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# GRADIENT CONTINUITY OF LOCAL MINIMIZERS OF A BORDERLINE DOUBLE-PHASE FUNCTIONAL WITH DMO COEFFICIENTS

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Abstract. Let $u \in W^{1,1}(\Omega)$. We study a local gradient continuity of the minimizers of functionals with borderline double-phase growth,

$$\mathcal{F}(u;\Omega) := \int_\Omega \Big( |Du|_{\mathbb{A}}^p + a(x)\, |Du|_{\mathbb{A}}^p \log(e + |Du|_{\mathbb{A}}) \Big)\, dx$$

for $1 < p < \infty$ and $|Du|_{\mathbb{A}} := \langle \mathbb{A}(x)Du, Du\rangle^{1/2}$, where the matrix $\mathbb{A}(x) = \{A^{\alpha\beta}(x)\}$ is symmetric with positive upper and lower bounds of its eigenvalues. We prove that $Du \in C^1_{\rm loc}(\Omega)$ based mainly on the exit-time approach, provided that the moderating factor $a(x)$ and coefficient $\mathbb{A}(x)$ satisfy the so-called Dini-type mean oscillation conditions.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ be a bounded domain and $1 < p < \infty$. We consider the local minimizers $u \in W^{1,1}(\Omega)$ for the functional with borderline double phase growth,

$$\mathcal{F}(u;\Omega) := \int_\Omega \Big( |Du|_{\mathbb{A}}^p + a(x)\, |Du|_{\mathbb{A}}^p \log(e + |Du|_{\mathbb{A}}) \Big)\, dx, \tag{1.1}$$

where the moderating factor $a : \Omega \to [0, +\infty)$ is bounded, and $|Du|_{\mathbb{A}} := \big\langle \mathbb{A}(x)\, Du, Du \big\rangle^{1/2}$. The matrix $\big(A^{\alpha\beta}(x)\big)$ for $\alpha, \beta = 1, 2, \dots, n$ is assumed to be symmetric with $A^{\alpha\beta}(x) = A^{\beta\alpha}(x)$; moreover, there exists a constant $K \geq 1$ such that

$$K^{-1}|\xi|^2 \leq \sum_{\alpha,\beta=1}^{n} A^{\alpha\beta}(x)\, \xi_\alpha \xi_\beta \leq K\, |\xi|^2, \quad (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{1.2}$$

For a given $\gamma \geq 1$, we define the $\gamma$-moduli of mean oscillation as

$$\bar\omega_{a,\gamma}(\rho) := \sup_{y\in\Omega} \Big( \fint_{B_\rho(y)\cap\Omega} |a(x) - \bar a_\rho|^\gamma\, dx \Big)^{1/\gamma} \quad \text{and} \quad \bar\omega_{\mathbb{A},\gamma}(\rho) := \sup_{y\in\Omega} \Big( \fint_{B_\rho(y)\cap\Omega} |\mathbb{A}(x) - \bar{\mathbb{A}}_\rho|^\gamma\, dx \Big)^{1/\gamma},$$

where $\bar a_\rho = \fint_{B_\rho(y)\cap\Omega} a(x)\, dx$ and $\bar{\mathbb{A}}_\rho = \fint_{B_\rho(y)\cap\Omega} \mathbb{A}(x)\, dx$. Moreover, it is supposed that both $a(x)$ and $\mathbb{A}(x)$ satisfy the so-called Dini-type mean oscillation (DMO) introduced first by Spanne [30], which require the moduli to satisfy

$$\int_0^1 \bar\omega_{a,\gamma}(\rho) \log\big(\frac{1}{\rho}\big)\, \frac{d\rho}{\rho} < \infty \quad \text{and} \quad \int_0^1 \bar\omega_{\mathbb{A},\gamma}(\rho)\, \frac{d\rho}{\rho} < \infty. \tag{1.3}$$

Now we give the definition of local minimizers to the functional (1.1).

**Definition 1.1.** We say that $u \in W^{1,1}(\Omega)$ is a local minimizer of the functional (1.1) if $\mathcal{F}(u;\Omega) < \infty$ and for any $\Omega' \Subset \Omega$ we have

$$\mathcal{F}(u;\Omega') \leq \mathcal{F}(u+\phi;\Omega'), \quad \phi \in W^{1,1}_0(\Omega').$$

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Such a non-uniform functional (1.1) was originated from Zhikov's work [32] on the homogenization of strongly anisotropic materials, and the regularity study for elliptic problems with $(p,q)$-growth has drawn extensive attention over the past few decades. Marcellini proved the Lipschitz regularity for local minimizers of double-phase energy functionals in [25], and also got the same regularity for non-uniformly elliptic equations in [26]. A typical $(p,q)$-growth functional with

$$\mathcal{F}_{p,q}(u;\Omega) := \int_\Omega \Big(|Du|^p + a(x)\,|Du|^q\Big)\,dx, \tag{1.4}$$

Colombo-Mingione [8] recently proved the local Hölder continuity of the gradient for local minimizers under the assumptions that

$$a(x)\in C^{0,\alpha}(\Omega) \quad \text{for } \alpha\in(0,1], \quad \text{and} \quad \frac{q}{p} < 1+\frac{\alpha}{n}; \tag{1.5}$$

while Colombo-Mingione [9] showed the $C^{1,\gamma}_{\rm loc}$-regularity for the bounded minimizers of (1.4) under a relaxed gap condition $q \le p+\alpha$. Furthermore, Baroni-Colombo-Mingione [4] also got the same gradient Hölder continuity to the local minimizers of functional (1.4) based on harmonic type approximation by extending this constraint (1.5) to $q/p \le 1+\alpha/n$.

On the other hand, as a borderline case of $(p,q)$-growth with functional of the form

$$\mathcal{F}_{\log}(u;\Omega) := \int_\Omega \Big(|Du|^p + a(x)\,|Du|^p \log\big(e+|Du|\big)\Big)\,dx, \tag{1.6}$$

Baroni-Colombo-Mingione [3] proved that the local minimizers of $\mathcal{F}_{\log}(u;\Omega)$ can attain an optimal Hölder continuity provided $\limsup_{r\to 0^+}\omega_a(r)\log\left(\frac{1}{r}\right)=0$ with $\omega_a(r)$ as the modulus of continuity for $a(\cdot)$, and the gradient Hölder continuity by adding $\omega_a(r)\lesssim r^\alpha$ with $0<\alpha\le 1$. Later, Byun-Youn [6] employed the Riesz potential theory to obtain the pointwise gradient estimate and $C^1$-regularity to such an elliptic equation with borderline double-phase growth under assumption of the so-called log-Dini continuity with the form $\int_0^1 \omega_a(\rho)\log\left(\frac{1}{\rho}\right)\frac{d\rho}{\rho} < \infty$. Recently, Baroni-Coscia [5] also got the gradient continuity of local minimizers to such a functional $\mathcal{F}_{\log}(u;\Omega)$ under assumption of log-Dini continuity of $a(\cdot)$. More regularity results involving the double-phase problems, we can refer to [1, 2, 11, 17, 24, 29, 31] and the references therein.

Associated with $C^1$-regularity of elliptic problems, let us briefly recall some recent progresses. For such a linear elliptic equation: $-\operatorname{div}(\mathbb{A}(x)Du)=0$, Lieberman [23] first proved the gradient continuity if $\mathbb{A}(x)$ satisfies the so-called $\alpha$-increasing Dini continuity which means that $\int_0^1 \omega_{\mathbb{A}}(r)\,\frac{dr}{r}<\infty$ and $\omega_{\mathbb{A}}(r)/r^\alpha$ is non-decreasing for $\alpha>0$ with $\omega_{\mathbb{A}}(r)$ as a modulus of continuity of $\mathbb{A}(x)$. Instead of the Dini mean oscillation from Spanne in [30], Huang in [19] proved the $\mathrm{BMO}_\psi$-boundedness and continuity of $Du$ to linear elliptic systems if the leading coefficient $\mathbb{A}(x)\in \mathrm{BMO}_\psi$ with $\psi$ satisfying the Dini continuity and almost increasing of $\psi(r)\big(1+|\log\int_0^r(\psi(t)/t)|\big)$. Li in [22] gave an open problem for linear elliptic equations: does the $C^1$-regularity hold of solutions if one merely adds DMO condition on the coefficients? Dong-Kim in [14] gave a positive answer to Li's question, and they proved the $C^1$-regularity of solutions to linear elliptic equations under the DMO condition $\int_0^1 \bar\omega_{\mathbb{A},1}(\rho)\,\frac{d\rho}{\rho}<\infty$. However, it is insufficient one only assumes the continuity of coefficients $\mathbb{A}(x)$, since Jin-Maz'ya-Schaftingen [20] constructed a counterexample to show that the gradient may be discontinuous to the homogeneous linear elliptic equation if $\mathbb{A}(x)$ is merely continuous. Further, Duzaar-Mingione [12] used non-linear Wolff potential argument to show the gradient continuity for $p$-Laplace equations under assumption of Dini continuous coefficients. Recently, for $p(x)$-Laplace system

$$-\operatorname{div}\big(|Du|^{p(x)-2}Du\big)=f,$$

Ok [28] obtained a local gradient continuity under assumptions that $f\in L(n,1)$ (the so-called Lorentz spaces), and $p(x)$ is log-Dini continuous with $\int_0^1 \omega_p(r)\log\left(\frac{1}{r}\right)\frac{dr}{r}<\infty$, where $\omega_p(\cdot)$ is the modulus of continuity of $p(x)$. Baroni-Coscia in [5] showed the $C^1$-regularity of functional (1.6) if $a(x)$ satisfies the log-Dini condition. Dong-Lee-Kim [15] proved a global $C^1$-regularity of conormal and oblique derivative boundary problem to linear elliptic equations with DMO coefficients, while

Dong-Kwon [16] obtained the $C^1$-regularity of solutions to non-stationary Stokes equations with DMO coefficients.

Motivated by the above-mentioned papers, we are interested in the $C^1$-regularity to the local minimizers of functional (1.1). This is an extensive generalization of Baroni-Coscia's work in [5] by way of general matrix function $\mathbb{A}(x)$ instead of $Id$, and relaxing the regular assumption of $a(x)$ from the log-Dini continuity to the Dini-type mean oscillation. To this end, let us write

$$\Psi(x,t) := t^p + a(x)\, t^p \log(e+t);$$

and we are now in a position to state our main result.

**Theorem 1.2.** *Let $u \in W^{1,1}(\Omega)$ be a local minimizer of the functional* (1.1) *with $\Psi(x,Du) \in L^1(\Omega)$. If there exists a constant $\gamma = \gamma\,(n,p,K,\|a\|_{L^\infty(\Omega)},\|Du\|_{L^p(\Omega)}) \geq 1$ such that $a(x)$ and $\mathbb{A}(x)$ satisfy* (1.2) *and* (1.3)*, then we conclude that $Du$ is locally continuous in $\Omega$.*

The rest of this paper is organized as follows. We devote Section 2 to notations, basic inequalities and a few useful lemmas. In Section 3, we employ the perturbation argument to perform the comparison and decay estimates in the $L^1$ sense based on the freezing coefficients. We mainly focus Section 4 on the proofs of local boundedness and continuity of gradient to the local minimizers of functional (1.1) by way of the stop-time approach.

## 2. Preliminaries

Throughout this paper, we use $c$ to denote a generic positive constant that may vary from line to line. Important constants are distinguished by subscripts (e.g., $c_1, c_2$). To indicate dependence on specific parameters, we use parentheses; for example, $c(n,p,K)$ indicates that the constant depends only on $n,p,K$. We use the notation $a \lesssim b$ to explain that there exists a constant $c>0$ such that $a \leq cb$; similarly, $a \gtrsim b$ means $a \geq cb$; $a \approx b$ indicates that there exist two positive constants $c_1, c_2$ satisfying $c_1 a \leq b \leq c_2 a$.

Let us first recall related basic notations. We write $\varphi_{\log}(t) = t^p \log(e+t)$ for $p \in (1,\infty)$, and observe that

$$\varphi'_{\log}(\cdot) = pt^{p-1}\log(e+t) + \frac{t^p}{e+t}, \quad \varphi''_{\log}(\cdot) = p(p-1)t^{p-2}\log(e+t) + \frac{2pt^{p-1}}{e+t} - \frac{t^p}{(e+t)^2}.$$

A direct calculation yields that

$$t^2\varphi''_{\log}(t) \approx t\varphi'_{\log}(t) \approx \varphi_{\log}(t) \quad \forall t>0, \tag{2.1}$$

where the implied constants depend only on $p$. Since $\log(e+Lt) \leq L\log(e+t)$ for $L \geq 1$ and $t>0$, and $\log(e+\nu t) \geq \nu \log(e+t)$ for $\nu \in (0,1)$ and $t>0$, then we obtain the following facts:

$$\varphi_{\log}(Lt) \leq L^{p+1}\varphi_{\log}(t), \quad \varphi_{\log}(\nu t) \geq \nu^{p+1}\varphi_{\log}(t), \quad \text{for any } L \geq 1,\ \nu \in (0,1),\ t>0; \tag{2.2}$$

moreover, it follows from the expression of $\varphi'_{\log}(\cdot)$ that

$$\varphi'_{\log}(Lt) \leq L^p\varphi'_{\log}(t), \quad \varphi'_{\log}(\nu t) \geq \nu^p\varphi'_{\log}(t), \quad \text{for any } L \geq 1, \nu \in (0,1), t>0. \tag{2.3}$$

Let $\mathcal{D} \subset \mathbb{R}^n$ be a measurable set with positive measure. For any locally integrable function $f \in L^1(\mathcal{D},\mathbb{R}^m)$, we define the integral average of $f$ over $\mathcal{D}$ by

$$\langle f\rangle_{\mathcal{D}} = \fint_{\mathcal{D}} f\,dx = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} f\,dx.$$

A direct calculation yields that for any $\xi \in \mathbb{R}^n$ and $f \in L^q(\mathcal{D},\mathbb{R}^n)$ with $q \geq 1$,

$$\fint_{\mathcal{D}} |f-\langle f\rangle_{\mathcal{D}}|^q\,dx \leq c(n,q)\fint_{\mathcal{D}} |f-\xi|^q\,dx.$$

In the following context, let $B_r(x_0)$ denote the open ball in $\mathbb{R}^n$ centered at $x_0$ with radius $r>0$; we simply write $B_r$ while the center is clear from the context. The following lemma provides an estimate associated with the logarithmic function, for details see [5, Lemma 2.1].

