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WEAK-PULLBACK MEAN-RANDOM ATTRACTORS FOR NON-AUTONOMOUS STOCHASTIC REACTION-DIFFUSION EQUATIONS WITH DYNAMICAL BOUNDARY CONDITIONS

SISI LIU, MINGLI HONG, LU YANG

Abstract. In this article, we study the long-time behavior of solutions of the non-autonomous stochastic reaction-diffusion equations with dynamical boundary conditions. When the diffusion terms and drift terms are general nonlinear functions, we prove the existence of the weak pullback mean random attractors for deterministic equations with random initial data and stochastic equations, respectively.

1. Introduction

In this article, we consider the non-autonomous stochastic reaction-diffusion equations with dynamical boundary conditions:

$$\begin{gathered} du-\Delta u dt=(f(u)+g(x,t))dt+\epsilon G(t,u)dW, \quad \text{in } \mathcal{O}\times\mathbb{R}^+,\\ \frac{\partial u}{\partial t}+\frac{\partial u}{\partial \vec{n}}=h(u), \quad \text{on } \Gamma\times\mathbb{R}^+,\\ u(x,\tau)=u_0(x), \quad \text{in } \mathcal{O},\\ u(x,\tau)=v_0(x), \quad \text{on } \Gamma, \end{gathered} \tag{1.1}$$

where $\mathcal{O}\subset\mathbb{R}^n$ is a bounded domain with a smooth boundary Γ, $g\in L^2_{\rm loc}(\mathbb{R},L^2(\mathcal{O}))$, f, h and G are given nonlinear functions. $\epsilon G(t,u)dW$ is the noise term, and constant $\epsilon\in(0,1]$ represents the intensity of noise.

The reaction-diffusion equation with dynamical boundary condition arises in hydrodynamics and heat transfer theory. This problem has strong background in mathematical physics (see [1, 2, 3, 5, 6, 7, 8, 22, 13, 21, 23, 24, 14, 4] and the references therein). The dynamical behavior of reaction-diffusion equation with dynamical boundary condition have been extensively studied in the literature, see [1, 2, 3, 5, 7, 8, 22, 13, 21, 23, 14, 4] for the deterministic case and [6, 24] for the stochastic case. For example, the authors in [6] proved the existence of random attractor in $L^2(\mathcal{O})\times L^2(\Gamma)$, [24] obtained the existence of random attractors in $L^p(\mathcal{O})\times L^p(\Gamma)$ and $H^1(\mathcal{O})\times H^{\frac12}(\Gamma)$.

In recent years the random attractors have been considered widely by many authors, see [5, 6, 9, 10, 11, 15, 16, 17, 18, 19, 20, 24]. Most of these results require that the diffusion term $G(t,u)$ be linear in u. When $G(t,u)$ is a general nonlinear function, the situation becomes more complex, we even do not know whether system (1.1) generates a random dynamical system or not. To deal with the nonlinear diffusion term $G(t,u)$, The authors in [11] proposed the mean-square random attractor theory. However, the mean-square random attractor theory seems hard to tackle the general nonlinear drift term. Recently, Wang [18, 19] introduced a new type of weak mean-square random attractor (i.e., weak pullback mean random attractor) in appropriate Bochner spaces, to handle both the nonlinear drift terms and nonlinear diffusion terms. Different

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from [11], the weak mean-square random attractor is minimal instead of being invariant in [11]. Note that the theory of weak mean-square random attractor has been successfully applied to a variety of stochastic equations with general nonlinear drift and diffusion terms, for example, [18] considered the reaction-diffusion equation and [19] discussed the Navier-Stokes equations with a general Lipschitz nonlinear diffusion term. Later, Gu [9] applied the theory of weak pullback mean random dynamical systems to the stochastic p-Laplacian equations, and in [10] for abstract stochastic equations.

In this article, inspired by the ideas in [18, 19], we use the theory of weak pullback mean random attractors to investigate the long time behavior of solutions for the stochastic equation (1.1) with nonlinear drift terms and nonlinear diffusion terms. To be precise, we first prove the existence and uniqueness of weak pullback mean random attractors for the deterministic equation (1.1) (i.e., $\epsilon = 0$) with random initial data, see Theorem 3.6, which can be considered as a special case of the stochastic equations with random initial data. And then, if the intensity ϵ is small enough, we prove the existence of weak pullback mean random attractors for system (1.1) with general nonlinear drift terms and nonlinear diffusion terms, see Theorem 4.4.

This paper is organized as follows: In Section 2, we review some preliminaries of function space and the weak pullback mean random attractors. In Section 3, we consider the deterministic case of (1.1) with random initial data, we define a mean random dynamical system and prove the existence of weak pullback mean random attractors. In Section 4, we explore the stochastic equation (1.1) with random initial data, and establish the weak pullback mean random attractors for (1.1) with a small enough intensity of noise but general nonlinear diffusion terms and drift terms.

