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EXISTENCE OF SOLUTIONS TO PERTURBED CRITICAL BIHARMONIC EQUATIONS WITH HARDY POTENTIAL

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Abstract. This article studies a perturbed critical biharmonic equation with Hardy potential. The main challenge arises from the combined effects of critical Sobolev nonlinearity and singular Hardy potential, which induce a double loss of compactness in the variational framework. Through delicate analysis of fibering maps and the mountain pass lemma, the existence of at least one nontrivial mountain pass solution is obtained under appropriate growth conditions on the nonlinearity.

1. Introduction

This article concerns the biharmonic elliptic problem

$$\begin{gathered} \Delta^2 u - \lambda \frac{|u|^{q-2}u}{|x|^s} = |u|^{2^{**}-2}u + f(x,u), \quad x \in \Omega, \\ u = \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ is a bounded domain containing 0, with smooth boundary $\partial\Omega$, $2 \leq q \leq 2^{**}(s) := \frac{2(N-s)}{N-4} < 2^{**} := \frac{2N}{N-4}$ for $0 < s \leq 4$, $2 \leq q < 2^{**}$ for $s = 0$, λ is a positive parameter, Δ^2 is the biharmonic operator, and $f(x,t)$: $\overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and is odd with respect to t which satisfies the following two assumptions:

- (A1) $\lim_{t\to 0^+} \frac{f(x,t)}{t} = 0$ and $\lim_{t\to+\infty} \frac{f(x,t)}{t^{2^{**}-1}} = 0$ uniformly in $x \in \Omega$;
- (A2) there exists a constant $\rho \in (q, 2^{**})$ such that $0 < \rho F(x,t) \leq t f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}\backslash\{0\}$, where $F(x,t) = \int_0^t f(x,\tau)\mathrm{d}\tau$. This condition is usually referred to as the Ambrosetti-Rabinowitz superlinear condition.

From (A1), we observe that, for any $\varepsilon > 0$, there exist $C_i(\varepsilon) > 0, (i = 1, 2, 3)$ such that

$$|f(x,t)| \leq \varepsilon t^{2^{**}-1} + C_1(\varepsilon)t, \quad x \in \Omega,\ t \in \mathbb{R}, \tag{1.2}$$

$$|f(x,t)| \leq \varepsilon t + C_2(\varepsilon)t^{2^{**}-1}, \quad x \in \Omega,\ t \in \mathbb{R}, \tag{1.3}$$

$$|F(x,t)| \leq \frac{1}{2}\varepsilon t^2 + C_3(\varepsilon)t^{2^{**}}, \quad x \in \Omega,\ t \in \mathbb{R}. \tag{1.4}$$

As a consequence of (A2), one sees that there exists a constant $C > 0$ such that

$$F(x,t) \geq C|t|^\rho, \quad x \in \Omega,\ t \in \mathbb{R}\,. \tag{1.5}$$

It is worth mentioning that there are many interesting works on nonlinear elliptic problems with critical exponents. Among the huge amount of literature, we refer the interested reader to [1, 2, 7, 16, 19, 24, 25] and the references therein. A fundamental challenge in these problems stems from the loss of compactness caused by critical Sobolev exponents, which has been extensively studied in the seminal works cited above. For instance, Bernis et al. [3] proved that problem (1.1) with $s = 0$, $1 < q < 2$ and $f(x,u) \equiv 0$ has infinitely many solutions for some $\lambda > 0$. In

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[15] we studied problem (1.1) with $q = 2$, $s = 0$ and $f(x, u) = \mu u \ln u^2$, obtaining existence and nonexistence results under appropriate parameter assumptions. Later, Zhang et al. [23] weakened part of the existence condition and specified the types and energy levels of solutions. Feng and Su [8] established existence of ground state solutions for the critical biharmonic equation using generalized Lions-type theorems.

Beyond this classical difficulty, singularity constitutes another core challenge in the study of such equations. For instance, singularities arising from the nonlinearity itself have been investigated systematically. Li et al. [13, 14] studied elliptic equations with strong singularity, namely the $u^{-\gamma}$ term. Their work provides a necessary and sufficient condition for the existence of weak solutions and proves uniqueness under appropriate conditions.

Another important class of singularities emerges from the differential operator, particularly through Hardy-type terms $\frac{|u|^{q-2}u}{|x|^s}$ which exhibit singular behavior at $x = 0$. These terms present significant analytical challenges in variational frameworks and have motivated considerable research activity, especially for biharmonic problems with critical Sobolev growth. For example, for the case $q = 2$, Kang et al. [9] considered problem (1.1) with $\lambda < \lambda_{s,2}$ (see (3.1) below), $0 \leq s \leq 2$ and $f(x, u) \equiv 0$. By using the Sobolev-Hardy inequality and variational method, they showed that problem (1.1) has at least one nontrivial solution when $N \geq 8 - s$. Later, Li et al. [12] generalized these results by investigating problem (1.1) with $\lambda > 0$, $0 \leq s \leq 4$ and $f(x, u) = f(x)$. They proved that there exist at least two nontrivial solutions when the norm of f is appropriately small and the parameters satisfy certain conditions. Chen and Chen [5] investigated critical biharmonic problems involving Hardy-type singularities, establishing existence of multiple normalized solutions including ground states and excited states. Yu et al. [22] studied the critical p-biharmonic equation with Hardy potential, proving existence of ground state solutions via the Nehari manifold method. Su and Shi [17] investigated critical biharmonic equations with Hardy potential coupled with an additional p-Laplacian term in $\mathbb{R}^N$, and established the existence of ground state solution. The interplay between Hardy terms and critical growth reveals particularly rich phenomena. D'Ambrosio and Jannelli [6] studied problem (1.1) with $q = 2$, $s = 4$ and a linear perturbation term $f(x, u) = \mu u$. Their seminal work identified a striking critical phenomenon: there exists a threshold value $\widetilde{\lambda}$ such that when the Hardy coefficient exceeds this value, solutions exist only for μ bounded away from zero, while non-existence occurs for sufficiently small μ.

Despite these advances, several fundamental challenges persist. The main difficulties stem from the combined effects of critical Sobolev nonlinearity and singular Hardy potential, which cause a double loss of compactness in the variational setting. This becomes particularly acute when considering general nonlinear perturbations $f(x, u)$ that depend on both spatial variables and the solution itself, especially under superlinear growth conditions. Motivated by the aforementioned works, it is natural to consider the existence of weak solutions to problem (1.1) with $q \geq 2$ and general nonlinearity f that depends on both x and the unknown function u.

In this paper, the above difficulties are overcome by combining a careful analysis of the fibering maps of the energy functional associated with the problem with the Mountain Pass Lemma [18] and Brézis-Lieb's lemma [4], and a mountain pass type solution to problem (1.1) is obtained. Let us explain our strategy in a more detailed way. When $0 \leq s < 4$, with the help of Brézis-Lieb's lemma, we prove that the energy functional associated with problem (1.1) satisfies the $(PS)_c$ condition for $c < c(S)$ (Lemma 3.5). Then, after some careful estimates on the norms of the truncated Talenti functions, we show that the energy functional has a mountain pass geometry around 0 and that the corresponding mountain pass level is small than $c(S)$. Finally, on the basis of the above two steps, a mountain pass type solution to problem (1.1) follows from a standard variational approach. When $s = 4$, by introducing an equivalent norm in $H_0^2(\Omega)$ (Lemma 3.3) and applying a variant of Brézis-Lieb's lemma, we can also show that the energy functional satisfies the $(PS)_c$ condition for $c < c(S_\lambda)$ (Lemma 3.8). Then the mountain pass level is proved to be strictly smaller than $c(S_\lambda)$ with the help of another group of Talenti functions and the existence of a mountain pass type solution follows.

The remainder of this paper is organized as follows. In Section 2 we give some notations, definitions and introduce some necessary lemmas. The main results of this paper are also stated in this section. In Section 3 we give the detailed proof of the main results.