**Lemma 2.1.** *Let $q > 1$ and $\sigma, \beta, \theta \geq 0$ be constants, and let $f \in L^q(B_r)$ with $r \leq e^{-1}$. Then there exists a positive constant $c = c(n, \beta, \sigma, \theta, q)$ such that*

$$\fint_{B_r} |f| \log^\beta (e + |f|^\sigma)\, dx \leq c\big(1 + r^\theta \|f\|_{L^1(B_r)}\big)\Big(\log\big(\frac{1}{r}\big)\Big)^\beta \Big(\fint_{B_r} |f|^q\, dx\Big)^{1/q}.$$

It is important to observe that any function satisfying the Dini mean oscillation condition must be continuous, see [30, Corollary 1].

**Lemma 2.2.** *Let $f(x) \in L^1(\Omega)$ satisfy the Dini mean oscillation condition with the form*

$$\int_0^1 \bar{\omega}_f(\rho)\frac{d\rho}{\rho} < \infty \quad \text{with} \quad \bar{\omega}_f(\rho) := \sup_{y\in\Omega} \fint_{B_\rho(y)\cap\Omega} |f(x) - \langle f\rangle_{B_\rho\cap\Omega}|\, dx.$$

*Then $f(x)$ has to be continuous in $\Omega$ with modulus $\omega_f(r) = c\int_0^r \bar{\omega}_f(\rho)\,\frac{d\rho}{\rho}$ for some $c > 0$.*

With Lemma 2.2 in hand, we readily see that $a(x)$ is log-Hölder continuous and $\mathbb{A}(x)$ is continuous provided that $a(x)$ and $\mathbb{A}(x)$ meet (1.3).

**Lemma 2.3.** *If both $a(x)$ and $\mathbb{A}(x)$ satisfy (1.3) for $\gamma > 1$, then $a(x)$ is log-Hölder continuous, and $\mathbb{A}(x)$ is continuous in $\Omega$, respectively, with*

$$\sup_{r\in(0,1]} \omega_a(r) \log\big(\frac{1}{r}\big) < \infty, \tag{2.4}$$

*where $\omega_a(r)$ is the continuity modulus of $a(x)$ in $B_r$.*

*Proof.* It follows from the boundedness of $a(x)$ and (1.3) that

$$\begin{aligned}\int_0^1 \bar{\omega}_a(\rho)\frac{d\rho}{\rho} &= \int_0^{1/e} \bar{\omega}_a(\rho)\frac{d\rho}{\rho} + \int_{1/e}^1 \bar{\omega}_a(\rho)\frac{d\rho}{\rho}\\ &\leq \int_0^{1/e} \bar{\omega}_a(\rho)\log\big(\frac{1}{\rho}\big)\frac{d\rho}{\rho} + 2\|a\|_{L^\infty(\Omega)} < \infty.\end{aligned}$$

By Lemma 2.2 we conclude that $a(x)$ is continuous and satisfies $\omega_a(r) = c\int_0^r \bar{\omega}_a(\rho)\frac{d\rho}{\rho}$. Then we use (1.3) again to obtain that

$$\sup_{r\in(0,1]} \omega_a(r)\log\big(\frac{1}{r}\big) \leq \int_0^1 \bar{\omega}_{a,\gamma}(\rho)\log\big(\frac{1}{\rho}\big)\,\frac{d\rho}{\rho} < \infty.$$

The continuity of $\mathbb{A}(x)$ is readily obtained due to (1.3) and Lemma 2.2. □

For the sake of convenience, we collect the parameters in the context and denote them by

$$\texttt{data} := \Big(n, p, K, \|a\|_{L^\infty(\Omega)}, \|Du\|_{L^p(\Omega)}\Big).$$

In what follows, we present the higher integrability result of the gradient to the local minimizers of functional (1.1), for its proof see [3, Theorem 4.1].

**Lemma 2.4.** *Let $u \in W^{1,1}(\Omega)$ be a local minimizer of the functional (1.1). Assume that $\mathbb{A}(x)$ satisfies (1.2) and $a(x)$ is log-Hölder continuous with the modulus $\omega_a(r)$ satisfying (2.4). Then there exist constants $\varepsilon = \varepsilon(\texttt{data}) > 0$ and $c = c(\texttt{data}) \geq 1$ such that*

$$\Big(\fint_{B_{r/2}} \Psi(x, |Du|)^{1+\varepsilon}\, dx\Big)^{\frac{1}{1+\varepsilon}} \leq c \fint_{B_r} \Psi(x, |Du|)\, dx$$

*for all $B_r \Subset \Omega$.*

The following lemma describes a self-improving property of the reverse Hölder inequality based on the backward iteration, see [18, Remark 6.12] or [21, Lemma 3.1].

**Lemma 2.5.** *Let $\mathcal{D}$ be a domain. If for any function $g \in L^{1+\varepsilon}_{\rm loc}(\mathcal{D})$, there exist $b>0$ and $A \ge 0$ such that for any $B_r \Subset \mathcal{D}$,*

$$\Big(\fint_{B_{r/2}} g^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} \le b \fint_{B_r} g\,dx + A,$$

*then we have*

$$\Big(\fint_{B_{r/2}} g^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} \le B\Big(\Big(\fint_{B_r} g^{\tau}\,dx\Big)^{1/\tau} + A\Big)$$

*for any $\tau>0$, where $B = B\big(n,b,\tau\big) > 0$.*

Then, the conclusion of Lemma 2.4 can be improved by way of Lemma 2.5. To this end, let

$$\bar{\Psi}_r(t) := t^p + \bar{a}_r\, t^p \log(e+t)$$

with $\bar{a}_r = \fint_{B_r} a(x)\,dx$.

**Lemma 2.6.** *Under the same assumptions as Lemma 2.4, there exist an $R_1 = R_1\left(\|Du\|_{L^p(\Omega)}\right) > 0$ and $c = c(\texttt{data}) \ge 1$ such that*

$$\Big(\fint_{B_{r/2}} \Psi(x,|Du|)^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} \le c\bar{\Psi}_r\Big(\fint_{B_r} |Du|\,dx\Big),$$

*for any $r \in (0,R_1)$ with $B_r \Subset \Omega$.*

*Proof.* Employing the self-improving property of Lemma 2.5, we obtain

$$\Big(\fint_{B_{r/2}} \Big(\Psi\big(x,|Du|\big)\Big)^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} \le c\Big(\fint_{B_r} \Big(\Psi\big(x,|Du|\big)\Big)^{\frac{1}{p+1}}\,dx\Big)^{p+1}. \tag{2.5}$$

Note that

$$\begin{aligned}\Big(\Psi(x,|Du|)\Big)^{\frac{1}{p+1}} &= \Big(|Du|^p + \big(\bar{a}_r - \bar{a}_r + a(x)\big)\,\varphi_{\log}(|Du|)\Big)^{\frac{1}{p+1}} \\ &\le |Du|^{\frac{p}{p+1}} + \big(\bar{a}_r + \omega_a(r)\big)^{\frac{1}{p+1}}\varphi_{\log}^{\frac{1}{p+1}}(|Du|)\big).\end{aligned}$$

Since both $t^{\frac{p}{p+1}}$ and $\varphi_{\log}^{\frac{1}{p+1}}(t)$ are concave functions, by the Jensen inequality, we conclude that

$$\begin{aligned}&\Big(\fint_{B_r} \Big(\Psi\big(x,|Du|\big)\Big)^{\frac{1}{p+1}}\,dx\Big)^{p+1} \\ &\le c\Big(\fint_{B_r} |Du|^{\frac{p}{p+1}}\,dx\Big)^{p+1} + c\Big(\bar{a}_r + \omega_a(r)\Big)\Big(\fint_{B_r} \Big(\varphi_{\log}\big(|Du|\big)\Big)^{\frac{1}{p+1}}\,dx\Big)^{p+1} \\ &\le c\Big(\fint_{B_r} |Du|\,dx\Big)^p + c\Big(\bar{a}_r + \omega_a(r)\Big)\varphi_{\log}\Big(\fint_{B_r} |Du|\,dx\Big) \\ &\le c\bar{\Psi}_r\Big(\fint_{B_r} |Du|\,dx\Big) + c\omega_a(r)\,\varphi_{\log}\Big(\fint_{B_r} |Du|\,dx\Big).\end{aligned} \tag{2.6}$$

Let us take $R_1 = \frac{1}{e+\omega_n\|Du\|_{L^p(\Omega)}}$ with $\omega_n$ as the measure of the unit ball in $\mathbb{R}^n$. For $r \in (0,R_1)$, the second term of the right-hand side in (2.6) is estimated as follows:

$$\begin{aligned}\omega_a(r)\varphi_{\log}\Big(\fint_{B_r} |Du|\,dx\Big) &= \omega_a(r)\log\Big(e + \fint_{B_r} |Du|\,dx\Big)\Big(\fint_{B_r} |Du|\,dx\Big)^p \\ &\le \omega_a(r)\,\log\Big(\frac{e+\omega_n\|Du\|_{L^1(B_r)}}{r^n}\Big)\,\bar{\Psi}_r\Big(\fint_{B_r} |Du|\,dx\Big) \\ &\le \omega_a(r)\,\log\Big(\frac{1}{r^{n+1}}\Big)\,\bar{\Psi}_r\Big(\fint_{B_r} |Du|\,dx\Big) \\ &\le c\bar{\Psi}_r\Big(\fint_{B_r} |Du|\,dx\Big).\end{aligned} \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we obtain the desired estimate. □

## 3. Comparison estimates

In this section, we mainly perform the perturbation estimates between the original equation and a local frozen problem over a sequence of concentric balls. For $B_r = B_r(y) \Subset \Omega$ with $r \in (0, R_1)$ and $\theta \in \big(0, 1/4\big)$, we write

$$r_j = \theta^j r \quad \text{and} \quad B_j = B_{r_j}(y) \quad \text{for} \quad j = 0, 1, 2, \dots.$$

Moreover, we denote $\bar{a}_j$ and $\bar{\mathbb{A}}_j$ as the integral averages of $a(x)$ and $\mathbb{A}(x)$ over $B_j$ with

$$\bar{a}_j = \fint_{B_j} a(x)\,dx \quad \text{and} \quad \bar{\mathbb{A}}_j = \fint_{B_j} \mathbb{A}(x)\,dx,$$

and introduce the frozen double phase operator $\bar{\Psi}_j$ defined by

$$\bar{\Psi}_j\,(t) := t^p + \bar{a}_j\, t^p \log(e+t).$$

By (2.1) we readily see that $t\bar{\Psi}_j'(t) \approx \bar{\Psi}_j(t)$ for $t > 0$. First, we give two inequalities for $\bar{\Psi}_j$ which is useful for the further analysis.

**Lemma 3.1.** *For $\bar{\Psi}_j\,(t)$, we have the following results:*

(1) *For any $\xi_1, \xi_2 \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and $\mathbb{A}$ satisfying (1.2), we have*

$$\Big\langle \frac{\bar{\Psi}_j'(|\xi_1|_{\mathbb{A}})}{|\xi_1|_{\mathbb{A}}}\mathbb{A}\xi_1 - \frac{\bar{\Psi}_j'(|\xi_2|_{\mathbb{A}})}{|\xi_2|_{\mathbb{A}}}\mathbb{A}\xi_2, \xi_1 - \xi_2\Big\rangle \gtrsim \frac{\bar{\Psi}_j'\big(|\xi_1| + |\xi_2|\big)}{|\xi_1| + |\xi_2|}|\xi_1 - \xi_2|^2; \tag{3.1}$$

(2) *For any $\xi \in \mathbb{R}^m \setminus \{\mathbf{0}\}$, both $\mathbb{A}_1$ and $\mathbb{A}_2$ satisfying (1.2), it holds*

$$\Big|\frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_1})}{|\xi|_{\mathbb{A}_1}}\mathbb{A}_1\xi - \frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_2})}{|\xi|_{\mathbb{A}_2}}\mathbb{A}_2\xi\Big| \lesssim \bar{\Psi}_j'(|\xi|)\cdot|\mathbb{A}_1 - \mathbb{A}_2|; \tag{3.2}$$

*where the constants of the inequalities depend only on $p$ and $K$.*

*Proof.* We begin with the proof of (1). Without loss of generality, we assume that $|\xi_1|_{\mathbb{A}} \le |\xi_2|_{\mathbb{A}}$. By (2.3) we have that

$$\Big(\frac{|\xi_1|_{\mathbb{A}}}{|\xi_2|_{\mathbb{A}}}\Big)^p \varphi_{\log}'(|\xi_2|_{\mathbb{A}}) \le \varphi_{\log}'(|\xi_1|_{\mathbb{A}}) \le \Big(\frac{|\xi_1|_{\mathbb{A}}}{|\xi_2|_{\mathbb{A}}}\Big)^{p-1} \varphi_{\log}'(|\xi_2|_{\mathbb{A}}).$$

