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VARIABLE-EXPONENT DOUBLE-PHASE PROBLEMS WITH PARAMETRIC LOGISTIC REACTION

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Abstract. We prove the existence and multiplicity of solutions for a variable-exponent double-phase problem with parametric logistic reaction term. Combining variational and truncation methods with homological critical group theory, we prove the existence of at least one or two solutions with respect to a positive parameter, i.e., a bifurcation result.

1. Introduction

A strong interest in double-phase problems has arisen in recent years, since Marcellini [38, 39] and Zhikov [52, 53] introduced the operator

$$\mathcal{G}(u) = \operatorname{div}\Big(a(z)|\nabla u|^{p(z)-2}\nabla u + |\nabla u|^{q(z)-2}\nabla u\Big) \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega) \tag{1.1}$$

to study elasticity phenomena. Many authors used this operator in other applications, we refer to Bahrouni-Rădulescu-Repovš [4] in transonic flow models, Benci-D'Avenia-Fortunato-Pisani [5] for applications in quantum physics, Bonheure-D'Avenia-Pomponio [6] investigated models in electromagnetism, Charkaoui-Ben Loghfyry-Zeng [9, 10] and Harjulehto-Hästö [30] for restoration and denoising of images, Cherfils-Il'yasov [11] for reaction-diffusion system models, and Cohen-Keller [12] for positone problems.

Double phase problems were also considered from a mathematical point of view, see the recent works of Amoroso-Bonanno-D'Aguì-Winkert [1], Amoroso-Crespo Blanco-Pucci-Winkert [2], Amoroso-Morabito [3], Crespo Blanco-Gasiński-Harjuletho-Winkert [13], Crespo Blanco-Winkert [14], Guarnotta-Livrea-Winkert [28], Ho-Winkert [32], Leonardi-Papageorgiou [36] and the references therein for existence and multiplicity of solutions and Ragusa-Tachikawa [49] and the related references for regularity results. Finally, we refer the interested reader to the recent surveys of Mingione-Rădulescu [40] and Papageorgiou [41].

The reaction term is of logistic type, similar to those in Gasiński-Papageorgiou [24, 25, 26] and Papageorgiou-Vetro-Vetro [45]. Here, inspired by the recent paper of Da Silva-Failla-Gasiński-Papageorgiou [15], we study a double phase variable exponent problem with logistic type reaction where both the parametric term and the perturbation are sublinear. In particular, we obtain a bifurcation result.

The aim of this work is to study the problem

$$\begin{gathered} -\Delta^a_{p(z)}u - \Delta_{q(z)}u = \lambda u^{\tau(z)-1} - f(z,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with C^2 boundary $\partial\Omega$.

For $m \in C(\overline{\Omega})$ we put $m_+ = \max_{\overline{\Omega}} m$, $m_- = \min_{\overline{\Omega}} m$. We will use the following hypotheses:

(H1) $p, q, \tau \in C^{0,1}(\overline{\Omega})$ such that $1 < \tau_- \leq \tau(z) \leq \tau_+ < q_- \leq q(z) \leq q_+ < p_- \leq p(z) \leq p_+$, $\frac{1}{q_-+1} \leq p_+ - q_-$ and $\frac{p_+}{q_-} < 1 + \frac{1}{N}$;

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(H2) there exists $\ell \in \mathbb{R}^N \setminus \{0\}$ such that for any $z \in \Omega$, the function $t \mapsto q(z + t\ell)$ is monotone for $t \in I = \{t : z + t\ell \in \Omega\}$;

(H3) $a \in C^{0,1}(\overline{\Omega})$, $a \geq 0$ for all $z \in \overline{\Omega}$, $a \neq 0$.

(H4) On the reaction term, we will assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function, such that $f(\cdot, x)$ is a measurable function for all $x \in \mathbb{R}$ and $f(z, \cdot)$ is locally Lipschitz for a.a. $z \in \Omega$, with $L^\infty(\Omega)$-local Lipschitz constant and

(i) for every $\kappa > 0$ there exists $\widehat{a}_\kappa \in L^\infty(\Omega)$ such that

$$0 \leq f(z, x) \leq \widehat{a}_\kappa \quad \text{for a.a. } z \in \Omega \text{ for all } 0 \leq x \leq \kappa.$$

(ii) $\lim_{x \to +\infty} \frac{f(z,x)}{x^{\tau(z)-1}} = 0$ uniformly with respect to $z \in \Omega$.

(iii) $\lim_{x \to 0^+} \frac{f(z,x)}{x^{\tau(z)-1}} = +\infty$ uniformly with respect to $z \in \Omega$ and there exists $\beta \in C^{0,1}(\Omega)$, $1 < \beta_- \leq \beta(z) \leq \beta_+ < \tau_-$ such that

$$0 \leq \liminf_{x \to 0^+} \frac{f(z, x)}{x^{\beta(z)-1}} \leq \limsup_{x \to 0^+} \frac{f(z, x)}{x^{\beta(z)-1}} < +\infty$$

uniformly with respect to $z \in \Omega$.

The rest of the article is organized as follows. In Section 2, we introduce the mathematical background and we recall some useful properties of the variable exponent double phase operator. Moreover, a brief review of homological groups is provided. In Section 3, we prove our main results.

2. Mathematical background

We recall some properties and definitions of variable exponent space; see Diening-Harjulehto-Hästö-Růžička [17], Fan-Shen-Zhao [20], Fan-Zhao [22], Kováčik-Rákosník [35], Rădulescu-Repovš [48], and Růžička [50] for more details. We indicate with $L^0(\Omega)$ the set of measurable functions. Let us consider $r \in C(\overline{\Omega})$ and

$$1 < r_- = \min_{z \in \overline{\Omega}} r(z) \leq r(z) \leq \max_{z \in \overline{\Omega}} r(z) = r_+.$$

The variable exponent Lebesgue space $L^{r(z)}(\Omega)$ is defined as

$$L^{r(z)}(\Omega) = \left\{ u \in L^0(\Omega) : \rho_{r(z)}(u) < +\infty \right\},$$

where

$$\rho_{r(z)}(u) := \int_\Omega |u|^{r(z)} dz$$

is the modular of $L^{r(z)}(\Omega)$. We equip this space with the Luxemburg norm, i.e.,

$$\|u\|_{L^{r(z)}(\Omega)} = \inf \Big\{ \mu > 0 : \int_\Omega |\frac{u(z)}{\mu}|^{r(z)} dz \leq 1 \Big\}.$$

Moreover, we consider the Sobolev space

$$W^{1,r(z)}(\Omega) = \Big\{ u \in L^{r(z)}(\Omega) : |\nabla u| \in L^{r(z)}(\Omega) \Big\},$$

with the norm

$$\|u\|_{W^{1,r(z)}(\Omega)} = \|u\|_{L^{r(z)}(\Omega)} + \||\nabla u|\|_{L^{r(z)}(\Omega)}.$$

We point out that on $W_0^{1,r(z)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,r(z)}(\Omega)}}$, we use the equivalent norm (see Fan-Zhang-Zhao [21])

$$\|u\|_{W_0^{1,r(z)}(\Omega)} = \||\nabla u|\|_{L^{r(z)}(\Omega)}.$$

Now, we state the following modular relations.