2. Preliminaries

In this section, we introduce some notation and definitions that will be used throughout this paper. In what follows, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm for $1 \le p \le \infty$, and denote the norm of the weighted space $L^p(\Omega, |x|^{-s})$ by

$$\|\cdot\|_{L^p(\Omega,|x|^{-s})} = \Big(\int_\Omega |x|^{-s}|\cdot|^p \mathrm{d}x\Big)^{1/p}, \quad 1 \le p < \infty.$$

The Sobolev space $H_0^2(\Omega)$ will be equipped with the norm $\|u\| := \|u\|_{H_0^2(\Omega)} = \|\Delta u\|_2$, which is equivalent to the norm due to Poincaré's inequality, and its dual space is denoted by $H^{-2}(\Omega)$. We always use $\to$ and $\rightharpoonup$ to denote the strong and weak convergence in each Banach space, respectively, and use $C, C_1, C_2, \dots$ to denote generic positive constants. $B_r(x_0)$ is a ball of radius r centered at x_0. For each $t > 0$, $O(t)$ denotes the quantity satisfying $|O(t)/t| \le C$, $O_1(t)$ means that there exist two positive constants C_1 and C_2 such that $C_1 t \le O_1(t) \le C_2 t$, $o(t)$ means that $|o(t)/t| \to 0$ as $t \to 0$, and $o_n(1)$ denotes an infinitesimal as $n \to \infty$. In this paper, we consider weak solutions to problem (1.1) in the following sense.

Definition 2.1 (Weak solution)**.** A function $u \in H_0^2(\Omega)$ is called a weak solution to (1.1), if for all $\varphi \in H_0^2(\Omega)$, it holds that

$$\int_\Omega \Delta u \Delta\varphi \mathrm{d}x - \lambda\int_\Omega \frac{|u|^{q-2}u\varphi}{|x|^s}\mathrm{d}x - \int_\Omega |u|^{2^{**}-2}u\varphi \mathrm{d}x - \int_\Omega f(x,u)\varphi \mathrm{d}x = 0.$$

The energy functional associated with problem (1.1) is

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^s}\mathrm{d}x - \frac{1}{2^{**}}\|u\|_{2^{**}}^{2^{**}} - \int_\Omega F(x,u)\mathrm{d}x, \ \forall\ u \in H_0^2(\Omega).$$

From hypotheses (A1) and (A2) and $2 \le q < 2^{**}$, it is directly verified that $I_\lambda(u)$ is a C^1 functional in $H_0^2(\Omega)$ (see [12]).

We then introduce a compactness condition known as local (PS) condition or the $(PS)_c$ condition, which will assist us in finding weak solutions to problem (1.1).

Definition 2.2 ($(PS)_c$ condition)**.** Assume that X is a real Banach space, $I : X \to \mathbb{R}$ is a C^1 functional and $c \in \mathbb{R}$. We say that I satisfies the $(PS)_c$ condition if each sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c \text{ in } \mathbb{R} \text{ and } I'(u_n) \to 0 \text{ in } X^{-1}(\Omega) \text{ as } n \to \infty$$

has a convergent subsequence, where X^{-1} is the dual space of X.

The following three lemmas are crucial in our analysis. The first one is the famous Mountain Pass Lemma; the second one is the Brézis-Lieb's lemma; and the third one is its variant.

Lemma 2.3 (Mountain Pass Lemma [18])**.** *Assume that $(X, \|\cdot\|_X)$ is a real Banach space, $I : X \to \mathbb{R}$ is a C^1 functional and there exist $\beta > 0$ and $r > 0$ such that I satisfies the following mountain pass geometry:*

- (i) *$I(u) \ge \beta > 0$ if $\|u\|_X = r$;*
- (ii) *there exists a $\overline{u} \in X$ such that $\|\overline{u}\|_X > r$ and $I(\overline{u}) < 0$.*

Then there exist a sequence $\{u_n\} \subset X$ such that $I(u_n) \to c_0$ in $\mathbb{R}$ and $I'(u_n) \to 0$ in X^{-1} as $n \to \infty$, where X^{-1} is the dual space of X and

$$c_0 := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \ge \beta, \quad \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = \overline{u}\},$$

which is called the mountain level. Furthermore, c_0 is a critical vale of I if I satisfies the $(PS)_{c_0}$ condition.

Lemma 2.4 (Brézis-Lieb's lemma [4])**.** *Let* $p \in (1,\infty)$*. Suppose that* $\{u_n\}$ *is a bounded sequence in* $L^p(\Omega)$ *and* $u_n \to u$ *a.e. in* Ω*. Then*

$$\lim_{n\to\infty}(\|u_n\|_p^p - \|u_n - u\|_p^p) = \|u\|_p^p.$$

Lemma 2.5 ([11])**.** *Let* $r > 1$*,* $q \in [1, r]$ *and* $\delta \in [0, \frac{Nq}{r})$*. If* $\{u_n\}$ *is a bounded sequence in* $L^r(\mathbb{R}^N, |x|^{-\frac{\delta r}{q}})$ *and* $u_n \to u$ *a.e. in* $\mathbb{R}^N$*. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big|\frac{|u_n|^q}{|x|^\delta} - \frac{|u_n-u|^q}{|x|^\delta} - \frac{|u|^q}{|x|^\delta}\Big|^{\frac{r}{q}}\mathrm{d}x = 0,$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\Big|\frac{|u_n|^{q-1}u_n}{|x|^\delta} - \frac{|u_n-u|^{q-1}(u_n-u)}{|x|^\delta} - \frac{|u|^{q-1}u}{|x|^\delta}\Big|^{\frac{r}{q}}\mathrm{d}x = 0.$$

The main results of this paper are summarized in the theorem below,.

Theorem 2.6. *Assume that both* (A1) *and* (A2) *hold.*

- (i) *For* $0 \le s < 4$ *and* $q = 2$*, if* $N \ge 8 - s$ *or* $\max\{\frac{N}{N-4}, \frac{8}{N-4}\} < \rho < 2^{**}$*, then* (1.1) *has at least one nontrivial weak solution for* $0 < \lambda < \lambda_{s,2}$*, where* $\lambda_{s,2}$ *is given in* (3.1)*.*
- (ii) *For* $0 \le s < 4$ *and* $2 < q < 2^{**}(s)$*, if* $q > \max\{\frac{N-s}{N-4}, \frac{2(4-s)}{N-4}\}$ *or* $\max\{\frac{N}{N-4}, \frac{8}{N-4}\} < \rho < 2^{**}$*, then* (1.1) *has at least one nontrivial weak solution for all* $\lambda > 0$*.*
- (iii) *For* $s = 4$ *and* $q = 2$*, if* $\max\{\frac{N}{\gamma_\lambda}, \frac{4(N-2-\gamma_\lambda)}{N-4}\} < \rho < 2^{**}$*, then* (1.1) *has at least one nontrivial weak solution for* $\lambda \in (0, \lambda_{4,2})$*, where* γ_λ *and* $\lambda_{4,2}$ *are given in Lemma 3.9 and Remark 3.2, respectively.*

Remark 2.7. A typical example of a function that meets conditions (A1) and (A2) is

$$f(x,t) = \sum_{i=1}^{k} C_i(x)|t|^{q_i-2}t, \quad \text{for } x \in \overline{\Omega},\ t \in \mathbb{R},$$

where $k \in \mathbb{N}$, $q < q_i \le 2^{**}(s)$ for $0 < s \le 4$, $q < q_i < 2^{**}$ for $s = 0$, and $C_i \in C(\overline{\Omega})$ is positive.

3. Proofs of the main results

This section starts off with a lemma that provides the Sobolev-Hardy inequality, which is crucial for getting the $(PS)_c$ condition of functional I_λ. The proof of this result can be found in [21].

Lemma 3.1. *Assume that* $2 \le q \le 2^{**}(s) = \frac{2(N-s)}{N-4}$ $(0 \le s \le 4)$*. Then*

- (i) *there exists a constant* $C > 0$ *such that* $C\Big(\int_\Omega \frac{|u|^q}{|x|^s}\mathrm{d}x\Big)^{\frac{1}{q}} \le \|u\|$ *for all* $u \in H_0^2(\Omega)$*;*
- (ii) *if* $2 \le q < 2^{**}(s)$*, the mapping* $u \mapsto \frac{u}{|x|^{\frac{s}{q}}}$ *from* $H_0^2(\Omega)$ *into* $L^q(\Omega)$ *is compact.*

Remark 3.2. We denote the best Sobolev-Hardy constant by

$$\lambda_{s,q} = \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\|u\|^2}{(\int_\Omega \frac{|u|^q}{|x|^s}\mathrm{d}x)^{2/q}}. \tag{3.1}$$

In particular, $\lambda_{4,2} = \frac{1}{16}N^2(N-4)^2$. We always write $\lambda_{0,2^{**}}$ as S for simplicity, which satisfies $\|u\|_{2^{**}}^{2^{**}} \le S^{-\frac{2^{**}}{2}}\|u\|^{2^{**}}$.

To prove Theorem 2.6, the following result, which follows immediately from (3.1) with $q = 2$, is needed.