2. Preliminaries

In this section, we review some basic concepts, properties, and theorems that we will use later. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$ is a complete filtered probability space, and $\{\mathscr{F}_t\}_{t\in\mathbb{R}}$ is an increasing right continuous family of sub-σ-algebras of $\mathscr{F}$ that contains all P-null sets. Let X be a separable real Banach space with norm $\|\cdot\|_X$. We denote the norm in $L^p(\mathcal{O})$, $L^p(\Gamma)$ by $\|\cdot\|_{L^p(\mathcal{O})}$ and $\|\cdot\|_{L^p(\Gamma)}$ respectively. In particular, $\|\cdot\|_{\mathcal{O}}$ and $\|\cdot\|_\Gamma$ stand for the norms in $L^2(\mathcal{O})$ and $L^2(\Gamma)$ with $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ denoting the corresponding inner product in $L^2(\mathcal{O})$ and $L^2(\Gamma)$ respectively. Denote $\|\cdot\|_{\mathcal{O}\times\Gamma}$ as the norm in $L^2(\mathcal{O}) \times L^2(\Gamma)$.

A function $\Phi : \Omega \to X$ is called strong measurable, if there exists a sequence of simple functions $\Phi_n : \Omega \to X$, such that P-a.s. $\lim_{n\to\infty} \|\Phi_n - \Phi\|_X = 0$. Moreover, a function Φ is called Bochner integrable if Φ is strong measurable and there exists a sequence of simple function $\Phi_n : \Omega \to X$, such that

$$\lim_{n\to\infty} \int_\Omega \|\Phi_n - \Phi\|_X \, dP = 0.$$

The Bochner integral of Φ on Ω is defined by

$$\int_\Omega \Phi \, dP = \lim_{n\to\infty} \int_\Omega \Phi_n \, dP,$$

for every $p \in (1,\infty)$, the Banach space $L^p(\Omega, \mathscr{F}; X)$ include all(equivalence classes of)Bochner integrable functions $\Phi : \Omega \to X$ such that

$$\|\Phi\|_{L^p(\Omega,\mathscr{F};X)} = \Big(\int_\Omega \|\Phi\|_X \, dP \Big)^{1/p} < \infty.$$

Suppose $\mathscr{D}$ is the collection of all families of nonempty bounded subsets of $L^p(\Omega, \mathscr{F}_\tau; X)$ parametrized by $\tau \in \mathbb{R}$, i.e.

$$\begin{gathered} D(\tau) \subseteq L^p(\Omega, \mathscr{F}_\tau; X) : \text{bounded}, D(\tau) \neq \emptyset, \tau \in \mathbb{R}\} \\ \mathscr{D} = \{D : D \text{ satisfies certain conditions}\}. \end{gathered} \tag{2.1}$$

If $D = \{D(\tau) : \tau \in \mathbb{R}\} \in \mathscr{D}$ then every family $D_0 = \{D_0(\tau) : \emptyset \neq D_0(\tau) \subseteq D(\tau), \forall \tau \in \mathbb{R}\} \in \mathscr{D}$, and the collection $\mathscr{D}$ is called inclusion-closed.

Definition 2.1. A family $\Phi = \{\Phi(t,\tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R}\}$ of mappings is called a mean random dynamical system on $L^p(\Omega, \mathscr{F}; X)$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$ if for all $\tau \in \mathbb{R}$ and $t, s \in \mathbb{R}^+$, the following conditions are satisfied:

(1) $\Phi(t,\tau)$ maps $L^p(\Omega, \mathscr{F}_\tau; X)$ to $L^p(\Omega, \mathscr{F}_{t+\tau}; X)$,
(2) $\Phi(0,\tau) = I$ on $L^p(\Omega, X)$,
(3) $\Phi(t+s,\tau) = \Phi(t, \tau+s) \circ \Phi(s,\tau)$.

Definition 2.2. A family $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \mathscr{D}$ is called a $\mathscr{D}$-pullback absorbing set for Φ on $L^p(\Omega, \mathscr{F}; X)$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$ if for every $\tau \in \mathbb{R}$ and $D \in \mathscr{D}$, there exists $T = T(\tau, D) > 0$ such that

$$\Phi(t,\tau-t)(D(\tau-t)) \subseteq K(\tau) \textit{ for all } t \geq T.$$

If $K(\tau)$ is a weakly compact nonempty subset of $L^p(\Omega, \mathscr{F}_\tau; X)$ for every $\tau \in \mathbb{R}$, then $K = \{K(\tau) : \tau \in \mathbb{R}\}$ is called a weakly compact $\mathscr{D}$-pullback absorbing set for Φ.