Lemma 2.1. *Let $u \in W_0^{1,r(z)}(\Omega)$, then the following hold:*

(1) $\|u\|_{W_0^{1,r(z)}(\Omega)} < 1 (= 1; > 1) \iff \rho_{r(z)}(\nabla u) < 1 (= 1; > 1)$;

(2) *if* $\|u\|_{W_0^{1,r(z)}(\Omega)} > 1$, *then* $\|u\|^{r_-}_{W_0^{1,r(z)}(\Omega)} \leq \rho_{r(z)}(\nabla u) \leq \|u\|^{r_+}_{W_0^{1,r(z)}(\Omega)}$;

(3) *if* $\|u\|_{W_0^{1,r(z)}(\Omega)} < 1$, *then* $\|u\|^{r_+}_{W_0^{1,r(z)}(\Omega)} \leq \rho_{r(z)}(\nabla u) \leq \|u\|^{r_-}_{W_0^{1,r(z)}(\Omega)}$.

Assumption (H2) guarantees the existence and positivity of the first eigenvalue $\lambda_{1,q(z)}$ of $\big(-\Delta_{q(z)}, W_0^{1,q(z)}(\Omega)\big)$ (see Fan [19] and Fan-Zhang-Zhao [21, Proposition 3.3]). Moreover, the following property holds (see Fan-Zhang-Zhao [21, Lemma 3.1] and Crespo Blanco-Winkert [14]).

Proposition 2.2. *Assume that* (H1) *and* (H2) *hold. Then, there exists $\widetilde{\mu}_* > 0$, such that*

$$\widetilde{\mu}_* \rho_{q(z)}(u) \leq \rho_{q(z)}(\nabla u) \quad \forall u \in W_0^{1,q(z)}(\Omega).$$

For our purpose, we introduce the notion of Musielak-Orlicz space and Musielak-Orlicz-Sobolev space (see Crespo Blanco-Gasiński-Harjulehto-Winkert [13] and Harjulehto-Hästö [31] for a complete overview on this topic). Let us define the nonlinear function $\mathcal{H}\colon \Omega \times [0,+\infty] \to [0,+\infty]$, by

$$\mathcal{H}(z,t) = a(z)t^{p(z)} + t^{q(z)} \quad \forall (z,t) \in \Omega \times [0+\infty],$$

and let us consider the related modular function

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(z,|u|)dz. \tag{2.1}$$

Then, we introduce the so called Musielak-Orlicz space as

$$L^{\mathcal{H}}(\Omega) = \big\{u \in L^0(\Omega) : \rho_{\mathcal{H}}(u) < +\infty\big\}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{\mathcal{H}}(\Omega)} = \inf\big\{\mu > 0 : \rho_{\mathcal{H}}\big(\frac{u}{\mu}\big) \leq 1\big\}.$$

Remark 2.3. Note that the density function $\mathcal{H}(z,t)$ associated to the double phase operator has an unbalanced growth, that is, there exists $b > 0$ such that

$$|\xi|^{q(z)} \leq \mathcal{H}(z,|\xi|) \leq b(1 + |\xi|^{p(z)}) \quad \text{for a.a. } z \in \Omega,\ \forall \xi \in \mathbb{R}^N,$$

i.e., $\mathcal{H}(z,\cdot)$ is trapped between two different powers of $|\xi|$.

Also in this case we have a useful result (see e.g., Crespo Blanco-Gasiński-Harjulehto-Winkert [13, Proposition 2.13] or Harjulehto-Hästö [31, Chapter 3]).

Lemma 2.4. *For $u \in L^{\mathcal{H}}(\Omega)$, we have*

1. *if $u \neq 0$ and $\lambda \in \mathbb{R}$, then, $\|u\|_{L^{\mathcal{H}}(\Omega)} = \lambda \Leftrightarrow \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) = 1$;*
2. *$\|u\|_{L^{\mathcal{H}}(\Omega)} < 1 (= 1; > 1) \Leftrightarrow \rho_{\mathcal{H}}(u) < 1 (= 1; > 1)$;*
3. *if $\|u\|_{L^{\mathcal{H}}(\Omega)} < 1$, then $\|u\|^{p_+}_{L^{\mathcal{H}}(\Omega)} \leq \rho_{\mathcal{H}}(u) \leq \|u\|^{q_-}_{L^{\mathcal{H}}(\Omega)}$;*
4. *if $\|u\|_{L^{\mathcal{H}}(\Omega)} > 1$, then $\|u\|^{q_-}_{L^{\mathcal{H}}(\Omega)} \leq \rho_{\mathcal{H}}(u) \leq \|u\|^{p_+}_{L^{\mathcal{H}}(\Omega)}$;*
5. *$\|u\|_{L^{\mathcal{H}}(\Omega)} \to 0$ if and only if $\rho_{\mathcal{H}}(u) \to 0$;*
6. *$\|u\|_{L^{\mathcal{H}}(\Omega)} \to +\infty$ if and only if $\rho_{\mathcal{H}}(u) \to +\infty$;*
7. *$\|u\|_{L^{\mathcal{H}}(\Omega)} \to 1$ if and only if $\rho_{\mathcal{H}}(u) \to 1$.*

Analogously, we introduce the Musielak-Orlicz-Sobolev space by

$$W^{1,\mathcal{H}}(\Omega) = \big\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\big\},$$

with the norm

$$\|u\|_{W^{1,\mathcal{H}}(\Omega)} = \|u\|_{L^{\mathcal{H}}(\Omega)} + \|\nabla u\|_{L^{\mathcal{H}}(\Omega)}.$$

Also we define

$$W_0^{1,\mathcal{H}}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,\mathcal{H}}(\Omega)}}.$$

For this space the Poincaré inequality holds (see Crespo Blanco-Gasiński-Harjulehto-Winkert [13, Proposition 2.18]). So we can consider the following equivalent norm

$$\|u\| = \|\nabla u\|_{L^{\mathcal{H}}(\Omega)}.$$

The density function $\mathcal{H}(z,\cdot)$ is uniformly convex and so the spaces $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W_0^{1,\mathcal{H}}(\Omega)$ are uniformly convex, in particular, by Milman-Pettis Theorem, reflexive (see e.g., Brézis [7, Theorem 3.31]).

