominus\,{\scriptstyle [|];\emptyset});\ \big(\ominus\,[|];\binom{1}{1}SN_3\big), N \ \to\ (189)$;
(197) $(\ominus\,{\scriptstyle [|];\emptyset});\ \big(\ominus\,[|];\emptyset\big), \binom{1}{2}E{-}H \ \to\ (180)$;
(198) $(\ominus\,{\scriptstyle [|];\emptyset});\ \big(\ominus\,[|];N\big), \binom{1}{1}SN \ \to\ (185)$;
(199) $(\ominus{\scriptstyle [\,)(\,];\emptyset});\big(\ominus\,[\,)(\,];N,\emptyset\big) \ \to\ (181)$;
(201) $a, (\ominus\,{\scriptstyle [|];\emptyset});\ \big(\ominus\,[|];\binom{0}{2}SN_3\big) \ \to\ (189)$.

From this set we point out the configuration (174) that does not have a center and by perturbation produces configuration (172) $c, (\ominus\,{\scriptstyle [|];\emptyset}); \big(\ominus\,[|];\emptyset\big)$, so logically the codimension of (172) (which corresponds to phase portrait Vul_{29} from [39]) could be 5.

4: Codimension 7 configurations. The codimension 7 degenerate configurations are:

(194) $(\ominus\,{\scriptstyle [|||];\emptyset});\ \big(\ominus\,[|||];\emptyset\big), N \ \to\ (191)$;
(195) $(\ominus\,{\scriptstyle [|];\emptyset});\ \big(\ominus\,[|];\binom{1}{1}SN_2\big), N \ \to\ (193)$;
(200) $(\ominus\,{\scriptstyle [\times];\emptyset});\ \big(\ominus\,[\times];N,\emptyset\big) \ \to\ (182)$;
(202) $(\ominus\,{\scriptstyle [|];n^d});\ \big(\ominus\,[|];\binom{0}{2}SN_2\big) \ \to\ (201)$.
(203) $(\ominus\,{\scriptstyle [|];\emptyset});\ \big(\ominus\,[|];\binom{1}{2}E{-}H\big) \ \to\ (197)$;
(204) $(\ominus\,{\scriptstyle [\cup];\emptyset});\ \big(\ominus\,[\cup];N\big) \ \to\ (199)$;
(207) $a, (\ominus\,{\scriptstyle [|];\emptyset}); \big[\infty; \big(\ominus\,[|];\emptyset_3\big)\big] \ \to\ (201)$.

Remark 5.2 (Important)**.** In this codimension we should have had to put the configuration (192) $(\ominus\,{\scriptstyle [||^c];\emptyset}); \big(\ominus\,[||^c];\emptyset\big), N$ which may be perturbed into degenerate configuration (191) of codimension 6. However we have detected a mistake in [9] because configuration (192) is topologically equivalent to configuration (137). The reason is that the intersection of two parallel complex lines produces a real infinite singularity which behaves like an intricate singularity with two hyperbolic sectors.

We have decided not to shift the codes of the configurations from (193) up to (208) to one less number, and leave the gap in (192) as an empty set. We think that if a reader sees in some future papers a code above (192) and wants to refer to the original paper [9], she/he does not need to take the shift into consideration.

5: Codimension 8 configurations. The codimension 8 degenerate configurations are:

(196) $({\scriptstyle\ominus[|^{2}];\emptyset})$; $\big(\ominus\,[|^2];\emptyset\big), N \;\rightarrow\; (194)$;
(205) $({\scriptstyle\ominus[|||];\emptyset})$; $\big(\ominus\,[|||];N\big) \;\rightarrow\; (204)$;
(208) $({\scriptstyle\ominus[||];n^{*}})$; $\big[\infty;\big(\ominus\,[||];\emptyset_2\big)\big] \;\rightarrow\; (207)$.

6: Codimension 9 configuration. And finally there is just one codimension 9 degenerate configuration:

(206) $({\scriptstyle\ominus[|^{2}];\emptyset})$; $\big(\ominus\,[|^2];N\big) \;\rightarrow\; (196)$.

