Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 32, pp. 1–14.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.32

LINEARIZED HIGH-ORDER AND CONVERGENT SCHEME FOR THE KURAMOTO-SIVASHINSKY EQUATION

YIRAN ZHANG, GUOHUI WANG, YUANFENG JIN

Abstract. In this article, we provide a linearized compact scheme for the Kuramoto-Sivashinsky equation with the periodic boundary condition. By applying the double reduction order method and a fourth-order compact operator, the scheme achieves second-order convergence in time and fourth-order convergence in space. We present proofs of the conservation, unique solvability, and convergence of the scheme.

1. Introduction

The Kuramoto-Sivashinsky equation is a representative nonlinear dissipative partial differential equation which are commonly used to describe the dynamical behavior of unsteady fluctuations and turbulence [14, 18]. Kuramoto and Tsuzuki derived it from a general two-component reaction-diffusion system and regarded it as a one-dimensional representation of the ring-like propagating pattern observed experimentally in the Belousov-Zhabotinsky reaction. Sivashinsky obtained the equation by analyzing the spontaneous instability of a laminar flame's planar front. It was also obtained by Cohen et al. [8] through the formulation of a rigorous mathematical model that investigates the nonlinear saturation of the dissipative trapped-ion mode. Kudryashov et al. [12] employed the Kuramoto-Sivashinsky equation to capture nonlinear long-wave propagation in a viscous-elastic tube. In addition, it has a wide range of applications in spatiotemporally chaotic dynamics [6], free surface film-flows and hydrodynamic instabilities [4, 19].

Many scholars have investigated its analytical solutions and related properties from a theoretical perspective. Lan and Cvitanović [17] explored unstable recurrent patterns in the Kuramoto-Sivashinsky system, employing Newton descent method to identify periodic solutions and characterizing chaotic dynamics via low-dimensional return maps and symbolic dynamics. Dong and Lan [9] systematically investigated spatially periodic steady solutions of the Kuramoto-Sivashinsky equation through the lens of dynamical systems, employing symbolic dynamics for classification and analyzing bifurcations of fundamental cycles. Kudryashov and Soukharev [13] demonstrated that many different ansatz methods for solving the Kuramoto-Sivashinsky equation lead to equivalent results. They analyzed various 'new' solitary wave solutions and showed that these can be reduced to two known solutions through algebraic transformations. Frisch et al. [10] analyzed the Kuramoto-Sivashinsky equation and demonstrated that the stability of cellular solutions is related to their viscoelastic behavior under large-scale perturbations. Cerpa and Mercado [5] proved the local exact controllability of the Kuramoto-Sivashinsky equation's trajectories using Carleman estimates for the linear system and a local inversion theorem for the nonlinear system.

The Kuramoto-Sivashinsky equation involves the interaction of several physical processes, such as nonlinear convection, diffusion, and higher-order dissipation effects. Together, these factors lead to wave instability and complex spatiotemporal structures, making the exact analytical solution difficult to obtain. Hence, it is especially meaningful to devise efficient algorithms for the Kuramoto-Sivashinsky equation. Christov and Bekyarov [7] introduced a Fourier-Galerkin method

---

2020 *Mathematics Subject Classification.* 65M06, 65M12.
Key words and phrases. Kuramoto-Sivashinsky equation; linearized compact scheme; reduction order method; conservation law; convergence.

Submitted November 19, 2025. Published April 30, 2026.

for solving the soliton solution of the Kuramoto-Sivashinsky equation and demonstrated its high accuracy and efficiency. Akrivis [1] presented a Crank-Nicolson-type finite difference scheme for the Kuramoto-Sivashinsky equation with periodic boundary conditions, discussed the linearization of the nonlinearity, derived second-order error estimates, and proved the existence and uniqueness of the scheme. Akrivis [2] also investigated semidiscrete and fully discrete finite element methods with second-order time accuracy to solve the Kuramoto-Sivashinsky equation. Akrivis and Smyrlis [3] performed a numerical analysis using implicit-explicit BDF schemes for temporal discretization and pseudo-spectral methods for spatial discretization. Xu and Shu [22] developed a local discontinuous Galerkin method for the Kuramoto-Sivashinsky equation, achieving L^2 stability and high-order accuracy. Uddin et al. [21] proposed a mesh-free numerical method using radial basis functions to solve the Kuramoto-Sivashinsky equations, showing its accuracy and robustness in comparison to traditional methods. Researchers have developed numerical methods not only for the Kuramoto-Sivashinsky equation, but also for its generalized form [11, 15, 16].

In this article, we consider the following version of Kuramoto-Sivashinsky equation with the periodical boundary condition [1]

$$u_t + \alpha u_{xx} + \beta u_{xxxx} + \gamma u u_x = 0, \quad x \in \mathbb{R}, \ \ t \in [0, T], \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

$$u(x, t) = u(x + L, t), \quad x \in \mathbb{R}, \quad t \in [0, T], \tag{1.3}$$

where α, β, γ, L and T are positive real constants, L denotes the spatial period, and $u_0(x)$ is a given smooth function. The purpose of the study is to present a linearized compact difference scheme for the Kuramoto-Sivashinsky equation via the order reduction method, which has fourth-order in space and second-order in time. This provides more methodical options for the study of the Kuramoto-Sivashinsky equation.

The rest of this article is organized as follows. In Section 2, employing the double reduction order method, we design a three-level linearized finite difference scheme for the Kuramoto-Sivashinsky equation. In Section 3, we obtain a conservative invariant of the difference scheme. Existence and uniqueness of the solution are established with a rigorous argument in Section 4. Finally, we demonstrate the convergence of the difference scheme in Section 5.

2. Construction of the difference scheme

This section applies the double order-reduction approach to construct a three-level linear difference scheme for the problem (1.1)-(1.3) and provides a detailed analysis of the truncation errors.

We divide the domain $[0, L] \times [0, T]$ and take positive integers M and N. Let $h = L/M$ and $\tau = T/N$. Denote index sets $I_h = \{j : 1 \le j \le M\}$ and $I_\tau = \{k : 0 \le k \le N\}$. We denote $\Omega_h = \{x_j = jh : j \in I_h \cup \{0\}\}$, $\Omega_\tau = \{t_k = k\tau : k \in I_\tau\}$ and $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$.

For any grid function $u = \{u_j^k : j \in I_h \cup \{0\}, k \in I_\tau\}$ on $\Omega_{h\tau}$, we denote

$$\begin{gathered}
\delta_x^+ u_j^k = \frac{1}{h}(u_{j+1}^k - u_j^k), \quad \delta_x^2 u_j^k = \frac{1}{h}(\delta_x^+ u_j^k - \delta_x^+ u_{j-1}^k), \\
\Delta_x u_j^k = \frac{1}{2h}(u_{j+1}^k - u_{j-1}^k), \quad u_{j+\frac{1}{2}}^k = \frac{1}{2}(u_j^k + u_{j+1}^k), \\
u_j^{k+\frac{1}{2}} = \frac{1}{2}(u_j^k + u_j^{k+1}), \quad u_j^{\bar{k}} = \frac{1}{2}(u_j^{k-1} + u_j^{k+1}), \\
\delta_t u_j^{k+\frac{1}{2}} = \frac{1}{\tau}(u_j^{k+1} - u_j^k), \quad \Delta_t u_j^k = \frac{1}{2\tau}(u_j^{k+1} - u_j^{k-1}).
\end{gathered}$$

Let $\mathcal{U}_h = \{u : u = \{u_j\}, \ u_{j+M} = u_j\}$. For any grid functions $u, z \in \mathcal{U}_h$, we define the discrete inner product $\langle u, z \rangle = h \sum_{j=1}^M u_j z_j$ and the corresponding norms (seminorm)

$$\|u\| = \sqrt{\langle u, u \rangle}, \quad |u|_1 = \sqrt{\langle \delta_x^+ u, \delta_x^+ u \rangle}, \quad \|u\|_\infty = \max_{1 \le j \le M} |u_j|.$$

In addition, define the function

$$\phi(u, z)_j = \frac{1}{3}\left[\Delta_x (uz)_j + u_j \Delta_x z_j\right], \quad j \in I_h.$$

Lemma 2.1 ([23, Lemma 2.3])**.** *Let $k(x)$ belongs to $C^5[x_{j-1}, x_{j+1}]$ and denote $K_j = k(x_j)$ and $G_j = k''(x_j)$, then for $j \in I_h$, we have*

$$k(x_j)k'(x_j) = \phi(K, K)_j - \frac{h^2}{2}\phi(G, K)_j + \mathcal{O}(h^4),$$
$$k''(x_j) = \delta_x^2 K_j - \frac{h^2}{12}\delta_x^2 G_j + \mathcal{O}(h^4).$$

We rewrite (1.1)-(1.3) as

$$u_t + \alpha z + \beta w + \gamma u u_x = 0, \quad x \in \mathbb{R}, \quad t \in [0, T], \tag{2.1}$$
$$z = u_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T], \tag{2.2}$$
$$w = z_{xx}, \quad x \in \mathbb{R}, \quad t \in [0, T], \tag{2.3}$$
$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \tag{2.4}$$
$$\begin{gathered} u(x,t) = u(x+L,t), \quad z(x,t) = z(x+L,t), \quad w(x,t) = w(x+L,t), \\ x \in \mathbb{R}, \quad t \in [0,T]. \end{gathered} \tag{2.5}$$