Combined with

$$\Big(\frac{|\xi_1|_{\mathbb{A}}}{|\xi_2|_{\mathbb{A}}}\Big)^p |\xi_2|_{\mathbb{A}}^{p-1} \le |\xi_1|_{\mathbb{A}}^{p-1} = \Big(\frac{|\xi_1|_{\mathbb{A}}}{|\xi_2|_{\mathbb{A}}}\Big)^{p-1} |\xi_2|_{\mathbb{A}}^{p-1},$$

we conclude that

$$\Big(\frac{|\xi_1|_{\mathbb{A}}}{|\xi_2|_{\mathbb{A}}}\Big)^p \bar{\Psi}_j'(|\xi_2|_{\mathbb{A}}) \le \bar{\Psi}_j'(|\xi_1|_{\mathbb{A}}) \le \Big(\frac{|\xi_1|_{\mathbb{A}}}{|\xi_2|_{\mathbb{A}}}\Big)^{p-1} \bar{\Psi}_j'(|\xi_2|_{\mathbb{A}}).$$

Thanks to the mean value theorem, there exists a $\theta \in (p, p+1)$ such that

$$\bar{\Psi}_j'(|\xi_1|_{\mathbb{A}}) = \Big(\frac{|\xi_1|_{\mathbb{A}}}{|\xi_2|_{\mathbb{A}}}\Big)^{\theta-1} \bar{\Psi}_j'(|\xi_2|_{\mathbb{A}}).$$

Then we have

$$\begin{aligned}
&\Big\langle \frac{\bar{\Psi}_j'(|\xi_1|_{\mathbb{A}})}{|\xi_1|_{\mathbb{A}}}\mathbb{A}\xi_1 - \frac{\bar{\Psi}_j'(|\xi_2|_{\mathbb{A}})}{|\xi_2|_{\mathbb{A}}}\mathbb{A}\xi_2, \xi_1 - \xi_2\Big\rangle \\
&= \Big\langle \frac{\bar{\Psi}_j'(|\xi_2|_{\mathbb{A}})}{|\xi_2|_{\mathbb{A}}^{\theta}}|\xi_1|_{\mathbb{A}}^{\theta-1}\mathbb{A}\xi_1 - \frac{\bar{\Psi}_j'(|\xi_2|_{\mathbb{A}})}{|\xi_2|_{\mathbb{A}}^{\theta}}|\xi_2|_{\mathbb{A}}^{\theta-1}\mathbb{A}\xi_2, \xi_1 - \xi_2\Big\rangle \\
&= \frac{\bar{\Psi}_j'(|\xi_2|_{\mathbb{A}})}{|\xi_2|_{\mathbb{A}}^{\theta}}\big\langle |\xi_1|_{\mathbb{A}}^{\theta-1}\mathbb{A}\xi_1 - |\xi_2|_{\mathbb{A}}^{\theta-1}\mathbb{A}\xi_2, \xi_1 - \xi_2\big\rangle.
\end{aligned} \tag{3.3}$$

On the other hand, it follows from $|\xi_2|_{\mathbb{A}} \le |\xi_2|_{\mathbb{A}} + |\xi_1|_{\mathbb{A}} \le 2|\xi_2|_{\mathbb{A}}$ and (1.2) that

$$\frac{\bar{\Psi}_j'(|\xi_2|_{\mathbb{A}})}{|\xi_2|_{\mathbb{A}}^{\theta}} \approx \frac{\bar{\Psi}_j'\big(|\xi_1|_{\mathbb{A}} + |\xi_2|_{\mathbb{A}}\big)}{\big(|\xi_1|_{\mathbb{A}} + |\xi_2|_{\mathbb{A}}\big)^{\theta}} \approx \frac{\bar{\Psi}_j'\big(|\xi_1| + |\xi_2|\big)}{\big(|\xi_1| + |\xi_2|\big)^{\theta}}. \tag{3.4}$$

Moreover, an algebraic inequality in [27, (2.27)] gives

$$\big\langle |\xi_1|_{\mathbb{A}}^{\theta-1}\mathbb{A}\xi_1 - |\xi_2|_{\mathbb{A}}^{\theta-1}\mathbb{A}\xi_2, \xi_1-\xi_2\big\rangle \gtrsim \Big(|\xi_1|+|\xi_2|\Big)^{\theta-1}|\xi_1-\xi_2|^2 \tag{3.5}$$

Combining (3.3), (3.4) (3.5) we obtain the desired result (1).

We now turn to the proof of (2). Without loss of generality, we assume $|\xi|_{\mathbb{A}_1} \le |\xi|_{\mathbb{A}_2}$. A direct calculation yields that

$$\begin{aligned}\Big|\frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_1})}{|\xi|_{\mathbb{A}_1}}\mathbb{A}_1\xi - \frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_2})}{|\xi|_{\mathbb{A}_2}}\mathbb{A}_2\xi\Big| &\lesssim \Big|\Big(\frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_1})}{|\xi|_{\mathbb{A}_1}} - \frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_2})}{|\xi|_{\mathbb{A}_2}}\Big)\mathbb{A}_1\xi + \frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_2})}{|\xi|_{\mathbb{A}_2}}(\mathbb{A}_1-\mathbb{A}_2)\xi\Big| \\ &\lesssim \Big|\frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_1})}{|\xi|_{\mathbb{A}_1}} - \frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_2})}{|\xi|_{\mathbb{A}_2}}\Big| \cdot |\xi| + \bar{\Psi}_j'(|\xi|)\cdot|\mathbb{A}_1-\mathbb{A}_2|.\end{aligned} \tag{3.6}$$

For the first term of the right-hand side, we employ the mean value theorem and $t\bar{\Psi}_j''(t) \approx \bar{\Psi}_j'(t)$ to obtain that there exists a $z \in (|\xi|_{\mathbb{A}_1}, |\xi|_{\mathbb{A}_2})$ such that

$$\Big|\frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_1})}{|\xi|_{\mathbb{A}_1}} - \frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_2})}{|\xi|_{\mathbb{A}_2}}\Big| = \Big|\frac{z\bar{\Psi}_j''(z)-\bar{\Psi}_j'(z)}{z^2}\Big|\cdot\big||\xi|_{\mathbb{A}_1}-|\xi|_{\mathbb{A}_2}\big| \lesssim \frac{\bar{\Psi}_j'(z)}{z^2}\big||\xi|_{\mathbb{A}_1}-|\xi|_{\mathbb{A}_2}\big|. \tag{3.7}$$

It follows from (1.2) that $|\xi|^2 \approx |\xi|_{\mathbb{A}_1}^2 \approx |\xi|_{\mathbb{A}_2}^2$, by $z \in (|\xi|_{\mathbb{A}_1}, |\xi|_{\mathbb{A}_2})$ which implies that

$$\begin{aligned}\frac{\bar{\Psi}_j'(z)}{z^2}\big||\xi|_{\mathbb{A}_1}-|\xi|_{\mathbb{A}_2}\big| &\approx \frac{\bar{\Psi}_j'(|\xi|)}{|\xi|^2}\frac{\big||\xi|_{\mathbb{A}_1}^2-|\xi|_{\mathbb{A}_2}^2\big|}{|\xi|_{\mathbb{A}_1}+|\xi|_{\mathbb{A}_2}} \\ &\lesssim \frac{\bar{\Psi}_j'(|\xi|)}{|\xi|^2}\frac{\big|\langle\mathbb{A}_1\xi,\xi\rangle-\langle\mathbb{A}_2\xi,\xi\rangle\big|}{|\xi|} \\ &\lesssim \frac{\bar{\Psi}_j'(|\xi|)}{|\xi|}\cdot\big|\mathbb{A}_1-\mathbb{A}_2\big|.\end{aligned} \tag{3.8}$$

Let us put (3.7) and (3.8) together, to obtain

$$\Big|\frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_1})}{|\xi|_{\mathbb{A}_1}} - \frac{\bar{\Psi}_j'(|\xi|_{\mathbb{A}_2})}{|\xi|_{\mathbb{A}_2}}\Big|\cdot|\xi| \lesssim \bar{\Psi}_j'(|\xi|)\cdot\big|\mathbb{A}_1-\mathbb{A}_2\big|. \tag{3.9}$$

We insert (3.9) into (3.6) to conclude the desired result (2). □

Let $v_j \in u + W_0^{1,1}\big(\frac12 B_j\big)$ be a minimizer of the local frozen functional

$$\mathcal{F}\big(v_j;\tfrac12 B_j\big) := \int_{\frac12 B_j}\bar{\Psi}_j\big(|Dv_j|_{\mathbb{A}}\big)\,dx. \tag{3.10}$$

Note that $Dv_j$ as a minimizer of the local frozen functional, which enjoys the higher integrability governed by the boundary data, see [10, Lemma 4.3] or [5, Theorem 3.4]. More precisely, there exists a constant $c = c(n,p,K) \ge 1$ such that

$$\Big(\fint_{\frac12 B_j}\Big(\bar{\Psi}_j\big(|Dv_j|\big)\Big)^{1+\varepsilon}dx\Big)^{\frac{1}{1+\varepsilon}} \le c\Big(\fint_{\frac12 B_j}\Big(\bar{\Psi}_j\big(|Du|\big)\Big)^{1+\varepsilon}dx\Big)^{\frac{1}{1+\varepsilon}} \tag{3.11}$$

with $\varepsilon$ as the same in Lemma 2.6.

Next we give a comparison estimate between $Du$ and $Dv_j$. To this end, we introduce an auxiliary vector field $V_{\bar{\Psi}_j}(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$V_{\bar{\Psi}_j}(\xi) := \Big(\frac{\bar{\Psi}_j'(|\xi|)}{|\xi|}\Big)^{1/2}\xi.$$

We immediately know that $|V_{\bar{\Psi}_j}(\xi)|^2 \approx \bar{\Psi}_j(|\xi|)$ for $\xi \in \mathbb{R}^n\setminus\{\mathbf{0}\}$; moreover, for any $\xi_1, \xi_2 \in \mathbb{R}^n\setminus\{\mathbf{0}\}$ it holds

$$\big|V_{\bar{\Psi}_j}(\xi_1) - V_{\bar{\Psi}_j}(\xi_2)\big|^2 \approx \frac{\bar{\Psi}_j'(|\xi_1|+|\xi_2|)}{|\xi_1|+|\xi_2|}|\xi_1-\xi_2|^2, \tag{3.12}$$

where the implied constant depends only on $p$, see [13, Lemma 2.4].

**Lemma 3.2.** *Let $u \in W^{1,1}(\Omega)$ be a local minimizer of the functional (1.1), $v_j \in u + W_0^{1,1}\big(\frac{1}{2}B_j\big)$ be a minimizer of the local frozen functional (3.10), and $\gamma = \frac{2(2+\varepsilon)}{\varepsilon}$. If $a(x)$ is log-Hölder continuous as shown in (2.4) and $\mathbb{A}(x)$ satisfies (1.2), then there exists a constant $c = c(\texttt{data}) \geq 1$ with*

$$\fint_{\frac{1}{2}B_j} \big|V_{\bar{\Psi}_j}(Du) - V_{\bar{\Psi}_j}(Dv_j)\big|^2\,dx \leq c\Big(\bar{\omega}_{a,\gamma}(r_j)\log(\tfrac{1}{r_j})\Big)^2 \bar{\Psi}_j\Big(\fint_{B_j}|Du|\,dx\Big).$$

*Proof.* From the Euler-Lagrange equations of functionals (1.1) and (3.10), respectively, we use $\phi = u - v_j \in W_0^{1,1}\big(\frac{1}{2}B_j\big)$ as the test function to have that

$$\int_{\frac{1}{2}B_j} \Big\langle \frac{\Psi'(x,|Du|_{\mathbb{A}})}{|Du|_{\mathbb{A}}}\mathbb{A}Du, Du - Dv_j\Big\rangle\,dx = 0,$$
$$\int_{\frac{1}{2}B_j} \Big\langle \frac{\bar{\Psi}_j'(|Dv_j|_{\mathbb{A}})}{|Dv_j|_{\mathbb{A}}}\mathbb{A}Dv_j, Du - Dv_j\Big\rangle\,dx = 0.$$

It follows from (3.1) of Lemma 3.1 that

$$\begin{aligned}
&\fint_{\frac{1}{2}B_j} \frac{\bar{\Psi}_j'(|Du|+|Dv_j|)}{|Du|+|Dv_j|}\,|Du - Dv_j|^2\,dx \\
&\leq c\fint_{\frac{1}{2}B_j} \Big\langle \frac{\bar{\Psi}_j'(|Du|_{\mathbb{A}})}{|Du|_{\mathbb{A}}}\mathbb{A}Du - \frac{\bar{\Psi}_j'(|Dv_j|_{\mathbb{A}})}{|Dv_j|_{\mathbb{A}}}\mathbb{A}Dv_j, Du - Dv_j\Big\rangle\,dx \\
&= c\fint_{\frac{1}{2}B_j} \Big\langle \frac{\bar{\Psi}_j'(|Du|_{\mathbb{A}})}{|Du|_{\mathbb{A}}}\mathbb{A}Du - \frac{\Psi'(x,|Du|_{\mathbb{A}})}{|Du|_{\mathbb{A}}}\mathbb{A}Du, Du - Dv_j\Big\rangle\,dx \\
&\leq c\fint_{\frac{1}{2}B_j} \big|a(x) - \bar{a}_j\big|\,\big|Du\big|^{p-1}\,\log\big(e + |Du|\big)\,\big|Du - Dv_j\big|\,dx.
\end{aligned}$$