Lemma 3.3. *For* $\lambda \in (0, \lambda_{s,2})$ $(0 \le s \le 4)$*, there exists a* $\mu_s > 0$ *such that*

$$\int_\Omega \Big(|\Delta u|^2 - \lambda\frac{|u|^2}{|x|^s}\Big)\mathrm{d}x \ge \mu_s\|u\|^2, \tag{3.2}$$

for all $u \in H_0^2(\Omega)$*.*

Remark 3.4. From Lemma 3.3 it follows that the following best Sobolev constant is well defined for $\lambda \in (0, \lambda_{4,2})$,

$$S_\lambda = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\Delta u|^2 - \lambda \frac{|u|^2}{|x|^4}) \mathrm{d}x}{\big(\int_\Omega |u|^{2^{**}} \mathrm{d}x\big)^{2/2^{**}}} > 0. \tag{3.3}$$

3.1. **Case** $0 \le s < 4$**.** In general, the functional $I_\lambda(u)$ does not satisfy the $(PS)_c$ condition for all $c \in \mathbb{R}$, because of the appearance of the critical term. However, with the help of Brézis-Lieb's lemma, we can find a constant $c(S)$ such that the $(PS)_c$ condition holds for all $c < c(S)$. This will be essential in revealing the main results.

Lemma 3.5. *Assume that $f(x,t)$ satisfies* (A1) *and* (A2), *and $0 \le s < 4$. Let $\{u_n\} \subset H_0^2(\Omega)$ be a sequence such that $I_\lambda(u_n) \to c < c(S)$ and $I'_\lambda(u_n) \to 0$ in $H^{-2}(\Omega)$ as $n \to \infty$, where $c(S) := \frac{2}{N} S^{N/4}$. Then, $I_\lambda(u)$ satisfies the $(PS)_c$ condition if $q = 2$ and $0 < \lambda < \lambda_{s,2}$, or $2 < q < 2^{**}(s)$ and $\lambda > 0$.*

Proof. We begin by showing the boundedness of $\{u_n\}$ in $H_0^2(\Omega)$. Depending on whether or not q is equal to 2, the proof will be divided into two cases. When $q > 2$, by (A2), we obtain, for any $\lambda > 0$, that

$$\begin{aligned}
c + 1 + o_n(1)\|u_n\| &\ge I_\lambda(u_n) - \frac{1}{q}\langle I'_\lambda(u_n), u_n\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|^2 + \Big(\frac{1}{q} - \frac{1}{2^{**}}\Big)\|u_n\|_{2^{**}}^{2^{**}} + \int_\Omega \Big(\frac{1}{q} f(x,u_n)u_n - F(x,u_n)\Big)\mathrm{d}x \\
&\ge \Big(\frac{1}{2} - \frac{1}{q}\Big)\|u_n\|^2, \quad n \to \infty.
\end{aligned}$$

When $q = 2$, in accordance with (A2) and (3.2), we obtain, for $\lambda \in (0, \lambda_{s,2})$, that

$$\begin{aligned}
c + 1 + o_n(1)\|u_n\| &\ge I_\lambda(u_n) - \frac{1}{\rho}\langle I'_\lambda(u_n), u_n\rangle \\
&= \Big(\frac{1}{2} - \frac{1}{\rho}\Big)\int_\Omega \Big(|\Delta u_n|^2 - \lambda\frac{|u_n|^2}{|x|^s}\Big)\mathrm{d}x + \Big(\frac{1}{\rho} - \frac{1}{2^{**}}\Big)\|u_n\|_{2^{**}}^{2^{**}} \\
&\quad + \int_\Omega \Big(\frac{1}{\rho} f(x,u_n)u_n - F(x,u_n)\Big)\mathrm{d}x \\
&\ge \Big(\frac{1}{2} - \frac{1}{\rho}\Big)\int_\Omega \Big(|\Delta u_n|^2 - \lambda\frac{|u_n|^2}{|x|^s}\Big)\mathrm{d}x \\
&\ge \Big(\frac{1}{2} - \frac{1}{\rho}\Big)\mu_s\|u_n\|^2, \quad n \to \infty.
\end{aligned} \tag{3.4}$$

It is obvious that in either case $\{u_n\}$ is a bounded sequence in $H_0^2(\Omega)$. Consequently, by recalling Lemma 3.1 one sees that there is a subsequence of $\{u_n\}$ (which we still denote by $\{u_n\}$) such that, as $n \to \infty$,

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } H_0^2(\Omega), \\
u_n \to u \quad \text{in } L^r(\Omega) \quad (1 \le r < 2^{**}), \\
u_n \to u \quad \text{in } H_0^1(\Omega), \\
u_n \to u \quad \text{in } L^q(\Omega, |x|^{-s}) \quad (1 \le q < 2^{**}(s)), \\
|u_n|^{2^{**}-2}u_n \rightharpoonup |u|^{2^{**}-2}u \quad \text{in } L^{\frac{2^{**}}{2^{**}-1}}(\Omega), \\
u_n \to u \quad \text{a.e. in } \Omega.
\end{gathered} \tag{3.5}$$

In view of (1.2) (with $\varepsilon = 1$), there exists a positive constant C, independent of n, such that

$$\Big|\int_\Omega f(x,u_n)u_n \mathrm{d}x\Big| \le \int_\Omega |f(x,u_n)||u_n|\mathrm{d}x \le \|u_n\|_{2^{**}}^{2^{**}} + C_1(1)\|u_n\|_2^2 \le C.$$

For each $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{C_1(\varepsilon)^{\frac{2^{**}}{2^{**}-2}}}$, where $C_1(\varepsilon) > 0$ is given in (1.2). Then for any measurable subset $E \subset \Omega$ with $\operatorname{meas} E < \delta$, we obtain, by recalling (1.2) again and applying Hölder's inequality, that

$$\begin{aligned} \Big|\int_E f(x,u_n)u_n \mathrm{d}x\Big| &\le \varepsilon \int_E |u_n|^{2^{**}}\mathrm{d}x + C_1(\varepsilon)\int_E |u_n|^2 \mathrm{d}x \\ &\le \varepsilon \|u_n\|_{2^{**}}^{2^{**}} + C_1(\varepsilon)\|u_n\|_{2^{**}}^2 (mesE)^{\frac{2^{**}-2}{2^{**}}} \\ &\le C_1\varepsilon + C_2\varepsilon^{\frac{2^{**}-2}{2^{**}}}, \end{aligned}$$

uniformly with respect to $n \in \mathbb{N}$, where C_1, C_2 are positive constants independent of n. Hence the family of functions $\{f(x,u_n)u_n\}$ is equi-absolutely-continuous. In addition, recalling that $u_n \to u$ a.e. in Ω as $n \to \infty$ and $f(x,t)$ is continuous, we obtain $f(x,u_n)u_n \to f(x,u)u$ a.e. in Ω as $n \to \infty$. This, together with the fact that $\operatorname{meas}\Omega < \infty$, implies that $f(x,u_n)u_n \to f(x,u)u$ in measure. Therefore, by the Vitali convergence theorem we obtain

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)u_n \mathrm{d}x = \int_\Omega f(x,u)u \mathrm{d}x. \tag{3.6}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\int_\Omega F(x,u_n)\mathrm{d}x = \int_\Omega F(x,u)\mathrm{d}x. \tag{3.7}$$

By recalling (1.2) (with $\varepsilon = 1$) and (3.5), we have

$$\begin{aligned} \Big|\int_\Omega f(x,u_n)u \mathrm{d}x\Big| &\le \int_\Omega |f(x,u_n)||u|\mathrm{d}x \\ &\le \int_\Omega |u_n|^{2^{**}-1}|u|\mathrm{d}x + C_1(1)\int_\Omega |u_n||u|\mathrm{d}x \\ &\to \|u\|_{2^{**}}^{2^{**}} + C_1(1)\|u\|_2^2, \quad n\to\infty, \end{aligned}$$

which, together with the fact that $f(x,t)$ is a continuous function and $u_n \to u$ a.e. in Ω as $n \to \infty$, shows that $f(x,u_n)u \to f(x,u)u$ a.e. in Ω as $n \to \infty$. Then, according to the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)u \mathrm{d}x = \int_\Omega f(x,u)u \mathrm{d}x. \tag{3.8}$$