We denote

$$\hat{c}_0 = \max_{0 \le x \le L, t \in [0,T]} \{|u(x,t)|, |u_x(x,t)|, |u_{xx}(x,t)|, |u_{xxx}(x,t)|\}$$

and define grid functions

$$U_j^k = u(x_j, t_k), \quad Z_j^k = z(x_j, t_k), \quad W_j^k = w(x_j, t_k), \quad j \in I_h, \;\; k \in I_\tau.$$

Considering (2.1) at the point $(x_j, t_{1/2})$, we have

$$u(x_j, t_{1/2}) + \alpha z(x_j, t_{1/2}) + \beta w(x_j, t_{1/2}) + \gamma u(x_j, t_{1/2}) u_x(x_j, t_{1/2}) = 0, \quad j \in I_h. \tag{2.6}$$

From the Taylor expansion and (1.1), we obtain

$$\begin{aligned} u(x_j, t_{1/2}) =& u(x_j, t_0) + \frac{\tau}{2} u_t(x_j, t_0) + \mathcal{O}(\tau^2) \\ =& u(x_j, t_0) + \frac{\tau}{2}(-\alpha u_{xx}(x_j, t_0) - \beta u_{xxxx}(x_j, t_0) - \gamma u(x_j, t_0) u_x(x_j, t_0)) + \mathcal{O}(\tau^2) \\ =& u_0(x_j) - \frac{\tau}{2}(\alpha u_0''(x_j) + \beta u_0''''(x_j) + \gamma u_0(x_j) u_0'(x_j)) + \mathcal{O}(\tau^2). \end{aligned}$$

We denote

$$\hat{u}(x) = u_0(x) - \frac{\tau}{2}(\alpha u_0''(x) + \beta u_0''''(x) + \gamma u_0(x) u_0'(x)), \quad \hat{z}(x) = \hat{u}''(x),$$
$$\hat{u}_j = \hat{u}(x_j), \quad \hat{z}_j = \hat{z}(x_j), \quad j \in I_h.$$

Evidently, there exists a constant $\tilde{c}_0 > 0$ such that

$$\max_{0 \le x \le L} \{|\hat{u}(x)|, |\hat{u}'(x)|, |\hat{z}(x)|, |\hat{z}'(x)|\} \le \tilde{c}_0.$$

Then we denote

$$c_0 = \max\{\hat{c}_0, \tilde{c}_0\}. \tag{2.7}$$

Lemma 2.1 leads to

$$u(x_j, t_{1/2}) u_x(x_j, t_{1/2}) = \phi(\hat{u}, U^{1/2})_j - \frac{h^2}{2}\phi(\hat{z}, U^{1/2})_j + \mathcal{O}(\tau^2 + h^4). \tag{2.8}$$

Substituting (2.8) into (2.6) and using Taylor expansions, we obtain

$$\delta_t U_j^{1/2} + \alpha Z_j^{1/2} + \beta W_j^{1/2} + \gamma \big[\phi(\hat{u}, U^{1/2})_j - \frac{h^2}{2}\phi(\hat{z}, U^{1/2})_j\big] = Q_j^0, \quad j \in I_h, \tag{2.9}$$

where

$$|Q_j^0| \le c_1(\tau^2 + h^4), \quad j \in I_h, \tag{2.10}$$

with c_1 being a positive constant.

For $j \in I_h$ and $k \in I_\tau \setminus \{0, N\}$, from (2.1) at (x_j, t_k) we derive

$$\Delta_t U_j^k + \alpha Z_j^{\bar{k}} + \beta W_j^{\bar{k}} + \gamma \big[\phi(U^k, U^{\bar{k}})_j - \frac{h^2}{2}\phi(Z^k, U^{\bar{k}})_j\big] = Q_j^k, \tag{2.11}$$

where

$$|Q_j^k| \le c_2(\tau^2 + h^4), \tag{2.12}$$

with c_2 being a positive constant.

For $j \in I_h$ and $k \in I_\tau \setminus \{N\}$, from (2.2) and (2.3) at (x_j, t_k) we derive

$$Z_j^k = \delta_x^2 U_j^k - \frac{h^2}{12}\delta_x^2 Z_j^k + R_j^k, \quad W_j^k = \delta_x^2 Z_j^k - \frac{h^2}{12}\delta_x^2 W_j^k + S_j^k. \tag{2.13}$$

There are two positive constant c_3 and c_4 such that the local truncation errors satisfying

$$|R_j^k| \le c_3 h^4, \quad |\Delta_t R_j^k| \le c_3(\tau^2 + h^4), \tag{2.14}$$

$$|S_j^k| \le c_4 h^4, \quad |\Delta_t S_j^k| \le c_4(\tau^2 + h^4). \tag{2.15}$$

Taking into account the initial and boundary conditions (2.4)-(2.5), we obtain

$$U_j^0 = u_0(x_j), \quad j \in I_h, \tag{2.16}$$

$$U_j^k = U_{j+M}^k, \quad Z_j^k = Z_{j+M}^k, \quad W_j^k = W_{j+M}^k, \quad j \in I_h,\ k \in I_\tau. \tag{2.17}$$

Omitting the small terms Q_j^k, R_j^k and S_j^k, replacing the grid functions U_j^k, Z_j^k, W_j^k by u_j^k, z_j^k, w_j^k in (2.9), (2.11), (2.13) respectively, and noticing the initial and boundary conditions (2.16)-(2.17), we construct a finite difference scheme for (2.1)-(2.5) as follows

$$\delta_t u_j^{1/2} + \alpha z_j^{1/2} + \beta w_j^{1/2} + \gamma\big[\phi(\hat{u}, u^{1/2})_j - \frac{h^2}{2}\phi(\hat{z}, u^{1/2})_j\big] = 0, \quad j \in I_h, \tag{2.18}$$

$$\begin{gathered}\Delta_t u_j^k + \alpha z_j^{\bar{k}} + \beta w_j^{\bar{k}} + \gamma\big[\phi(u^k, u^{\bar{k}})_j - \frac{h^2}{2}\phi(z^k, u^{\bar{k}})_j\big] = 0, \\ j \in I_h,\ k \in I_\tau \setminus \{0, N\},\end{gathered} \tag{2.19}$$

$$z_j^k = \delta_x^2 u_j^k - \frac{h^2}{12}\delta_x^2 z_j^k, \quad j \in I_h,\ k \in I_\tau, \tag{2.20}$$

$$w_j^k = \delta_x^2 z_j^k - \frac{h^2}{12}\delta_x^2 w_j^k, \quad j \in I_h,\ k \in I_\tau, \tag{2.21}$$

$$u_j^0 = u_0(x_j), \quad j \in I_h, \tag{2.22}$$

$$u_j^k = u_{j+M}^k, \quad z_j^k = z_{j+M}^k, \quad w_j^k = w_{j+M}^k, \quad j \in I_h,\ k \in I_\tau. \tag{2.23}$$

3. Conservation law

This section explores the conservation law of the difference scheme (2.18)-(2.23). We introduce two lemmas that are used throughout the proof.

Lemma 3.1 ([20, Chapter 1])**.** *For arbitrary grid functions $u, z \in \mathcal{U}_h$, we have*

$$\|z\|_\infty \le \frac{\sqrt{L}}{2}|z|_1, \quad |z|_1 \le \frac{2}{h}\|z\|, \quad \|z\| \le \frac{L}{\sqrt{6}}|z|_1,$$
$$\langle \delta_x^2 u, z\rangle = -\langle \delta_x^+ u, \delta_x^+ z\rangle, \quad \langle \phi(u, z), z\rangle = 0.$$

Lemma 3.2. *For any grid functions $u, z, w, R, S \in \mathcal{U}_h$, satisfying*

$$z_j = \delta_x^2 u_j - \frac{h^2}{12}\delta_x^2 z_j + R_j, \quad w_j = \delta_x^2 z_j - \frac{h^2}{12}\delta_x^2 w_j + S_j, \quad j \in I_h,$$
$$u_j = u_{j+M}, \quad z_j = z_{j+M}, \quad w_j = w_{j+M}, \quad j \in I_h \cup \{0\},$$

we have

$$\langle z, u\rangle = -|u|_1^2 - \frac{h^2}{12}\|z\|^2 + \frac{h^4}{144}|z|_1^2 + \frac{h^2}{12}\langle z, R\rangle + \langle R, u\rangle, \tag{3.1}$$

$$\langle z, u\rangle \le -|u|_1^2 - \frac{h^2}{18}\|z\|^2 + \frac{h^2}{12}\langle z, R\rangle + \langle R, u\rangle, \tag{3.2}$$

$$\langle w, u\rangle = \|z\|^2 - \langle z, R\rangle + \frac{h^2}{12}\langle w, R\rangle - \frac{h^2}{12}\langle S, z\rangle + \langle S, u\rangle. \tag{3.3}$$

Proof. The proof of (3.1)-(3.2) comes from [23]. The proof of (3.3) is as follows

$$\begin{aligned}\langle w,u\rangle &= \big\langle \delta_x^2 z-\frac{h^2}{12}\delta_x^2 w+P,u\big\rangle\\ &= \|z\|^2-\frac{h^2}{12}\,|z|_1^2-\langle z,R\rangle-\frac{h^2}{12}\big\langle\delta_x^2 z-\frac{h^2}{12}\delta_x^2 w+P,z\big\rangle-\frac{h^4}{144}\,\langle w,\delta_x^2 z\rangle+\frac{h^2}{12}\,\langle w,R\rangle+\langle P,u\rangle\\ &= \|z\|^2-\langle z,R\rangle+\frac{h^2}{12}\,\langle w,R\rangle-\frac{h^2}{12}\,\langle P,z\rangle+\langle P,u\rangle\,. \end{aligned}$$ □