6. NOTATION

During the previous decades mathematicians have classified particular subfamilies of quadratic phase portraits and have assigned to them various labels. Many times a phase portrait appears in different papers having different labels. At this moment we need to choose in a consistent way labels for all phase portraits of quadratic systems.

Here is what we propose: We have 207 distinct topological configurations of singularities [9]. This implies that two phase portraits having different configurations of singularities cannot be topologically equivalent. So the 207 configurations of singularities provide a nice skeleton for our topological classification of the phase portraits. Many of the topological configurations will produce just one phase portrait, some may have several realizable phase portraits, and a few configurations may have dozens of phase portraits. We propose to call each phase portrait as $QSr_a^{(b)}$ where QS stands for "quadratic differential system", 'r' is the number of the configuration of singularities from [9], 'b' is the topological codimension of the phase portrait (except in the case where centers are present), and 'a' is simply an integer to enumerate the different phase portraits which have the same configuration and codimension.

The use of the codimension for the notation allows us to reduce the size of a, but more importantly, it helps us to link the different phase portraits and to detect which ones can (or cannot) bifurcate from others. That is, it helps us to locate the "neighbors" of the phase portraits.

We already have the complete set of phase portraits (modulo limit cycles) for some of the configurations. Others have been partially studied. For example, configuration (1) $s, s, s, a; N, N, N$ has exactly 13 possible phase portraits: $QS1_1^{(0)}$, $QS1_2^{(0)}$, $QS1_3^{(0)}$ and $QS1_4^{(0)}$ which correspond to the structurally stable cases $\mathbb{S}^2_{7,1}$, $\mathbb{S}^2_{7,2}$, $\mathbb{S}^2_{7,3}$ and $\mathbb{S}^2_{7,4}$ from [6] respectively; the phase portraits $QS1_1^{(1)}$, $QS1_2^{(1)}$, $QS1_3^{(1)}$, $QS1_4^{(1)}$, $QS1_5^{(1)}$ and $QS1_6^{(1)}$, which correspond to the structurally unstable topological codimension 1 phase portraits $\mathbb{U}^1_{D,15}$, $\mathbb{U}^1_{D,16}$, $\mathbb{U}^1_{D,17}$, $\mathbb{U}^1_{D,18}$, $\mathbb{U}^1_{D,19}$ and $\mathbb{U}^1_{D,20}$ from [7], respectively; and the phase portraits $QS1_1^{(2)}$, $QS1_2^{(2)}$ and $QS1_3^{(2)}$, which correspond to the structurally unstable topological codimension 2 phase portraits b, $C1a$ and $C1b$ of [40]. These last three phase portraits have two separatrix connections without forcing the anti-saddle to become a center. It is easy to see that if we try to force a third connection in this configuration, the anti-saddle becomes a center and we are in fact in configuration (2) and in the phase portrait Vul_{10}.

The configuration for which all its realizable phase portraits have already been found, and up to now, it has the largest number of phase portraits is (39) $s, a, sn; \binom{0}{2}SN, N$. There are 99 distinct realizable phase portraits (modulo limit cycles), which split in 46 of topological codimension 2, 47 of codimension 3 and 6 of codimension 4. We know that this list is complete because the study of all the family with a finite saddle-node and a $\binom{0}{2}SN$ has already been done [13, 14], and also the study of all potential phase portraits of codimension 2 of this class is also done [15]. Again, it is not possible to have 3 separatrix connections in this family.

There are many topological configurations of singularities which have a single phase portrait like (47) $cp; N$ whose only possible phase portrait $QS47_1^{(2)}$ can be seen labeled as Fig. 12(a) in [27] as well as in many other papers with different other labels.

Regarding the phase portraits with centers we have already mentioned that we decided not to assign to them a codimension. So the phase portraits Vul_{11}, Vul_8, Vul_9 and Vul_{10} that can be obtained from configuration (2) may be labeled $QS2_1$, $QS2_2$, $QS2_3$ and $QS2_4$, respectively.