To complete the proof, let $w_n = u_n - u$. Then $\{w_n\}$ is also a bounded sequence in $H_0^2(\Omega)$. So there exists a subsequence of $\{w_n\}$ (still denoted by $\{w_n\}$) such that

$$\lim_{n\to\infty}\|w_n\|^2 = l \ge 0. \tag{3.9}$$

We claim that $l = 0$. Indeed, according to (3.5), we have

$$\begin{aligned} \|u_n\|^2 &= \int_\Omega |\Delta w_n + \Delta u|^2 \mathrm{d}x \\ &= \int_\Omega |\Delta w_n|^2\mathrm{d}x + \int_\Omega |\Delta u|^2 \mathrm{d}x + 2\int_\Omega \Delta w_n \Delta u \mathrm{d}x \\ &= \|w_n\|^2 + \|u\|^2 + o_n(1), \quad n\to\infty. \end{aligned} \tag{3.10}$$

Moreover, since $\|u_n\|_{2^{**}} \le C$ and $u_n \to u$ a.e. in Ω as $n \to \infty$, one sees, by using Brézis-Lieb's lemma, that

$$\|u_n\|_{2^{**}}^{2^{**}} = \|w_n\|_{2^{**}}^{2^{**}} + \|u\|_{2^{**}}^{2^{**}} + o_n(1), \quad n\to\infty. \tag{3.11}$$

Therefore, in accordance with (3.5), (3.6), (3.8), (3.10), (3.11) and the assumption that $I'_\lambda(u_n) \to 0$ in $H^{-2}(\Omega)$ as $n \to \infty$, we have

$$\begin{aligned} 0 &= \langle I'_\lambda(u_n), u_n\rangle + o_n(1) \\ &= \int_\Omega \Big(|\Delta u_n|^2 - \lambda\frac{|u_n|^q}{|x|^s}\Big)\mathrm{d}x - \|u_n\|_{2^{**}}^{2^{**}} - \int_\Omega f(x,u_n)u_n \mathrm{d}x + o_n(1) \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Big(|\Delta u|^2 - \lambda\frac{|u|^q}{|x|^s}\Big)\mathrm{d}x - \|u\|_{2^{**}}^{2^{**}} - \int_\Omega f(x,u)u\mathrm{d}x + \|w_n\|^2 - \|w_n\|_{2^{**}}^{2^{**}} + o_n(1)\\
&= \langle I_\lambda'(u),u\rangle + \|w_n\|^2 - \|w_n\|_{2^{**}}^{2^{**}} + o_n(1), \quad n\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \langle I_\lambda'(u_n),u\rangle + o_n(1)\\
&= \int_\Omega \Big(|\Delta u|^2 - \lambda\frac{|u|^q}{|x|^s}\Big)\mathrm{d}x - \|u\|_{2^{**}}^{2^{**}} - \int_\Omega f(x,u)u\mathrm{d}x + o_n(1)\\
&= \langle I_\lambda'(u),u\rangle + o_n(1), \quad n\to\infty.
\end{aligned}
$$

Combining the above two equalities one obtains

$$\langle I_\lambda'(u),u\rangle = 0, \tag{3.12}$$

$$\|w_n\|_{2^{**}}^{2^{**}} - \|w_n\|^2 = o_n(1), \quad n\to\infty. \tag{3.13}$$

In addition, from the Sobolev embedding one has

$$\|w_n\|_{2^{**}}^{2^{**}} \le S^{-\frac{2^{**}}{2}}\|w_n\|^{2^{**}} < C, \quad \forall\, n\in\mathbb{N}. \tag{3.14}$$

It follows from (3.9), (3.13) and (3.14) that there is a subsequence of $\{w_n\}$ such that

$$\lim_{n\to\infty}\|w_n\|_{2^{**}}^{2^{**}} = \lim_{n\to\infty}\|w_n\|^2 = l. \tag{3.15}$$

Letting $n\to\infty$ in (3.14), we have $l \le S^{-\frac{2^{**}}{2}} l^{\frac{2^{**}}{2}}$. If $l>0$, then

$$l \ge S^{\frac{2^{**}}{2^{**}-2}} = S^{N/4}. \tag{3.16}$$

On one hand, in view of (3.5), (3.7), (3.10), (3.11) and the fact that $I_\lambda(u_n) = c + o(1)$ as $n\to\infty$, we have

$$
\begin{aligned}
c &= I_\lambda(u_n) + o_n(1)\\
&= \frac12\|u_n\|^2 - \frac{\lambda}{q}\int_\Omega \frac{|u_n|^q}{|x|^s}\mathrm{d}x - \frac{1}{2^{**}}\|u_n\|_{2^{**}}^{2^{**}} - \int_\Omega F(x,u_n)\mathrm{d}x + o_n(1)\\
&= \frac12\|u\|^2 - \frac{\lambda}{q}\int_\Omega \frac{|u|^q}{|x|^s}\mathrm{d}x - \frac{1}{2^{**}}\|u\|_{2^{**}}^{2^{**}} - \int_\Omega F(x,u)\mathrm{d}x\\
&\quad + \frac12\|w_n\|^2 - \frac{1}{2^{**}}\|w_n\|_{2^{**}}^{2^{**}} + o_n(1)\\
&= I_\lambda(u) + \frac12\|w_n\|^2 - \frac{1}{2^{**}}\|w_n\|_{2^{**}}^{2^{**}} + o_n(1), \quad n\to\infty,
\end{aligned}
$$

which yields that

$$I_\lambda(u) = c - \frac12\|w_n\|^2 + \frac{1}{2^{**}}\|w_n\|_{2^{**}}^{2^{**}} + o_n(1), \quad n\to\infty.$$

Recalling (3.15) and (3.16), we obtain from the above equality that

$$I_\lambda(u) = c - \Big(\frac12 - \frac{1}{2^{**}}\Big)l \le c - \frac{2}{N}S^{N/4} < 0.$$

On the other hand, by (3.12) and (A2), we can derive

$$
\begin{aligned}
I_\lambda(u) &= I_\lambda(u) - \frac1q\langle I_\lambda'(u),u\rangle\\
&= \Big(\frac12 - \frac1q\Big)\|u\|^2 + \Big(\frac1q - \frac{1}{2^{**}}\Big)\|u_n\|_{2^{**}}^{2^{**}} + \int_\Omega \Big(\frac1q f(x,u)u - F(x,u)\Big)\mathrm{d}x \ge 0,
\end{aligned} \tag{3.17}
$$

a contradiction. Thus, $\lim_{n\to\infty}\|w_n\|^2 = l = 0$, which implies that $u_n\to u$ in $H_0^2(\Omega)$ as $n\to\infty$. The proof is complete. □

Before going further, we list some well-known estimates on the Talenti functions, which will play a crucial role in estimating the mountain pass level of I_λ around 0. For any $\varepsilon > 0$, we define

$$U_\varepsilon(x) = [N(N-4)(N^2-4)]^{\frac{N-4}{8}} \frac{\varepsilon^{\frac{N-4}{2}}}{[\varepsilon^2 + |x|^2]^{\frac{N-4}{2}}}, \quad x \in \mathbb{R}^N.$$

Then $U_\varepsilon(x)$ is a solution of the critical problem

$$\Delta^2 u = u^{2^{**}-1}, \quad x \in \mathbb{R}^N, \quad N \geq 5,$$

and $\|U_\varepsilon\|^2 = \|U_\varepsilon\|_{2^{**}}^{2^{**}} = S^{N/4}$, where

$$S = \inf_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_{2^{**}}^2} = \frac{\|U_\varepsilon\|^2}{\|U_\varepsilon\|_{2^{**}}^2}$$

is given in Remark 3.2.

The Talenti functions, after being truncated, are estimated in the following (see [12], [20] and [21]).