Theorem 3.3. *Suppose that* $\{u_j^k,z_j^k,w_j^k : j\in I_h, k\in I_\tau\}$ *are solutions of* (2.18)-(2.23). *Then*

$$\frac{1}{2}(\|u^1\|^2+\|u^0\|^2)+\beta\tau\|z^{1/2}\|^2-\alpha\tau\Big(|u^{1/2}|_1^2+\frac{h^2}{12}\|z^{1/2}\|^2-\frac{h^4}{144}|z^{1/2}|_1^2\Big)=\|u^0\|^2, \tag{3.4}$$

$$E(u^{k+1},u^k)=E(u^1,u^0),\quad k\in I_\tau\setminus\{0,N\}, \tag{3.5}$$

where

$$E(u^{k+1},u^k)=\frac{1}{2}(\|u^{k+1}\|^2+\|u^k\|^2)+2\beta\tau\sum_{l=1}^{k}\|z^{\bar l}\|^2-2\alpha\tau\sum_{l=1}^{k}\Big(|u^{\bar l}|_1^2+\frac{h^2}{12}\|z^{\bar l}\|^2-\frac{h^4}{144}|z^{\bar l}|_1^2\Big).$$

Proof. Taking the inner product of (2.18) with $u^{1/2}$ and applying Lemma 3.1 leads to

$$\langle\delta_t u^{1/2},u^{1/2}\rangle+\alpha\langle z^{1/2},u^{1/2}\rangle+\beta\langle w^{1/2},u^{1/2}\rangle=0. \tag{3.6}$$

Averaging (2.20) and (2.21) with superscripts $k=0$ and $k=1$ yields

$$z_j^{1/2}=\delta_x^2u_j^{1/2}-\frac{h^2}{12}\delta_x^2z_j^{1/2},\quad w_j^{1/2}=\delta_x^2z_j^{1/2}-\frac{h^2}{12}\delta_x^2w_j^{1/2},\quad j\in I_h.$$

For the second and third terms of (3.6), applying (3.1) and (3.3) in Lemma 3.2, we obtain

$$\langle z^{1/2},u^{1/2}\rangle=-|u^{1/2}|_1^2-\frac{h^2}{12}\|z^{1/2}\|^2+\frac{h^4}{144}|z^{1/2}|_1^2,\quad \langle w^{1/2},u^{1/2}\rangle=\|z^{1/2}\|^2.$$

Substituting above equations into (3.6) gives

$$\frac{1}{2\tau}(\|u^1\|^2-\|u^0\|^2)-\alpha\Big(|u^{1/2}|_1^2+\frac{h^2}{12}\|z^{1/2}\|^2-\frac{h^4}{144}|z^{1/2}|_1^2\Big)+\beta\|z^{1/2}\|^2=0.$$

Rearranging the above equation, we obtain (3.4).

Taking the inner product of (2.19) with $u^{\bar k}$ and applying Lemma 3.1 gives

$$\langle\delta_t u^k,u^{\bar k}\rangle+\alpha\langle z^{\bar k},u^{\bar k}\rangle+\beta\langle w^{\bar k},u^{\bar k}\rangle=0,\quad k\in I_\tau\setminus\{0,N\}. \tag{3.7}$$

According to (2.20) and (2.21), we have

$$z_j^{\bar k}=\delta_x^2u_j^{\bar k}-\frac{h^2}{12}\delta_x^2z_j^{\bar k},\quad w_j^{\bar k}=\delta_x^2z_j^{\bar k}-\frac{h^2}{12}\delta_x^2w_j^{\bar k},\quad j\in I_h,\ \ k\in I_\tau\setminus\{0,N\}.$$

Similar to the derivation of $\langle z^{1/2},u^{1/2}\rangle$ and $\langle w^{1/2},u^{1/2}\rangle$, for $k\in I_\tau\setminus\{0,N\}$, we have

$$\langle z^{\bar k},u^{\bar k}\rangle=-|u^{\bar k}|_1^2-\frac{h^2}{12}\|z^{\bar k}\|^2+\frac{h^4}{144}|z^{\bar k}|_1^2,\quad \langle w^{\bar k},u^{\bar k}\rangle=\|z^{\bar k}\|^2.$$

Substituting above equations into (3.7) yields

$$\frac{1}{4\tau}(\|u^{k+1}\|^2-\|u^{k-1}\|^2)-\alpha\Big(|u^{\bar k}|_1^2+\frac{h^2}{12}\|z^{\bar k}\|^2-\frac{h^4}{144}|z^{\bar k}|_1^2\Big)+\beta\|z^{\bar k}\|^2=0,$$

for $k\in I_\tau\setminus\{0,N\}$. Replacing the superscript k with l and summing for l from 1 to k, we obtain (3.5). □

4. Existence and uniqueness

This section investigates the conditions for the unique solvability of the difference scheme presented in (2.18)-(2.23).

Theorem 4.1. *When $h < \sqrt{12\beta/\alpha}$ and $\tau < h^2/4\alpha$, the finite difference scheme* (2.18)-(2.23) *is uniquely solvable.*

Proof. From (2.20)-(2.23), u^0, z^0 and w^0 are determined. From (2.18), (2.20) and (2.21), the first level yields a linear system of equations about u^1, z^1 and w^1. Now we consider its homogeneous linear system of equations

$$\frac{1}{\tau}u_j^1 + \frac{\alpha}{2}z_j^1 + \frac{\beta}{2}w_j^1 + \frac{\gamma}{2}\phi(u^0,u^1)_j - \frac{\gamma h^2}{4}\phi(z^0,u^1)_j = 0, \quad j \in I_h, \tag{4.1}$$

$$z_j^1 = \delta_x^2 u_j^1 - \frac{h^2}{12}\delta_x^2 z_j^1, \quad j \in I_h, \tag{4.2}$$

$$w_j^1 = \delta_x^2 z_j^1 - \frac{h^2}{12}\delta_x^2 w_j^1, \quad j \in I_h. \tag{4.3}$$

Taking the inner product of (4.1) with u^1 and combining Lemma 3.1, we have

$$\frac{1}{\tau}\|u^1\|^2 + \frac{\alpha}{2}\langle z^1, u^1\rangle + \frac{\beta}{2}\langle w^1, u^1\rangle = 0. \tag{4.4}$$

Applying (3.1) and (3.3) in Lemma 3.2 yields

$$\langle z^1, u^1\rangle = -|u^1|_1^2 - \frac{h^2}{12}\|z^1\|^2 + \frac{h^4}{144}|z^1|_1^2,$$
$$\langle w^1, u^1\rangle = -\|z^1\|^2.$$

Substituting above into (4.4) and combining Lemma 3.1, we obtain

$$\begin{aligned} 0 &= \frac{1}{\tau}\|u^1\|^2 + \frac{\alpha}{2}\Big(-|u^1|_1^2 - \frac{h^2}{12}\|z^1\|^2 + \frac{h^4}{144}|z^1|_1^2\Big) + \frac{\beta}{2}\|z^1\|^2 \\ &\geq \Big(\frac{1}{\tau} - \frac{2\alpha}{h^2}\Big)\|u^1\|^2 + \Big(\frac{\beta}{2} - \frac{\alpha h^2}{24}\Big)\|z^1\|^2 + \frac{\alpha h^4}{288}|z^1|_1^2. \end{aligned}$$

Hence, it follows that $\|u^1\| = 0$, $\|z^1\| = 0$, when $h^2 < 12\beta/\alpha$ and $\tau < h^2/2\alpha$. Therefore, (4.1)-(4.3) only allow zeros solutions, which implies that (2.18), (2.20) and (2.21) determine u^1, z^1 and w^1 uniquely.

Now we suppose that u^{k-1}, u^k, z^{k-1}, z^k, w^{k-1}, w^k have been determined. From (2.19)-(2.21) a linear system of equations with respect to u^{k+1}, z^{k+1} and w^{k+1} is obtained. Then we consider the homogeneous system of equations

$$\frac{1}{2\tau}u_j^{k+1} + \frac{\alpha}{2}z_j^{k+1} + \frac{\beta}{2}w_j^{k+1} + \frac{\gamma}{2}\phi(u^k,u^{k+1})_j - \frac{\gamma h^2}{4}\phi(z^k,u^{k+1})_j = 0, \quad j \in I_h, \tag{4.5}$$

$$z_j^{k+1} = \delta_x^2 u_j^{k+1} - \frac{h^2}{12}\delta_x^2 z_j^{k+1}, \quad j \in I_h, \tag{4.6}$$

$$w_j^{k+1} = \delta_x^2 z_j^{k+1} - \frac{h^2}{12}\delta_x^2 w_j^{k+1}, \quad j \in I_h. \tag{4.7}$$

Taking the inner product of (4.5) with u^{k+1} and applying Lemma 3.1 gives

$$\frac{1}{2\tau}\|u^{k+1}\|^2 + \frac{\alpha}{2}\langle z^{k+1}, u^{k+1}\rangle + \frac{\beta}{2}\langle w^{k+1}, u^{k+1}\rangle = 0. \tag{4.8}$$