Definition 2.3. A family $K = \{K(\tau) : \tau \in \mathbb{R}\} \in \bar{\mathscr{D}}$ is called a $\bar{\mathscr{D}}$-pullback weakly attracting set for Φ on $L^p(\Omega, \mathscr{F}; X)$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$ if for every $\tau \in \mathbb{R}$, $D \in \bar{\mathscr{D}}$ and for every weak neighborhood $\mathcal{N}^\omega(K(\tau))$ of $K(\tau)$ in $L^p(\Omega, \mathscr{F}_\tau; X)$, there exists $T = T(\tau, D, \mathcal{N}^\omega(K(\tau))) > 0$ such that for all $t \geq T$

$$\Phi(t,\tau-t)(D(\tau-t)) \subseteq \mathcal{N}^\omega(K(\tau)).$$

Moreover, if $K(\tau)$ is a weakly compact subset of $L^p(\Omega, \mathscr{F}_\tau; X)$ for every $\tau \in \mathbb{R}$, then $K = \{K(\tau) : \tau \in \mathbb{R}\}$ is called a $\mathscr{D}$-pullback weakly compact weakly attracting set for Φ in $L^p(\Omega, \mathscr{F}_\tau; X)$.

Definition 2.4. A family $\mathscr{A} = \{\mathscr{A}(\tau) : \tau \in \mathbb{R}\} \in \bar{\mathcal{D}}$ is called a weak $\bar{\mathcal{D}}$-pullback mean random attractor for Φ in $L^p(\Omega, \mathscr{F}; X)$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$ if the following conditions are satisfied:

(1) $\mathscr{A}(\tau)$ is a weakly compact subset of $L^p(\Omega, \mathscr{F}; X)$ for every $\tau \in \mathbb{R}$,
(2) $\mathscr{A}$ is a $\bar{\mathcal{D}}$-pullback weakly attracting set of Φ,
(3) $\mathscr{A}$ is the minimal element of $\bar{\mathcal{D}}$ with properties (1) and (2); i.e., if there exists a set $B = \{B(\tau) : \tau \in \mathbb{R}\} \in \bar{\mathscr{D}}$ is a $\bar{\mathscr{D}}$-pullback weakly compact weakly attracting set for Φ, $\mathscr{A}(\tau) \subseteq B(\tau)$ for all $\tau \in \mathbb{R}$.

Theorem 2.5. *Let X be a reflexive Banach space and $p \in (1,\infty)$. Suppose $\bar{\mathcal{D}}$ is an inclusion-closed collection of some families of nonempty bounded subsets of $L^p(\Omega, \mathscr{F}; X)$ and Φ is a mean random dynamical system in $L^p(\Omega, \mathscr{F}; X)$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$. If Φ has a weakly compact $\bar{\mathscr{D}}$-pullback absorbing set $K \in \bar{\mathcal{D}}$ in $L^p(\Omega, \mathscr{F}; X)$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$, then Φ has a unique $\bar{\mathcal{D}}$-pullback mean random attractor $\mathscr{A} \in \bar{\mathcal{D}}$ in $L^p(\Omega, \mathscr{F}; X)$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$, which is given by, for each $\tau \in \mathbb{R}$,*

$$\mathscr{A}(\tau) = \Omega^\omega(K,\tau) = \cap_{r\geq 0}\overline{\cup_{t\geq r}\Phi(t,\tau-t)(K(\tau-t))}^{\omega},$$

where the closure is taken with respect to the weak topology of $L^p(\Omega, \mathscr{F}_\tau; X)$.

If we replace the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$ and $L^p(\Omega, \mathscr{F}; X)$ in Definition 2.1–2.4 with the ordinary probability space $(\Omega, \mathscr{F}, P)$ and $L^p(\Omega, \mathscr{F}; X)$, the related concepts still exist. Then we replace the space in Theorem 2.5, so we obtain the existence and uniqueness of weak pullback mean random attractors in $L^p(\Omega, \mathscr{F}; X)$ over probability space $(\Omega, \mathscr{F}, P)$. Moreover, in the following content, we write $L^p(\Omega, \mathscr{F}; X)$ as $L^p(\Omega, X)$ for convenience.

Theorem 2.6. *Let X be a reflexive Banach space and $p \in (1,\infty)$. Suppose $\mathcal{D}$ is an inclusion-closed collection of some families of nonempty bounded subsets of $L^p(\Omega, X)$. If Φ has a weakly compact $\mathscr{D}$-pullback absorbing set $K \in \mathcal{D}$, then Φ has a unique $\mathcal{D}$-pullback mean random attractor $\mathscr{A} \in \mathcal{D}$ which is given by, for each $\tau \in \mathbb{R}$*

$$\mathscr{A}(\tau) = \Omega^\omega(K,\tau) = \cap_{r\geq 0}\overline{\cup_{t\geq r}\Phi(t,\tau-t)(K(\tau-t))}^{\omega},$$

where the closure is taken with respect to the weak topology of $L^p(\Omega, X)$.