We look for weak solutions of (1.2), i.e., $u \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$\int_\Omega \Big(a(z)|\nabla u|^{p(z)-2}\nabla u + |\nabla u|^{q(z)-2}\nabla u\Big)\nabla v dz = \int_\Omega \big(\lambda u^{\tau(z)-1} - f(z,x)\big) v dz \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega).$$

In particular, by variational methods, each weak solution is a critical point of a suitable energy functional. For this, we highlight some properties of the double phase operator (see Crespo Blanco-Gasiński-Harjulehto-Winkert [13, Theorem 3.3]). Let $V : W_0^{1,\mathcal{H}}(\Omega) \to \big(W_0^{1,\mathcal{H}}(\Omega)\big)^*$ be defined by

$$\langle Vu, h\rangle = \int_\Omega \Big(a(z)|\nabla u|^{p(z)-2}\nabla u + |\nabla u|^{q(z)-2}\nabla u\Big)\nabla h dz \quad \forall u, h \in W_0^{1,\mathcal{H}}(\Omega).$$

Lemma 2.5. *Assume that hypotheses* (H1) *and* (H3) *hold. Then, V is a continuous, bounded, strictly monotone operator. Moreover, V is of type (S_+), i.e., if $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $\limsup_{n\to+\infty}\langle V(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.*

Definition 2.6. Let $u \in L^0(\Omega)$, we write $0 \prec u$ if for all $D \subset \Omega$ compact, there exists a positive constant $c_D > 0$ such that

$$0 < c_D \le u(z) \text{ for a.a. } z \in \Omega.$$

Note that if $0 \prec u$ then $0 < u(z)$ for a.a. $z \in \Omega$.

In the last part of this section, we recall some results from homological group theory see Hu-Papageorgiou [34, Chapter 3] and Perera-Schechter [46, Chapter 1] for details. Let X be a Banach space. We use the following notation to indicate the critical point set of $\varphi \in C^1(X)$,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Furthermore, let $c \in \mathbb{R}$, we use

$$\varphi^c = \{u \in X : \varphi(u) \le c\}.$$

We introduce the homological critical group of φ. Let $Y_2 \subseteq Y_1 \subseteq X$. For all $k \in \mathbb{N}_0$, we denote with $H_k(Y_1, Y_2)$ the k-th relative singular homological group with real coefficients for the pair (Y_1, Y_2). Let $u_0 \in K_\varphi$ be isolated and $c = \varphi(u_0)$. Then, the critical group of φ at u_0 is defined as

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u_0\}) \quad \forall k \in N_0,$$

where U is a neighbourhood of u_0 such that $\varphi^c \cap K_\varphi = \{u_0\}$. The excision property of singular homotopy implies that the definition of C_k is independent of the choice of isolating neighbourhood U. If u_0 is an isolated local minimizer of φ, then

$$C_k(\varphi, u_0) = \delta_{k,0}\mathbb{R} \quad \forall k \in \mathbb{N}_0,$$

being $\delta_{k,0}$ the Kronecker delta, i.e.,

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m; \\ 0 & \text{if } k \ne m. \end{cases} \tag{2.2}$$

3. Positive solutions

In this section we prove our main results. In particular, we obtain a bifurcation theorem with respect to $\lambda > 0$. We use the following sets

$$\mathcal{L} = \{\lambda > 0 : (1.2) \text{ has a positive solution}\}, \tag{3.1}$$

$$S_\lambda = \text{ set of positive solutions of (1.2)}. \tag{3.2}$$

Proposition 3.1. *If hypotheses* (H1–(H4) *hold, then for all $\lambda > 0$ small, we have $\lambda \notin \mathcal{L}$.*

Proof. Let $\eta \in (0, \widetilde{\mu}_*)$ (see Proposition 2.2). By assumption (H4), for $\lambda_0 > 0$ small enough, we have

$$\lambda x^{\tau(z)-1} - f(z,x) \le \eta x^{q(z)-1} \text{ for a.a. } z \in \Omega \text{ all } x \ge 0 \text{ and } 0 < \lambda < \lambda_0. \tag{3.3}$$

Suppose, by contradiction, that $\lambda \in (0, \lambda_0) \cap \mathcal{L}$. Then, there exists $u \in S_\lambda$ solution of (1.2). By (3.3), and choosing $v = u$ as a test function in the weak formulation, we have

$$\rho_{\mathcal{H}}(\nabla u) = \int_\Omega \Big(\lambda u^{\tau(z)-1} - f(z,u)\Big) u dz \le \eta \int_\Omega u^{q(z)} dz = \eta \rho_{q(z)}(u). \tag{3.4}$$

On the other hand, by Proposition 2.2, it follows that
$$\widetilde{\mu}_* \rho_{q(z)}(u) \le \rho_{q(z)}(\nabla u) \le \rho_{\mathcal{H}}(\nabla u). \tag{3.5}$$
Then, by (3.4) and (3.5), we infer
$$(\widetilde{\mu}_* - \eta)\, \rho_{q(z)}(u) \le 0, \tag{3.6}$$
so $u = 0$. A contradiction since $u \in S_\lambda$. □

Proposition 3.2. *If hypotheses* (H1)–(H4) *hold, then, for $\lambda > 0$ large enough, $\lambda \in \mathcal{L}$.*

Proof. Let $\lambda > 0$. Consider the C^1-functional
$$\varphi(u) = \int_\Omega \Big(\frac{a(z)}{p(z)} |\nabla u|^{p(z)} + \frac{1}{q(z)} |\nabla u|^{q(z)} \Big) dz + \int_\Omega F(z, u^+) dz - \lambda \int_\Omega \frac{1}{\tau(z)} (u^+)^{\tau(z)} dz \tag{3.7}$$
defined for $u \in W_0^{1,\mathcal{H}}(\Omega)$, where $F(z,t) = \int_0^t f(z,s) ds$. By hypotheses (H1) and (H4), φ is coercive. Moreover, φ is sequentially lower semicontinuous. The Tonelli-Weierstrass Theorem (see Hu-Papageorgiou [34, Theorem 7.67]) implies that there exists $u_\lambda \in W_0^{1,\mathcal{H}}(\Omega)$, global minimum of φ. Moreover, if $u \in W_0^{1,\mathcal{H}}(\Omega)$ and $u(z) > 0$ for a.a. $z \in \Omega$, then, $\varphi(u_\lambda) < 0 = \varphi(0)$, for λ large enough. So, $u_\lambda \ne 0$ and, clearly,
$$\langle \varphi'(u_\lambda), h \rangle = 0 \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega).$$
Choosing $h = -u_\lambda^- \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function, we have
$$\rho_{\mathcal{H}}(\nabla u_\lambda^-) = 0.$$
Lemma 2.4 implies that $u_\lambda \ge 0$ and $u_\lambda \not\equiv 0$. Thus, $u_\lambda \in S_\lambda$ and $\lambda \in \mathcal{L}$. □

Note that, Propositions 3.1 and 3.2 imply that
$$\mathcal{L} \ne \emptyset \text{ and } \inf \mathcal{L} = \lambda_* > 0.$$

Proposition 3.3. *If hypotheses* (H1)–(H4) *hold and $\lambda \in \mathcal{L}$, then*
$$\emptyset \ne S_\lambda \subseteq W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega).$$

Proof. Let $u \in S_\lambda$. Since u solves (1.2), $u \in W_0^{1,\mathcal{H}}(\Omega)$. Moreover, by Crespo Blanco-Winkert [14, Theorem 3.1] (see also Ho-Winkert [32, Theorem 4.2]), $u \in L^\infty(\Omega)$. Then, $\emptyset \ne S_\lambda \subset W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. □

Proposition 3.4. *If hypotheses* (H1)– (H4) *hold, $\lambda \in \mathcal{L}$, and $\mu > \lambda$, then $\mu \in \mathcal{L}$. In particular, $\mathcal{L}$ is a half-line.*