Applying (3.1) and (3.3) in Lemma 3.2, we find that

$$\langle z^{k+1}, u^{k+1}\rangle = -|u^{k+1}|_1^2 - \frac{h^2}{12}\|z^{k+1}\|^2 + \frac{h^4}{144}|z^{k+1}|_1^2,$$
$$\langle w^{k+1}, u^{k+1}\rangle = -\|z^{k+1}\|^2.$$

Substituting above equations into (4.8) and applying Lemma 3.1 yields

$$
\begin{aligned}
0 &= \frac{1}{\tau}\|u^{k+1}\|^2 + \alpha\Big(- |u^{k+1}|_1^2 - \frac{h^2}{12}\|z^{k+1}\|^2 + \frac{h^4}{144}|z^{k+1}|_1^2\Big) + \beta\|z^{k+1}\|^2 \\
&\geq \Big(\frac{1}{\tau} - \frac{4\alpha}{h^2}\Big)\|u^{k+1}\|^2 + \Big(\beta - \frac{\alpha h^2}{12}\Big)\|z^{k+1}\|^2 + \frac{\alpha h^4}{144}|z^{k+1}|_1^2.
\end{aligned}
$$

Then it holds that $\|u^{k+1}\| = 0$, $\|z^{k+1}\| = 0$, when $h^2 < 12\beta/\alpha$ and $\tau < h^2/4\alpha$. Therefore, (4.5)-(4.7) have only the zero solution, implying that (2.19), (2.20) and (2.21) determine u^{k+1}, z^{k+1} and w^{k+1} uniquely. Using mathematical induction, we obtain the conclusion. □

5. Convergence

This section analyzes the convergence of scheme (2.18)-(2.23).

Lemma 5.1 ([23, Lemma 2.5])**.** *For an arbitrary grid functions $u, z \in \mathcal{U}_h$, we have*

$$
\Delta_x(uz)_j = \frac{1}{2}(\delta_x^+ u_j)z_{j+1} + \frac{1}{2}(\delta_x^+ u_{j-1})z_{j-1} + u_j\Delta_x z_j.
$$

Let h_0 and τ_0 be two positive constants. We denote

$$
\begin{aligned}
c_5 &= \frac{8}{\beta}Lc_1^2 + 12Lc_3^2 + \Big(\frac{17h_0^4}{72} + 2\beta\Big)Lc_4^2, \\
c_6 &= 2\alpha^2 + \frac{\beta}{8} + \frac{10}{27}\gamma^2L^2(Lc_0 + \sqrt{L})^2 + \frac{5}{54}\gamma^2L^2\Big(Lc_0h_0^2 + 3\sqrt{8L + 2L^2c_3^2h_0^{10}}\Big)^2 \\
&\quad + \frac{2}{9}\gamma^2(c_0h_0^2 + c_0h_0)^2 + \frac{40}{243}\gamma^2L^2(Lc_0 + c_0)^2, \\
c_7 &= 2Lc_2^2 + \Big[\frac{4}{27}\gamma^2L^2(Lc_0 + c_0)^2 + \frac{1}{3}\gamma^2L^2(Lc_0 + \sqrt{L})^2 \\
&\quad + \frac{1}{12}\gamma^2L^2\Big(Lc_0h_0^2 + 3\sqrt{8L + 2L^2c_3^2h_0^{10}}\Big)^2 + 3\beta + \frac{h_0^4}{128}\Big]Lc_3^2 + \Big(3\beta^2 + \frac{\beta h_0^4}{128}\Big)Lc_4^2, \\
c_8 &= \exp\Big(\frac{6c_6T}{\beta}\Big)\Big(c_5 + \frac{c_7}{c_6}\Big), \\
c_9 &= \max\Big\{\sqrt{\frac{20L^2c_5}{27} + \frac{2L^2}{3}}, \sqrt{\frac{20L^2c_8}{27} + \frac{2L^3c_3^2}{3}}\Big\}.
\end{aligned}
$$

The following theorem shows that the method converges.

Theorem 5.2 (Convergence)**.** *Suppose that $\{U_j^k, Z_j^k, W_j^k : j \in I_h,\ k \in I_\tau\}$ are solutions of (2.1)-(2.5) and $\{u_j^k, z_j^k, w_j^k : j \in I_h,\ k \in I_\tau\}$ are solutions of (2.18)-(2.23). We denote $\eta_j^k = U_j^k - u_j^k$, $\zeta_j^k = z_j^k - z_j^k$, $\xi_j^k = W_j^k - w_j^k$, when $h \leq h_0$, $\tau \leq \tau_0$ and $\tau^2 + h^4 \leq 1/c_9$, the estimate for the error is given.*

$$
|\eta^k|_1 \leq c_9(\tau^2 + h^4), \quad k \in I_\tau. \tag{5.1}
$$

Proof. Subtracting (2.18)-(2.23) from (2.9), (2.11), (2.13), (2.16), (2.17), the error system can be expressed as

$$
\delta_t\eta_j^{1/2} + \alpha\zeta_j^{1/2} + \beta\xi_j^{1/2} + \gamma\phi(\hat{u}, \eta^{1/2})_j - \frac{\gamma h^2}{2}\phi(\hat{z}, \eta^{1/2})_j = Q_j^0, \quad j \in I_h, \tag{5.2}
$$

$$
\begin{aligned}
&\Delta_t\eta_j^k + \alpha\zeta_j^{\bar{k}} + \beta\xi_j^{\bar{k}} + \gamma\Big(\phi(U^k, U^{\bar{k}})_j - \phi(u^k, u^{\bar{k}})_j\Big) \\
&- \frac{\gamma h^2}{2}\Big(\phi(Z^k, U^{\bar{k}})_j - \phi(z^k, u^{\bar{k}})_j\Big) = Q_j^k, \quad j \in I_h,\ k \in I_\tau \setminus \{N\},
\end{aligned} \tag{5.3}
$$

$$
\zeta_j^k = \delta_x^2\eta_j^k - \frac{h^2}{12}\delta_x^2\zeta_j^k + R_j^k, \quad j \in I_h,\ k \in I_\tau, \tag{5.4}
$$

$$
\xi_j^k = \delta_x^2\zeta_j^k - \frac{h^2}{12}\delta_x^2\xi_j^k + S_j^k, \quad j \in I_h,\ k \in I_\tau, \tag{5.5}
$$

$$
\eta_j^0 = 0, \quad \zeta_j^0 = 0, \quad \xi_j^0 = 0, \quad j \in I_h, \tag{5.6}
$$

$$\eta_j^k = \eta_{j+M}^k, \quad \zeta_j^k = \zeta_{j+M}^k, \quad \xi_j^k = \xi_{j+M}^k, \quad j \in I_h, \ k \in I_\tau. \tag{5.7}$$

We prove the convergence of the difference scheme using mathematical induction. Prior to this, we present several preliminary results essential for the proof. It follows from (2.7) that

$$|U^k|_1 \le \sqrt{L}c_0, \quad \|U^k\|_\infty \le c_0, \quad k \in I_\tau, \tag{5.8}$$

$$\|Z^k\| \le \sqrt{L}c_0, \quad \|Z^k\|_\infty \le c_0, \quad |Z^k|_1 \le \sqrt{L}c_0, \quad k \in I_\tau. \tag{5.9}$$

Taking the inner product of (5.4) with ζ^k yields

$$\begin{aligned}
\|\zeta^k\|^2 &= \langle \delta_x^2\eta^k, \zeta^k\rangle + \frac{h^2}{12}|\zeta^k|_1^2 + \langle R^k, \zeta^k\rangle \\
&\le \frac{1}{6}\|\zeta^k\|^2 + \frac{3}{2}\|\delta_x^2\eta^k\|^2 + \frac{1}{3}\|\zeta^k\|^2 + \frac{1}{6}\|\zeta^k\|^2 + \frac{3}{2}\|R^k\|^2 \\
&\le \frac{2}{3}\|\zeta^k\|^2 + \frac{24}{h^4}\|\eta^k\|^2 + \frac{3}{2}\|R^k\|^2.
\end{aligned}$$

Thus, we have

$$\|\zeta^k\|^2 \le \frac{72}{h^4}\|\eta^k\|^2 + \frac{9}{2}\|R^k\|^2. \tag{5.10}$$

Similarly, weobtain

$$\|\Delta_t\zeta^k\|^2 \le \frac{72}{h^4}\|\Delta_t\eta^k\|^2 + \frac{9}{2}\|\Delta_t R^k\|^2, \tag{5.11}$$

$$\|\xi^k\|^2 \le \frac{72}{h^4}\|\zeta^k\|^2 + \frac{9}{2}\|S^k\|^2. \tag{5.12}$$

Taking the inner product of (5.4) with $\delta_x^2\eta^k$ and utilizing Lemma 3.1 gives

$$\begin{aligned}
\|\delta_x^2\eta^k\|^2 &= \langle \zeta^k, \delta_x^2\eta^k\rangle + \frac{h^2}{12}\langle \delta_x^2\zeta^k, \delta_x^2\eta^k\rangle - \langle R^k, \delta_x^2\eta^k\rangle \\
&\le \|\zeta^k\|^2 + \frac{1}{4}\|\delta_x^2\eta^k\|^2 + \frac{h^4}{144}\|\delta_x^2\zeta^k\|^2 + \frac{1}{4}\|\delta_x^2\eta^k\|^2 + \|R^k\|^2 + \frac{1}{4}\|\delta_x^2\eta^k\|^2 \\
&\le \frac{3}{4}\|\delta_x^2\eta^k\|^2 + \frac{10}{9}\|\zeta^k\|^2 + \|R^k\|^2,
\end{aligned}$$

then we have

$$\|\delta_x^2\eta^k\|^2 \le \frac{40}{9}\|\zeta^k\|^2 + 4\|R^k\|^2. \tag{5.13}$$