3. Weak mean random attractors for deterministic equations with random initial data

In this section, we prove the existence of the weak $\mathscr{D}_0$-pullback mean random attractors for the deterministic equation (1.1) (i.e., $\epsilon = 0$) with random initial data.

3.1. Generation of mean random system.

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ. For every $\tau \in \mathbb{R}$ and $t > \tau$, we consider the following problem

$$\begin{gathered}\frac{\partial u}{\partial t} - \Delta u = f(u) + g(x,t), \quad \text{in } \mathcal{O} \times \mathbb{R}^+,\\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial \vec{n}} = h(u), \quad \text{on } \Gamma \times \mathbb{R}^+,\\ u(0) = u_0, \quad \text{on } \mathcal{O},\\ u(0) = v_0, \quad \text{on } \Gamma.\end{gathered} \tag{3.1}$$

The function $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathcal{O}))$. f and $h : \mathbb{R} \to \mathbb{R}$ are smooth nonlinear functions such that for all $m \in \mathbb{R}$,

$$f(m)m \le -\alpha_1 |m|^p + \beta, \tag{3.2}$$

$$|f(m_1) - f(m_2)| \le \alpha_2 |m_1 - m_2|(1 + |m_1|^{p-2} + |m_2|^{p-2}), \tag{3.3}$$

$$h(m)m \le -\tilde{\alpha}_1 |m|^q + \tilde{\beta}, \tag{3.4}$$

$$|h(m_1) - h(m_2)| \le \tilde{\alpha}_2 |m_1 - m_2|(1 + |m_1|^{q-2} + |m_2|^{q-2}), \tag{3.5}$$

$$f(0) = 0, \quad h(0) = 0, \tag{3.6}$$

$$f'(m) \le l, \quad h'(m) \le k, \tag{3.7}$$

where α_1, $\tilde{\alpha}_1$, α_2, $\tilde{\alpha}_2$, β, $\tilde{\beta}$, l, k are some positive numbers and p, $q \ge 2$. We use the notation $\gamma_0 : u \to u|_\Gamma$ as the trace operator and $v(t) := u(t)|_\Gamma$, the trace operator belongs to $\mathcal{L}(H^1(\mathcal{O}), H^{\frac{1}{2}}(\Gamma))$.

For convenience, we firstly give the result for problem (3.1) when the initial condition (u_0, v_0) is deterministic.

Definition 3.1. Let $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathcal{O}))$ and (3.2)–(3.7) hold. Then for every $\tau \in \mathbb{R}$ and every deterministic $(u_0, v_0) \in L^2(\mathcal{O}) \times L^2(\Gamma)$, problem (3.1) has a unique solution

$$\begin{gathered}u \in C([\tau,\infty), L^2(\mathcal{O})) \cap L^2_{\rm loc}((\tau,\infty), H^1(\mathcal{O})) \cap L^p_{\rm loc}((\tau,\infty), L^p(\mathcal{O})),\\ v \in C([\tau,\infty), L^2(\Gamma)) \cap L^2_{\rm loc}((\tau,\infty), H^{\frac{1}{2}}(\Gamma)) \cap L^q_{\rm loc}((\tau,\infty), L^q(\Gamma)).\end{gathered} \tag{3.8}$$

Then when the initial data (u_0, v_0) is random, for the well-posedness of problem (3.1), the definition of the solution for system (3.1) with random initial condition is given below.

Definition 3.2. Let $\tau \in \mathbb{R}$ and $(u_0, v_0) \in L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$. We say that a continuous mapping $(u(\cdot,\tau,u_0), v(\cdot,\tau,v_0)) : [\tau,\infty) \to L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$ is called a solution of problem (3.1) if

$$\begin{gathered}u(\cdot,\tau,u_0) \in C([\tau,\infty), L^2(\Omega, L^2(\mathcal{O}))) \cap L^2_{\rm loc}((\tau,\infty), L^2(\Omega, H^1(\mathcal{O}))) \cap L^p_{\rm loc}((\tau,\infty), L^p(\Omega, L^p(\mathcal{O}))),\\ v(\cdot,\tau,v_0) \in C([\tau,\infty), L^2(\Omega, L^2(\Gamma))) \cap L^2_{\rm loc}((\tau,\infty), L^2(\Omega, H^{\frac{1}{2}}(\Gamma))) \cap L^p_{\rm loc}((\tau,\infty), L^q(\Omega, L^q(\Gamma))),\\ \gamma_0(u(t)) = v(t) \quad \text{a.e. } t \ge \tau,\end{gathered} \tag{3.9}$$

and for every $t > \tau$, $(\eta, \gamma_0\eta) \in (H^1(\mathcal{O}) \cap L^p(\mathcal{O})) \times (H^{1/2}(\Gamma) \cap L^q(\mathcal{O}))$, u satisfies, P-a.s.,