Since the total amount of topologically distinct phase portraits of quadratic systems modulo limit cycles is expected to be between 1200 and 2000 (or even more), we do not attempt to label all of those already found in a paper. From now on we will begin using this notation, and in the future, the complete collection of them may appear in a web form page.

This notation was originally designed to cover the phase portraits of quadratic systems without limit cycles. It is of course possible to extend it to phase portraits with limit cycles. In order to do this one must take into account that some configurations with 2 or 3 anti-saddles, may have different phase portraits with the same number of limit cycles depending which anti-saddle they surround. For example, configuration (3) $s, a, a, a; S$ has three finite anti-saddles and there is just one possible phase portrait of codimension 0 without limit cycles labeled $QS3_1^{(0)}$ which corresponds to $\mathbb{S}^2_{5,1}$ from [6]. In this phase portrait, one anti-saddle receives (or emits) exactly one separatrix, another anti-saddle receives (or emits) exactly two separatrices and the third anti-saddle receives (or emits) exactly three separatrices. So it is clear that if a system has a phase portrait topologically equivalent modulo limit cycles to $QS3_1^{(0)}$ and has some limit cycles, one must specify around which ones of the anti-saddles they occur. We propose to denote these possibilities as $QS3^{(0)}_{1(i,j,k)LC}$ where i, j, k are respectively the numbers of limit cycles around the anti-saddles with decreasing number of separatrices, i.e. three, two, one. Collecting from many different papers, we have already been able to corroborate the existence of $QS3^{(0)}_{1(0,0,1)LC}$, $QS3^{(0)}_{1(0,0,2)LC}$, $QS3^{(0)}_{1(0,1,0)LC}$, $QS3^{(0)}_{1(1,0,0)LC}$ and $QS3^{(0)}_{1(1,0,1)LC}$.

7. Concluding comments

Constructing phase portraits for polynomial differential systems began with studies of rather limited families of them. It took a very long time to find all the 31 phase portraits of quadratic systems with a center (obtained in 1983 [39]) even though the conditions for having a center were given by Dulac in 1908. Since then the number of phase portraits for QS has very much increased and in the literature we find over 5000 phase portraits, not necessarily topologically distinct. How many of these are in fact distinct and how many of these are topologically distinct modulo limit cycles? We need to respond to these questions and furthermore how do we go about finding what exactly are the ones still missing.

By assigning codimensions to each one of the 207 topological configurations of singularities we split the family QS in ten subfamilies. Each one of these subfamilies can then be further split by using the tool of codimension of phase portraits. We thus obtain a powerful tool for the classification problem modulo limit cycles.This tool has been very helpful recently in understanding whether or not among the over 5000 topological phase portraits of quadratic systems which appear in various publications in the literature some can be identified. By splitting these over 5000 phase portraits according to their codimensions we thus have have far fewer of them to compare.

A comprehensive database is currently under development and this database systematically compares the phase portraits and has identified approximately 1150 topologically distinct phase portraits (modulo limit cycles), plus around 150 more that include limit cycles. This resource enables quick identification of whether a phase portrait from a new study already exists in the database or not, and references where it has previously appeared. It also assists in detecting redundancies and omissions across various publications.

In codimension 1 two types of phase portraits have been identified: those whose instability originates from the presence of a saddle-node (either finite or at infinity), and those whose instability results from the existence of a separatrix connection. Based on this classification we can begin assembling a table that organizes the phase portraits within each class, as illustrated in Table 1.

Table 1. Number of phase portraits according to codimension

CS	Global cod. of PhaseP							
	0	**1**	**2**	**3**	**4**	**5**	**6**	...
0	44	61	?	?	?	?	?	?
1	x	141	?	?	?	?	?	?
2	x	x	?	?	?	?	?	?
3	x	x	x	?	?	?	?	?
⋮	x	x	x	x	?	?	?	?

In the first column of Table 1, we present the codimension contributed by the configuration of singularities, while the label row indicates the global codimension of the phase portrait. Naturally the global codimension cannot be smaller than the codimension arising from the singularities.