By the Young inequality with $\epsilon > 0$ to be chosen later, we deduce that

$$\begin{aligned}
&\fint_{\frac{1}{2}B_j} \frac{\bar{\Psi}_j'(|Du|+|Dv_j|)}{|Du|+|Dv_j|}\,|Du - Dv_j|^2\,dx \\
&\leq \fint_{\frac{1}{2}B_j} |a(x) - \bar{a}_j|\,|Du|^{p-1}\,\log(e+|Du|)\,|Du - Dv_j|\,dx \\
&\leq c\fint_{\frac{1}{2}B_j} |a(x) - \bar{a}_j|\,\Big(|Du| + |Dv_j|\Big)^{\frac{p}{2}+\frac{p-2}{2}}\,\log(e+|Du|)\,|Du - Dv_j|\,dx \\
&\leq c_\epsilon\fint_{\frac{1}{2}B_j} |a(x) - \bar{a}_j|^2\Big(|Du| + |Dv_j|\Big)^p\Big(\log(e+|Du|)\Big)^2\,dx \\
&\quad + \epsilon\fint_{\frac{1}{2}B_j} \Big(|Du| + |Dv_j|\Big)^{p-2}\,|Du - Dv_j|^2\,dx \\
&= c_\epsilon\, I_1 + \epsilon I_2.
\end{aligned} \tag{3.13}$$

To estimate $I_1$, we employ the Hölder inequality and Lemma 2.1 to obtain that

$$\begin{aligned}
I_1 &\leq \Big(\fint_{\frac{1}{2}B_j} |a(x) - \bar{a}_j|^{\frac{2(2+\varepsilon)}{\varepsilon}}\,dx\Big)^{\frac{\varepsilon}{2+\varepsilon}}\Big(\fint_{\frac{1}{2}B_j}\Big(|Du| + |Dv_j|\Big)^{\frac{p(2+\varepsilon)}{2}}\Big(\log(e+|Du|)\Big)^{2+\varepsilon}\,dx\Big)^{\frac{2}{2+\varepsilon}} \\
&\leq c\big(\bar{\omega}_{a,\gamma}(r_j)\big)^2\Big(\Big(\log\Big(\frac{1}{r_j}\Big)\Big)^{2+\varepsilon}\Big(\fint_{\frac{1}{2}B_j}\Big(|Du| + |Dv_j|\Big)^{p(1+\varepsilon)}\,dx\Big)^{\frac{2+\varepsilon}{2(1+\varepsilon)}}\Big)^{\frac{2}{2+\varepsilon}} \\
&\leq c\Big(\bar{\omega}_{a,\gamma}(r_j)\log(\tfrac{1}{r_j})\Big)^2\Big(\fint_{\frac{1}{2}B_j}\Big(\bar{\Psi}_j\big(|Du|\big)\Big)^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} \\
&\leq c\Big(\bar{\omega}_{a,\gamma}(r_j)\log(\tfrac{1}{r_j})\Big)^2\,\bar{\Psi}_j\Big(\fint_{B_j}|Du|\,dx\Big),
\end{aligned}$$

where we used Lemma 2.6 in the last inequality. For $I_2$, it is obvious that

$$I_2 \le c \fint_{\frac12 B_j} \frac{\bar\Psi_j'(|Du|+|Dv_j|)}{|Du|+|Dv_j|}\,|Du-Dv_j|^2\,dx.$$

Let us now insert the estimates of $I_1$ and $I_2$ into (3.13), and pick $\epsilon$ small enough so as to absorb the term $\epsilon I_2$ into the left-hand side of (3.13), then we conclude the desired estimate from (3.12). □

In what follows, we introduce the local limiting frozen functional. Let $w_j \in v_j + W_0^{1,1}\big(\frac14 B_j\big)$ be a minimizer of the local functional

$$\mathcal{F}(w_j;\tfrac14 B_j) := \int_{\frac14 B_j} \bar\Psi_j\left(|Dw_j|_{\bar{\mathbb{A}}_j}\right) dx. \tag{3.14}$$

It is obvious that the minimizer $w_j$ also has the higher integrability governed by the boundary value. To be concrete, we have

$$\begin{aligned}\Big(\fint_{\frac14 B_j}\Big(\bar\Psi_j(|Dw_j|)\Big)^{1+\varepsilon}dx\Big)^{\frac{1}{1+\varepsilon}} &\lesssim \Big(\fint_{\frac14 B_j}\Big(\bar\Psi_j(|Dv_j|)\Big)^{1+\varepsilon}dx\Big)^{\frac{1}{1+\varepsilon}}\\ &\lesssim \Big(\fint_{\frac12 B_j}\Big(\bar\Psi_j(|Du|)\Big)^{1+\varepsilon}dx\Big)^{\frac{1}{1+\varepsilon}}\end{aligned} \tag{3.15}$$

where $\varepsilon$ is the same as in Lemma 2.6. Moreover, by considering $w_j$ as the minimizer of frozen functional with constant coefficients, it enjoys the local gradient Hölder continuity, for detail see [7, Lemma 4.2].

**Lemma 3.3.** *Let $w_j \in W^{1,\bar\Psi_j}(B_j)$ be a minimizer of the limiting frozen functional* (3.14) *with $\bar{\mathbb{A}}_j$ satisfying* (1.2). *Then there are positive constants $\alpha=\alpha(n,p)\in(0,1)$, $c_1=c_1(n,p)$ and $\tau_1=\tau_1(n,p)$ such that*

$$\operatorname{osc}_{\tau B_j} Dw_j \le c_1\tau^\alpha \fint_{B_j} |Dw_j - \langle Dw_j\rangle_{B_j}|\,dx \quad \textit{for any } \tau\in(0,\tau_1). \tag{3.16}$$

We are now in a position to present the comparison estimate between $Dv_j$ and $Dw_j$. More precisely, we have the following lemma.

**Lemma 3.4.** *Let $v_j \in u + W_0^{1,1}\big(\frac12 B_j\big)$ be a minimizer of local functional* (3.10), *$w_j \in v_j + W_0^{1,1}\big(\frac14 B_j\big)$ be a minimizer of limiting functional*(3.14), *and $\gamma = \frac{2(2+\varepsilon)}{\varepsilon}$. If $\mathbb{A}(x)$ satisfies* (1.2), *then there exists a constant $c=c(\texttt{data})\ge 1$ with*

$$\fint_{\frac14 B_j} \left|V_{\bar\Psi_j}(Dv_j) - V_{\bar\Psi_j}(Dw_j)\right|^2 dx \le c\big(\bar\omega_{\mathbb{A},\gamma}(r_j)\big)^2\,\bar\Psi_j\Big(\fint_{B_j}|Du|\,dx\Big).$$

*Proof.* We use $\phi = v_j - w_j \in W_0^{1,1}\big(\frac14 B_j\big)$ as the test function to the Euler-Lagrange equations of functionals (3.10) and (3.14), respectively, to conclude that

$$\int_{\frac14 B_j} \Big\langle \frac{\bar\Psi_j'(|Dv_j|_{\mathbb{A}})}{|Dv_j|_{\mathbb{A}}}\mathbb{A}Dv_j, Dv_j - Dw_j\Big\rangle\,dx = 0,$$

$$\int_{\frac14 B_j} \Big\langle \frac{\bar\Psi_j'(|Dw_j|_{\bar{\mathbb{A}}_j})}{|Dw_j|_{\bar{\mathbb{A}}_j}}\bar{\mathbb{A}}_j Dw_j, Dv_j - Dw_j\Big\rangle\,dx = 0.$$

By (3.1) we have that

$$\begin{aligned}&\fint_{\frac14 B_j} \frac{\bar\Psi_j'(|Dv_j|+|Dw_j|)}{|Dv_j|+|Dw_j|}\,|Dv_j-Dw_j|^2\,dx\\ &\le c\fint_{\frac14 B_j}\Big\langle \frac{\bar\Psi_j'(|Dv_j|_{\bar{\mathbb{A}}_j})}{|Dv_j|_{\bar{\mathbb{A}}_j}}\bar{\mathbb{A}}_j Dv_j - \frac{\bar\Psi_j'(|Dw_j|_{\bar{\mathbb{A}}_j})}{|Dw_j|_{\bar{\mathbb{A}}_j}}\bar{\mathbb{A}}_j Dw_j, Dv_j - Dw_j\Big\rangle\,dx\\ &= c\fint_{\frac14 B_j}\Big\langle \frac{\bar\Psi_j'(|Dv_j|_{\bar{\mathbb{A}}_j})}{|Dv_j|_{\bar{\mathbb{A}}_j}}\bar{\mathbb{A}}_j Dv_j - \frac{\bar\Psi_j'(|Dv_j|_{\mathbb{A}})}{|Dv_j|_{\mathbb{A}}}\mathbb{A} Dv_j, Dv_j - Dw_j\Big\rangle\,dx.\end{aligned}$$

By (3.2), $t\bar{\Psi}_j'(t) \approx \bar{\Psi}_j(t)$ and the Young inequality, we obtain that

$$\begin{aligned}
&\fint_{\frac14 B_j} \frac{\bar{\Psi}_j'(|Dv_j|+|Dw_j|)}{|Dv_j|+|Dw_j|}\,|Dv_j-Dw_j|^2\,dx\\
&\le c\fint_{\frac14 B_j} \bar{\Psi}_j'\big(|Dv_j|+|Dw_j|\big)\,|\bar{\mathbb{A}}_j-\mathbb{A}|\,|Dv_j-Dw_j|\,dx\\
&\le c\fint_{\frac14 B_j} |\bar{\mathbb{A}}_j-\mathbb{A}|\Big(\bar{\Psi}_j\big(|Dv_j|+|Dw_j|\big)\Big)^{1/2}\Big(\frac{\bar{\Psi}_j'\big(|Dv_j|+|Dw_j|\big)}{|Dv_j|+|Dw_j|}\Big)^{1/2}|Dv_j-Dw_j|\,dx \qquad (3.17)\\
&\le c\fint_{\frac14 B_j} |\bar{\mathbb{A}}_j-\mathbb{A}|^2\,\Big(\bar{\Psi}_j(|Dv_j|)+\bar{\Psi}_j(|Dw_j|)\Big)\,dx\\
&\quad+\frac14\fint_{\frac14 B_j} \frac{\bar{\Psi}_j'\big(|Dv_j|+|Dw_j|\big)}{|Dv_j|+|Dw_j|}\,|Dv_j-Dw_j|^2\,dx.
\end{aligned}$$

To estimate the first term of the right-hand side, we employ the Hölder inequality, (3.15) and (3.11) to show that

$$\begin{aligned}
&\fint_{\frac14 B_j} |\bar{\mathbb{A}}_j-\mathbb{A}|^2\,\Big(\bar{\Psi}_j(|Dv_j|)+\bar{\Psi}_j(|Dw_j|)\Big)\,dx\\
&\le c\Big(\fint_{\frac14 B_j} |\bar{\mathbb{A}}_j-\mathbb{A}|^{\frac{2(2+\varepsilon)}{\varepsilon}}\,dx\Big)^{\frac{\varepsilon}{2+\varepsilon}}\,\Big(\fint_{\frac14 B_j}\Big(\bar{\Psi}_j(|Dv_j|)\Big)^{\frac{2+\varepsilon}{2}}\,dx\Big)^{\frac{2}{2+\varepsilon}}\\
&\quad+c\Big(\fint_{\frac14 B_j} |\bar{\mathbb{A}}_j-\mathbb{A}|^{\frac{2(2+\varepsilon)}{\varepsilon}}\,dx\Big)^{\frac{\varepsilon}{2+\varepsilon}}\,\Big(\fint_{\frac14 B_j}\Big(\bar{\Psi}_j(|Dw_j|)\Big)^{\frac{2+\varepsilon}{2}}\,dx\Big)^{\frac{2}{2+\varepsilon}} \qquad (3.18)\\
&\le c\big(\bar{\omega}_{\mathbb{A},\gamma}(r_j)\big)^2\,\Big(\fint_{\frac14 B_j}\Big(\bar{\Psi}_j(|Du|)\Big)^{\frac{2+\varepsilon}{2}}\,dx\Big)^{\frac{2}{2+\varepsilon}}\\
&\le c\big(\bar{\omega}_{\mathbb{A},\gamma}(r_j)\big)^2\,\bar{\Psi}_j\Big(\fint_{Bj}|Du|\,dx\Big),
\end{aligned}$$

where we used Lemma 2.6 in the last step. Putting (3.18) into (3.17), we obtain the desired estimate in accordance with (3.12). ∎

With Lemmas 3.2 and 3.4 in hand, we devote the following to the comparison estimate between $Du$ and $Dw_j$ in the $L^1$ senses. Let

$$\Upsilon(\rho)=\bar{\omega}_{\mathbb{A},\gamma}(\rho)+\bar{\omega}_{a,\gamma}(\rho)\log(\tfrac{1}{\rho}).$$

**Lemma 3.5.** *Let $u\in W^{1,1}(\Omega)$ be a local minimizer of the functional (1.1), and $w_j\in v_j+W_0^{1,1}\big(\frac14 B_j\big)$ be a minimizer of the limit functional (3.14). If $a(x)$ is log-Hölder continuous as (2.4) and $\mathbb{A}(x)$ satisfies (1.2), then there exist a constant $c_2=c_2\,(\texttt{data})\ge 1$ such that*

$$\fint_{\frac14 B_j}|Du-Dw_j|\,dx\le c_2\big(\Upsilon(r_j)\big)^{\frac{1}{p+1}}\fint_{B_j}|Du|\,dx.$$

*Proof.* Note that the Young inequality yields

$$\begin{aligned}
|Du-Dw_j|&=\Big(\Upsilon(r_j)\,\big(|Du|+|Dw_j|\big)\Big)^{1/2}\cdot\Big(\Upsilon(r_j)\,\big(|Du|+|Dw_j|\big)\Big)^{-1/2}|Du-Dw_j|\\
&\le\frac12\Big(\Upsilon(r_j)\,\big(|Du|+|Dw_j|\big)+\Big(\Upsilon(r_j)\,\big(|Du|+|Dw_j|\big)\Big)^{-1}\,|Du-Dw_j|^2\Big).
\end{aligned}$$