$$\begin{aligned}&(u(t),\eta)_{\mathcal{O}} + (v(t),\gamma_0\eta)_\Gamma + \int_\tau^t (\nabla u, \nabla\eta)_{\mathcal{O}}\,ds\\ &= (u_0,\eta)_{\mathcal{O}} + (v_0,\gamma_0\eta)_\Gamma + \int_\tau^t\int_{\mathcal{O}} f(u)\eta\,dx\,ds + \int_\tau^t\int_\Gamma h(v)\gamma_0\eta\,dS\,ds + \int_\tau^t\int_{\mathcal{O}} g(x,s)\eta\,dx\,ds.\end{aligned} \tag{3.10}$$

Now, we prove the existence and uniqueness of the solution for system (3.1) in the sense of Definition 3.2.

Theorem 3.3. *Let $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathcal{O}))$ and condition (3.2)–(3.7) hold. Then for every $\tau \in \mathbb{R}$ and $(u_0, v_0) \in L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$, problem (3.1) has a unique solution $(u(\cdot, \tau, u_0), v(\cdot, \tau, v_0))$ in the sense of Definition 3.2. Also, (u, v) satisfies the energy equation:*

$$\begin{aligned} &\frac{d}{dt} E\left(\|u(t,\tau,u_0)\|^2_{\mathcal{O}}\right) + \frac{d}{dt} E\left(\|v(t,\tau,v_0)\|^2_{\Gamma}\right) + 2E\left(\|\nabla u(t,\tau,u_0)\|^2_{\mathcal{O}}\right) \\ &= 2E\Big(\int_{\mathcal{O}} f(u)u\,dx\Big) + 2E\Big(\int_{\Gamma} h(v)v dS\Big) + 2E\left(g(t), u\right)_{\mathcal{O}}. \end{aligned} \tag{3.11}$$

Proof. The proof is divided into four steps. We first use the Galerkin method to construct a sequence of approximate solutions.

1.Approximate solutions. We define an operator $A = -\Delta$ maps $H^1(\mathcal{O})$ to $(H^1(\mathcal{O}))^*$, then through [18], we know A has a family of eigenfunctions $\{e_j\}_{j=1}^\infty$, which is the orthonormal basis of $L^2(\mathcal{O}) \times L^2(\Gamma)$ and the corresponding family of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ satisfies

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \to \infty \ as\ j \to \infty.$$

Let $n \in \mathbb{N}$, denote X_n the space spanned by $\{e_j : j = 1, \dots, n\}$ and $P_n : L^2(\mathcal{O}) \times L^2(\Gamma) \to X_n$ the projection given by

$$P_n u = \sum_{j=1}^n \varsigma_j e_j, \quad P_n v = \sum_{j=1}^n \iota_j e_j \quad \forall u \in L^2(\mathcal{O}), \forall v \in L^2(\Gamma).$$

Then extend P_n to $(L^p(\mathcal{O}))^*$, $(L^q(\Gamma))^*$ and $(H^1(\mathcal{O}))^*$ by

$$P_n \eta = \sum_{j=1}^n (\eta(e_j)) e_j, \quad \text{for } \eta \in (L^p(\mathcal{O}))^* \text{ or } (H^1(\mathcal{O}))^*,$$

$$P_n \xi = \sum_{j=1}^n (\xi(e_j)) e_j, \quad \text{for } \xi \in (L^q(\Gamma))^*.$$

Let $(u_0, v_0) \in L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$ such that $E(\|(u_0, v_0)\|^2_{\mathcal{O}\times\Gamma}) < \infty$. Then for every fixed ω, we study the following deterministic system for $(u_n(t, \tau, \omega), v_n(t, \tau, \omega)) \in X_n$,

$$\begin{gathered} \frac{du_n}{dt} + P_n A(u_n) = P_n f(u_n) + P_n g, \quad t > \tau, \\ \frac{dv_n}{dt} + \frac{\partial P_n v_n}{\partial \vec{n}} = P_n h(v_n), \quad t > \tau, \\ (u_n(\tau, \tau, \omega), v_n(\tau, \tau, \omega)) = P_n(u_0(\omega), v_0(\omega)). \end{gathered} \tag{3.12}$$

Under conditions (3.2)–(3.7) and $T > 0$, we find that for $\tau \in \mathbb{R}$ and every fixed $\omega \in \Omega$, problem (3.12) has a unique maximal solution $(u_n(\cdot, \tau, \omega), v_n(\cdot, \tau, \omega)) \in C^1([\tau, \tau + T), X_n)$. Moreover, when $t \ge \tau$, (u_n, v_n) is $\mathcal{F}$-measurable with respect to $\omega \in \Omega$. Then the uniform estimate given below imply $T = \infty$.