Many phase portraits are already known, allowing some of the "?" symbols in the table to be replaced with some lower bounds. In fact, with the current knowledge, many of the boxes already have a proved definitive number, but we have preferred to show the table as it was prior to the use of codimension to classify the more degenerate phase portraits.

Finally, the concept of codimension will enable us to create an encyclopedia that catalogs all topologically distinct phase portraits (modulo limit cycles). This encyclopedia will clarify which phase portraits of codimension $k+1$ serve as "borders" to some of codimension k, and identify the corresponding neighboring phase portraits across these boundaries. This will yield a bifurcation diagram, not in the parameter space, but within the moduli space of all topologically distinct phase portraits of QS modulo the group action of affine transformation and time rescaling.

To highlight the importance of the concept of codimension and to demonstrate the utility of the new notation, we reproduce several figures from previous works, now updated to incorporate the new notation.

Specifically we consider a structurally stable phase portrait, denoted $\mathbb{S}^2_{12,6}$ in [6], which can now be referred to as $QS5_6^{(0)}$ under the new nomenclature. We then apply to this phase portrait all possible coalescences of singularities and appearance of separatrix connections that correspond to codimension-one bifurcations. All such configurations are drawn from the examples presented in [7]. In Figures 11-14, we present the phase portraits according to the following criteria. In the left column we always display a phase portrait topologically equivalent to $QS5_6^{(0)}$. Some phase portraits of Figure 14 may appear different, because the way we show them makes it easier to visualize how the separatrix connections are formed. In the first three figures, we have also labeled certain singularities with the letters "S" and "N" to indicate those that will coalesce. The middle column in each figure shows the codimension-1 phase portrait that lies on the boundary of $QS5_6^{(0)}$. In the right column we exhibit the phase portraits obtained from the middle column when moving back into the 0-codimension portraits in a different way from the first bifurcation. Taken together, the middle columns of all these figures illustrate all the codimension-1 boundaries of the subspace of QS with phase portrait$QS5_6^{(0)}$.

Now we choose one of these borders, namely $\mathbb{U}^1_{D,36}$ in [7], which can now be referred to as $QS5_7^{(1)}$ and we visit [5] (see Figure 15) and detect all the phase portraits of codimension 2 that can be obtained from $QS5_7^{(1)}$ when producing the coalescence of a finite saddle with a finite anti-saddle. And we also see what phase portraits of codimension 1 appear when the border is crossed. In some cases the vanishing of the saddle-node produces also the disappearance of the separatrix connection, and thus we obtain a codimension 0 phase portrait.

In this way we can systematically construct the bifurcation diagram of the set of phase portraits (modulo limit cycles) of all QS, not in parameter space, but in the moduli space of their equivalence

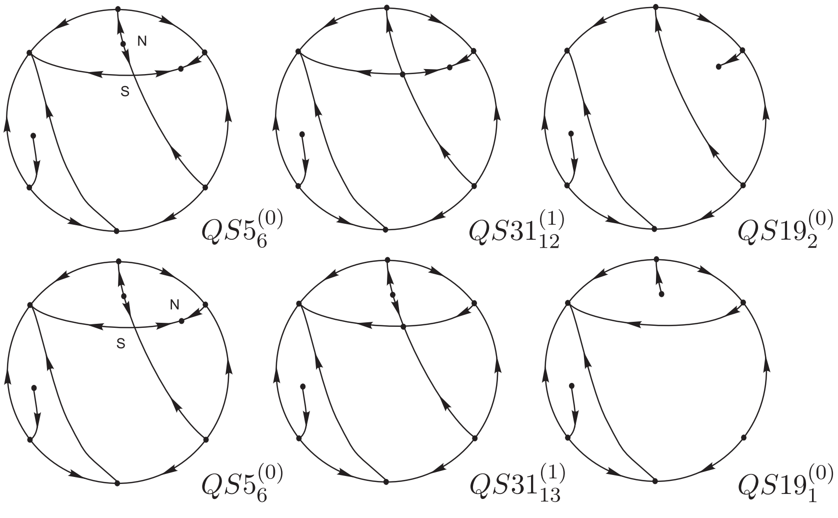


FIGURE 11. Borders of $QS5_6^{(0)}$ containing a finite saddle-node, extracted from [7, Fig. 5.36].