Lemma 3.6. *Let $\tau \in C_0^\infty(\Omega)$ be a cut-off function such that $\tau(x) = \tau(|x|)$, $0 \leq \tau(x) \leq 1$ for $x \in \Omega$, and*

$$\tau(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| > 2R, \end{cases}$$

where $R > 0$ is a constant such that $B_{2R}(0) \subset \Omega$. Set $u_\varepsilon(x) = \tau(x)U_\varepsilon(x)$. Suppose that $\varepsilon \to 0$. Then

$$\|u_\varepsilon\|^2 = S^{N/4} + O(\varepsilon^{N-4}),$$
$$\|u_\varepsilon\|_{2^{**}}^{2^{**}} = S^{N/4} + O(\varepsilon^N).$$

Set $v_\varepsilon(x) = \frac{u_\varepsilon}{\|u_\varepsilon\|_{2^{**}}}$. Then

$$\begin{gathered} \|v_\varepsilon\|^2 = S + O(\varepsilon^{N-4}), \\ \|v_\varepsilon\|_{2^{**}}^{2^{**}} = 1, \end{gathered} \tag{3.18}$$

$$\|v_\varepsilon\|_\rho^\rho = \begin{cases} O_1(\varepsilon^{\frac{(N-4)\rho}{2}}), & 1 < \rho < \frac{N}{N-4}, \\ O_1(\varepsilon^{N-\frac{(N-4)\rho}{2}}|\ln\varepsilon|), & \rho = \frac{N}{N-4}, \\ O_1(\varepsilon^{N-\frac{(N-4)\rho}{2}}), & \frac{N}{N-4} < \rho < 2^{**}, \end{cases} \tag{3.19}$$

$$\int_\Omega \frac{|v_\varepsilon|^q}{|x|^s} \mathrm{d}x = \begin{cases} O_1(\varepsilon^{\frac{(N-4)q}{2}}), & 1 < q < \frac{N-s}{N-4}, \\ O_1(\varepsilon^{N-\frac{(N-4)q}{2}-s}|\ln\varepsilon|), & q = \frac{N-s}{N-4}, \\ O_1(\varepsilon^{N-\frac{(N-4)q}{2}-s}), & \frac{N-s}{N-4} < q < 2^{**}(s). \end{cases} \tag{3.20}$$

With the help of the Talenti functions given above, we can show that the mountain pass level of I_λ around 0 is strictly less than $c(S)$.

Lemma 3.7. *Assume that* (A1), (A2)*, and the condition (i) or (ii) of Theorem 2.6 hold. Then there exists a $u^* \in H_0^2(\Omega)$ such that*

$$\sup_{t \geq 0} I_\lambda(tu^*) < c(S), \tag{3.21}$$

where $c(S) := \frac{2}{N} S^{N/4}$.

Proof. Define the fibering maps associated with the energy functional I_λ by

$$\begin{aligned} \psi_u(t) &= I_\lambda(tu) \\ &= \frac{1}{2}t^2\|u\|^2 - \frac{1}{q}\lambda t^q \int_\Omega \frac{|u|^q}{|x|^s}\mathrm{d}x - \frac{1}{2^{**}}t^{2^{**}}\|u\|_{2^{**}}^{2^{**}} - \int_\Omega F(x,tu)\mathrm{d}x, \ t \geq 0. \end{aligned} \tag{3.22}$$

Recalling (3.18) and the fact that $F(x,t) \geq 0$ for any $x \in \Omega$ and $t \geq 0$, one sees that as $t \to 0$,

$$\psi_{v_\varepsilon}(t) \leq \frac{1}{2}t^2\|v_\varepsilon\|^2 \to 0, \tag{3.23}$$

uniformly for $\varepsilon \in (0,\varepsilon_1)$, where $\varepsilon_1 > 0$ is a suitably small but fixed number and v_ε is given in Lemma 3.6. Therefore, there exists a $t_0 > 0$, independent of ε, such that

$$\psi_{v_\varepsilon}(t) < c(S), \quad t \in (0,t_0). \tag{3.24}$$

Setting $g(t) = \frac{1}{2}t^2\|v_\varepsilon\|^2 - \frac{1}{2^{**}}t^{2^{**}}$ we have

$$\psi_{v_\varepsilon}(t) = g(t) - \frac{1}{q}\lambda t^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x - \int_\Omega F(x,tv_\varepsilon)\mathrm{d}x, \quad t \geq 0.$$

According to (1.5), there exists a positive constant C such that

$$\begin{aligned}\psi_{v_\varepsilon}(t) &\leq \max_{t\geq t_0} g(t) - \frac{1}{q}\lambda t^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x - Ct^\rho\|v_\varepsilon\|_\rho^\rho \\ &\leq \max_{t\geq 0} g(t) - \frac{1}{q}\lambda t_0^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x - Ct_0^\rho\|v_\varepsilon\|_\rho^\rho, \quad t \in [t_0,\infty).\end{aligned} \tag{3.25}$$

Direct calculations show that g takes its maximum at $t_\varepsilon^* := \|v_\varepsilon\|^{\frac{2}{2^{**}-2}}$ and $g(t_\varepsilon^*) = \frac{2}{N}\|v_\varepsilon\|^{N/2}$. Therefore, in view of (3.18) and (3.25), one sees that for $t \geq t_0$,

$$\begin{aligned}\psi_{v_\varepsilon}(t) &\leq g(t_\varepsilon^*) - \frac{1}{q}\lambda t_0^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x - Ct_0^\rho\|v_\varepsilon\|_\rho^\rho \\ &= \frac{2}{N}\big(S + O(\varepsilon^{N-4})\big)^{N/4} - \frac{1}{q}\lambda t_0^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x - Ct_0^\rho\|v_\varepsilon\|_\rho^\rho \\ &= c(S) + O(\varepsilon^{N-4}) - \frac{1}{q}\lambda t_0^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x - Ct_0^\rho\|v_\varepsilon\|_\rho^\rho, \quad \varepsilon \to 0.\end{aligned}$$

Our goal is to show that for ε suitably small,

$$\psi_{v_\varepsilon}(t) < c(S), \quad t \in [t_0,\infty), \tag{3.26}$$

which is fulfilled once we can prove that

$$O(\varepsilon^{N-4}) - \frac{1}{q}\lambda t_0^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x - Ct_0^\rho\|v_\varepsilon\|_\rho^\rho < 0. \tag{3.27}$$

It is easy to see that (3.27) holds if either of the following 2 conditions is valid

(I) $O(\varepsilon^{N-4}) - Ct_0^\rho\|v_\varepsilon\|_\rho^\rho < 0$;

(II) $O(\varepsilon^{N-4}) - \frac{1}{q}\lambda t_0^q \int_\Omega \frac{|v_\varepsilon|^q}{|x|^s}\mathrm{d}x < 0$.

First, when $0 \leq s < 4$, $q \geq 2$ and $\max\{\frac{N}{N-4}, \frac{8}{N-4}\} < \rho < 2^{**}$, by recalling (3.19), we obtain

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{N-4}}{\varepsilon^{N-\frac{(N-4)\rho}{2}}} = \lim_{\varepsilon\to 0} \varepsilon^{\frac{(N-4)\rho}{2}-4} = 0,$$

which implies that (I) holds.

Next, we show that (II) is fulfilled for small ε when other cases in (i) or (ii) of Theorem 2.6 are satisfied, with the help of (3.20). Indeed, if $q = 2$ and $N = 8 - s$, then $q = \frac{N-s}{N-4}$, which, together with (3.20), implies that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{N-4}}{\varepsilon^{N-\frac{(N-4)q}{2}-s}|\ln\varepsilon|} = \lim_{\varepsilon\to 0} \frac{1}{|\ln\varepsilon|} = 0. \tag{3.28}$$

If $q = 2$ and $N > 8 - s$, then $q > \frac{N-s}{N-4}$ and

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{N-4}}{\varepsilon^{N-\frac{(N-4)q}{2}-s}} = \lim_{\varepsilon\to 0} \varepsilon^{N-(8-s)} = 0. \tag{3.29}$$

It follows from (3.28) and (3.29) that (II) is valid if $q = 2$ and $N \geq 8 - s$.

Similarly, if $q > 2$ and $\max\{\frac{N-s}{N-4}, \frac{2(4-s)}{N-4}\} < q < 2^{**}(s)$, we have

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{N-4}}{\varepsilon^{N-\frac{(N-4)q}{2}-s}} = \lim_{\varepsilon\to 0} \varepsilon^{\frac{1}{2}[(N-4)q-(8-2s)]} = 0,$$

which also implies that (II) holds.

In conclusion, if the condition (i) or (ii) of Theorem 2.6 holds, by combining (3.24) with (3.26), we have, for ε suitably small, that

$$\sup_{t\geq 0}\psi_{v_\varepsilon}(t) < c(S).$$

Fixing such an $\varepsilon > 0$ and taking $u^* \equiv v_\varepsilon$, the proof is complete. □

In what follows, we shall show the existence of the weak solutions to (1.1) on the basis of Lemmas 3.5 and 3.7 and the Mountain Pass Lemma.