2. Uniform estimates. Taking the inner product of (3.12) with u_n in $L^2(\mathcal{O})$ and v_n in $L^2(\Gamma)$, we obtain

$$\frac{d}{dt}\|u_n\|^2_{\mathcal{O}} + \frac{d}{dt}\|v_n\|^2_{\Gamma} + 2\|\nabla u_n\|^2_{\mathcal{O}} = 2\int_{\mathcal{O}} f(u_n) u_n dx + 2\int_{\Gamma} h(v_n) v_n dS + 2(g, u_n)_{\mathcal{O}}. \tag{3.13}$$

From (3.2), we obtain

$$2\int_{\mathcal{O}} f(u_n) u_n dx \le -2\alpha_1 \int_{\mathcal{O}} |u_n|^p dx + 2\beta |\mathcal{O}|, \tag{3.14}$$

in which $|\mathcal{O}|$ is the Lebesgue measure of $\mathcal{O}$. From (3.4), we obtain

$$2\int_{\Gamma} h(v_n) v_n dS \le -2\tilde{\alpha}_1 \int_{\Gamma} |v_n|^q dS + 2\tilde{\beta}|\Gamma|, \tag{3.15}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. Using the Young inequality, we have

$$2(g,u_n)_{\mathcal{O}} \le \|u_n\|_{\mathcal{O}}^2 + \|g(t)\|_{\mathcal{O}}^2. \tag{3.16}$$

By (3.13)–(3.16), we obtain

$$\begin{aligned} \frac{d}{dt}\|u_n\|_{\mathcal{O}}^2 + \frac{d}{dt}\|v_n\|_{\Gamma}^2 + 2\|\nabla u_n\|_{\mathcal{O}}^2 &\le -2\alpha_1 \int_{\mathcal{O}} |u_n|^p dx - 2\tilde{\alpha}_1 \int_{\Gamma} |v_n|^q dS \\ &\quad + 2\tilde{\beta}|\Gamma| + 2\beta|\mathcal{O}| + \|u_n\|_{\mathcal{O}}^2 + \|g(t)\|_{\mathcal{O}}^2. \end{aligned} \tag{3.17}$$

Multiplying (3.17) by e^{-t} and integrating over (τ,t) for $t>\tau$, then we have

$$\begin{aligned} &\|u_n(t,\tau,\omega)\|_{\mathcal{O}}^2 + \|v_n(t,\tau,\omega)\|_{\Gamma}^2 + 2\int_\tau^t e^{t-s}\|\nabla u_n(s,\tau,\omega)\|_{\mathcal{O}}^2 ds \\ &+ \int_\tau^t e^{t-s}\|v_n(s,\tau,\omega)\|_{\Gamma}^2 ds + 2\tilde{\alpha}_1 \int_\tau^t e^{t-s}\|v_n(s,\tau,\omega)\|_{L^q(\Gamma)}^q ds \\ &+ 2\alpha_1 \int_\tau^t e^{t-s}\|u_n(s,\tau,\omega)\|_{L^p(\mathcal{O})}^p ds \\ &\le e^{t-\tau}(\|(u_0(\omega),v_0(\omega))\|_{\mathcal{O}\times\Gamma}^2 + \int_\tau^t e^{t-s}\|g(s)\|_{\mathcal{O}}^2 ds + 2\beta e^{t-\tau}|\mathcal{O}| + 2\tilde{\beta}e^{t-\tau}|\Gamma|. \end{aligned} \tag{3.18}$$

Therefore, for every fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$,

$$\|u_n(t,\tau,\omega)\|_{\mathcal{O}}^2 \le C_T, \quad \forall t\in[\tau,\tau+T] \tag{3.19}$$

$$\|v_n(t,\tau,\omega)\|_{\Gamma}^2 \le C_T, \quad \forall t\in[\tau,\tau+T], \tag{3.20}$$

$$\int_\tau^{\tau+T} \|\nabla u_n(s,\tau,\omega)\|_{\mathcal{O}}^2 ds \le \frac{1}{2}C_T, \tag{3.21}$$

$$\int_\tau^{\tau+T} \|u_n(s,\tau,\omega)\|_{L^p(\mathcal{O})}^p ds \le \frac{1}{2\alpha_1}C_T, \tag{3.22}$$

$$\int_\tau^{\tau+T} \|v_n(s,\tau,\omega)\|_{L^q(\Gamma)}^q ds \le \frac{1}{2\tilde{\alpha}_1}C_T, \tag{3.23}$$

in which

$$C_T = e^T\Big(\|(u_0(\omega),v_0(\omega)\|_{\mathcal{O}\times\Gamma}^2 + \int_\tau^{\tau+T} \|g(s)\|_{\mathcal{O}}^2 + 2\beta|\mathcal{O}| + 2\tilde{\beta}|\Gamma|\Big). \tag{3.24}$$