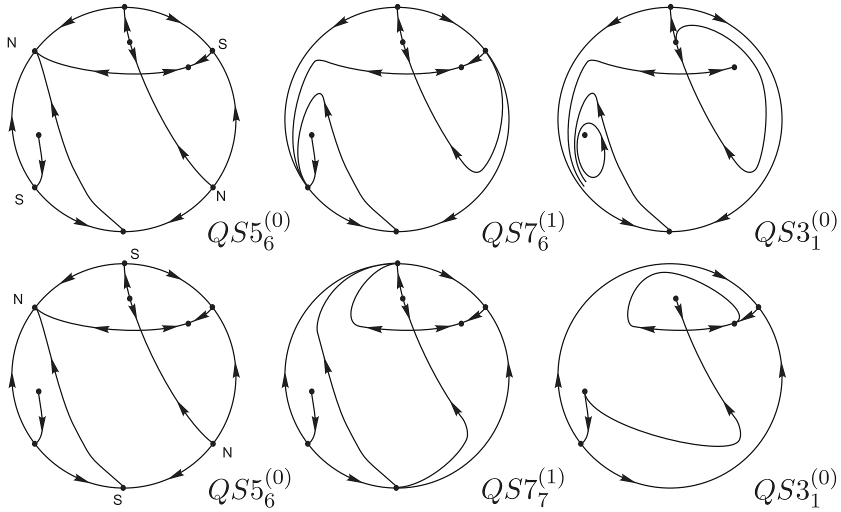


FIGURE 12. Borders of $QS5_6^{(0)}$ containing a $\binom{0}{2}SN$, extracted from [7, Fig. 5.66].

classes. Work is in progress for replacing the question marks In Table 1 with information extending the one given in the first 2×2 matrix where we have specific numbers (44,61,141).

8. APPENDIX

In this appendix we prove that all the geometrical configurations of singularities shown in FIGURE 5 are realizable for quartic differential systems. In the same way that the quadratic system $x' = x^2, \quad y' = 2x^2 + y^2$ has the configuration of singularities $hh_{(4)}; N^*$, Ⓒ, Ⓒ, it is a simple exercise to check that the quartic system

$$x' = x^2 + x^4, \quad y' = 2x^2 + y^2 + 2x^4 + y^4, \tag{8.1}$$

has the geometric configuration $hh_{(4)}$, ©, ©, ©, ©, ©, ©, ©, ©, ©, ©, ©, ©; N^*, Ⓒ, Ⓒ, Ⓒ, Ⓒ, this means that we have the same singularities as in the quadratic case plus 12 finite and 2 infinite complex singularities. The infinite node is located at $[0 : 1 : 0]$. For compactness we will avoid writing the complex singularities in this appendix.