Applying Lemma 3.1 leads to

$$|\eta^k|_1^2 \le \frac{L^2}{6}|\delta_x^+\eta^k|_1^2 \le \frac{20L^2}{27}\|\zeta^k\|^2 + \frac{2L^2}{3}\|R^k\|^2. \tag{5.14}$$

Firstly, we demonstrate that η^1 satisfies (5.1). Taking the inner product of (5.2) with $\delta_t\eta^{1/2}$, we have

$$\begin{aligned}
&\|\delta_t\eta^{1/2}\|^2 + \alpha\langle f^{1/2}, \delta_t\eta^{1/2}\rangle + \beta\langle \xi^{1/2}, \delta_t\eta^{1/2}\rangle + \gamma\big\langle \phi(\hat{u}, \eta^{1/2}), \delta_t\eta^{1/2}\big\rangle - \frac{\gamma h^2}{2}\big\langle \phi(\hat{z}, \eta^{1/2}), \delta_t\eta^{1/2}\big\rangle \\
&= \langle Q^0, \delta_t\eta^{1/2}\rangle.
\end{aligned} \tag{5.15}$$

It follows from (5.6) that

$$\|\delta_t\eta^{1/2}\|^2 = \frac{1}{\tau^2}\|\eta^1\|^2, \quad \langle \zeta^{1/2}, \delta_t\eta^{1/2}\rangle = \frac{1}{2\tau}\langle \zeta^1, \eta^1\rangle, \quad \langle Q^0, \delta_t\eta^{1/2}\rangle = \frac{1}{\tau}\langle Q^0, \eta^1\rangle. \tag{5.16}$$

Using (5.5) and applying (3.3) in Lemma 3.2 gives

$$\langle \xi^{1/2}, \delta_t\eta^{1/2}\rangle = \frac{1}{2\tau}\Big[\|\zeta^1\|^2 - \langle \zeta^1, R^1\rangle + \frac{h^2}{12}\langle \xi^1, R^1\rangle - \frac{h^2}{12}\langle S^1, \zeta^1\rangle + \langle S^1, \eta^1\rangle\Big]. \tag{5.17}$$

With the help of the definition of $\phi\langle u, z\rangle_j$ and applying Lemma 3.1, we obtain

$$\langle \phi(\hat{u}, \eta^{1/2}), \delta_t\eta^{1/2}\rangle = \frac{1}{2\tau}\langle \phi(\hat{u}, \eta^1), \eta^1\rangle = 0, \tag{5.18}$$

$$\langle \phi(\hat{z}, \eta^{1/2}), \delta_t \eta^{1/2} \rangle = \frac{1}{2\tau} \langle \phi(\hat{z}, \eta^1), \eta^1 \rangle = 0. \tag{5.19}$$

Substituting (5.16)-(5.19) into (5.15) and using (5.12) yields

$$\begin{aligned}
\frac{1}{\tau}\|\eta^1\|^2 &= -\frac{\alpha}{2}\langle \zeta^1, \eta^1\rangle + \langle Q^0, \eta^1\rangle - \frac{\beta}{2}\Big[\|\zeta^1\|^2 - \langle f^1, R^1\rangle + \frac{h^2}{12}\langle \xi^1, R^1\rangle \\
&\quad - \frac{h^2}{12}\langle S^1, \zeta^1\rangle + \langle S^1, \eta^1\rangle\Big] \\
&\le \frac{\alpha}{2}\|\zeta^1\|\|\eta^1\| + \|Q^0\|\|\eta^1\| - \frac{\beta}{2}\|\zeta^1\|^2 + \frac{\beta}{2}\|\zeta^1\|\|R^1\| \\
&\quad + \frac{\beta h^2}{24}\|\xi^1\|\|R^1\| + \frac{\beta h^2}{24}\|\zeta^1\|\|S^1\| + \frac{\beta}{2}\|\eta^1\|\|S^1\| \\
&\le -\frac{\beta}{2}\|\zeta^1\|^2 + \frac{\beta}{8}\|\zeta^1\|^2 + \frac{\alpha^2}{2\beta}\|\eta^1\|^2 + \frac{\beta}{8}\|\zeta^1\|^2 + \frac{\beta}{2}\|R^1\|^2 + \frac{\beta h^4}{576}\|\xi^1\|^2 + \frac{\beta}{4}\|R^1\|^2 \\
&\quad + \frac{\beta}{16}\|\zeta^1\|^2 + \frac{\beta h^4}{144}\|S^1\|^2 + \frac{\beta^2}{8}\|S^1\|^2 + \frac{1}{2}\|\eta^1\|^2 + \frac{1}{2}\|Q^0\|^2 + \frac{1}{2}\|\eta^1\|^2 \\
&\le -\frac{3\beta}{16}\|\zeta^1\|^2 + \Big(\frac{\alpha^2}{2\beta} + 1\Big)\|\eta^1\|^2 + \frac{1}{8}\|\zeta^1\|^2 + \frac{\beta h^4}{128}\|S^1\|^2 + \Big(\frac{\beta h^4}{144} \\
&\quad + \frac{\beta^2}{8}\Big)\|S^1\|^2 + \frac{3\beta}{4}\|R^1\|^2 + \frac{1}{2}\|Q^0\|^2.
\end{aligned}$$

Rearranging the terms above inequality and using (2.10), (2.14), (2.15), when $h \le h_0$ and $\tau \le \frac{2\beta}{\alpha^2+2\beta}$, we have

$$\|\zeta^1\|^2 \le \frac{8}{\beta}\|Q^0\|^2 + 12\|R^1\|^2 + \Big(\frac{17h_0^4}{72} + 2\beta\Big)\|S^1\|^2 \le c_5(\tau^2 + h^4)^2. \tag{5.20}$$

With the help of (5.14), we obtain

$$|\eta^1|_1^2 \le \frac{20L^2}{27}\|\zeta^1\|^2 + \frac{2L^2}{3}\|R^1\|^2 \le \Big(\frac{20L^2 c_5}{27} + \frac{2L^2}{3}\Big)(\tau^2 + h^4)^2.$$

That is, $|\eta^1|_1 \le c_9(\tau^2 + h^4)$.

We denote

$$F^k = \frac{1}{2}(\|\zeta^k\|^2 + \|\zeta^{k-1}\|^2), \quad k \in I_\tau. \tag{5.21}$$

From (5.6), (5.21) and (5.20), we have

$$F^1 = \frac{1}{2}(\|\zeta^1\|^2 + \|\zeta^0\|^2) = \frac{1}{2}\|\zeta^1\|^2 \le \frac{c_5}{2}(\tau^2 + h^4)^2. \tag{5.22}$$

Taking the inner product of (5.3) with $\Delta_t e^k$ leads to

$$\begin{aligned}
&\|\Delta_t \eta^k\|^2 + \alpha\langle \zeta^{\bar{k}}, \Delta_t \eta^k\rangle + \beta\langle \xi^{\bar{k}}, \Delta_t \eta^k\rangle + \gamma\big\langle \phi(U^k, U^{\bar{k}}) - \phi(u^k, u^{\bar{k}}), \Delta_t \eta^k\big\rangle \\
&- \frac{\gamma h^2}{2}\big\langle \phi(Z^k, U^{\bar{k}}) - \phi(z^k, u^{\bar{k}}), \Delta_t \eta^k\big\rangle \\
&= \langle Q^k, \Delta_t \eta^k\rangle, \quad k \in I_\tau \setminus \{N\}.
\end{aligned} \tag{5.23}$$

Secondly, we suppose that (5.1) holds for $k = 1, 2, \dots, l$ with $1 \le l \le N-1$. When $(\tau^2 + h^4) \le 1/c_9$, using (5.8)-(5.10) yields

$$|u^k|_1 \le |U^k|_1 + |\eta^k|_1 \le \sqrt{L}c_0 + 1, \tag{5.24}$$

$$\|u^k\|_\infty \le \frac{\sqrt{L}}{2}|u^k|_1 \le \frac{\sqrt{L}}{2}\Big(\sqrt{L}c_0 + 1\Big), \tag{5.25}$$

$$|z^k|_1 \le |Z^k|_1 + |\zeta^k|_1 \le \sqrt{L}c_0 + \frac{2}{h}\|\zeta^k\| \le \sqrt{L}c_0 + 3\sqrt{\frac{8}{h^4} + 2Lc_3^2 h^6}, \tag{5.26}$$

$$\|z^k\|_\infty \le \frac{\sqrt{L}}{2}|z^k|_1 \le \frac{Lc_0}{2} + \frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4} + 2Lc_3^2 h^6}. \tag{5.27}$$