By $\bar{\Psi}_j(t) \approx t\,\bar{\Psi}_j'(t)$, then we conclude that

$$\begin{aligned}
&\bar{\Psi}_j\big(|Du - Dw_j|\big)\\
&\le c\bar{\Psi}_j'\big(|Du| + |Dw_j|\big)\,|Du - Dw_j|\\
&\le c\bar{\Psi}_j'\big(|Du| + |Dw_j|\big)\Big(\Upsilon(r_j)\,\big(|Du| + |Dw_j|\big) + \Big(\Upsilon(r_j)\,\big(|Du| + |Dw_j|\big)\Big)^{-1}|Du - Dw_j|^2\Big)\\
&= c\Big(\Upsilon(r_j)\bar{\Psi}_j\big(|Du| + |Dw_j|\big) + \big(\Upsilon(r_j)\big)^{-1}\frac{\bar{\Psi}_j'\big(|Du| + |Dw_j|\big)}{|Du| + |Dw_j|}|Du - Dw_j|^2\Big)\,.
\end{aligned}\tag{3.19}$$

For the second term of the right-hand side, by (3.12), Lemmas 3.2 and 3.4 we deduce that

$$\begin{aligned}
&\fint_{\frac14 B_j}\frac{\bar{\Psi}_j'(|Du| + |Dw_j|)}{|Du| + |Dw_j|}\,|Du - Dw_j|^2\,dx\\
&\le c\fint_{\frac14 B_j}\big|V_{\bar{\Psi}_j}(Du) - V_{\bar{\Psi}_j}(Dw_j)\big|^2\,dx\\
&\le c\fint_{\frac14 B_j}\big|V_{\bar{\Psi}_j}(Du) - V_{\bar{\Psi}_j}(Dv_j)\big|^2\,dx + c\fint_{\frac14 B_j}\big|V_{\bar{\Psi}_j}(Dv_j) - V_{\bar{\Psi}_j}(Dw_j)\big|^2\,dx\\
&\le c\big(\Upsilon(r_j)\big)^2\,\bar{\Psi}_j\Big(\fint_{B_j}|Du|\,dx\Big).
\end{aligned}\tag{3.20}$$

We integrate (3.19) over $\frac14 B_j$, and use (3.15) (3.20) to obtain that

$$\begin{aligned}
\fint_{\frac14 B_j}\bar{\Psi}_j\big(|Du - Dw_j|\big)\,dx &\le c\Upsilon(r_j)\fint_{\frac14 B_j}\bar{\Psi}_j\big(|Du| + |Dw_j|\big)\,dx\\
&\quad + c\big(\Upsilon(r_j)\big)^{-1}\fint_{\frac14 B_j}\frac{\bar{\Psi}_j'\big(|Du| + |Dw_j|\big)}{|Du| + |Dw_j|}|Du - Dw_j|^2\,dx\\
&\le c\Upsilon(r_j)\fint_{\frac12 B_j}\bar{\Psi}_j\big(|Du|\big)\,dx + c\Upsilon(r_j)\,\bar{\Psi}_j\,\Big(\fint_{B_j}|Du|\,dx\Big)\\
&\le c\Upsilon(r_j)\,\bar{\Psi}_j\,\Big(\fint_{B_j}|Du|\,dx\Big),
\end{aligned}\tag{3.21}$$

We used Lemma 2.6 in the last inequality. Since $\Upsilon(r_j) \le c$ due to the boundness of $\mathbb{A}$ and (2.3), we employ the Jensen inequality, (3.21) and (2.2) to have

$$\begin{aligned}
\bar{\Psi}_j\Big(\fint_{\frac14 B_j}|Du - Dw_j|\,dx\Big) &\le \fint_{\frac14 B_j}\bar{\Psi}_j\big(|Du - Dw_j|\big)\,dx\\
&\le c\Upsilon(r_j)\,\bar{\Psi}_j\,\Big(\fint_{B_j}|Du|\,dx\Big)\\
&\le \bar{\Psi}_j\,\Big(c\big(\Upsilon(r_j)\big)^{\frac{1}{p+1}}\fint_{B_j}|Du|\,dx\Big)\,,
\end{aligned}$$

which yields the desired estimate in accordance with the non-decreasing property of $\bar{\Psi}_j$. □

On the basis of Lemma 3.5, we give a decay estimate of the energy functional.

**Lemma 3.6.** *Let $u \in W^{1,1}(\Omega)$ be a local minimizer of the functional* (1.1)*. If $a(x)$ is log-Hölder continuous as shown in* (2.4) *and $\mathbb{A}(x)$ satisfies* (1.2)*, then for any $\theta \in (0, \min\{\frac14, \tau_1\})$ there exists $c_3 = c_3\,(\texttt{data}) \ge 1$ such that*

$$\fint_{B_{j+1}}|Du - \langle Du\rangle_{B_{j+1}}|\,dx \le c_3\Big(\theta^{\alpha}\fint_{B_j}|Du - \langle Du\rangle_{B_j}|\,dx + \theta^{-n}\big(\Upsilon(r_j)\big)^{\frac{1}{p+1}}\fint_{B_j}|Du|\,dx\Big)\,.$$

*Proof.* Note that

$$\begin{aligned} &\fint_{B_{j+1}} |Du - \langle Du\rangle_{B_{j+1}}|\,dx \\ &\le 2\fint_{B_{j+1}} |Du - \langle Dw_j\rangle_{B_{j+1}}|\,dx \\ &\le 2\fint_{B_{j+1}} |Du - Dw_j|\,dx + 2\fint_{B_{j+1}} |Dw_j - \langle Dw_j\rangle_{B_{j+1}}|\,dx \\ &\le c\theta^{-n}\fint_{\frac14 B_j} |Du - Dw_j|\,dx + c\theta^{\alpha}\fint_{\frac14 B_j} |Dw_j - \langle Dw_j\rangle_{\frac14 B_j}|\,dx \\ &\le \theta^{-n}\fint_{\frac14 B_j} |Du - Dw_j|\,dx + c\theta^{\alpha}\Big(\fint_{\frac14 B_j} |Dw_j - Du|\,dx + \fint_{\frac14 B_j} |Du - \langle Du\rangle_{\frac14 B_j}|\,dx\Big), \end{aligned} \tag{3.22}$$

where we used Lemma 3.3 in the third inequality. By confining $0 < \theta < \frac14$ we obtain that

$$\begin{aligned} \fint_{B_{j+1}} |Du - \langle Du\rangle_{B_{j+1}}|\,dx &\le c\theta^{-n}\fint_{\frac14 B_j} |Du - Dw_j|\,dx + c\theta^{\alpha}\fint_{\frac14 B_j} |Du - \langle Du\rangle_{\frac14 B_j}|\,dx \\ &\le c\theta^{-n}\big(\Upsilon(r_j)\big)^{\frac{1}{p+1}}\fint_{B_j} |Du|\,dx + c\theta^{\alpha}\fint_{B_j} |Du - \langle Du\rangle_{B_j}|\,dx, \end{aligned}$$

where we used Lemma 3.5 in the last inequality, and we complete the proof. □

In the sequel, we show a comparison estimate between $Du$ and $Dw_j$ so as to be used to prove the iterating decay of $Du$.

**Lemma 3.7.** *Let $u \in W^{1,1}(\Omega)$ be a local minimizer of the functional* (1.1)*, and $w_j \in v_j + W_0^{1,1}\big(\frac14 B_j\big)$ be a minimizer of the limit functional* (3.14)*. Assume that $a(x)$ is log-Hölder continuous as shown in* (2.4) *and $\mathbb{A}(x)$ satisfies* (1.2)*. If for $\lambda > 0$ and $\Lambda \ge 1$ one has*

$$\frac{\lambda}{\Lambda} \le \inf_{\frac14 B_{j+1}} |Dw_j| \quad \textit{and} \quad \fint_{B_j} |Du|\,dx \le \lambda, \tag{3.23}$$

*then there exists $c = c(\texttt{data}) \ge 1$ such that*

$$\fint_{\frac14 B_{j+1}} |Du - Dw_j|\,dx \le c\theta^{-n}\Lambda^p\,\Upsilon(r_j)\,\lambda.$$

*Proof.* By (3.15) and Lemma 2.6, we have

$$\bar\Psi_j\Big(\fint_{\frac14 B_j} |Dw_j|\,dx\Big) \le \fint_{\frac14 B_j} \bar\Psi_j\left(|Dw_j|\right)dx \le c\bar\Psi_j\Big(\fint_{B_j} |Du|\,dx\Big),$$

which follows from (2.2) that

$$\fint_{\frac14 B_j} |Dw_j|\,dx \le c\fint_{B_j} |Du|\,dx\,. \tag{3.24}$$

Therefore, by the Young inequality and $0 < \theta < \frac14$ we obtain that

$$\begin{aligned} \fint_{\frac14 B_{j+1}} |Du - Dw_j|\,dx &\le \Upsilon(r_j)\fint_{\frac14 B_{j+1}} \Big(|Du| + |Dw_j|\Big)\,dx + \frac{1}{\Upsilon(r_j)}\fint_{\frac14 B_{j+1}} \frac{|Du - Dw_j|^2}{|Du| + |Dw_j|}\,dx \\ &\le c\theta^{-n}\Upsilon(r_j)\fint_{B_j} |Du|\,dx + \frac{1}{\Upsilon(r_j)}\fint_{\frac14 B_{j+1}} \frac{|Du - Dw_j|^2}{|Du| + |Dw_j|}\,dx \\ &\le c\theta^{-n}\Upsilon(r_j)\lambda + \frac{1}{\Upsilon(r_j)}\fint_{\frac14 B_{j+1}} \frac{|Du - Dw_j|^2}{|Du| + |Dw_j|}\,dx. \end{aligned} \tag{3.25}$$

To estimate the second term of the right-hand side, we notice that

$$1 \le \frac{\bar{\Psi}_j'(|Dw_j|)}{\bar{\Psi}_j'(\lambda/\Lambda)} \le \frac{\bar{\Psi}_j'(|Du|+|Dw_j|)}{\Lambda^{-p}\bar{\Psi}_j'(\lambda)}$$

in $\frac{1}{4}B_{j+1}$ because of (3.23) and (2.3). Then we employ (3.20) to obtain that

$$\begin{aligned}
\fint_{\frac{1}{4}B_{j+1}} \frac{|Du-Dw_j|^2}{|Du|+|Dw_j|}\,dx &\le \fint_{\frac{1}{4}B_{j+1}} \frac{\bar{\Psi}_j'(|Du|+|Dw_j|)}{\Lambda^{-p}\bar{\Psi}_j'(\lambda)} \frac{|Du-Dw_j|^2}{|Du|+|Dw_j|}\,dx \\
&\le \frac{\theta^{-n}\Lambda^p}{\bar{\Psi}_j'(\lambda)} \fint_{\frac{1}{4}B_j} \frac{\bar{\Psi}_j'(|Du|+|Dw_j|)}{|Du|+|Dw_j|}|Du-Dw_j|^2\,dx \\
&\le \frac{c\theta^{-n}\Lambda^p}{\bar{\Psi}_j'(\lambda)} \big(\Upsilon(r_j)\big)^2 \bar{\Psi}_j\Big(\fint_{B_j}|Du|\,dx\Big) \\
&\le \frac{c\theta^{-n}\Lambda^p}{\bar{\Psi}_j'(\lambda)} \big(\Upsilon(r_j)\big)^2 \bar{\Psi}_j(\lambda) \\
&\le c\theta^{-n}\Lambda^p(\Upsilon(r_j))^2\lambda,
\end{aligned} \tag{3.26}$$

where we used $\bar{\Psi}_j(\lambda) \approx \lambda\bar{\Psi}_j'(\lambda)$ in the last step. By inserting (3.26) into (3.25) we complete the proof. □

With Lemma 3.7, we are able to show the iterating decay associated to $Du$

**Lemma 3.8.** *Let $u \in W^{1,1}(\Omega)$ be a local minimizer of the functional (1.1). Assume that $a(x)$ is log-Hölder continuous as (2.4) and $\mathbb{A}(x)$ satisfies (1.2). For a given $\Lambda \ge 1$, if there is a positive constant $\theta_1 = \theta_1(\texttt{data}, \Lambda)$, such that for any $\theta \in (0, \theta_1)$ and $\lambda > 0$ it holds*

$$\frac{\lambda}{\Lambda} \le \fint_{B_{j+2}} |Du|\,dx\,, \quad \fint_{B_j} |Du|\,dx \le \lambda \quad \text{and} \quad \Upsilon(r_j) \le \Big(\frac{\theta^{3n}}{c_2\Lambda}\Big)^{p+1}, \tag{3.27}$$

*then there exists a constant $c_1^* = c_1^*(\texttt{data}, \theta, \Lambda) \ge 1$ such that*

$$\fint_{B_{j+2}} |Du - \langle Du\rangle_{B_{j+2}}|\,dx \le c_4\ \theta^\alpha \fint_{B_{j+1}} |Du - \langle Du\rangle_{B_{j+1}}|\,dx + c_1^*\Upsilon(r_j)\,\lambda$$

*with $c_4$ being independent of $\theta$.*

*Proof.* Let us use Lemma 3.5 to obtain that

$$\begin{aligned}
\fint_{B_{j+2}} |Du - Dw_j|\,dx &\le \theta^{-2n} \fint_{\frac{1}{4}B_j} |Du-Dw_j|\,dx \\
&\le c_2\theta^{-2n}\big(\Upsilon(r_j)\big)^{\frac{1}{p+1}} \fint_{B_j} |Du|\,dx \\
&\le \frac{\lambda}{2\Lambda}
\end{aligned}$$

with the last inequality because of (3.27). Therefore, by $\fint_{B_{j+2}} |Du|\,dx \ge \frac{\lambda}{\Lambda}$ as in (3.27) we conclude that

$$\begin{aligned}
\sup_{\frac{1}{4}B_{j+1}} |Dw_j| &\ge \fint_{B_{j+2}} |Dw_j|\,dx \\
&\ge \fint_{B_{j+2}} |Du|\,dx - \fint_{B_{j+2}} |Du-Dw_j|\,dx \\
&\ge \frac{\lambda}{\Lambda} - \fint_{B_{j+2}} |Du-Dw_j|\,dx \\
&\ge \frac{\lambda}{2\Lambda}.
\end{aligned} \tag{3.28}$$

On the other hand, let us confine $0<\theta_1\le\tau_1$ and $c\theta_1^{\alpha}\le\frac{1}{4\Lambda}$ with $\tau_1$ as in Lemma 3.3. For any $0<\theta\le\theta_1$, then by (3.16) and (3.24) we obtain that

$$\Big|\operatorname*{osc}_{\frac14 B_{j+1}} Dw_j\Big|\le c\theta^{\alpha}\fint_{\frac14 B_j}|Dw_j|\,dx\le c\theta^{\alpha}\fint_{B_j}|Du|\,dx\le c\theta^{\alpha}\lambda\le\frac{\lambda}{4\Lambda},\tag{3.29}$$

where we used $\fint_{B_j}|Du|\,dx\le\lambda$ in (3.27) in the last second inequality.