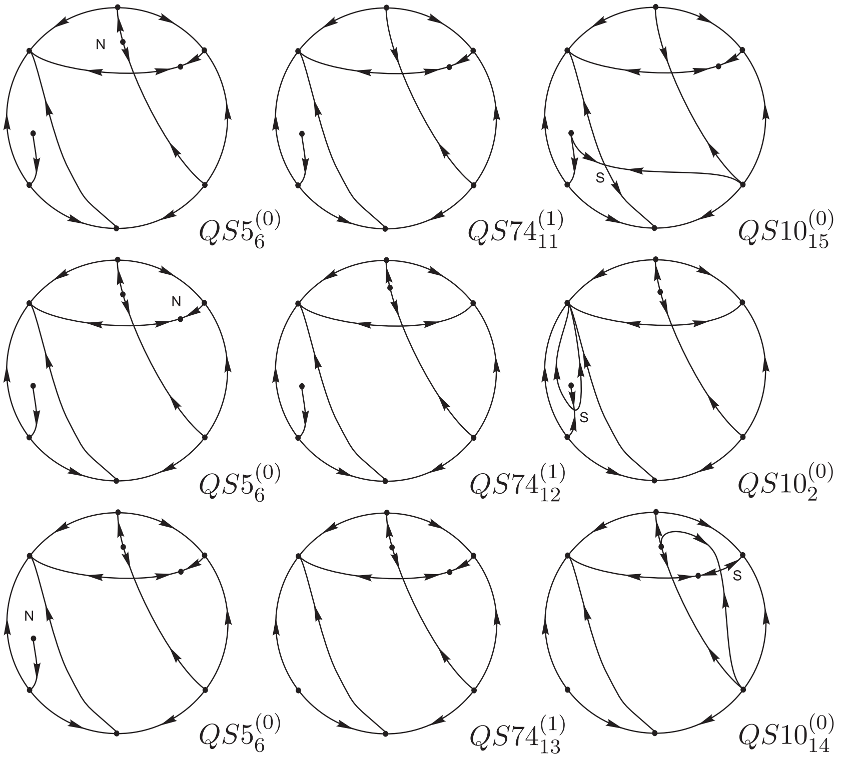


FIGURE 13. Borders of $QS5_6^{(0)}$ containing a $\binom{1}{1}SN$, extracted from [7, Fig. 5.91].

Now we make the following perturbation of system (8.1)

$$\begin{aligned} x' &= x^2 + x^4 + \varepsilon_d xy^3 + \varepsilon_*\Big(\frac{(\varepsilon_i\varepsilon_s - \varepsilon_*\varepsilon_i^3 + 2\varepsilon_i^2 + 2\varepsilon_s^2)xy^2}{\varepsilon_i\varepsilon_s} + \varepsilon_i xy + y^3\Big), \\ y' &= \varepsilon_i x + \varepsilon_s y + 2x^2 + y^2 + 2x^4 + y^4. \end{aligned} \tag{8.2}$$

As we will see the perturbation just in ε_* turns the infinite star node into an N^d, the perturbation just in ε_d turns the infinite star node into a generic node, the perturbation just in ε_i turns the finite intricate singularity into a nilpotent one and the perturbation just in ε_s turns the finite singularity into a semi-elemental one. In fact the perturbation in ε_d produces a reduction of the codimension in two steps because it perturbs the infinite star node into a generic node. And the perturbation in ε_s produces another reduction of codimension of two steps because it perturbs the finite intricate singularity into a semi-elemental saddle-node of multiplicity 4. But any path in FIGURE 5 consisting in 4 steps from the bottom to the top can be followed using the corresponding perturbation $\varepsilon_*, \varepsilon_d, \varepsilon_i, \varepsilon_s$ in the proper order.

It is easy to check that if $\varepsilon_* \neq 0 = \varepsilon_d = \varepsilon_i = \varepsilon_s$ the finite singular point remains intricate with the same geometrical configuration, no other finite singularities appear, no other infinite singularities appear, the infinite node at $[0:1:0]$ has not changed the position, however it is no longer a star node but a one-direction node N^d (i.e. we have configuration $hh_{(4)}; N^d$). And if we make the perturbation $\varepsilon_d \neq 0 = \varepsilon_i = \varepsilon_s$ (ε_* is irrelevant) again nothing has changed except that the infinite node at $[0:1:0]$ is generic, more precisely we have N^f if $\varepsilon_d > 0$ or N^∞ if $\varepsilon_d < 0$ (i.e. we have configuration $hh_{(4)}; N^f$ or $hh_{(4)}; N^\infty$).