Using (5.4) and (5.5), for $1 \le k \le l$, we obtain

$$
\begin{aligned}
&\langle \xi^{\bar k}, \Delta_t \eta^k \rangle \\
&= \langle \delta_x^2 \zeta^{\bar k} - \frac{h^2}{12} \delta_x^2 \xi^{\bar k} + S^{\bar k}, \Delta_t \eta^k \rangle \\
&= \langle \zeta^{\bar k}, \Delta_t (\zeta^k + \frac{h^2}{12} \delta_x^2 \zeta^k - R^k) \rangle - \frac{h^2}{12} \big\langle \xi^{\bar k}, \Delta_t (\zeta^k + \frac{h^2}{12} \delta_x^2 \zeta^k - R^k) \big\rangle + \langle S^{\bar k}, \Delta_t \eta^k \rangle \\
&= \langle \zeta^{\bar k}, \Delta_t \zeta \rangle + \frac{h^2}{12} \langle \delta_x^2 \zeta^{\bar k}, \Delta_t \zeta^k \rangle - \langle \zeta^{\bar k}, \Delta_t R^k \rangle - \frac{h^2}{12} \langle \delta_x^2 \zeta^{\bar k} - \frac{h^2}{12} \delta_x^2 \xi^{\bar k} + S^{\bar k}, \Delta_t \zeta^k \rangle \\
&\quad - \frac{h^4}{144} \langle \delta_x^2 \xi^{\bar k}, \Delta_t \zeta^k \rangle + \frac{h^2}{12} \langle \xi^{\bar k}, \Delta_t R^k \rangle + \langle S^{\bar k}, \Delta_t \eta^k \rangle \\
&= \frac{1}{4\tau} (\|\zeta^{k+1}\|^2 - \|\zeta^{k-1}\|^2) - \langle \zeta^{\bar k}, \Delta_t R^k \rangle - \frac{h^2}{12} \langle S^{\bar k}, \Delta_t \zeta^k \rangle + \frac{h^2}{12} \langle \xi^{\bar k}, \Delta_t R^k \rangle + \langle S^{\bar k}, \Delta_t \eta^k \rangle .
\end{aligned}
\tag{5.28}
$$

Noticing that

$$
\begin{aligned}
\phi(U^k, U^{\bar k})_j - \phi(u^k, u^{\bar k})_j &= \phi(u^k, \eta^{\bar k})_j + \phi(\eta^k, U^{\bar k})_j \\
&= \frac{1}{3} \big[u_j^k \Delta_x \eta_j^{\bar k} + \Delta_x (u^k \eta^{\bar k})_j \big] + \frac{1}{3} \big[\eta_j^k \Delta_x U_j^{\bar k} + \Delta_x (\eta^k U^{\bar k})_j \big],
\end{aligned}
$$

and applying Lemma 5.1, we find that

$$
\begin{aligned}
&\phi(U^k, U^{\bar k})_j - \phi(u^k, u^{\bar k})_j \\
&= \frac{1}{3} \Big[u_j^k \Delta_x \eta_j^{\bar k} + \frac{1}{2} \Big(\delta_x^+ \eta_j^{\bar k} \Big) u_{j+1}^k + \frac{1}{2} \Big(\delta_x^+ \eta_{j-1}^{\bar k} \Big) u_{j-1}^k + \eta_j^{\bar k} \Delta_x u_j^k \Big] \\
&\quad + \frac{1}{3} \Big[\eta_j^k \Delta_x U_j^{\bar k} + \frac{1}{2} \left(\delta_x^+ \eta_j^k \right) U_{j+1}^{\bar k} + \frac{1}{2} \left(\delta_x^+ \eta_{j-1}^k \right) U_{j-1}^{\bar k} + \eta_j^k \Delta_x U_j^{\bar k} \Big].
\end{aligned}
\tag{5.29}
$$

Combining (5.8), (5.24), (5.25) with (5.29) leads to

$$
\begin{aligned}
&- \big\langle \phi(U^k, U^{\bar k}) - \phi(u^k, u^{\bar k}), \Delta_t \eta^k \big\rangle \\
&= -\frac{h}{3} \sum_{j=1}^{M} \Big[u_j^k \Delta_x \eta_j^{\bar k} + \frac{1}{2} \Big(\delta_x^+ \eta_j^{\bar k} \Big) u_{j+1}^k + \frac{1}{2} \Big(\delta_x^+ \eta_{j-1}^{\bar k} \Big) u_{j-1}^k + \eta_j^{\bar k} \Delta_x u_j^k \Big] \Delta_t \eta_j^k \\
&\quad - \frac{h}{3} \sum_{j=1}^{M} \Big[\eta_j^k \Delta_x U_j^{\bar k} + \frac{1}{2} \Big(\delta_x^+ \eta_j^k \Big) U_{j+1}^{\bar k} + \frac{1}{2} \Big(\delta_x^+ \eta_{j-1}^k \Big) U_{j-1}^{\bar k} + \eta_j^k \Delta_x U_j^{\bar k} \Big] \Delta_t \eta_j^k \\
&\le \frac{1}{3} \Big(\|u^k\|_\infty |\eta^{\bar k}|_1 + \frac{1}{2} |\eta^{\bar k}|_1 \|u^k\|_\infty + \frac{1}{2} |\eta^{\bar k}|_1 \|u^k\|_\infty + \|\eta^{\bar k}\|_\infty |u^k|_1 \Big) \|\Delta_t \eta^k\| \\
&\quad + \frac{1}{3} \Big(\|\eta^k\|_\infty |U^{\bar k}|_1 + \frac{1}{2} |\eta^k|_1 \|U^{\bar k}\|_\infty + \frac{1}{2} |\eta^k|_1 \|U^{\bar k}\|_\infty + \|\eta^k\|_\infty |U^{\bar k}|_1 \Big) \|\Delta_t \eta^k\| \\
&\le \frac{1}{3} \Big[\frac{\sqrt{L}}{2} \left(\sqrt{L} c_0 + 1 \right) |\eta^{\bar k}|_1 + \frac{\sqrt{L}}{2} \left(\sqrt{L} c_0 + 1 \right) |\eta^{\bar k}|_1 + \left(\sqrt{L} c_0 + 1 \right) \frac{\sqrt{L}}{2} |\eta^{\bar k}|_1 \Big] \|\Delta_t \eta^k\| \\
&\quad + \frac{1}{3} \Big[\sqrt{L} c_0 \frac{\sqrt{L}}{2} |\eta^k|_1 + c_0 |\eta^k|_1 + \sqrt{L} c_0 \frac{\sqrt{L}}{2} |\eta^k|_1 \Big] \|\Delta_t \eta^k\| \\
&= \frac{L c_0 + \sqrt{L}}{2} |\eta^{\bar k}|_1 \|\Delta_t \eta^k\| + \frac{L c_0 + c_0}{3} |\eta^k|_1 \|\Delta_t \eta^k\|, \quad 1 \le k \le l.
\end{aligned}
\tag{5.30}
$$

Similarly, we concluded that

$$
\begin{aligned}
\phi(Z^k, U^{\bar k})_j - \phi(z^k, u^{\bar k})_j &= \frac{1}{3} \Big[z_j^k \Delta_x \eta_j^{\bar k} + \frac{1}{2} \left(\delta_x^+ \eta_j^{\bar k} \right) z_{j+1}^k + \frac{1}{2} \left(\delta_x^+ \eta_{j-1}^{\bar k} \right) z_{j-1}^k + \eta_j^{\bar k} \Delta_x z_j^k \Big] \\
&\quad + \frac{1}{3} \Big[\zeta_j^k \Delta_x U_j^{\bar k} + \frac{1}{2} \left(\delta_x^+ \zeta_j^k \right) U_{j+1}^{\bar k} + \frac{1}{2} \left(\delta_x^+ \zeta_{j-1}^k \right) U_{j-1}^{\bar k} + \zeta_j^k \Delta_x U_j^{\bar k} \Big].
\end{aligned}
\tag{5.31}
$$

Combining (5.9), (5.26), (5.27) with (5.31), for $1 \le k \le l$, we have

$$
\begin{aligned}
&\big\langle \phi(Z^k, U^{\bar{k}}) - \phi(z^k, u^{\bar{k}}), \Delta_t \eta^k \big\rangle \\
&= -\frac{h}{3}\sum_{j=1}^{M}\Big[z_j^k \Delta_x \eta_j^{\bar{k}} + \frac{1}{2}\left(\delta_x^+ \eta_j^{\bar{k}}\right) z_{j+1}^k + \frac{1}{2}\left(\delta_x^+ \eta_{j-1}^{\bar{k}}\right) z_{j-1}^k + \eta_j^{\bar{k}} \Delta_x z_j^k\Big]\Delta_t \eta_j^k \\
&\quad - \frac{h}{3}\sum_{j=1}^{M}\Big[\zeta_j^k \Delta_x U_j^{\bar{k}} + \frac{1}{2}\left(\delta_x^+ \zeta_j^k\right) U_{j+1}^{\bar{k}} + \frac{1}{2}\left(\delta_x^+ \zeta_{j-1}^k\right) U_{j-1}^{\bar{k}} + \zeta_j^k \Delta_x U_j^{\bar{k}}\Big]\Delta_t \eta_j^k \\
&\le \frac{1}{3}\Big(\|z^k\|_\infty |\eta^{\bar{k}}|_1 + \frac{1}{2}|\eta^{\bar{k}}|_1 \|z^k\|_\infty + \frac{1}{2}|\eta^{\bar{k}}|_1\|z^k\|_\infty + \|\eta^{\bar{k}}\|_\infty |z^k|_1\Big)\|\Delta_t \eta^k\| \\
&\quad + \frac{1}{3}\Big(\|\zeta^k\|\|\Delta_x U^{\bar{k}}\|_\infty + \frac{1}{2}|\zeta^k|_1\|U^{\bar{k}}\|_\infty + \frac{1}{2}|\zeta^k|_1\|U^{\bar{k}}\|_\infty + \|\zeta^k\|\|\Delta_x U^{\bar{k}}\|_\infty\Big)\|\Delta_t \eta^k\| \\
&\le \frac{1}{3}\Big[2\Big(\frac{Lc_0}{2} + \frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4} + 2Lc_3^2 h^6}\Big) + \frac{\sqrt{L}}{2}\Big(\sqrt{L}c_0 + 3\sqrt{\frac{8}{h^4} + 2Lc_3^2h^6}\Big)\Big]|\eta^{\bar{k}}|_1\|\Delta_t \eta^k\| \\
&\quad + \frac{1}{3}\Big(2c_0 + \frac{2}{h}c_0\Big)\|\zeta^k\|\|\Delta_t \eta^k\| \\
&= \Big(\frac{Lc_0}{2} + \frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4} + 2Lc_3^2h^6}\Big)|\eta^{\bar{k}}|_1\|\Delta_t\eta^k\| + \frac{2}{3}\left(c_0 + \frac{c_0}{h}\right)\|f^k\|\|\Delta_t \eta^k\|.
\end{aligned} \tag{5.32}
$$