Proof of Theorem 2.6 (i) and (ii). We first show that the functional I_λ satisfies the mountain pass geometry in both cases. If $q > 2$, according to (1.4) and (3.1), for any $\varepsilon > 0$, there exists a $C_3(\varepsilon) > 0$ such that for any $u \in H_0^2(\Omega)\backslash\{0\}$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda_{s,q}^{-\frac{q}{2}}}{q}\lambda\|u\|^q - \frac{S^{-\frac{2^{**}}{2}}}{2^{**}}\|u\|^{2^{**}} - \frac{\lambda_{0,2}^{-1}}{2}\varepsilon\|u\|^2 - C_3(\varepsilon)S^{-\frac{2^{**}}{2}}\|u\|^{2^{**}} \\ &= (1-\lambda_{0,2}^{-1}\varepsilon)\frac{\|u\|^2}{2} - \frac{\lambda_{s,q}^{-\frac{q}{2}}}{q}\lambda\|u\|^q - \Big(\frac{1}{2^{**}} + C_3(\varepsilon)\Big)S^{-\frac{2^{**}}{2}}\|u\|^{2^{**}}. \end{aligned} \tag{3.30}$$

If $q = 2$, by using (1.4), (3.1) again and (3.2), we obtain that for any $u \in H_0^2(\Omega)\backslash\{0\}$,

$$\begin{aligned} I_\lambda(u) &\geq \frac{\mu_s}{2}\|u\|^2 - \frac{S^{-\frac{2^{**}}{2}}}{2^{**}}\|u\|^{2^{**}} - \frac{\lambda_{0,2}^{-1}}{2}\varepsilon\|u\|^2 - C_3(\varepsilon)S^{-\frac{2^{**}}{2}}\|u\|^{2^{**}} \\ &= (\mu_s - \lambda_{0,2}^{-1}\varepsilon)\frac{\|u\|^2}{2} - \Big(\frac{1}{2^{**}} + C_3(\varepsilon)\Big)S^{-\frac{2^{**}}{2}}\|u\|^{2^{**}}. \end{aligned} \tag{3.31}$$

Choosing $\varepsilon > 0$ so small that $1 - \lambda_{0,2}^{-1}\varepsilon > 0$ and $\mu_s - \lambda_{0,2}^{-1}\varepsilon > 0$, and noticing $2 \leq q < 2^{**}$, one sees that there exist $\beta,\ r > 0$ such that $I_\lambda(u) \geq \beta$ for all $\|u\| = r$.

On the other hand, recalling that $F(x,t) \geq 0$ for any $x \in \Omega$ and $t \geq 0$ by (A2), we have for any $u \in H_0^2(\Omega)\backslash\{0\}$

$$\psi_u(t) \leq \frac{1}{2}t^2\|u\|^2 - \frac{1}{2^{**}}t^{2^{**}}\|u\|_{2^{**}}^{2^{**}}, \tag{3.32}$$

which implies that $\lim_{t\to\infty}\psi_u(t) = -\infty$. Therefore, there exists a $t_u > 0$ suitably large such that $\|t_u u\| > r$ and $\psi_u(t_u) = I_\lambda(t_u u) < 0$. Thus, I_λ satisfies the mountain pass geometry around 0, and there exists a sequence $\{u_n\} \subset H_0^2(\Omega)$ such that $I_\lambda(u_n) \to c_0 \geq \beta$ and $I'_\lambda(u_n) \to 0$ in $H^{-2}(\Omega)$ as $n \to \infty$, where

$$c_0 = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \quad \text{and} \quad \Gamma = \{\gamma \in C([0,1], H_0^2(\Omega)) : \gamma(0) = 0, \gamma(1) = t_{u^*}u^*\},$$

and u^* is given in Lemma 3.7. In view of (3.30) and (3.21), one sees

$$c_0 \leq \max_{t\in[0,1]} I_\lambda(t t_{u^*}u^*) \leq \sup_{t\geq 0} I_\lambda(tu^*) < c(S). \tag{3.33}$$

It then follows from Lemma 3.5 that $I_\lambda(u)$ satisfies the $(PS)_{c_0}$ condition. Consequently, there exists a convergent subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \to u$ in $H_0^2(\Omega)$ as $n \to \infty$, which implies that $I_\lambda(u) = c_0$ and $I'_\lambda(u) = 0$, i.e., u is a nontrivial weak solution to problem (1.1). The proof of (i) and (ii) of Theorem 2.6 is complete. □

3.2. **Case** $s = 4$.

In this case the difficulty is the lack of compactness of the mapping $u \mapsto \frac{u}{|x|^2}$ from $H_0^2(\Omega)$ to $L^2(\Omega)$ and the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{2^{**}}(\Omega)$, which prevents us from establishing the usual (PS) condition directly. However, inspired by some ideas from [12], we can show that I_λ satisfies the $(PS)_c$ condition for some c, and then the existence of a mountain pass type solution to (1.1) follows.

Lemma 3.8. *Assume that* $q = 2$, $s = 4$ *and that* $f(x,t)$ *satisfies* (A1) *and* (A2). *Let* $\{u_n\} \subset H_0^2(\Omega)$ *be a sequence such that* $I_\lambda(u_n) \to c < c(S_\lambda)$ *and* $I'_\lambda(u_n) \to 0$ *in* $H^{-2}(\Omega)$ *as* $n \to \infty$, *where* $c(S_\lambda) := \frac{2}{N}S_\lambda^{N/4}$. *Then* $I_\lambda(u)$ *satisfies the* $(PS)_c$ *condition provided that* $0 < \lambda < \lambda_{4,2}$.

Proof. Recalling (3.4) with $s = 4$, one sees that $\{u_n\}$ is bounded in $H_0^2(\Omega)$ when $0 < \lambda < \lambda_{4,2}$. Hence, according to Lemma 3.1, there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that, as $n \to \infty$,

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{in } H_0^2(\Omega), \\
u_n \to u \quad \text{in } L^r(\Omega)\ (1 \le r < 2^{**}), \\
u_n \to u \quad \text{in } H_0^1(\Omega), \\
u_n \rightharpoonup u \quad \text{in } \ L^2(\Omega, |x|^{-2}), \\
|u_n|^{2^{**}-2}u_n \rightharpoonup |u|^{2^{**}-2}u \quad \text{in } L^{\frac{2^{**}}{2^{**}-1}}(\Omega), \\
u_n \to u \quad \text{a.e. in } \Omega.
\end{gathered} \tag{3.34}$$

As was done in the proof of Lemma 3.5, we set $w_n = u_n - u$. Then $\{w_n\}$ is bounded in $H_0^2(\Omega)$, which ensures that there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, such that

$$\lim_{n\to\infty} \|w_n\|^2 = l \ge 0.$$

Next we show that $l = 0$. In view of (3.34) and Brézis-Lieb's lemma, we know that (3.10) and (3.11) are still valid. In addition, from Lemma 2.5, one see that as $n \to \infty$,

$$\int_\Omega \frac{|u_n|^2}{|x|^4}\mathrm{d}x = \int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x + \int_\Omega \frac{|u|^2}{|x|^4}\mathrm{d}x + o_n(1). \tag{3.35}$$

Combining (3.6), (3.8), (3.10), (3.11), (3.34), (3.35) with the assumption that $I'_\lambda(u_n) \to 0$ in $H^{-2}(\Omega)$ as $n \to \infty$, one obtains

$$\begin{aligned}
0 &= \langle I'_\lambda(u_n), u_n\rangle + o_n(1) \\
&= \int_\Omega \Big(|\Delta u_n|^2 - \lambda\frac{|u_n|^2}{|x|^4}\Big)\mathrm{d}x - \|u_n\|_{2^{**}}^{2^{**}} - \int_\Omega f(x,u_n)u_n\mathrm{d}x + o_n(1) \\
&= \int_\Omega \Big(|\Delta u|^2 - \lambda\frac{|u|^2}{|x|^4}\Big)\mathrm{d}x - \|u\|_{2^{**}}^{2^{**}} - \int_\Omega f(x,u)u\mathrm{d}x + \|w_n\|^2 - \lambda\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x - \|w_n\|_{2^{**}}^{2^{**}} + o_n(1) \\
&= \langle I'_\lambda(u), u\rangle + \|w_n\|^2 - \lambda\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x - \|w_n\|_{2^{**}}^{2^{**}} + o_n(1), \quad n \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
0 &= \langle I'_\lambda(u_n), u\rangle + o_n(1) \\
&= \int_\Omega \Big(\Delta u_n\Delta u - \lambda\frac{|u_n|u}{|x|^4}\Big)\mathrm{d}x - \int_\Omega |u_n|^{2^{**}-2}u_n u\mathrm{d}x - \int_\Omega f(x,u_n)u\mathrm{d}x + o_n(1) \\
&= \int_\Omega \Big(|\Delta u|^2 - \lambda\frac{|u|^2}{|x|^4}\Big)\mathrm{d}x - \|u\|_{2^{**}}^{2^{**}} - \int_\Omega f(x,u)u\mathrm{d}x + o_n(1) \\
&= \langle I'_\lambda(u), u\rangle + o_n(1), \quad n \to \infty.
\end{aligned}$$