If we make a perturbation $\varepsilon_i \neq 0 = \varepsilon_s = \varepsilon_* = \varepsilon_d$, in this case the finite intricate singular point becomes nilpotent, and applying Theorem 3.5 in [21] to study the nilpotent singularity, it is a saddle-node of multiplicity 4. The singularity at infinity remains a star node (i.e. we have configuration $\widehat{sn}_{(4)}; N^*$). If we make the perturbation $\varepsilon_* = \varepsilon_i \neq 0 = \varepsilon_d = \varepsilon_s$ again the finite singularity turns into a nilpotent saddle-node of multiplicity 4 and the infinite node at $[0:1:0]$ is

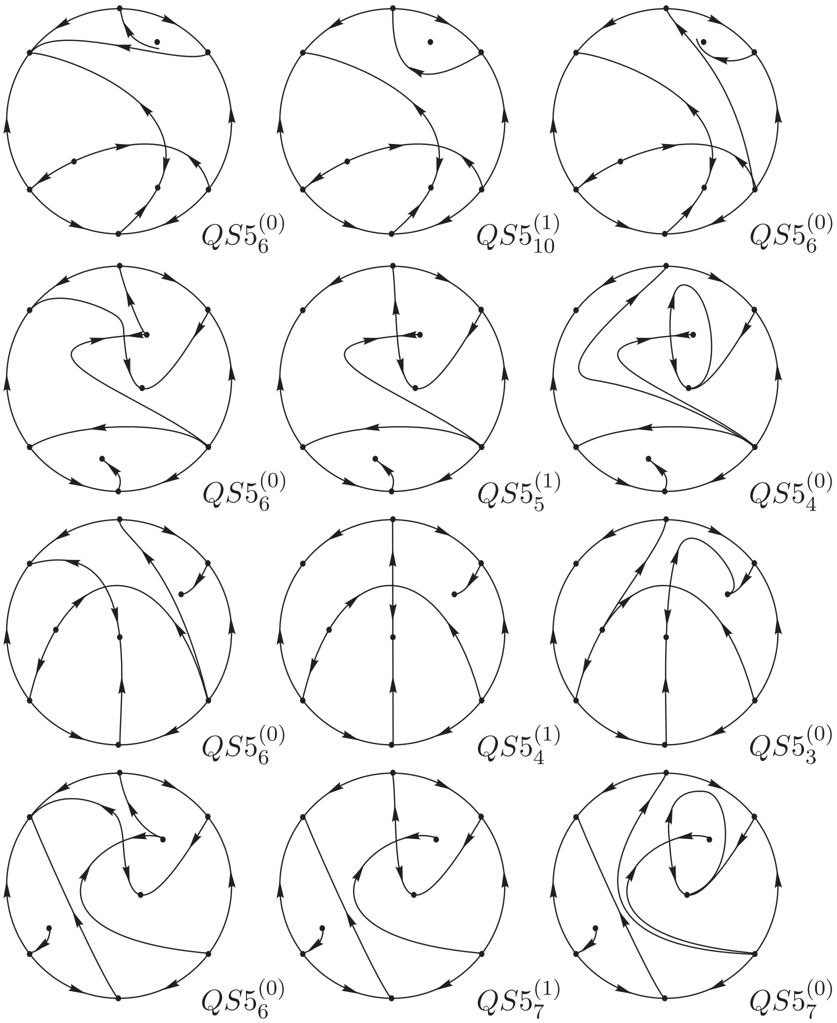


Figure 14. Borders of $QS5_6^{(0)}$ containing a separatrix connection, extract from [7, Figs. 5.149, 5.150, 5.152 and 5.155].

an N^d (i.e. we have configuration $\widehat{sn}_{(4)}; N^d$). And if we make the perturbation $\varepsilon_i = \varepsilon_d \neq 0$ (ε_* is irrelevant) again the finite singularity turns into a nilpotent saddle-node of multiplicity 4 and the infinite node at $[0:1:0]$ is generic (N^f if $\varepsilon_d > 0$ or N^∞ if $\varepsilon_d < 0$). Thus we have configuration $\widehat{sn}_{(4)}; N^f$ or $\widehat{sn}_{(4)}; N^\infty$.

In any of the cases the perturbation in ε_s turns the finite singularity into a semi-elemental one. The tricky coefficient of xy^2 in the first equation is the one that maintains the singularity with multiplicity 4.