Let us put (3.28) and (3.29) together, to obtain

$$\inf_{\frac14 B_{j+1}}|Dw_j|\ge\sup_{\frac14 B_{j+1}}|Dw_j|-\Big|\operatorname*{osc}_{\frac14 B_{j+1}}Dw_j\Big|\ge\frac{\lambda}{2\Lambda}-\frac{\lambda}{4\Lambda}=\frac{\lambda}{4\Lambda}.\tag{3.30}$$

According to the validity of (3.27) and (3.30), we use Lemma 3.7 to obtain

$$\fint_{\frac14 B_{j+1}}|Du-Dw_j|\,dx\le c\theta^{-n}\Lambda^{p}\,\Upsilon(r_j)\,\lambda.$$

Finally, we employ the same approach as in (3.22), to conclude that

$$\begin{aligned}\fint_{B_{j+2}}|Du-\langle Du\rangle_{B_{j+2}}|\,dx&\le c\theta^{-n}\fint_{\frac14 B_{j+1}}|Du-Dw_j|\,dx+c\theta^{\alpha}\fint_{\frac14 B_{j+1}}|Du-\langle Du\rangle_{\frac14 B_{j+1}}|\,dx\\&\le c\theta^{-2n}\Lambda^{p}\,\Upsilon(r_j)\,\lambda+c\theta^{\alpha}\fint_{\frac14 B_{j+1}}|Du-\langle Du\rangle_{\frac14 B_{j+1}}|\,dx,\end{aligned}$$

which yields the desired estimate. □

## 4. Proof of main result

In this section, we prove the continuity of gradient to the local minimizers of the functional (1.1) by imposing DMO condition on $a(x)$ and $\mathbb{A}(x)$ as in (1.3). First, we show a control relation by an integral to the DMO assumption on $a(x)$ and $\mathbb{A}(x)$ in (1.3).

**Lemma 4.1.** *Let $r_j=\theta^j r$ for $\theta\in\big(0,1/4\big)$ and $j=0,1,2,\dots$. If both $a(x)$ and $\mathbb{A}(x)$ are imposed on the DMO assumption as* (1.3)*, then there exists $c_2^*=c_2^*(\texttt{data},\theta)\ge1$ such that*

$$\sum_{j=0}^{\infty}\Upsilon(r_j)\le c_2^*\int_0^{2r}\Upsilon(\rho)\,\frac{d\rho}{\rho}.\tag{4.1}$$

*Proof.* A direct calculation yields that

$$\begin{aligned}\int_0^{2r}\Upsilon(\rho)\,\frac{d\rho}{\rho}&=\sum_{j=0}^{\infty}\int_{r_{j+1}}^{r_j}\bar\omega_{a,\gamma}(\rho)\log\big(\frac1\rho\big)\,\frac{d\rho}{\rho}+\int_r^{2r}\bar\omega_{a,\gamma}(\rho)\log\big(\frac1\rho\big)\,\frac{d\rho}{\rho}\\&\quad+\sum_{j=0}^{\infty}\int_{r_{j+1}}^{r_j}\bar\omega_{\mathbb{A},\gamma}(\rho)\,\frac{d\rho}{\rho}+\int_r^{2r}\bar\omega_{\mathbb{A},\gamma}(\rho)\,\frac{d\rho}{\rho}\\&:=I_1+I_2+I_3+I_4.\end{aligned}\tag{4.2}$$

For $I_1$, let $\rho\in(r_{j+1},r_j)$, and for any $y\in\Omega$ we obtain that

$$\begin{aligned}\Big(\fint_{B_{j+1}(y)\cap\Omega}|a-\langle a\rangle_{B_{j+1}\cap\Omega}|^{\gamma}\,dx\Big)^{1/\gamma}&\le c\Big(\fint_{B_{j+1}(y)\cap\Omega}|a-\langle a\rangle_{B_\rho\cap\Omega}|^{\gamma}\,dx\Big)^{1/\gamma}\\&\le c\Big(\frac{1}{r_{j+1}^{n}}\int_{B_\rho(y)\cap\Omega}|a-\langle a\rangle_{B_\rho\cap\Omega}|^{\gamma}\,dx\Big)^{1/\gamma}\\&\le c\theta^{\frac{-n}{\gamma}}\Big(\frac{1}{\rho^{n}}\int_{B_\rho(y)\cap\Omega}|a-\langle a\rangle_{B_\rho\cap\Omega}|^{\gamma}\,dx\Big)^{1/\gamma},\end{aligned}$$

which follows from Definition of $\bar\omega_{a,\gamma}$ that

$$\bar\omega_{a,\gamma}(r_{j+1})\le c\theta^{\frac{-n}{\gamma}}\bar\omega_{a,\gamma}(\rho).\tag{4.3}$$

Then we obtain that

$$\begin{aligned}\sum_{j=0}^{\infty}\int_{r_{j+1}}^{r_j}\bar{\omega}_{a,\gamma}(\rho)\log\big(\frac{1}{\rho}\big)\,\frac{d\rho}{\rho} &\geq c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{a,\gamma}(r_{j+1})\log\big(\frac{1}{r_j}\big)\int_{r_{j+1}}^{r_j}\frac{d\rho}{\rho}\\ &= c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{a,\gamma}(r_{j+1})\log\big(\frac{1}{r_j}\big)\log\big(\frac{1}{\theta}\big).\end{aligned}\tag{4.4}$$

Since $0<\theta<1/4$, a simple calculation yields that

$$\log\big(\frac{1}{r_{j+1}}\big)=\log\big(\frac{1}{\theta}\big)+\log\big(\frac{1}{r_j}\big)\leq 2\log\big(\frac{1}{\theta}\big)\log\big(\frac{1}{r_j}\big).\tag{4.5}$$

Together with (4.4) and (4.5), we arrive at our first claim

$$\sum_{j=0}^{\infty}\int_{r_{j+1}}^{r_j}\bar{\omega}_{a,\gamma}(\rho)\log\big(\frac{1}{\rho}\big)\,\frac{d\rho}{\rho}\geq c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{a,\gamma}(r_{j+1})\log\big(\frac{1}{r_{j+1}}\big)\,.\tag{4.6}$$

For $I_2$, by the same argument as (4.3), we see $\bar{\omega}_{a,\gamma}(r)\leq c\bar{\omega}_{a,\gamma}(\rho)$ for $\rho\in(r,2r)$, which implies that

$$\begin{aligned}\int_{r}^{2r}\bar{\omega}_{a,\gamma}(\rho)\log\big(\frac{1}{\rho}\big)\,\frac{d\rho}{\rho} &\geq c\bar{\omega}_{a,\gamma}(r)\log\Big(\frac{1}{2r}\Big)\int_{r}^{2r}\frac{d\rho}{\rho}\\ &\geq c\bar{\omega}_{a,\gamma}(r)\log\big(\frac{1}{r}\big).\end{aligned}\tag{4.7}$$

For $I_3$, we also have $\bar{\omega}_{\mathbb{A},\gamma}(r_{j+1})\leq c\theta^{\frac{-n}{\gamma}}\bar{\omega}_{\mathbb{A},\gamma}(\rho)$ for $\rho\in(r_{j+1},r_j)$, which implies that

$$\begin{aligned}\sum_{j=0}^{\infty}\int_{r_{j+1}}^{r_j}\bar{\omega}_{\mathbb{A},\gamma}(\rho)\,\frac{d\rho}{\rho} &\geq c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{\mathbb{A},\gamma}(r_{j+1})\int_{r_{j+1}}^{r_j}\frac{d\rho}{\rho}\\ &= c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{\mathbb{A},\gamma}(r_{j+1})\log\big(\frac{1}{\theta}\big)\\ &\geq c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{\mathbb{A},\gamma}(r_{j+1}).\end{aligned}\tag{4.8}$$

For $I_4$, we obtain that $\bar{\omega}_{\mathbb{A},\gamma}(r)\leq c\bar{\omega}_{\mathbb{A},\gamma}(\rho)$ for $\rho\in(r,2r)$, and it yields

$$\int_{r}^{2r}\bar{\omega}_{\mathbb{A},\gamma}(\rho)\,\frac{d\rho}{\rho}\geq c\bar{\omega}_{\mathbb{A},\gamma}(r).\tag{4.9}$$

Finally, let us put (4.6) (4.7) (4.8) and (4.9) into (4.2), to attain that

$$\begin{aligned}&\int_0^{2r}\Upsilon(\rho)\frac{d\rho}{\rho}\\ &\geq c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{a,\gamma}(r_{j+1})\log\big(\frac{1}{r_{j+1}}\big)+c\bar{\omega}_{a,\gamma}(r)\log\big(\frac{1}{r}\big)+c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\bar{\omega}_{\mathbb{A},\gamma}(r_{j+1})+c\bar{\omega}_{\mathbb{A},\gamma}(r)\\ &\geq c\theta^{\frac{n}{\gamma}}\sum_{j=0}^{\infty}\Upsilon(r_j),\end{aligned}$$

which completes the proof. □

We now show the local gradient boundedness to the local minimizer of the functional (1.1).

**Theorem 4.2.** *Let $u\in W^{1,1}(\Omega)$ be a local minimizer of the functional* (1.1). *Assume that $a(x)$ and $\mathbb{A}(x)$ satisfy* (1.3) *and* (1.2). *Then there exist $c=c(\texttt{data})\geq 1$ and $R=R(\texttt{data})>0$ such that for $r\in(0,R)$ with $B_{2r}\Subset\Omega$ one has*

$$\|Du\|_{L^\infty(B_r)}\leq c\fint_{B_{2r}}|Du|\,dx.\tag{4.10}$$

*Proof.* Let $y \in B_r$ be the Lebesgue point of $Du$. To specify some constants, we let $\Lambda = 100$, and

$$\theta = \min\Big\{\theta_1, \Big(\frac{1}{200c_3^2}\Big)^{\frac{1}{2\alpha}}, \Big(\frac{1}{4c_4}\Big)^{1/\alpha}\Big\}, \tag{4.11}$$

where $\theta_1 = \theta_1(\mathtt{data}, \Lambda)$ as in Lemma 3.8. By Lemma 4.1 we choose $R > 0$ sufficiently small such that for any $r \in (0, R)$ one has

$$\sum_{j=0}^{\infty} \Upsilon(r_j) \le c_2^* \int_0^{2R} \Upsilon(\rho)\,\frac{d\rho}{\rho} \le \min\Big\{\Big(\frac{\theta^{3n}}{100c_2}\Big)^{p+1}, \Big(\frac{\theta^{4n}}{400c_3}\Big)^{p+1}, \frac{\theta^{2n}}{20c_1^*}\Big\}. \tag{4.12}$$

Let

$$\lambda := 200\theta^{-2n} \fint_{B_r} |Du|\,dx,$$

and

$$C_j := \fint_{B_j} |Du|\,dx + \theta^{-2n} \fint_{B_j} |Du - \langle Du\rangle_{B_j}|\,dx. \tag{4.13}$$

We prove our main result using the exit-time method. To this end, let us define

$$\mathbb{J} := \big\{j \in \mathbb{N} \cup \{0\} : C_j \le \frac{\lambda}{50}\big\},$$

We see that $\mathbb{J} \neq \emptyset$ because $\{0\} \in \mathbb{J}$. First, if $\mathbb{J}$ is infinite, then we can find an infinite subsequence of $\{C_j\}$, denoted by $\{C_i\}$, such that $C_i \le \frac{\lambda}{50}$ for all $i \in \mathbb{N} \cup \{0\}$. Let $i \to \infty$, by the Lebesgue differentiation theorem we obtain that

$$|Du(y)| \le \liminf_{i\to\infty} \fint_{B_i(y)} |Du|\,dx \le \liminf_{i\to\infty} C_i \le \frac{\lambda}{50} \le c \fint_{B_r(y)} |Du|\,dx, \tag{4.14}$$

which makes (4.10) true.