By (3.19)-(3.24) and the definition of Bochner integration, we obtain bounds which are independent of n on the solutions $(u_n(\cdot,\tau,\omega),v_n(\cdot,\tau,\omega))$ in various spaces. Hence for every fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$,

$$\begin{aligned} &\{u_n(\cdot,\tau,\omega)\}_{n=1}^\infty \text{is bounded in} \\ &L^\infty((\tau,\tau+T),L^2(\mathcal{O})) \cap L^2((\tau,\tau+T),H^1(\mathcal{O})) \cap L^p((\tau,\tau+T),L^p(\mathcal{O})), \\ &\{v_n(\cdot,\tau,\omega)\}_{n=1}^\infty \text{ is bounded in} \\ &L^\infty((\tau,\tau+T),L^2(\Gamma)) \cap L^2((\tau,\tau+T),H^{1/2}(\Gamma)) \cap L^q((\tau,\tau+T),L^q(\Gamma)). \end{aligned} \tag{3.25}$$

From (3.25), we have that

$$\{A(u_n)\}_{n=1}^\infty \text{ is bounded in } L^2((\tau,\tau+T),(H^1(\mathcal{O}))^*). \tag{3.26}$$

Suppose p^* and q^* is the conjugate exponent of p and q respectively. By (3.3) and (3.5) we obtain

$$\begin{aligned}
&\int_\tau^{\tau+T}\int_{\mathcal{O}} |f(u_n(s,\tau,\omega))|^{p^*}\,dx\,ds \\
&\le \int_\tau^{\tau+T}\int_{\mathcal{O}} \big(\alpha_2|u_n(s,\tau,\omega)| + \alpha_2|u_n(s,\tau,\omega)|^{p-1}\big)^{p^*}\,dx\,ds \\
&\le 2^{(p^*-1)}\alpha_2^{p^*}\Big(\int_\tau^{\tau+T}\|u_n\|^{p^*}_{L^{p^*}(\mathcal{O})}ds + \int_\tau^{\tau+T}\|u_n\|^{p}_{L^{p}(\mathcal{O})}ds\Big) \\
&\le 2^{(p^*-1)}\alpha_2^{p^*}\Big(\int_\tau^{\tau+T}\|u_n\|^{2}_{\mathcal{O}}ds + \int_\tau^{\tau+T}\|u_n\|^{p}_{L^{p}(\mathcal{O})}ds\Big),
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&\int_\tau^{\tau+T}\int_{\mathcal{O}} |h(v_n(s,\tau,\omega))|^{q^*}\,dx\,ds \\
&\le \int_\tau^{\tau+T}\int_{\mathcal{O}} \big(\tilde\alpha_2|v_n(s,\tau,\omega)| + \tilde\alpha_2|v_n(s,\tau,\omega)|^{q-1}\big)^{q^*}\,dx\,ds \\
&\le 2^{(q^*-1)}\tilde\alpha_2^{q^*}\Big(\int_\tau^{\tau+T}\|v_n\|^{q^*}_{L^{q^*}(\Gamma)}ds + \int_\tau^{\tau+T}\|v_n\|^{q}_{L^{q}(\Gamma)}ds\Big) \\
&\le 2^{(q^*-1)}\tilde\alpha_2^{q^*}\Big(\int_\tau^{\tau+T}\|v_n\|^{2}_{\Gamma}ds + \int_\tau^{\tau+T}\|u_n\|^{q}_{L^{q}(\Gamma)}ds\Big).
\end{aligned} \tag{3.28}$$

From this, (3.19)-(3.20), (3.22)-(3.23), (3.27) and (3.28), then we can obtain

$$\{f(u_n(\cdot,\tau,\omega))\}_{n=1}^\infty \text{ is bounded in } L^{p^*}((\tau,\tau+T),L^{p^*}(\mathcal{O})), \tag{3.29}$$

$$\{h(v_n(\cdot,\tau,\omega))\}_{n=1}^\infty \text{ is bounded in } L^{q^*}((\tau,\tau+T),L^{q^*}(\Gamma)). \tag{3.30}$$

We infer from (3.12), (3.26), (3.29) and (3.30) that

$$\begin{aligned}
&\{\frac{du_n}{dt}\}_{n=1}^\infty \text{ is bounded in } L^2((\tau,\tau+T),(H^1(\mathcal{O}))^*) + L^{p^*}((\tau,\tau+T),L^{p^*}(\mathcal{O})), \\
&\{\frac{dv_n}{dt}\}_{n=1}^\infty \text{ is bounded in } L^2((\tau,\tau+T),(H^{1/2}(\Gamma))^*) + L^{q^*}((\tau,\tau+T),L^{q^*}(\Gamma)).
\end{aligned} \tag{3.31}$$