The way to check that the semi-elemental singularity is of multiplicity 4 is by applying Theorem 2.19 in [21]. Then the expression $g(x) = a_m x^m + o(x^m)$ needed to determine if the singularity is a node or a saddle or a saddle-node has $m = 4$, and thus we can add one more perturbation in order to obtain $m = 3$. And one more to get that $m = 2$, and a last one to split into two elemental singularities.

Since the perturbations ε_* and ε_d are independent of ε_i and ε_s all the different trips in Figure 5 are possible and all the configurations of singularities are realizable. So we can conclude that the geometrical configuration of singularities $hh_{(4)}; N^*$ (plus the required complex singularities of multiplicity 1), has codimension 7 inside the family of quartic systems, but codimension 6 inside

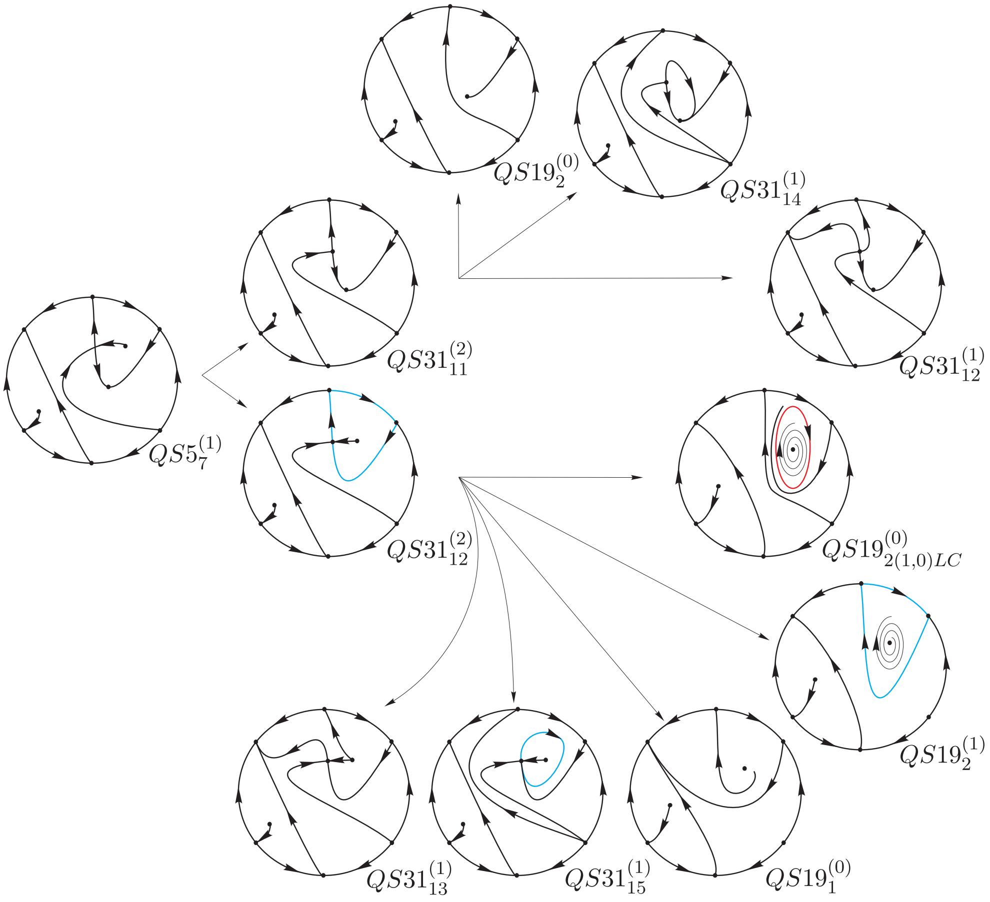


FIGURE 15. Borders of $QS5_7^{(1)}$ containing a finite saddle-node, extract from [5, Figs. 36].

the family of quadratic systems. As we have said at the beginning, codimension is a relative concept.

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