Proof. Let $u_\lambda \in S_\lambda$ and consider the following Carathéodory function:
$$k_\mu(z,x) = \begin{cases} \mu u_\lambda(z)^{\tau(z)-1} - f(z, u_\lambda(z)) & \text{if } x \le u_\lambda(z); \\ \mu x^{\tau(z)-1} - f(z,x) & \text{if } u_\lambda(z) \le x. \end{cases} \tag{3.8}$$
Let $K_\mu(z,t) = \int_0^t k_\mu(z,s) ds$ and let $\psi : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ be the C^1-functional, defined as
$$\psi(u) = \int_\Omega \Big(\frac{a(z)}{p(z)} |\nabla u|^{p(z)} + \frac{1}{q(z)} |\nabla u|^{q(z)} \Big) dz - \int_\Omega K_\mu(z,u) dz \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.9}$$
Clearly, ψ is coercive and sequentially weakly lower semicontinuous (remember that $\tau(z) < q(z) < p(z)$), then, by the Tonelli-Weierstrass Theorem (see Hu-Papageorgiou [34, Theorem 7.67]), there exists $u_\mu \in W_0^{1,\mathcal{H}}(\Omega)$ global minimum of ψ, that is,
$$\langle \psi'(u_\mu), h \rangle = 0 \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega).$$
Choosing $h = (u_\lambda - u_\mu)^+ \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function, we obtain
$$\begin{aligned} \langle V(u_\mu), (u_\lambda - u_\mu)^+ \rangle &= \int_\Omega \Big(\mu u_\lambda^{\tau(z)-1} - f(z, u_\lambda) \Big) (u_\lambda - u_\mu)^+ dz \\ &\ge \int_\Omega \Big(\lambda u_\lambda^{\tau(z)-1} - f(z, u_\lambda) \Big) (u_\lambda - u_\mu)^+ dz \end{aligned}$$

$$= \langle V(u_\lambda), (u_\lambda - u_\mu)^+ \rangle.$$

By monotonicity of V (Lemma 2.5), we infer that

$$u_\lambda \le u_\mu. \tag{3.10}$$

By (3.8) and (3.10), it follows that $\mu \in \mathcal{L}$ and $u_\mu \in S_\mu$. This means that $(\lambda^*, +\infty) \subseteq \mathcal{L}$. ☐

Consider $\beta(z)$ as in (H4). For $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$\lambda x^{\tau(z)-1} - f(z,x) \le (\lambda+\varepsilon)x^{\tau(z)-1} + c_\varepsilon x^{\beta(z)-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.11}$$

Clearly,

$$(\lambda+\varepsilon)x^{\tau(z)-1} + c_\varepsilon x^{\beta(z)-1} \le (\lambda+\varepsilon)\max\{x^{\tau_+-1}, x^{\tau_--1}\} + c_\varepsilon \max\{x^{\beta_+-1}, x^{\beta_--1}\} \tag{3.12}$$

for almost all $z \in \Omega$ and all $x \ge 0$. To simplify the notation, we set

$$g_\lambda(z,x) = (\lambda+\varepsilon)\max\{x^{\tau_+-1}, x^{\tau_--1}\} + c_\varepsilon \max\{x^{\beta_+-1}, x^{\beta_--1}\} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0. \tag{3.13}$$

Consider the problem

$$\begin{gathered} -\Delta^a_{p(z)}u - \Delta_{q(z)}u = g_\lambda(z,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.14}$$

Proposition 3.5. *If hypotheses* (H1)–(H4) *hold and* $\lambda > 0$*, then problem* (3.14) *admits a unique solution* $\widehat{u} \in W^{1,\mathcal{H}}_0(\Omega) \cap L^\infty(\Omega)$ *such that* $0 \prec \widehat{u}$.

Proof. Consider the associated energy functional $\sigma_\lambda : W^{1,\mathcal{H}}_0(\Omega) \to \mathbb{R}$ defined as

$$\sigma_\lambda(u) = \int_\Omega \Big(\frac{a(z)}{p(z)}|\nabla u|^{p(z)} + \frac{1}{q(z)}|\nabla u|^{q(z)}\Big)dz - \int_\Omega G_\lambda(z,u^+)dz \quad \forall u \in W^{1,\mathcal{H}}_0(\Omega), \tag{3.15}$$

where $G_\lambda(z,t) = \int_0^s g_\lambda(z,s)ds$. Clearly, σ_λ is coercive and sequentially lower semi continuous, then arguing as in Propositions 3.2 and 3.3, there exists a global minimum $\widehat{u} \in W^{1,\mathcal{H}}_0 \cap L^\infty(\Omega)$, that is a nonnegative weak solution of (3.14). Moreover, by Failla-Gasiński-Papageorgiou [18, Lemma 3.1], (see also Hu-Papageorgiou [34, Theorem 5.9]), we infer that $0 \prec \widehat{u}$. Finally, we prove uniqueness (cf. Papageorgiou-Repovš-Vetro [44, Proposition 4.1]). Set

$$j(u) = \begin{cases} \int_\Omega \Big(\frac{a(z)}{p(z)}|\nabla u^{\frac{1}{\tau_+}}|^{p(z)} + \frac{1}{q(z)}|\nabla u^{\frac{1}{\tau_+}}|^{q(z)}\Big)dz & \text{if } u \ge 0, \ u^{\frac{1}{\tau_+}} \in W^{1,\mathcal{H}}_0(\Omega); \\ +\infty & \text{otherwise.} \end{cases} \tag{3.16}$$

Let $\operatorname{dom}(j) = \{u \in L^1(\Omega) : j(u) < +\infty\}$ be the effective domain of j. Note that, by Takáč-Giacomoni [51, Theorem 2.2], j is convex. Assume that $\widehat{v} \in W^{1,\mathcal{H}}_0(\Omega) \cap L^\infty(\Omega)$, $0 \prec \widehat{v}$ be another solution of (3.14). Let $\delta > 0$ and set $\widehat{u}_\delta = \widehat{u} + \delta$ and $\widehat{v}_\delta = \widehat{v} + \delta$. By Hu-Papageorgiou [34, Proposition 2.86], we infer that

$$\frac{\widehat{u}_\delta}{\widehat{v}_\delta} \in L^\infty(\Omega) \text{ and } \frac{\widehat{v}_\delta}{\widehat{u}_\delta} \in L^\infty(\Omega).$$

Moreover, setting $h = (\widehat{u}_\delta)^{\tau_+} - (\widehat{v}_\delta)^{\tau_+} \in W^{1,\mathcal{H}}_0(\Omega) \cap L^\infty(\Omega)$ and $t \in (0,1)$ small, we obtain $(\widehat{u}_\delta)^{\tau_+} + th \in \operatorname{dom}(j)$ and $(\widehat{v}_\delta)^{\tau_+} + th \in \operatorname{dom}(j)$.