Otherwise, we next consider that $\mathbb{J}$ is finite. Let $j_m = \max \mathbb{J}$ be the exit time, by the definition of $\mathbb{J}$ we see that

$$C_{j_m} \le \frac{\lambda}{50} \quad \text{and} \quad C_j > \frac{\lambda}{50} \quad \text{for } j \ge j_m + 1. \tag{4.15}$$

We set

$$a_j = |\langle Du\rangle_{B_j}| \quad \text{and} \quad E_j = \fint_{B_j} |Du - \langle Du\rangle_{B_j}|\,dx.$$

In what follows, we use the method of induction to show that

$$a_j + E_j \le \lambda \quad \text{for } j \ge j_m. \tag{4.16}$$

In the case of $j = j_m$, it is obvious that $a_j + E_j \le C_{j_m} \le \frac{\lambda}{50} \le \lambda$. Assume $a_j + E_j \le \lambda$ for all $j \in \{j_m, \dots, k\}$, we prove that $a_{k+1} + E_{k+1} \le \lambda$. To this end, by the elementary calculations we have that

$$\begin{aligned} a_{k+1} &= \big|\langle Du\rangle_{B_{k+1}} - \langle Du\rangle_{B_k} + \langle Du\rangle_{B_k} - \cdots - \langle Du\rangle_{B_{j_m}} + \langle Du\rangle_{B_{j_m}}\big| \\ &\le \fint_{B_{k+1}} |Du - \langle Du\rangle_{B_k}|\,dx + \cdots + \fint_{B_{j_m+1}} |Du - \langle Du\rangle_{B_{j_m}}|\,dx + \fint_{B_{j_m}} |Du|\,dx \\ &\le \theta^{-n} E_k + \cdots + \theta^{-n} E_{j_m+1} + C_{j_m} \\ &= \theta^{-n} \sum_{j=j_m+1}^{k} E_j + C_{j_m}. \end{aligned} \tag{4.17}$$

To estimate $\theta^{-n}\sum_{j=j_m+1}^{k} E_j$, let us verify Condition (3.27) of Lemma 3.8. Applying Lemma 3.6 to $E_{j+2}$ repeatedly, together with $c_3^2\theta^{2\alpha} \le \frac{1}{200}$ in (4.11) we obtain that

$$\begin{aligned}
&E_{j+2}\\
&\le (c_3\theta)^{\alpha(j+2-j_m)}E_{j_m} + c_3\theta^{-n}\sum_{i=j_m}^{j+1}\big(\Upsilon(r_i)\big)^{\frac{1}{p+1}} \fint_{B_i}|Du|\,dx\\
&\le (c_3\theta)^{2\alpha}E_{j_m} + c_3\theta^{-n}\Big(\sum_{i=j_m}^{j}\big(\Upsilon(r_i)\big)^{\frac{1}{p+1}} \fint_{B_i}|Du|\,dx\Big) + c_3\theta^{-n}\big(\Upsilon(r_{j+1})\big)^{\frac{1}{p+1}} \fint_{B_{j+1}}|Du|\,dx\\
&\le \frac{E_{j_m}}{200} + c_3\theta^{-n}\Big(\sum_{i=j_m}^{j}\big(\Upsilon(r_i)\big)^{\frac{1}{p+1}} \fint_{B_i}|Du|\,dx\Big) + c_3\theta^{-2n}\big(\Upsilon(r_{j+1})\big)^{\frac{1}{p+1}} \fint_{B_j}|Du|\,dx\\
&\le \frac{E_{j_m}}{200} + c_3\theta^{-2n}\sum_{i=j_m}^{j}\Big(\big(\Upsilon(r_i)\big)^{\frac{1}{p+1}} + \big(\Upsilon(r_{i+1})\big)^{\frac{1}{p+1}}\Big)\fint_{B_i}|Du|\,dx.
\end{aligned} \tag{4.18}$$

Since

$$E_{j_m} \le \theta^{2n}C_{j_m} \le \theta^{2n}\frac{\lambda}{50} \tag{4.19}$$

from (4.13) and (4.15). Moreover,

$$\begin{aligned}
&c_3\theta^{-2n}\sum_{i=j_m}^{j}\Big(\big(\Upsilon(r_i)\big)^{\frac{1}{p+1}} + \big(\Upsilon(r_{i+1})\big)^{\frac{1}{p+1}}\Big)\\
&\le 2c_3\theta^{-2n}\sum_{i=0}^{\infty}\big(\Upsilon(r_i)\big)^{\frac{1}{p+1}} \le 2c_3\theta^{-2n}\Big(\sum_{i=0}^{\infty}\Upsilon(r_i)\Big)^{\frac{1}{p+1}} \le \frac{\theta^{2n}}{200}
\end{aligned} \tag{4.20}$$

because of (4.12);

$$\fint_{B_i}|Du|\,dx \le a_i + E_i \le \lambda \quad \text{for } i \in \{j_m,\dots,k\} \tag{4.21}$$

from the assumption of induction. Then we put (4.19) (4.20) (4.21) into (4.18) to derive that

$$E_{j+2} \le \frac{\theta^{2n}\lambda}{200} + \frac{\theta^{2n}\lambda}{200} \le \frac{\theta^{2n}\lambda}{100} \quad \text{for } j \in \{j_m,\dots,k\},$$

which follows from $C_{j+2} > \frac{\lambda}{50}$ that

$$\fint_{B_{j+2}}|Du|\,dx = C_{j+2} - \theta^{-2n}E_{j+2} \ge \frac{\lambda}{50} - \frac{\lambda}{100} \ge \frac{\lambda}{100} \quad \text{for } j \in \{j_m,\dots,k\}. \tag{4.22}$$

Inequalities (4.22) (4.21) and (4.12) ensure the validity of Condition (3.27), then we employ Lemma 3.8 to obtain

$$E_{j+2} \le \frac{1}{4}E_{j+1} + c_1^*\Upsilon(r_j)\,\lambda \quad \text{for } j \in \{j_m,\dots,k\},$$

where we used $c_4\theta^{\alpha} \le 1/4$ in (4.11). This implies that

$$\sum_{j=j_m+2}^{k+2}E_j \le \frac{1}{4}\sum_{j=j_m+1}^{k+1}E_j + c_1^*\,\lambda\sum_{j=j_m}^{k}\Upsilon(r_j).$$

We add $E_{j_m+1}$ to both sides, and use $C_{j_m} \le \frac{\lambda}{50}$ in (4.15) and $c_1^*\sum_{j=0}^{\infty}\Upsilon(r_j) \le \frac{\theta^n}{20}$ in (4.12) to obtain that

$$\begin{aligned}
\sum_{j=j_m+1}^{k+2}E_j &\le \frac{1}{4}\sum_{j=j_m+1}^{k+1}E_j + E_{j_m+1} + c_1^*\,\lambda\sum_{j=j_m}^{k}\Upsilon(r_j)\\
&\le \frac{1}{4}\sum_{j=j_m+1}^{k+2}E_j + 2\theta^{n}\,C_{j_m} + c_1^*\,\lambda\sum_{j=j_m}^{k}\Upsilon(r_j)
\end{aligned}$$

$$\le \frac{1}{4}\sum_{j=j_m+1}^{k+2} E_j + \frac{\lambda}{20}\theta^n + \frac{\lambda}{20}\theta^n$$

Thus,

$$\sum_{j=j_m+1}^{k+2} E_j \le \frac{\lambda}{4}\theta^n, \tag{4.23}$$

which implies that $E_{k+1} \le \frac{\lambda}{2}$. Let us now put (4.23) into (4.17) to deduce that

$$a_{k+1} \le \frac{\lambda}{4} + C_{j_m} \le \frac{\lambda}{4} + \frac{\lambda}{50} \le \frac{\lambda}{2}.$$

This makes (4.16) true for any $j \ge j_m$, and it follows from the Lebesgue differentiation theorem that

$$|Du(y)| \le \liminf_{j\to\infty} \fint_{B_j(y)} |Du|\,dx \le \liminf_{j\to\infty} \Big(a_j + E_j\Big) \le \lambda \le c \fint_{B_r(y)} |Du|\,dx, \tag{4.24}$$

which completes the proof. □

*Proof of Theorem 1.2.* Let $\Omega_0 \Subset \Omega' \Subset \Omega$ and $\mathrm{dist}(\Omega_0,\Omega') = \mathrm{dist}(\Omega',\Omega) = \frac{1}{2}\mathrm{dist}(\Omega_0,\Omega)$. By Theorem 4.2 we see that $Du$ is locally bounded in $\Omega$, and denote

$$\lambda = \|Du\|_{L^\infty(\Omega')} + 1. \tag{4.25}$$

It suffices to prove that $\langle Du\rangle_{B_r(y)}$ for fixed point $y \in \Omega_0$ is a Cauchy sequence in $r$, which means that for any $\epsilon > 0$ there is a radius $R_\epsilon \le \mathrm{dist}\{\Omega_0,\Omega\}/8$ such that for any $\rho_1,\rho_2 \in (0,R_\epsilon]$ one has

$$|\langle Du\rangle_{B_{\rho_1}(y)} - \langle Du\rangle_{B_{\rho_2}(y)}| \le \frac{\epsilon\lambda}{3}. \tag{4.26}$$

Let $\rho_1 = R_\epsilon, \rho_2 \to 0$ we obtain

$$|\langle Du\rangle_{B_{R_\epsilon}(y)} - Du(y)| \le \frac{\epsilon\lambda}{3}.$$

In accordance with the absolute continuity of the integral in $B_r(y)$, we see that $y \mapsto \langle Du\rangle_{B_{R_\epsilon}(y)}$ is continuous, and find a small $\tilde R_\epsilon \in (0,R_\epsilon)$ such that for $|x-y| < \tilde R_\epsilon$ one has

$$|\langle Du\rangle_{B_{R_\epsilon}(x)} - \langle Du\rangle_{B_{R_\epsilon}(y)}| \le \frac{\epsilon\lambda}{3},$$

which leads to

$$\begin{aligned}|Du(x) - Du(y)| &\le |Du(x) - \langle Du\rangle_{B_{R_\epsilon}(x)}| + |\langle Du\rangle_{B_{R_\epsilon}(x)} - \langle Du\rangle_{B_{R_\epsilon}(y)}| + |\langle Du\rangle_{B_{R_\epsilon}(y)} - Du(y)|\\ &\le \epsilon\lambda.\end{aligned}$$

To prove $\langle Du\rangle_{B_r(y)}$ is a Cauchy sequence, we divide this into two steps as follows.

**Step 1.** VMO type estimate of $Du$. It suffices to prove that for any $\epsilon \in (0,1)$, there exists an $r_\epsilon \in (0,1/4)$ such that for $i \ge 2$ one has

$$\fint_{B_{\theta^i r_\epsilon}} |Du - \langle Du\rangle_{B_{\theta^i r_\epsilon}}|\,dx \le \epsilon\lambda. \tag{4.27}$$

Let $\Lambda = \frac{200}{\epsilon}$ and $\theta_1 = \theta_1(\mathtt{data},\Lambda)$ as in Lemma 3.8. We choose

$$\theta = \min\Big\{\theta_1, \Big(\frac{\epsilon}{4c_4}\Big)^{1/\alpha}\Big\}. \tag{4.28}$$

By Lemma 4.1 we can take $r_\epsilon$ sufficient small such that

$$\sum_{i=0}^{\infty} \Upsilon(r_i) \le c_2^* \int_0^{2r_\epsilon} \Upsilon(\rho)\,\frac{d\rho}{\rho} \le \min\Big\{\Big(\frac{\epsilon\theta^{3n}}{200c_2}\Big)^{p+1}, \frac{\theta^{2n}\epsilon}{200c_1^*}\Big\}. \tag{4.29}$$

We set $r_i = \theta^i r_\epsilon$ and $B_i = B_{r_i}$ for $i = 0,1,2,\dots$, and

$$\mathbb{I} := \big\{2 \le i \in \mathbb{N} : \fint_{B_i} |Du|\,dx < \frac{\epsilon\lambda}{200}\big\}. \tag{4.30}$$

If $i \in \mathbb{I}$, then we obtain that

$$\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx \le 2\fint_{B_i} |Du|\,dx \le \frac{2\epsilon\lambda}{200} \le \epsilon\lambda.$$

Otherwise, if $i \notin \mathbb{I}$, by (4.30) and (4.25) we see that

$$\frac{\epsilon\lambda}{200} \le \fint_{B_i} |Du|\,dx \quad\text{and}\quad \fint_{B_{i-2}} |Du|\,dx \le \lambda,$$

which combined with (4.29) just meets Condition (3.27), then we employ Lemma 3.8 to obtain

$$\begin{aligned}\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx &\le c_4\theta^{\alpha}\fint_{B_{i-1}} |Du - \langle Du\rangle_{B_{i-1}}|\,dx + c_1^*\Upsilon(r_{i-2})\lambda\\ &\le 2c_4\,\theta^{\alpha}\lambda + c_1^*\Upsilon(r_{i-2})\,\lambda.\end{aligned}$$

Since $2c_4\theta^{\alpha} \le \frac{\epsilon}{2}$ is confined in (4.28) and $c_1^*\Upsilon(r_{i-2}) \le \epsilon$ in (4.29), we obtain that (4.27).

**Step 2.** Cauchy sequence of $\langle Du\rangle_{B_k}$ for $k \ge 3$. More precisely, we prove that

$$\big|\langle Du\rangle_{B_k} - \langle Du\rangle_{B_h}\big| \le \frac{\epsilon\lambda}{10} \quad\text{for } 3 \le k < h \text{ with } k,h \in \mathbb{N}. \tag{4.31}$$

By the VMO type estimate of $Du$, it yields that there exists a new $r_\epsilon \in (0, 1/4)$ such that for any $i \ge 2$ it holds

$$\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx \le \frac{\theta^n\epsilon}{200}\lambda. \tag{4.32}$$

We denote

$$i_m := \begin{cases}\min\mathbb{I}, & \text{for } \mathbb{I} \ne \emptyset,\\ \infty, & \text{for } \mathbb{I} = \emptyset,\end{cases}$$

with $\mathbb{I}$ as in (4.30). Let us consider the proof of (4.31) in three cases.