Substituting (5.28), (5.30), (5.32) into (5.23) and using (5.11)-(5.12) gives

$$
\begin{aligned}
&\|\Delta_t \eta^k\|^2 \\
&\le -\alpha\langle \zeta^{\bar{k}}, \Delta_t \eta^k\rangle - \frac{\beta}{4\tau}(\|\zeta^{k+1}\|^2 - \|\zeta^{k-1}\|^2) \\
&\quad + \beta\langle \zeta^{\bar{k}}, \Delta_t R^k\rangle + \frac{\beta h^2}{12}\langle S^{\bar{k}}, \Delta_t \zeta^k\rangle - \frac{\beta h^2}{12}\langle \xi^{\bar{k}}, \Delta_t R^k\rangle - \beta\langle S^{\bar{k}}, \Delta_t \eta^k\rangle \\
&\quad + \gamma\Big(\frac{Lc_0 + \sqrt{L}}{2}|\eta^{\bar{k}}|_1\|\Delta_t \eta^k\| + \frac{Lc_0 + c_0}{3}|\eta^k|_1\|\Delta_t \eta^k\|\Big) \\
&\quad + \frac{\gamma h^2}{2}\Big[\Big(\frac{Lc_0}{2} + \frac{3\sqrt{L}}{2}\sqrt{\frac{8}{h^4} + 2Lc_3^2h^6}\Big)|\eta^{\bar{k}}|_1\|\Delta_t \eta^k\| \\
&\quad + \frac{2}{3}\left(c_0 + \frac{c_0}{h}\right)\|\zeta^k\|\|\Delta_t \eta^k\|\Big] + (Q^k, \Delta_t \eta^k) \\
&\le -\frac{\beta}{4\tau}(\|\zeta^{k+1}\|^2 - \|\zeta^{k-1}\|^2) + \alpha\|\zeta^{\bar{k}}\|\|\Delta_t \eta^k\| + \beta\|\zeta^{\bar{k}}\|\|\Delta_t R^k\| + \frac{\beta h^2}{12}\|S^{\bar{k}}\|\|\Delta_t \zeta^k\| \\
&\quad + \frac{\beta h^2}{12}\|\xi^{\bar{k}}\|\|\Delta_t R^k\| + \beta\|S^{\bar{k}}\|\|\Delta_t \eta^k\| + \frac{\gamma(Lc_0 + \sqrt{L})}{2}|\eta^{\bar{k}}|_1\|\Delta_t \eta^k\| \\
&\quad + \frac{\gamma(Lc_0 + c_0)}{3}|\eta^k|_1\|\Delta_t\eta^k\| + \frac{\gamma}{2}\Big(\frac{Lc_0h^2}{2} + \frac{3\sqrt{L}}{2}\sqrt{8 + 2Lc_3^2h^{10}}\Big)|\eta^{\bar{k}}|_1\|\Delta_t \eta^k\| \\
&\quad + \frac{\gamma}{3}\left(c_0h^2 + c_0h\right)\|\zeta^k\|\|\Delta_t\eta^k\| + \|Q^k\|\|\Delta_t \eta^k\| \\
&\le -\frac{\beta}{4\tau}(\|\zeta^{k+1}\|^2 - \|\zeta^{k-1}\|^2) + 2\alpha^2\|\zeta^{\bar{k}}\|^2 + \frac{1}{8}\|\Delta_t\eta^k\|^2 + \frac{\beta}{8}\|\zeta^{\bar{k}}\|^2 + 2\beta\|\Delta_t R^k\|^2 \\
&\quad + \beta^2\|S^{\bar{k}}\|^2 + \frac{h^4}{576}\|\Delta_t \zeta^k\|^2 + \frac{\beta h^4}{576}\|\xi^{\bar{k}}\|^2 + \beta\|\Delta_t R^k\|^2 + 2\beta^2\|S^{\bar{k}}\|^2 + \frac{1}{8}\|\Delta_t \eta^k\|^2 \\
&\quad + \frac{\gamma^2(Lc_0 + \sqrt{L})^2}{2}|\eta^{\bar{k}}|_1^2 + \frac{1}{8}\|\Delta_t\eta^k\|^2 + \frac{2\gamma^2(Lc_0 + c_0)^2}{9}|\eta^k|_1^2 + \frac{1}{8}\|\Delta_t \eta^k\|^2 \\
&\quad + \frac{\gamma^2}{2}\Big(\frac{Lc_0h^2}{2} + \frac{3\sqrt{L}}{2}\sqrt{8 + 2Lc_3^2h^{10}}\Big)^2|\eta^{\bar{k}}|_1^2 + \frac{1}{8}\|\Delta_t \eta^k\|^2 \\
&\quad + \frac{2\gamma^2}{9}\left(c_0h^2 + c_0h\right)^2\|\zeta^k\|^2 + \frac{1}{8}\|\Delta_t\eta^k\|^2 + 2\|Q^k\|^2 + \frac{1}{8}\|\Delta_t \eta^k\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le -\frac{\beta}{4\tau}(\|\zeta^{k+1}\|^2-\|\zeta^{k-1}\|^2)+\frac{7}{8}\|\Delta_t\eta^k\|^2+2\alpha^2\|\zeta^{\bar k}\|^2+\frac{\beta}{8}\|\zeta^{\bar k}\|^2+2\beta\|\Delta_t R^k\|^2\\
&\quad+\beta^2\|S^{\bar k}\|^2+\frac{1}{8}\|\Delta_t\eta^k\|^2+\frac{h^4}{128}\|\Delta_t R^k\|^2+\frac{\beta}{8}\|\zeta^{\bar k}\|^2+\frac{\beta h^4}{128}\|S^{\bar k}\|^2+\beta\|\Delta_t R^k\|^2\\
&\quad+2\beta^2\|S^{\bar k}\|^2+\Big[\frac{\gamma^2(Lc_0+\sqrt{L})^2}{2}+\frac{\gamma^2\left(Lc_0h^2+3\sqrt{8L+2L^2c_3^2h^{10}}\right)^2}{8}\Big]|\eta^{\bar k}|_1^2\\
&\quad+\frac{2\gamma^2(Lc_0+c_0)^2}{9}|\eta^k|_1^2+\frac{2\gamma^2}{9}\left(c_0h^2+c_0h\right)^2\|\zeta^k\|^2+2\|Q^k\|^2\\
&\le -\frac{\beta}{4\tau}(\|\zeta^{k+1}\|^2-\|\zeta^{k-1}\|^2)+\|\Delta_t\eta^k\|^2+\left(2\alpha^2+\frac{\beta}{8}\right)\|\zeta^{\bar k}\|^2\\
&\quad+\frac{2\gamma^2(c_0h^2+c_0h)^2}{9}\|\zeta^k\|^2+\frac{2\gamma^2(Lc_0+c_0)^2}{9}|\eta^k|_1^2\\
&\quad+\Big[\frac{\gamma^2(Lc_0+\sqrt{L})^2}{2}+\frac{\gamma^2\left(Lc_0h^2+3\sqrt{8L+2L^2c_3^2h^{10}}\right)^2}{8}\Big]|\eta^{\bar k}|_1^2\\
&\quad+2\|Q^k\|^2+\Big(3\beta+\frac{h^4}{128}\Big)\|\Delta_t R^k\|^2+\Big(3\beta^2+\frac{\beta h^4}{128}\Big)\|S^{\bar k}\|^2.
\end{aligned}
$$