Next, by Green's identity (see Hu-Papageorgiou [33, Theorem 4.106]), we have

$$\begin{aligned} j'\left((\widehat{u}_\delta)^{\tau_+}\right)(h) &= \frac{1}{\tau_+}\int_\Omega \frac{-\Delta^a_{p(z)}\widehat{u} - \Delta_{q(z)}\widehat{u}}{(\widehat{u}_\delta)^{\tau_+-1}}h\,dz = \frac{1}{\tau_+}\int_\Omega \frac{g_\lambda(z,\widehat{u})}{(\widehat{u}_\delta)^{\tau_+-1}}h\,dz, \\ j'\left((\widehat{v}_\delta)^{\tau_+}\right)(h) &= \frac{1}{\tau_+}\int_\Omega \frac{-\Delta^a_{p(z)}\widehat{v} - \Delta_{q(z)}\widehat{v}}{(\widehat{v}_\delta)^{\tau_+-1}}h\,dz = \frac{1}{\tau_+}\int_\Omega \frac{g_\lambda(z,\widehat{v})}{(\widehat{v}_\delta)^{\tau_+-1}}h\,dz. \end{aligned}$$

By Takáč-Giacomoni [51, Theorem 2.5] (Díaz-Saá-type inequality [16]), we obtain

$$0 \le \int_\Omega \Big(\frac{g_\lambda(z,\widehat{u})}{(\widehat{u}_\delta)^{\tau_+-1}} - \frac{g_\lambda(z,\widehat{v})}{(\widehat{v}_\delta)^{\tau_+-1}}\Big)\left((\widehat{u}_\delta)^{\tau_+} - (\widehat{v}_\delta)^{\tau_+}\right)dz.$$

Passing to the limits as $\delta \to 0$, by the Lebesgue dominated convergence Theorem (see e.g., Brézis [7, Theorem 4.2]) and combining the nonincreasing of the map $t \to \frac{g_\lambda(z,t)}{t^{\tau_+-1}}$ (see (3.13)), we conclude that $\widehat{u} \equiv \widehat{v}$. □

The unique solution of (3.14) is an upper bound for S_λ.

Proposition 3.6. *If hypotheses* (H1)–(H4) *hold and* $\lambda \in \mathcal{L}$, *then for all* $u \in S_\lambda$, *we have* $u \le \widehat{u}$.

Proof. Let $\lambda \in \mathcal{L}$ and $u \in S_\lambda$. Consider the Carathéodory function

$$e_\lambda(z,x) = \begin{cases} g_\lambda(z,u(z)) & \text{if } x \le u(z); \\ g_\lambda(z,x) & \text{if } u(z) < x, \end{cases} \tag{3.17}$$

and $E_\lambda(x,t) = \int_0^t e_\lambda(x,s)ds$. Consider the C^1-functional

$$\phi_\lambda(u) = \int_\Omega \Big(\frac{a(z)}{p(z)}|\nabla u|^{p(z)} + \frac{1}{q(z)}|\nabla u|^{q(z)}\Big)dz - \int_\Omega E_\lambda(z,u)dz \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.18}$$

Clearly, ϕ_λ is coercive and sequentially lower semicontinuous (since $1 < \beta_- \le \beta_+ < \tau_- \le \tau_+$). Then the Tonelli-Weierstrass Theorem implies that there exists $\widetilde{u} \in W_0^{1,\mathcal{H}}(\Omega)$, global minimum of ϕ, that is

$$\langle \phi_\lambda'(\widetilde{u}), h\rangle = 0 \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega).$$

Then

$$\langle V(\widetilde{u}), h\rangle = \int_\Omega e_\lambda(z,\widetilde{u})dz \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega).$$

Choose $h = (u-\widetilde{u})^+ \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function. By (3.13), (3.17) and since $u \in S_\lambda$, we infer that

$$\begin{aligned} \langle V(\widetilde{u}), (u-\widetilde{u})^+\rangle &= \int_\Omega g_\lambda(z,u)(u-\widetilde{u})^+dz \\ &\ge \int_\Omega \Big(\lambda u^{\tau(z)-1} - f(z,u)\Big)(u-\widetilde{u})^+dz \\ &= \langle V(u), (u-\widetilde{u})^+\rangle. \end{aligned}$$

By the monotonicity of V (see Lemma 2.5), we obtain $u \le \widetilde{u}$. Finally, by (3.17) and Proposition 3.5, it follows that $\widetilde{u} \equiv \widehat{u}$. Then, we conclude that

$$u \le \widehat{u} \quad \forall u \in S_\lambda. \tag{3.19}$$

□

Proposition 3.7. *If hypotheses* (H1)–(H4) *hold and* $\lambda \in \mathcal{L}$, *then there exists* $u_\lambda^* \in S_\lambda$ *such that*

$$u \le u_\lambda^* \quad \forall u \in S_\lambda.$$

Moreover, the map $\lambda \mapsto u_\lambda^*$ *is nondecreasing, i.e., if* $\lambda < \mu$ *follows that* $u_\lambda^* \le u_\mu^*$.

Proof. First, note that S_λ is upward directed, that is, if $u_1, u_2 \in S_\lambda$, then there exists $u \in S_\lambda$ such that $u_1 \le u$ and $u_2 \le u$ (see Filippakis-Papageorgiou [23, Lemma 4.1]). By Hu-Papageorgiou [33, Theorem 5.109], there exists a nondecreasing sequence $\{u_n\} \subseteq S_\lambda$ such that

$$\sup S_\lambda = \sup_{n\in\mathbb{N}} u_n.$$

Then, we have

$$\langle V(u_n), h\rangle = \int_\Omega \Big(\lambda u_n^{\tau(z)-1} - f(z,u_n)\Big)h\,dz \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega), \forall n \in \mathbb{N}. \tag{3.20}$$

By Proposition 3.6, and since the sequence $\{u_n\}$ is nondecreasing, we have

$$0 \le u_1 \le u_n \le \widehat{u} \quad \forall n \in \mathbb{N}. \tag{3.21}$$

Choose $h = u_n \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function in (3.20). By assumption (H4), Proposition 3.3 and (3.21), we infer that the sequence $\{u_n\}$ is bounded. Then, by the compactness of the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^{\tau(z)}(\Omega)$, we may assume that

$$u_n \rightharpoonup u_\lambda^* \text{ in } W_0^{1,\mathcal{H}}(\Omega) \text{ and } u_n \to u_\lambda^* \text{ in } L^{\tau(z)}(\Omega).$$

Choosing $h = u_n - u_\lambda^* \in W_0^{1,\mathcal{H}}(\Omega)$ in (3.20) as a test function and by the above convergence, we infer that

$$\lim_{n\to+\infty} \langle V(u_n), (u_n - u_\lambda^*)\rangle = 0.$$

By the (S_+) property of V (Lemma 2.5), we obtain $u_n \to u_\lambda^*$ in $W_0^{1,\mathcal{H}}(\Omega)$. Then, passing to the limit in (3.20), we have

$$\langle V(u_\lambda^*), h\rangle = \int_\Omega \Big(\lambda (u_\lambda^*)^{\tau(z)-1} - f(z, u_\lambda^*)\Big)\, h\, dz \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega).$$

Then $0 \le u_1 \le u_\lambda^* \le \widehat{u}$. Thus,

$$u_\lambda^* \in S_\lambda \text{ and } u_\lambda^* = \sup S_\lambda. \tag{3.22}$$