According to the two equalities above, we obtain (3.12) and

$$\|w_n\|^2 - \lambda\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x - \|w_n\|_{2^{**}}^{2^{**}} = o_n(1), \quad n \to \infty. \tag{3.36}$$

In addition, in view of (3.3), one has

$$\|w_n\|_{2^{**}}^{2^{**}} \le S_\lambda^{-\frac{2^{**}}{2}}\Big(\|w_n\|^2 - \lambda\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x\Big)^{\frac{2^{**}}{2}} \le S_\lambda^{-\frac{2^{**}}{2}}\|w_n\|^2 < C. \tag{3.37}$$

It follows from (3.36) and (3.37) that we can take a subsequence of $\{w_n\}$ such that

$$\lim_{n\to\infty}\Big(\|w_n\|^2 - \lambda\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x\Big) = \lim_{n\to\infty}\|w_n\|_{2^{**}}^{2^{**}} = k \ge 0. \tag{3.38}$$

Letting $n \to \infty$ in (3.37), we obtain that $k \le S_\lambda^{-\frac{2^{**}}{2}} k^{\frac{2^{**}}{2}}$.

If $k > 0$, then

$$k \geq S_\lambda^{\frac{2^{**}}{2^{**}-2}} = S_\lambda^{N/4}. \tag{3.39}$$

In accordance with (3.7), (3.10), (3.11), (3.34), (3.35) and $I_\lambda(u_n) = c + o(1)$ *as* $n \to \infty$, we have

$$\begin{aligned} c &= I_\lambda(u_n) + o_n(1) \\ &= \frac{1}{2}\int_\Omega \Big(|\Delta u_n|^2 - \lambda\frac{|u_n|^2}{|x|^4}\Big)\mathrm{d}x - \frac{1}{2^{**}}\|u_n\|_{2^{**}}^{2^{**}} - \int_\Omega F(x,u_n)\mathrm{d}x + o_n(1) \\ &= \frac{1}{2}\int_\Omega \Big(|\Delta u|^2 - \lambda\frac{|u|^2}{|x|^4}\Big)\mathrm{d}x - \frac{1}{2^{**}}\|u\|_{2^{**}}^{2^{**}} - \int_\Omega F(x,u)\mathrm{d}x \\ &\quad + \frac{1}{2}\|w_n\|^2 - \frac{1}{2}\lambda\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x - \frac{1}{2^{**}}\|w_n\|_{2^{**}}^{2^{**}} + o_n(1) \\ &= I_\lambda(u) + \frac{1}{2}\|w_n\|^2 - \frac{1}{2}\lambda\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x - \frac{1}{2^{**}}\|w_n\|_{2^{**}}^{2^{**}} + o_n(1), \quad n \to \infty, \end{aligned}$$

which then yields that

$$I_\lambda(u) = c - \Big(\frac{1}{2}\|w_n\|^2 - \frac{\lambda}{2}\int_\Omega \frac{|w_n|^2}{|x|^4}\mathrm{d}x - \frac{1}{2^{**}}\|w_n\|_{2^{**}}^{2^{**}}\Big) + o_n(1), \quad n \to \infty.$$

From the above equality and recalling (3.38) and (3.39), we have

$$I_\lambda(u) = c - \Big(\frac{1}{2} - \frac{1}{2^{**}}\Big)k \leq c - \frac{2}{N}S_\lambda^{N/4} < 0,$$

which contradicts (3.17). Thus, $k = 0$. Noticing that $0 < \lambda < \lambda_{4,2}$, we obtain from (3.2) that there exists a $\mu_4 > 0$ such that

$$\mu_4\|w_n\|^2 \leq \int_\Omega \Big(|\Delta w_n|^2 - \lambda\frac{|w_n|^2}{|x|^4}\Big)\mathrm{d}x = o_n(1), \quad n \to \infty,$$

which implies $\lim_{n\to\infty}\|w_n\|^2 = l = 0$, i.e., $u_n \to u$ in $H_0^2(\Omega)$ as $n \to \infty$. The proof is complete. $\square$

The following lemma shows that the mountain pass level of $I_\lambda(u)$ around 0 is strictly less than $c(S_\lambda)$; this is parallel to Lemma 3.7.

Lemma 3.9. *Assume that* (A1), (A2) *and the condition (iii) of Theorem 2.6 hold. Then there exists a* $\widetilde{u}_\lambda^* \in H_0^2(\Omega)$ *such that*

$$\sup_{t\geq 0} I_\lambda(t\widetilde{u}_\lambda^*) < c(S_\lambda), \tag{3.40}$$

where $c(S_\lambda) := \frac{2}{N}S_\lambda^{N/4}$.

Proof. To estimate the mountain pass level of I_λ around 0, we introduce the Talenti functions. For any $\varepsilon > 0$, define

$$\widetilde{U}_{\varepsilon,\lambda}(x) = \varepsilon^{2-\frac{N}{2}}\widetilde{U}_\lambda(\frac{x}{\varepsilon}), \quad \lambda \in [0, \lambda_{4,2}),$$

where $\widetilde{U}_\lambda(x) > 0$ is radially symmetric and is a solution to the critical problem

$$\Delta^2 u - \lambda\frac{u}{|x|^4} = u^{2^{**}-1}, \quad x \in \mathbb{R}^N, \quad N \geq 5.$$

Then, according to [10, Theorem 1.1], $\widetilde{U}_{\varepsilon,\lambda}(x)$ is an extremal function of S_λ and

$$\int_{\mathbb{R}^N}\Big(|\Delta\widetilde{U}_{\varepsilon,\lambda}(x)|^2 - \lambda\frac{|\widetilde{U}_{\varepsilon,\lambda}(x)|^2}{|x|^4}\Big)\mathrm{d}x = \int_{\mathbb{R}^N}|\widetilde{U}_{\varepsilon,\lambda}(x)|^{2^{**}}\mathrm{d}x = S_\lambda^{N/4},$$

where S_λ is given in Remark 3.4. Set $r = |x|$, then one also has

$$\begin{gathered} \widetilde{U}_\lambda(r) = O_1(r^{-\gamma_\lambda}), \quad \widetilde{U}_\lambda'(r) = O_1(r^{-\gamma_\lambda-1}), \quad r \to +\infty, \\ \widetilde{U}_\lambda(r) = O_1(r^{-\widetilde{\gamma}_\lambda}), \quad r \to 0, \end{gathered}$$

where $\gamma_\lambda = (N-4)(1-\frac{\alpha}{2})$, $\widetilde{\gamma}_\lambda = (N-4)\frac{\alpha}{2}$, and

$$\alpha := \varphi(\lambda) = 1 - \frac{\sqrt{N^2-4N+8-4\sqrt{(N-2)^2+\lambda}}}{N-4}, \quad \lambda \in (0,\lambda_{4,2}).$$

It directly verifies that $\varphi : [0,\lambda_{4,2}] \to [0,1]$ is continuous and strictly increasing. Therefore, $\alpha \in (0,1)$ and $0 < \widetilde{\gamma}_\lambda < \frac{N-4}{2} < \gamma_\lambda < N-4$ for $\lambda \in (0,\lambda_{4,2})$.