Simplifying and rearranging the above inequality, then using (2.10), (2.14), (2.15) and (5.14), we have

$$
\begin{aligned}
&\frac{\beta}{4\tau}\left[(\|\zeta^{k+1}\|^2+\|\zeta^k\|^2)-(\|\zeta^k\|^2+\|\zeta^{k-1}\|^2)\right]\\
&\le\Big(2\alpha^2+\frac{\beta}{8}\Big)\|\zeta^{\bar k}\|^2+\frac{2\gamma^2(c_0h^2+c_0h)^2}{9}\|\zeta^k\|^2+\frac{2\gamma^2(Lc_0+c_0)^2}{9}\Big(\frac{20L^2}{27}\|\zeta^k\|^2+\frac{2L^2}{3}\|R^k\|^2\Big)\\
&\quad+\Big[\frac{\gamma^2(Lc_0+\sqrt{L})^2}{2}+\frac{\gamma^2\Big(Lc_0h^2+3\sqrt{8L+2L^2c_3^2h^{10}}\Big)^2}{8}\Big]\Big(\frac{20L^2}{27}\|\zeta^{\bar k}\|^2+\frac{2L^2}{3}\|R^{\bar k}\|^2\Big)\\
&\quad+2\|Q^k\|^2+\Big(3\beta+\frac{h^4}{128}\Big)\|\Delta_t R^k\|^2+\Big(3\beta^2+\frac{\beta h^4}{128}\Big)\|S^{\bar k}\|^2\\
&=\Big[2\alpha^2+\frac{\beta}{8}+\frac{10\gamma^2L^2(Lc_0+\sqrt{L})^2}{27}+\frac{5\gamma^2L^2\left(Lc_0h^2+3\sqrt{8L+2L^2c_3^2h^{10}}\right)^2}{54}\Big]\|\zeta^{\bar k}\|^2\\
&\quad+\Big[\frac{2\gamma^2(c_0h^2+c_0h)^2}{9}+\frac{40\gamma^2L^2(Lc_0+c_0)^2}{243}\Big]\|\zeta^k\|^2+2\|Q^k\|^2+\frac{4\gamma^2L^2(Lc_0+c_0)^2}{27}\|R^k\|^2\\
&\quad+\Big[\frac{\gamma^2L^2(Lc_0+\sqrt{L})^2}{3}+\frac{\gamma^2L^2\Big(Lc_0h^2+3\sqrt{8L+2L^2c_3^2h^{10}}\Big)^2}{12}\Big]\|R^{\bar k}\|^2\\
&\quad+\Big(3\beta+\frac{h^4}{128}\Big)\|\Delta_t R^k\|^2+\Big(3\beta^2+\frac{\beta h^4}{128}\Big)\|S^{\bar k}\|^2\\
&\le c_6\Big[\frac{\|\zeta^{k+1}\|^2+\|\zeta^k\|^2}{2}+\frac{\|\zeta^k\|^2+\|\zeta^{k-1}\|^2}{2}\Big]+c_7(\tau^2+h^4)^2.
\end{aligned}
$$

Using (5.21), it follows that

$$
\frac{\beta}{2\tau}(F^{k+1}-F^k)\le c_6(F^{k+1}+F^k)+c_7(\tau^2+h^4)^2,\quad 1\le k\le l.
$$

With the help of the Gronwall inequality [20], when $2c_6\tau/\beta\le 1/3$, we have

$$
F^{k+1}\le\exp\Big(\frac{6c_6(k+1)\tau}{\beta}\Big)\Big[F^1+\frac{c_7}{2c_6}(\tau^2+h^4)^2\Big],\quad 1\le k\le l.
$$

Combining this with (5.22), we derive

$$
F^{k+1}\le\frac{c_8}{2}(\tau^2+h^4)^2,\quad 1\le k\le l.
$$

Noticing that $2F^{k+1} \geq \|\zeta^{k+1}\|^2$, we have

$$\|\zeta^{k+1}\|^2 \leq c_8(\tau^2 + h^4)^2, \quad 1 \leq k \leq l.$$

Then using (2.14) and (5.14) yields

$$|\eta^{k+1}|_1 \leq \sqrt{\frac{20L^2c_8}{27} + \frac{2L^3c_3^2}{3}}(\tau^2 + h^4) \leq c_9(\tau^2 + h^4), \quad 1 \leq k \leq l.$$

Therefore, with the help of mathematical induction, we conclude that

$$|\eta^k|_1 \leq c_9(\tau^2 + h^4), \quad k \in I_\tau \setminus \{0\}. \tag{5.33}$$

Finally, combining (5.6) and (5.33), we demonstrate the error estimate

$$|\eta^k|_1 \leq c_9(\tau^2 + h^4), \quad k \in I_\tau.$$

□

Based on Lemma 3.1 and Theorem 5.2, we find that

$$\|\eta^k\|_\infty \leq \frac{\sqrt{L}}{2}|\eta^k|_1 \leq \frac{c_9\sqrt{L}}{2}(\tau^2 + h^4), \quad k \in I_\tau.$$

Acknowledgments. This work was supported by the Natural Science Foundation of Jilin Province (No. YDZJ202501ZYTS584).

References

[1] G. Akrivis; Finite difference discretization of the Kuramoto-Sivashinsky equation, *Numer. Math.*, **63** (1992), 1-12.

[2] G. Akrivis, Finite element discretization of the Kuramoto-Sivashinsky equation, *Banach Center Publ.*, **29** (1994), 155-163.

[3] G. Akrivis, Y. S. Smyrlis; Implicit-explicit BDF methods for the Kuramoto-Sivashinsky equation, *Appl. Numer. Math.*, **51** (2004), 151-169.

[4] A. J. Babchin, A. L. Frenkel, B. G. Levich, G. I. Sivashinsky, Nonlinear saturation of Rayleigh-Taylor instability in thin films, *Phys. Fluids*, **26** (1983), 3159-3161.

[5] E. Cerpa, A. Mercado, Local exact controllability to the trajectories of the 1-D Kuramoto-Sivashinsky equation, *J. Differential Equations*, **250** (2011), 2024-2044.

[6] F. Christiansen, P. Cvitanovic, V. Putkaradze; Spatiotemporal chaos in terms of unstable recurrent patterns, *Nonlinearity*, **10** (1997), 55-70.

[7] C. I. Christov, K. L. Bekyarov; A Fourier-Series method for solving soliton problems, *SIAM J. Sci. Statist. Comput.*, **11** (1990), 631-647.

[8] B. I. Cohen, J. A. Krommes, W. Tang, M. N. Rosenbluth; Non-linear saturation of the dissipative trapped-ion mode by mode coupling, *Nucl. Fusion*, **16** (1976), 971-992.

[9] C. W. Dong, Y. H. Lan; Organization of spatially periodic solutions of the steady Kuramoto-Sivashinsky equation, *Commun. Nonlinear Sci. Numer. Sim.*, **19** (2014), 2140-2153.

[10] U. Frisch, Z. S. She, O. Thual; Viscoelastic behaviour of cellular solutions to the Kuramoto-Sivashinsky model, *J. Fluid Mech.*, **168** (1986), 221-240.

[11] A. H. Khater, R. S. Temsah; Numerical solutions of the generalized Kuramoto-Sivashinsky equation by Chebyshev spectral collocation methods, *Comput. Math. Appl.*, **56** (2008), 1465-1472.

[12] N. A. Kudryashov, D. I. Sinelschikov, I. Chernyavsky; Nonlinear evolution equations for description of perturbation in tube, *Nelin. Din.*, **4** (2008), 69-86.

[13] N. A. Kudryashov, M. B. Soukharev; Popular ansatz methods and solitary wave solutions of the Kuramoto-Sivashinsky equation, *Regul. Chaot. Dyn.*, **14** (2009), 407-419.

[14] Y. Kuramoto, T. Tsuzuki; Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Prog. Theor. Phys.*, **55** (1976), 356-369.

[15] H. L. Lai, C. F. Ma; Lattice Boltzmann method for the generalized Kuramoto-Sivashinsky equation, *Phys. A Stat. Mech. Appl.*, **388** (2009), 1405-1412.

[16] M. Lakestani, M. Dehghan; Numerical solutions of the generalized Kuramoto-Sivashinsky equation using B-spline functions, *Appl. Math. Model.*, **36** (2012), 605-617.

[17] Y. H. Lan, P. Cvitanovi'c; Unstable recurrent patterns in Kuramoto-Sivashinsky dynamics, *Phys. Rev. E*, **78** (2008), 026208

[18] G. I. Sivashinsky; Nonlinear analysis of hydrodynamic instability in laminar flames—I. Derivation of basic equations, *Acta Astronaut.*, **4** (1977), 1177-1206.

[19] G. I. Sivashinsky, D. M. Michelson; On irregular wavy flow of a liquid film down a vertical plane, *Prog. Theor. Phys.*, **63** (1980), 2112-2114.

[20] Z. Z. Sun; *Numerical Methods of Partial Differential Equations (in Chinese)*, 3^{rd} edition, Science Press, Beijing, 2022.
[21] M. Uddin, S. Haq, S. ul Islam; A mesh-free numerical method for solution of the family of Kuramoto-Sivashinsky equations, *Appl. Math. Comput.*, **212** (2009), 458-469.
[22] Y. Xu, C. W. Shu, Local discontinuous Galerkin methods for the Kuramoto-Sivashinsky equations and the Ito-type coupled KdV equations, *Comput. Methods Appl. Mech. Engrg.*, **195** (2006), 3430-3447.
[23] Q. F. Zhang, L. L. Liu; Convergence and stability in maximum norms of linearized fourth-order conservative compact scheme for Benjamin-Bona-Mahony-Burgers equation, *J. Sci. Comput.*, **87** (2021), 1-31.

Yiran Zhang
College of Science, Yanbian University, Yanji 133002, China
Email address: 3157889605@qq.com

Guohui Wang
College of Science, Yanbian University, Yanji 133002, China
Email address: 1547733468@qq.com

Yuanfeng Jin (corresponding author)
College of Science, Yanbian University, Yanji 133002, China
Email address: yfkim@ybu.edu.cn