Next, let $\lambda \in \mathcal{L}$ and $\mu > \lambda$, by Proposition 3.4, $\mu \in \mathcal{L}$. Consider the Carathéodory function

$$b(z,x) = \begin{cases} \mu (u_\lambda^*(z))^{\tau(z)-1} - f(z, u_\lambda^*(z)) & \text{if } x \le u_\lambda^*(z); \\ \mu x^{\tau(z)-1} - f(z,x) & \text{if } u_\lambda^*(z) < x, \end{cases} \tag{3.23}$$

and $B(z,t) = \int_0^t b(z,s) ds$. Consider the C^1-functional

$$\hat{\sigma}(u) = \int_\Omega \Big(\frac{a(z)}{p(z)} |\nabla u|^{p(z)} + \frac{1}{q(z)} |\nabla u|^{q(z)}\Big) dz - \int_\Omega B(z,u) dz \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.24}$$

This functional is coercive and sequentially lower semicontinuous, then, the Tonelli-Weierstrass Theorem ensures that there exists $u_\mu \in W_0^{1,\mathcal{H}}(\Omega)$, global minimum, that is

$$\langle \hat{\sigma}'(u_\mu), h\rangle = 0 \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega).$$

Choosing $h = (u_\lambda^* - u_\mu)^+ \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function in the weak formulation of u_μ, we obtain $u_\lambda^* \le u_\mu$ with $u_\mu \in S_\mu$. Clearly, $u_\mu \le u_\mu^*$. Thus

$$u_\lambda^* \le u_\mu^*.$$

Then, the maximal map $\lambda \mapsto u_\lambda^*$ is nondecreasing. □

The next result shows that for $\lambda > \lambda_*$, the problem (1.2) admits at least two solutions.

Proposition 3.8. *If hypotheses* (H1)–(H4) *hold and* $\lambda > \lambda_*$*, then problem* (1.2) *admits at least two solutions*

$$u_0, \bar{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega).$$

Proof. Let $\lambda_* < \eta < \lambda < \mu$. By Proposition 3.7,

$$u_\eta^* \le u_\mu^*.$$

So, consider the Carathéodory function

$$l(z,x) = \begin{cases} \lambda (u_\eta^*(z))^{\tau(z)-1} - f(z, u_\eta^*(z)) & \text{if } x < u_\eta^*(z); \\ \lambda x^{\tau(z)-1} - f(z,x) & \text{if } u_\eta^*(z) \le x \le u_\mu^*(z); \\ \lambda (u_\mu^*(z))^{\tau(z)-1} - f(z, u_\mu^*(z)) & \text{if } x > u_\mu^*(z). \end{cases} \tag{3.25}$$

Let $L(z,t) = \int_0^t l(z,s) ds$ and the C^1-functional $\Lambda : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ such that

$$\Lambda(u) = \int_\Omega \Big(\frac{a(z)}{p(z)} |\nabla u|^{p(z)} + \frac{1}{q(z)} |\nabla u|^{q(z)}\Big) dz - \int_\Omega L(z,u) dz \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.26}$$

As in the previous propositions, Λ is coercive and sequentially lower semi continuous, then there exists $u_0 \in W_0^{1,\mathcal{H}}(\Omega)$, critical point (global minimum) of Λ, thus, by (3.25),

$$u_0 \in K_\Lambda \subseteq [u_\eta^*, u_\mu^*] \tag{3.27}$$

(choose $v = (u^*_\eta - u_0)^+$ and $v = (u_0 - u^*_\mu)^+$ as test functions and recall Lemma 2.5). Then, $u_0 \in S_\lambda$.

Next, we consider the Carathéodory function

$$r(z,x) = \begin{cases} \lambda(x^+)^{\tau(z)-1} - f(z,x^+) & \text{if } x \le u^*_\mu(z), \\ \lambda(u^*_\mu(z))^{\tau(z)-1} - f(z,u^*_\mu(z)) & \text{if } x > u^*_\mu(z). \end{cases} \tag{3.28}$$

Let $R(z,t) = \int_0^t r(z,s)ds$ and consider the C^1-functional $\hat{\varrho} : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$, defined by

$$\hat{\varrho}(u) = \int_\Omega \Big(\frac{a(z)}{p(z)}|\nabla u|^{p(z)} + \frac{1}{q(z)}|\nabla u|^{q(z)}\Big)dz - \int_\Omega R(z,u)dz \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.29}$$

Moreover, let φ_λ be the energy functional for problem (1.2), defined by

$$\varphi_\lambda(u) = \int_\Omega \Big(\frac{a(z)}{p(z)}|\nabla u|^{p(z)} + \frac{1}{q(z)}|\nabla u|^{q(z)}\Big)dz + \int_\Omega F(z,u^+)dz - \lambda\int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \tag{3.30}$$

for $u \in W_0^{1,\mathcal{H}}(\Omega)$. Taking into account assumption (H4)(iii), there exist $M > \lambda$ and $\delta \in (0,1)$, such that

$$F(z,x) \ge \frac{M}{\tau(z)}x^{\tau(z)} \quad \text{for a.a. } z \in \Omega, \text{ for all } 0 \le x \le \delta. \tag{3.31}$$

Let $\rho \le 1$ and $0 < \|u\| \le \rho$. By (H1), we have

$$\begin{aligned} \varphi_\lambda(u) &= \int_\Omega \Big(\frac{a(z)}{p(z)}|\nabla u|^{p(z)} + \frac{1}{q(z)}|\nabla u|^{q(z)}\Big)dz + \int_\Omega F(z,u^+)dz - \lambda\int_\Omega \frac{1}{\tau(z)}(u^+)^{\tau(z)}dz \\ &\ge \frac{1}{p_+}\rho_{\mathcal{H}}(\nabla u) + \frac{(M-\lambda)}{\tau_+}\rho_{\tau(z)}(u) + \int_{\{\delta < u^+\}} \big[F(z,u^+) - \frac{M}{\tau(z)}(u^+)^{\tau(z)}\big]dz. \end{aligned}$$

By Lemmas 2.1 and 2.4 and since $M > \lambda$, we have that for $\rho > 0$ small enough

$$\varphi_\lambda(u) > 0 = \varphi_\lambda(0) \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega) : 0 < \|u\| \le \rho.$$

Clearly, from the above computation follows that 0 is a strict local minimizer for φ_λ, hence

$$C_k(\varphi_\lambda, 0) = \delta_{k,0}\mathbb{R} \quad \forall k \in \mathbb{N}_0.$$