Let $\tau(x)$ be given by Lemma 3.6 and set $\widetilde{u}_{\varepsilon,\lambda}(x) = \tau(x)\widetilde{U}_{\varepsilon,\lambda}(x)$. Then as $\varepsilon \to 0$, we have (see [10, 12])

$$\begin{gathered}\int_\Omega \Big(|\Delta \widetilde{u}_{\varepsilon,\lambda}(x)|^2 - \lambda \frac{|\widetilde{u}_{\varepsilon,\lambda}(x)|^2}{|x|^4}\Big)\mathrm{d}x = S_\lambda^{N/4} + O(\varepsilon^{2(\gamma_\lambda - \frac{N-4}{2})}),\\ \|\widetilde{u}_{\varepsilon,\lambda}\|_{2^{**}}^{2^{**}} = S_\lambda^{N/4} + O(\varepsilon^{2^{**}\gamma_\lambda - N}),\end{gathered}$$

and

$$\|\widetilde{u}_{\varepsilon,\lambda}\|_\rho^\rho = \begin{cases} O_1(\varepsilon^{\rho(\gamma_\lambda - \frac{N-4}{2})}), & 1 \le \rho < \frac{N}{\gamma_\lambda},\\ O_1(\varepsilon^{N - \frac{N-4}{2}\rho}|\ln \varepsilon|), & \rho = \frac{N}{\gamma_\lambda},\\ O_1(\varepsilon^{N-\frac{N-4}{2}\rho}), & \frac{N}{\gamma_\lambda} < \rho < 2^{**}. \end{cases} \tag{3.41}$$

As in the proof of Lemma 3.7, we consider the fibering maps $\psi_u(t)$ given in (3.22) with $u = \widetilde{u}_{\varepsilon,\lambda}$. Since (3.23) still holds for $\psi_{\widetilde{u}_{\varepsilon,\lambda}}(t)$, there exists a $\widetilde{t}_0 > 0$, independent of ε, such that

$$\psi_{\widetilde{u}_{\varepsilon,\lambda}}(t) < c(S_\lambda), \quad t \in (0,\widetilde{t}_0). \tag{3.42}$$

Next, we set

$$\widetilde{g}(t) = \frac{1}{2}t^2 \int_\Omega \Big(|\Delta \widetilde{u}_{\varepsilon,\lambda}|^2 - \lambda\frac{|\widetilde{u}_{\varepsilon,\lambda}|^2}{|x|^4}\Big)\mathrm{d}x - \frac{1}{2^{**}} t^{2^{**}} \|\widetilde{u}_{\varepsilon,\lambda}\|_{2^{**}}^{2^{**}}.$$

Then

$$\psi_{\widetilde{u}_{\varepsilon,\lambda}}(t) = \widetilde{g}(t) - \int_\Omega F(x, t\widetilde{u}_{\varepsilon,\lambda})\mathrm{d}x.$$

On recalling (1.5) one sees that there exists a positive constant C such that

$$\begin{aligned}\psi_{\widetilde{u}_{\varepsilon,\lambda}}(t) &\le \max_{t \ge \widetilde{t}_0} \widetilde{g}(t) - \int_\Omega F(x, t\widetilde{u}_{\varepsilon,\lambda})\mathrm{d}x\\ &\le \max_{t\ge 0} \widetilde{g}(t) - C\widetilde{t}_0^\rho \|\widetilde{u}_{\varepsilon,\lambda}\|_\rho^\rho, \quad t \in [\widetilde{t}_0, \infty).\end{aligned} \tag{3.43}$$

Direct computations show that $\widetilde{g}$ attains its maximum at

$$\widetilde{t}_\varepsilon^* := \Big(\frac{\int_\Omega (|\Delta \widetilde{u}_{\varepsilon,\lambda}|^2 - \lambda\frac{|\widetilde{u}_{\varepsilon,\lambda}|^2}{|x|^4})\mathrm{d}x}{\|\widetilde{u}_{\varepsilon,\lambda}\|_{2^{**}}^{2^{**}}}\Big)^{\frac{1}{2^{**}-2}}. \tag{3.44}$$

Therefore, by (3.43) and (3.44), one sees that for $t \in [\widetilde{t}_0, \infty)$,

$$\begin{aligned}\psi_{\widetilde{u}_{\varepsilon,\lambda}}(t) &\le \widetilde{g}(\widetilde{t}_\varepsilon^*) - C\widetilde{t}_0^\rho\|\widetilde{u}_{\varepsilon,\lambda}\|_\rho^\rho\\ &= \Big(\frac{1}{2} - \frac{1}{2^{**}}\Big)\frac{\big(\int_\Omega(|\Delta\widetilde{u}_{\varepsilon,\lambda}|^2 - \lambda\frac{|\widetilde{u}_{\varepsilon,\lambda}|^2}{|x|^4})\mathrm{d}x\big)^{N/4}}{\|\widetilde{u}_{\varepsilon,\lambda}\|_{2^{**}}^{\frac{(N-4)2^{**}}{4}}} - C\widetilde{t}_0^\rho\|\widetilde{u}_{\varepsilon,\lambda}\|_\rho^\rho\\ &= \frac{2}{N}\big(S_\lambda^{N/4} + O(\varepsilon^{2(\gamma_\lambda - \frac{N-4}{2})})\big)^{N/4}\big(S_\lambda^{N/4} + O(\varepsilon^{2^{**}\gamma_\lambda - N})\big)^{\frac{4-N}{4}} - C\widetilde{t}_0^\rho\|\widetilde{u}_{\varepsilon,\lambda}\|_\rho^\rho\\ &= c(S_\lambda) + O(\varepsilon^{2(\gamma_\lambda - \frac{N-4}{2})}) - C\widetilde{t}_0^\rho\|\widetilde{u}_{\varepsilon,\lambda}\|_\rho^\rho, \quad \text{as } \varepsilon \to 0,\end{aligned}$$

where in the last equality we have used the fact that

$$(2^{**}\gamma_\lambda - N)\frac{N-4}{4} - 2\Big(\gamma_\lambda - \frac{N-4}{2}\Big) = 4 - 4\alpha \ge 0.$$

To complete the proof, it remains to show that for ε suitably small,

$$\psi_{\widetilde{u}_{\varepsilon,\lambda}}(t) < c(S_\lambda), \quad t \in [\widetilde{t}_0, \infty), \tag{3.45}$$

which is fulfilled once we can prove that

$$O(\varepsilon^{2(\gamma_\lambda-\frac{N-4}{2})}) - C\widetilde{t}_0^{\rho}\|\widetilde{u}_{\varepsilon,\lambda}\|_\rho^\rho < 0. \tag{3.46}$$

In view of (3.41), if $\rho > \max\{\frac{N}{\gamma_\lambda}, \frac{4(N-2-\gamma_\lambda)}{N-4}\}$, one sees that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{2(\gamma_\lambda-\frac{N-4}{2})}}{\varepsilon^{N-\frac{N-4}{2}\rho}} = \lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{2}[\rho(N-4)-4(N-2-\gamma_\lambda)]} = 0,$$

which implies (3.46) is valid for suitably small ε. Therefore, for each $\lambda \in (0, \lambda_{4,2})$, (3.45) is valid for suitably small ε, which, together with (3.42), implies, for ε suitably small, that

$$\sup_{t\geq 0}\psi_{\widetilde{u}_{\varepsilon,\lambda}}(t) < c(S_\lambda).$$

Fixing such an $\varepsilon > 0$ and letting $\widetilde{u}_\lambda^* \equiv \widetilde{u}_{\varepsilon,\lambda}$, we obtain (3.40). The proof is complete. ☐

Proof of Theorem 2.6 (iii). With the help of Lemmas 3.8 and 3.9, we can prove Theorem 2.6 (iii) in a way similar to that of Theorem 2.6 (i) and (ii), and we only sketch the outline. First, by recalling (3.31) and (3.32) with $q = 2$ and $s = 4$, we see that I_λ satisfies the mountain pass geometry around 0 and there exists a sequence $\{u_n\} \subset H_0^2(\Omega)$ such that $I_\lambda(u_n) \to \widetilde{c}_0$ and $I_\lambda'(u_n) \to 0$ in $H^{-2}(\Omega)$ as $n \to \infty$, where

$$\widetilde{c}_0 = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}}\max_{t\in[0,1]} I_\lambda(\widetilde{\gamma}(t)) \quad \text{and} \quad \widetilde{\Gamma} = \{\widetilde{\gamma} \in C([0,1], H_0^2(\Omega)) : \widetilde{\gamma}(0) = 0, \widetilde{\gamma}(1) = t_{\widetilde{u}_\lambda^*}\widetilde{u}_\lambda^*\},$$

and $\widetilde{u}_\lambda^*$ is given in Lemma 3.9 which satisfies $I_\lambda(t_{\widetilde{u}_\lambda^*}\widetilde{u}_\lambda^*) < 0$. In view of Lemma 3.9, we have

$$\widetilde{c}_0 \leq \max_{t\in[0,1]} I_\lambda(tt_{\widetilde{u}_\lambda^*}\widetilde{u}_\lambda^*) \leq \sup_{t\geq 0} I_\lambda(t\widetilde{u}_\lambda^*) < c(S_\lambda).$$

Consequently, by Lemma 3.8, one sees that $I_\lambda(u)$ satisfies the $(PS)_{\widetilde{c}_0}$ condition. Then there exists a u such that $I_\lambda(u) = \widetilde{c}_0$ and $I_\lambda'(u) = 0$, i.e., u is a nontrivial weak solution to problem (1.1). The proof is complete. ☐

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