**Case 1.** $k < h \le i_m$. By (4.30) and (4.25) we see that

$$\frac{\epsilon\lambda}{200} \le \fint_{B_i} |Du|\,dx \quad\text{and}\quad \fint_{B_{i-2}} |Du|\,dx \le \lambda \quad\text{for } i \in \{k, \dots, h-1\}.$$

Employing Lemma 3.8, together with $2c_4\theta^{\alpha} \le \frac{1}{2}$ in (4.28), we obtain that

$$\begin{aligned}&\sum_{i=k}^{h-1}\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx\\ &\le c_4\theta^{\alpha}\sum_{i=k-1}^{h-2}\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx + c_1^*\,\lambda\sum_{i=k-2}^{h-3}\Upsilon(r_i)\\ &\le \frac{1}{2}\sum_{i=k}^{h-1}\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx + \frac{1}{2}\fint_{B_{k-1}} |Du - \langle Du\rangle_{B_{k-1}}|\,dx + c_1^*\,\lambda\sum_{i=k-2}^{h-3}\Upsilon(r_i),\end{aligned}$$

which yields

$$\sum_{i=k}^{h-1}\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx \le \fint_{B_{k-1}} |Du - \langle Du\rangle_{B_{k-1}}|\,dx + 2c_1^*\,\lambda\sum_{i=k-2}^{h-3}\Upsilon(r_i).$$

We use (4.32) and (4.29) to obtain

$$\sum_{i=k}^{h-1}\fint_{B_i} |Du - \langle Du\rangle_{B_i}|\,dx \le \frac{\theta^n\epsilon}{200}\lambda + \frac{\theta^n\epsilon}{80}\lambda \le \frac{\theta^n\epsilon}{40}\lambda.$$

Therefore,

$$\big|\langle Du\rangle_{B_k} - \langle Du\rangle_{B_h}\big| \le \sum_{i=k}^{h-1}\big|\langle Du\rangle_{B_i} - \langle Du\rangle_{B_{i+1}}\big|$$

$$\le \theta^{-n} \sum_{i=k}^{h-1} \fint_{B_i} |Du - \langle Du \rangle_{B_i}| \, dx \le \frac{\epsilon\lambda}{40}.$$

**Case 2.** $i_m \le k < h$ for $i_m < \infty$, we claim that

$$\big|\langle Du \rangle_{B_k}\big| \le \frac{\epsilon\lambda}{20} \quad \text{and} \quad \big|\langle Du \rangle_{B_h}\big| \le \frac{\epsilon\lambda}{20}, \tag{4.33}$$

which obviously leads to (4.31). It suffices to prove the first inequality of (4.33) in the following, since the second one can be obtained by the same way. In fact, if $k \in \mathbb{I}$, it is true obviously. If $k \notin \mathbb{I}$, we can find $k_0 \in \mathbb{I}$ satisfying $i_m \le k_0 < k$ and $\{k_0+1, \dots, k\} \cap \mathbb{I} = \emptyset$. In particular, we have

$$\frac{\epsilon\lambda}{200} \le \fint_{B_i} |Du| \, dx \quad \text{and} \quad \fint_{B_{i-2}} |Du| \, dx \le \lambda \quad \text{for } i \in \{k_0+1, \dots, k\}.$$

By Lemma 3.8 with $2c_4\theta^{\alpha} \le \frac{1}{2}$ as in (4.28) we obtain that

$$\begin{aligned}
&\sum_{i=k_0}^{k} \fint_{B_i} |Du - \langle Du \rangle_{B_i}| \, dx \\
&= \sum_{i=k_0+1}^{k} \fint_{B_i} |Du - \langle Du \rangle_{B_i}| \, dx + \fint_{B_{k_0}} |Du - \langle Du \rangle_{B_{k_0}}| \, dx \\
&\le c_4\theta^{\alpha} \sum_{i=k_0}^{k-1} \fint_{B_i} |Du - \langle Du \rangle_{B_i}| \, dx + c_1^* \lambda \sum_{i=k_0-1}^{k-2} \Upsilon(r_i) + \fint_{B_{k_0}} |Du - \langle Du \rangle_{B_{k_0}}| \, dx \\
&\le \frac{1}{2} \sum_{i=k_0}^{k} \fint_{B_i} |Du - \langle Du \rangle_{B_i}| \, dx + \frac{\theta^n \epsilon}{200}\lambda + \frac{\theta^n \epsilon}{200}\lambda,
\end{aligned}$$

where we used (4.29) and (4.32) in the last inequality. Hence

$$\sum_{i=k_0}^{k} \fint_{B_i} |Du - \langle Du \rangle_{B_i}| \, dx \le \frac{\theta^n \epsilon}{40}\lambda,$$

and

$$\begin{aligned}
\big|\langle Du \rangle_{B_k}\big| &\le \big|\langle Du \rangle_{B_{k_0}} - \langle Du \rangle_{B_k}\big| + \big|\langle Du \rangle_{B_{k_0}}\big| \\
&\le \sum_{i=k_0}^{k-1} \big|\langle Du \rangle_{B_i} - \langle Du \rangle_{B_{i+1}}\big| + \fint_{B_{k_0}} |Du| \, dx \\
&\le \theta^{-n} \sum_{i=k_0}^{k} \fint_{B_i} |Du - \langle Du \rangle_{B_i}| \, dx + \frac{\epsilon\lambda}{200} \le \frac{\epsilon\lambda}{20}.
\end{aligned}$$

**Case 3.** $k < i_m < h$ for $i_m < \infty$. From Case 2 with $h > i_m$ we obtain

$$\big|\langle Du \rangle_{B_h}\big| \le \frac{\epsilon\lambda}{20},$$

and by Case 1 with $k < i_m$ one has

$$\big|\langle Du \rangle_{B_k} - \langle Du \rangle_{B_{i_m}}\big| \le \frac{\epsilon\lambda}{40}.$$

Then we conclude that

$$\big|\langle Du \rangle_{B_k}\big| \le \big|\langle Du \rangle_{B_k} - \langle Du \rangle_{B_{i_m}}\big| + \big|\langle Du \rangle_{B_{i_m}}\big| \le \frac{\epsilon\lambda}{40} + \frac{\epsilon\lambda}{200} \le \frac{\epsilon\lambda}{20},$$

which yields (4.31) as Case 2. By combining **Cases 1-3**, we obtain that $\{\langle Du \rangle_{B_k}\}$ is a Cauchy sequence.

Finally, we prove (4.26) in the continuous way with respect to the radius $\rho$. For any $0 < \rho_1 < \rho_2 < \theta^3 r_\epsilon$, there exist two integers $k, h \geq 3$ such that $\theta^{k+1} r_\epsilon \leq \rho_1 < \theta^k r_\epsilon$ and $\theta^{h+1} r_\epsilon \leq \rho_2 < \theta^h r_\epsilon$. By (4.32) it implies that

$$\begin{aligned}\big|\langle Du\rangle_{B_k} - \langle Du\rangle_{B_{\rho_1}}\big| &\leq \fint_{B_{\rho_1}} \big|Du - \langle Du\rangle_{B_k}\big|\,dx \\ &\leq \Big(\frac{\theta^k r_\epsilon}{\rho_1}\Big)^n \fint_{B_k} |Du - \langle Du\rangle_{B_k}|\,dx \\ &\leq \theta^{-n}\cdot\frac{\theta^n\epsilon}{200}\lambda = \frac{\epsilon\lambda}{200}.\end{aligned}$$

By the same reasoning, we also have

$$\big|\langle Du\rangle_{B_h} - \langle Du\rangle_{B_{\rho_2}}\big| \leq \frac{\epsilon\lambda}{200}.$$

It follows from (4.31) that

$$\begin{aligned}\big|\langle Du\rangle_{B_{\rho_1}} - \langle Du\rangle_{B_{\rho_2}}\big| &\leq \big|\langle Du\rangle_{B_k} - \langle Du\rangle_{B_h}\big| + \big|\langle Du\rangle_{B_{\rho_1}} - \langle Du\rangle_{B_k}\big| + \big|\langle Du\rangle_{B_h} - \langle Du\rangle_{B_{\rho_2}}\big| \\ &\leq \frac{\epsilon\lambda}{10} + \frac{\epsilon\lambda}{200} + \frac{\epsilon\lambda}{200} \leq \frac{\epsilon\lambda}{5}.\end{aligned}$$

Let $R_\epsilon = \theta^3 r_\epsilon$, then (4.26) holds for $\rho_1, \rho_2 \in (0, R_\epsilon)$ and $y \in \Omega_0$. This completes the proof. □

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## References


[1] R. Arora, Á. Crespo-Blanco, P. Winkert; On logarithmic double phase problems, *J. Differ. Equ.*, **433** (2025), 113247.

[2] P. Baroni; A new condition ensuring gradient continuity for minimizers of non-autonomous functionals with mild phase transition, *Calc. Var. Partial Differ. Equ.*, **64** (2025), 112.

[3] P. Baroni, M. Colombo, G. Mingione; Non-autonomous functionals, borderline cases and related function classes, *St. Petersburg Math. J.*, **27** (2016), 347–379.

[4] P. Baroni, M. Colombo, G. Mingione; Regularity for general functionals with double phase, *Calc. Var. Partial Differ. Equ.*, **57** (2018), 1–48.

[5] P. Baroni, A. Coscia; Gradient regularity for non-autonomous functionals with Dini or non-Dini continuous coefficients, *Electron. J. Differ. Equ.*, **2022** (2022), 80.

[6] S. S. Byun, Y. Youn; Riesz potential estimates for a class of double phase problems, *J. Differ. Equ.*, **264** (2018), 1263–1316.

[7] I. Chlebicka, M. Kim, M. Weidner; Gradient Riesz potential estimates for a general class of measure data quasilinear systems, *Adv. Calc. Var.*, 2026. doi:10.1515/acv-2025-008.

[8] M. Colombo, G. Mingione; Regularity for double phase variational problems, *Arch. Ration. Mech. Anal.*, **215** (2015), 443–496.

[9] M. Colombo, G. Mingione; Bounded minimisers of double phase variational integrals, *Arch. Ration. Mech. Anal.*, **218** (2015), 219–273.

[10] C. De Filippis; On the regularity of the $\omega$-minima of $\varphi$-functionals, *Nonlinear Anal.*, **194** (2020), 111464.

[11] C. De Filippis, G. Mingione; Regularity for double phase problems at nearly linear growth, *Arch. Ration. Mech. Anal.*, **247** (2023), 85.

[12] F. Duzaar, G. Mingione; Gradient continuity estimates, *Calc. Var. Partial Differ. Equ.*, **39** (2010), 379–418.

[13] L. Diening, B. Stroffolini, A. Verde; Everywhere regularity of functionals with $\varphi$-growth, *Manuscr. Math.*, **129** (2009), 449–481.

[14] H. Dong, S. Kim; On $C^1$, $C^2$, and weak type-$(1,1)$ estimates for linear elliptic operators, *Commun. Partial Differ. Equ.*, **42** (2017), 417–435.

[15] H. Dong, J. Lee, S. Kim; On conormal and oblique derivative problem for elliptic equations with Dini mean oscillation coefficients, *Indiana Univ. Math. J.*, **69** (2020), 1815–1853.

[16] H. Dong, H. Kwon; Spatial $C^1$, $C^2$, and Schauder estimates for nonstationary Stokes equations with Dini mean oscillation coefficients, *Calc. Var. Partial Differ. Equ.*, **65** (2026), 43.

[17] Z. Feng, J. Zhang, S. Zheng, Weighted gradient estimates to nonlinearelliptic equations of p(x) -growth with measure data, *J. Math. Pures Appl.*, **203** (2025), Paper No. 103756.

[18] E. Giusti; *Direct methods in the calculus of variations*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2003.

[19] Q. Huang; Estimates on the generalized Morrey spaces $L_\phi^{2,\lambda}$ and $BMO_\phi$ for linear elliptic systems, *Indiana Univ. Math. J.*, **45** (1996), 397–439.
[20] T. Jin, V. Maz'ya, J. Van Schaftingen; Pathological solutions to elliptic problems in divergence form with continuous coefficients, *C. R. Math.*, **347** (2009), 773–778.
[21] T. Kuusi, G. Mingione; Vectorial nonlinear potential theory, *J. Eur. Math. Soc.*, **20** (2018), 929–964.
[22] Y. Li; On the $C^1$ regularity of solutions to divergence form elliptic systems with Dini-continuous coefficients, *Chin. Ann. Math. Ser. B*, **38** (2017), 489–496.
[23] G. M. Lieberman; Hölder continuity of the gradient of solutions of uniformly parabolic equations with conormal boundary conditions, *Ann. Mat. Pura Appl.*, **148** (1987), 77–99.
[24] H. Lv, S. Zheng, Z. Feng; Existence results for nonlinear Schrödinger equations involving the fractional $(p, q)$-Laplacian and critical nonlinearities, *Electron. J. Differ. Equ.* **2021** (2021), 1–24.
[25] P. Marcellini; Regularity of minimizers of integrals of the calculus of variations with non-standard growth conditions, *Arch. Ration. Mech. Anal.*, **105** (1989), 267–284.
[26] P. Marcellini; Regularity and existence of solutions of elliptic equations with $p, q$-growth conditions, *J. Differ. Equ.*, **90** (1991), 1–30.
[27] M. Misawa; Local Hölder regularity of gradients for evolutional $p$-Laplacian systems, *Annali di Matematica Pura ed Applicata*, **181** (2002), 389–405.
[28] J. Ok; Gradient continuity for $p(\cdot)$-Laplace systems, *Nonlinear Anal.*, **141** (2016), 139–166.
[29] M. Ragusa, A. Tachikawa; Regularity of minimizers for double phase functionals of borderline case with variable exponents, *Adv. Nonlinear Anal.*, **13** (2024), 20240017.
[30] S. Spanne; Some function spaces defined using the mean oscillation over cubes, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **19** (1965), 593–608.
[31] J. Zhang, S. Zheng, Z. Feng, Gradient estimates of general nonlinear singular elliptic equations with measure data, *J. Differ. Equ.*, **372** (2023), 402-457.
[32] V. V. Zhikov; Averaging of functionals of the calculus of variations and elasticity theory, *Izv. Akad. Nauk SSSR Ser. Mat.*, **29** (1987), 33.

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