Now, we use the C^1-continuity property (see Hu-Papageorgiou [34, Theorem 3.129]) to prove that $C_k(\widehat{\varrho}, 0) = C_k(\varphi_\lambda, 0)$ for all $k \in \mathbb{N}_0$. Now, we follow the ideas in Liu-Papageorgiou [37, Proposition 13]. First, note that arguing as above, using Lemma 2.1, we have

$$\begin{aligned} |\widehat{\varrho}(u) - \varphi_\lambda(u)| &\le \int_\Omega \big|R(z,u) + F(z,u^+) - \frac{\lambda}{\tau(z)}(u^+)^{\tau(z)}\big|dz \\ &\le \int_{\{u^*_\mu < u\}} \Big(\frac{\lambda}{\tau(z)}|(u^*_\mu)^{\tau(z)} - (u^+)^{\tau(z)}| + |F(z,u^*_\mu) - F(z,u^+)|\Big)dz \\ &\le c_1\rho_{\tau(z)}(u) \le c_2 \max\{\|u\|^{\tau_-}, \|u\|^{\tau_+}\} \end{aligned}$$

for some $c_1, c_2 > 0$. Let $\delta \in (0,1)$, and consider $0 < \|u\| \le \delta$, then

$$|\widehat{\varrho}(u) - \varphi_\lambda(u)| \le c_3\|u\|^{\tau_-}. \tag{3.32}$$

Furthermore, combining Lemma 2.1, (H4), Hölder's inequality and by the continuity of the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^s(\Omega)$, for $s \in [1, (q^*)_-)$, we have that for all $h \in W^{1,\mathcal{H}}(\Omega)$,

$$\begin{aligned} |\langle \widehat{\varrho}'(u) - \varphi'_\lambda(u), h\rangle| &= \big|\int_\Omega \Big(r(z,u) + f(z,u^+) - \lambda(u^+)^{\tau(z)-1}\Big)h dz\big| \\ &\le \int_{\{u^*_\mu < u\}} \Big(\lambda|(u^*_\mu)^{\tau(z)-1} - (u^+)^{\tau(z)-1}| + |f(z,u^+) - f(z,u^*_\mu)|\Big)h\,dz \\ &\le 2\lambda\int_\Omega (u^+)^{\tau(z)-1}h\,dz + 2c_3\int_\Omega (u^+)^{\beta(z)-1}h\,dz \\ &\le c_4\int_\Omega \Big((u^+)^{\tau(z)-1} + (u^+)^{\beta(z)-1}\Big)h\,dz \end{aligned}$$

$$\begin{aligned}&\leq c_5\int_\Omega\left(\max\left\{u^{\tau_--1},u^{\tau_+-1}\right\}+\max\left\{u^{\beta_--1},u^{\beta_+-1}\right\}\right)h\,dz\\&\leq c_6\left(\max\left\{\|u\|^{\tau_--1}_{L^{\tau_-}(\Omega)},\|u\|^{\tau_+-1}_{L^{\tau_+}(\Omega)}\right\}+\max\left\{\|u\|^{\beta_--1}_{L^{\beta_-}(\Omega)},\|u\|^{\beta_+-1}_{L^{\beta_+}(\Omega)}\right\}\right)\|h\|,\end{aligned}$$

for some $c_4, c_5, c_6 > 0$. For $\|u\| \leq 1$, we have

$$|\langle\widehat{\varrho}'(u)-\varphi_\lambda'(u)\rangle|\leq c_7\|u\|^{\beta_--1}\|h\|, \tag{3.33}$$

for some $c_7 > 0$. Then

$$\|\widehat{\varrho}'(u)-\varphi_\lambda'(u)\|\leq c_8\|u\|^{\beta_--1}, \tag{3.34}$$

for some $c_8 > 0$. From (3.32) and (3.34), for $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that

$$\|\widehat{\varrho}'(u)-\varphi_\lambda'(u)\|_{C^1(B_\delta)}\leq\varepsilon,$$

where $B_\delta = \{u \in W_0^{1,\mathcal{H}}(\Omega) : \|u\| \leq \delta\}$. Finally, $\widehat{\varrho}$ and φ_λ being coercive functionals, they satisfy the Cerami condition (see Papageorgiou-Rădulescu-Repovš [42, Proposition 5.1.15] and Papageorgiou- Rădulescu-Zhang [43, Proposition 4]). Then, by the C^1-property of the homological critical groups (see Gasiński-Papageorgiou [27, Theorem 5.126] and Hu-Papageorgiou [34, Theorem 3.129]), it follows that

$$C_k(\widehat{\varrho},0)=C_k(\varphi_\lambda,0)=\delta_{k,0}\mathbb{R}\quad\forall k\in\mathbb{N}_0. \tag{3.35}$$

Now, Chang [8, Theorem 4.6] implies that $u = 0$ is a local minimizer of $\widehat{\varrho}$. By (3.25) and (3.28), we have

$$\widehat{\varrho}=\Lambda+c,$$

for some $c \in \mathbb{R}$. We know that u_0 is a local minimum for Λ (see (3.27)), then, it is a local minimum also for $\widehat{\varrho}$. As above, $\widehat{\varrho}$ is coercive and satisfies the Cerami condition (see Hu-Papageorgiou [34, Proposition 3.19]) and by (3.28) follows that $K_{\widehat{\varrho}} \subseteq [0, u^*_\mu]$. If $K_{\widehat{\varrho}}$ is not finite, clearly the problem has at least two solutions and we are done. Assume $K_{\widehat{\varrho}}$ finite. Clearly

$$\widehat{\varrho}(u_0)\leq\widehat{\varrho}(0)=0.$$

Moreover, by the above computations, $u = 0$ is a local minimizer of $\widehat{\varrho}$. Then, by Hu-Papageorgiou [34, Proposition 3.132], it follows that there exists $\widehat{\delta} \in (0, \|u_0\|)$ such that

$$\widehat{\varrho}(u_0)\leq\widehat{\varrho}(0)<\inf\left\{\widehat{\varrho}(u):\|u\|=\widehat{\delta}\right\},$$

that is, $\widehat{\varrho}$ satisfies the Mountain Pass geometry. Then the Mountain Pass Theorem (see, e.g., Papageorgiou -Rădulescu-Repovš [42, Theorem 5.4.6]) ensures that there exists $\bar{u} \in W_0^{1,\mathcal{H}}(\Omega)$ with $\bar{u} \in K_{\widehat{\varrho}}$. Finally, note that, by the above computations, $\bar{u} \neq 0$, $\bar{u} \neq u_0$ and $\bar{u} \in S_\lambda$. □

Finally, we prove the admissibility of λ^*.

Proposition 3.9. *If hypotheses* (H1)–(H4) *hold, then* $\lambda^* \in \mathcal{L}$.

Proof. Let $\{\lambda_n\} \subseteq (\lambda_*, +\infty)$ be a nonincreasing sequence such that $\lambda_n \to \lambda^*$. By Proposition 3.7, we have

$$\langle V(u^*_{\lambda_n}),h\rangle=\int_\Omega\left(\lambda_n(u^*_{\lambda_n})^{\tau(z)-1}-f(z,u^*_{\lambda_n})\right)h\,dz\quad\forall h\in W_0^{1,\mathcal{H}}(\Omega),\forall n\in\mathbb{N}, \tag{3.36}$$

and

$$0\leq u^*_{\lambda_n}\leq u^*_{\lambda_1}\quad\forall n\in\mathbb{N}.$$

Choosing $h = u^*_{\lambda_n} \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function, we infer that the sequence $\{u^*_{\lambda_n}\}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Then, we can assume that

$$u^*_{\lambda_n}\rightharpoonup u^*\text{ in }W_0^{1,\mathcal{H}}(\Omega)\quad\text{and}\quad u^*_{\lambda_n}\to u^*\text{ in }L^{\tau(z)}(\Omega). \tag{3.37}$$

Choosing $h = u^*_{\lambda_n} - u^* \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function and by the (S_+) property of V (Lemma 2.5), we obtain

$$u^*_{\lambda_n}\to u^*\text{ in }W_0^{1,\mathcal{H}}(\Omega).$$

Passing to the limits as $n \to +\infty$ in (3.36), we have

$$\langle V(u^*), h\rangle = \int_\Omega \Big(\lambda^*(u^*)^{\tau(z)-1} - f(z,u^*)\Big)\, h\, dz \quad \forall h \in W_0^{1,\mathcal{H}}(\Omega). \tag{3.38}$$

Finally, we prove that $u^* \neq 0$. By contradiction, suppose that $u^* = 0$. So, we may assume that $u^*_{\lambda_n} \to 0$ in $W_0^{1,\mathcal{H}}(\Omega)$. Thus, without any loss of generality, we may assume that

$$\|u^*_{\lambda_n}\| \le 1 \quad \forall n \in \mathbb{N}.$$

Choosing $h = u^*$ in (3.38) and by Lemmas 2.1 and 2.4, we obtain

$$\|u^*\|^{p_+} \le \rho_{\mathcal{H}}(\nabla u^*) \le \frac{\lambda^*}{\tau_-}\rho_{\tau(z)}(u^*) - \int_\Omega f(z,u^*)u^* dz \le \frac{\lambda^*}{\tau_-}\|u^*\|^{\tau_-} - \int_\Omega f(z,u^*)u^* dz. \tag{3.39}$$

By (H4), we have

$$|f(z,x)| \le c_9\Big(x^{\beta(z)-1} + x^{\tau(z)-1}\Big) \quad \text{for a.a. } z \in \Omega \text{ all } x \ge 0.$$

for some $c_9 > 0$. Now Lemma 2.1 implies

$$|f(z,x)| \le c_9\left(\max\left\{x^{\beta_- -1}, x^{\beta_+ -1}\right\} + \max\left\{x^{\tau_- -1}, x^{\tau_+ -1}\right\}\right).$$

Next, we use a Guedda-Véron [29, Proposition 1.3] argument. Let $g \in L^s(\Omega)$, with $s > \frac{N}{q_-}$ and assume that is u satisfies

$$\int_\Omega \Big(a(x)|\nabla u|^{p(x)-2} + |\nabla u|^{q(x)-2}\Big)\nabla u \nabla v dx = \int_\Omega g(x) v dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega).$$

Then, we infer that

$$\|u\|_{L^\infty(\Omega)} \le \widetilde{C}\max\Big\{\|g\|_{L^s(\Omega)}^{\frac{1}{q_- -1}}, \|g\|_{L^s(\Omega)}^{\frac{1}{q_+ -1}}\Big\}.$$

Indeed, by Lemma 2.1, we can consider two cases: if $\|u\| \le 1$, by Perera-Squassina [47, Proposition 2.4], we have

$$\|u\|_{L^\infty(\Omega)} \le \bar{c}_1\|g\|_{L^s(\Omega)}^{\frac{1}{q_+ -1}}.$$

Similarly, for $\|u\| \ge 1$, again Perera-Squassina [47, Proposition 2.4] ensures that

$$\|u\|_{L^\infty(\Omega)} \le \bar{c}_2\|g\|_{L^s(\Omega)}^{\frac{1}{q_- -1}}.$$

Then, if $s \in (\frac{N}{q_-}, q_-^*)$ (see hypothesis (H_0)), with $q_-^* = \frac{Nq_-}{N-q_-}$, the Sobolev critical exponent, we have

$$\begin{aligned}
&\|u^*_{\lambda_n}\|_{L^\infty(\Omega)} \\
&\le c_{10}\max\Big\{\Big(\max\big\{\|u^*_{\lambda_n}\|^{\beta_- -1}_{L^s(\Omega)}, \|u^*_{\lambda_n}\|^{\beta_+ -1}_{L^s(\Omega)}\big\} + \max\big\{\|u^*_{\lambda_n}\|^{\tau_- -1}_{L^s(\Omega)}, \|u^*_{\lambda_n}\|^{\tau_+ -1}_{L^s(\Omega)}\big\}\Big)^{\frac{1}{q_- -1}}, \\
&\Big(\max\big\{\|u^*_{\lambda_n}\|^{\beta_- -1}_{L^s(\Omega)}, \|u^*_{\lambda_n}\|^{\beta_+ -1}_{L^s(\Omega)}\big\} + \max\big\{\|u^*_{\lambda_n}\|^{\tau_- -1}_{L^s(\Omega)}, \|u^*_{\lambda_n}\|^{\tau_+ -1}_{L^s(\Omega)}\big\}\Big)^{\frac{1}{q_+ -1}}\Big\}
\end{aligned}$$

for some $c_{10} > 0$. Then, since $1 < \beta_- \le \beta_+ < \tau_- \le \tau_+ < q_-$ and since $u^*_{\lambda_n} \to u^*$ in $W_0^{1,\mathcal{H}}(\Omega)$, we have

$$\|u^*_{\lambda_n}\|_{L^\infty(\Omega)} \to 0 \quad \text{as } n \to +\infty. \tag{3.40}$$

On the other hand, let $M > \lambda_1$. Arguing as in (3.31), there exists $\delta \in (0,1)$ such that

$$f(z,x)x \ge Mx^{\tau(z)} \text{ for a.a. } z \in \Omega \text{ and all } 0 \le x \le \delta. \tag{3.41}$$

So, by (3.40), there exists $n_0 \in \mathbb{N}$ such that

$$f(z, u^*_{\lambda_n}(z))u^*_{\lambda_n}(z) \ge M(u^*_{\lambda_n}(z))^{\tau(z)} \quad \text{for a.a. } z \in \Omega \text{ and all } n \ge n_0. \tag{3.42}$$

Finally, by this last inequality and (3.39), it follows that

$$\|u^*_{\lambda_n}\|^{p_+} \le (\lambda_1 - M)\rho_\tau(u^*_{\lambda_n}) < 0 \quad \forall n \ge n_0, \tag{3.43}$$

a contradiction. Then, $u^* \neq 0$ and $u^* \in S_\lambda$. □

To sum up, we have the following bifurcation theorem.

Theorem 3.10. *If hypotheses* (H1)–(H4) *hold, then there exists* $\lambda^* > 0$ *such that*

- *for all* $\lambda > \lambda^*$*, there exist at least two positive solutions* $u_0, \bar{u}$ *of* (1.2)*, with* $u_0, \bar{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$*;*
- *for* $\lambda = \lambda^*$*, there exists at least one positive solution* u^* *of the problem* (1.2) *with* $u^* \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$*;*
- *for all* $0 < \lambda < \lambda^*$ *the problem* (1.2) *does not admit any solution;*
- *for all* $\lambda \geq \lambda^*$*, the problem* (1.2) *has a maximal solution* $u_\lambda^* \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ *and the maximal solution map* $\lambda \mapsto u_\lambda^*$ *is nondecreasing.*

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