3. Existence of solutions. By Alaoglu compactness theorem and (3.25)-(3.31), we find that there exist a $u(\cdot,\tau,\omega)\in L^\infty((\tau,\tau+T),L^2(\mathcal{O}))\cap L^2((\tau,\tau+T),H^1(\mathcal{O}))\cap L^p((\tau,\tau+T),L^p(\mathcal{O}))$, $v(\cdot,\tau,\omega)\in L^\infty((\tau,\tau+T),L^2(\Gamma))\cap L^2((\tau,\tau+T),H^{1/2}(\Gamma))\cap L^q((\tau,\tau+T),L^q(\Gamma))$, $M_1\in L^{p^*}((\tau,\tau+T),L^{p^*}(\mathcal{O}))$, $M_2\in L^{q^*}((\tau,\tau+T),L^{q^*}(\Gamma))$, $\bar u\in L^2(\mathcal{O})$, $\bar v\in L^2(\Gamma)$, a subsequence $\{u_{n_j}\}_{j=1}^\infty$ of $\{u_n\}_{n=1}^\infty$, and a subsequence $\{v_{n_j}\}_{j=1}^\infty$ of $\{v_n\}_{n=1}^\infty$, such that

$$u_{n_j}(\cdot,\tau,\omega)\overset{*}{\rightharpoonup} u(\cdot,\tau,\omega) \text{ in } L^\infty((\tau,\tau+T),L^2(\mathcal{O})), \tag{3.32}$$

$$v_{n_j}(\cdot,\tau,\omega)\overset{*}{\rightharpoonup} v(\cdot,\tau,\omega) \text{ in } L^\infty((\tau,\tau+T),L^2(\Gamma)), \tag{3.33}$$

$$u_{n_j}(\cdot,\tau,\omega)\rightharpoonup u(\cdot,\tau,\omega) \text{ in } L^2((\tau,\tau+T),H^1(\mathcal{O})), \tag{3.34}$$

$$v_{n_j}(\cdot,\tau,\omega)\rightharpoonup v(\cdot,\tau,\omega) \text{ in } L^2((\tau,\tau+T),H^{1/2}(\Gamma)), \tag{3.35}$$

$$u_{n_j}(\cdot,\tau,\omega)\rightharpoonup u(\cdot,\tau,\omega) \text{ in } L^p((\tau,\tau+T),L^p(\mathcal{O})), \tag{3.36}$$

$$v_{n_j}(\cdot,\tau,\omega)\rightharpoonup v(\cdot,\tau,\omega) \text{ in } L^q((\tau,\tau+T),L^q(\Gamma)), \tag{3.37}$$

$$Au_{n_j}\rightharpoonup Au \text{ in } L^2((\tau,\tau+T),(H^1(\mathcal{O}))^*), \tag{3.38}$$

$$f(u_{n_j})\rightharpoonup M_1 \text{ in } L^{p^*}((\tau,\tau+T),L^{p^*}(\mathcal{O})), \tag{3.39}$$

$$h(v_{n_j})\rightharpoonup M_2 \text{ in } L^{q^*}((\tau,\tau+T),L^{q^*}(\Gamma)), \tag{3.40}$$

$$\frac{du_{n_j}}{dt}\rightharpoonup\frac{du}{dt} \text{ in } L^2((\tau,\tau+T),(H^1(\mathcal{O}))^*) + L^{p^*}((\tau,\tau+T),L^{p^*}(\mathcal{O})), \tag{3.41}$$

$$\frac{dv_{n_j}}{dt}\rightharpoonup\frac{dv}{dt} \text{ in } L^2((\tau,\tau+T),(H^{1/2}(\Gamma))^*) + L^{q^*}((\tau,\tau+T),L^{q^*}(\Gamma)), \tag{3.42}$$

$$u_{n_j}(t_0,\tau,\omega)\rightharpoonup\bar{u} \text{ in } L^2(\mathcal{O}); \quad v_{n_j}(t_0,\tau,\omega)\rightharpoonup\bar{v} \text{ in } L^2(\Gamma). \tag{3.43}$$

Let $p,q\geq 2$, it follows that

$$\{\frac{du_{n_j}}{dt}\}_{n=1}^{\infty} \text{ is bounded in } L^{p^*}((\tau,\tau+T),(H^1(\mathcal{O})\cap L^p(\mathcal{O}))^*),$$
$$\{\frac{dv_{n_j}}{dt}\}_{n=1}^{\infty} \text{ is bounded in } L^{q^*}((\tau,\tau+T),(H^{1/2}(\Gamma)\cap L^q(\Gamma))^*).$$

Moreover, we notice that the embedding $H^1(\mathcal{O})\hookrightarrow L^2(\mathcal{O})\times L^2(\Gamma)$ is compact, and $L^2(\mathcal{O})\times L^2(\Gamma)\hookrightarrow (H^1(\mathcal{O})\cap L^p(\mathcal{O}))^*\times(H^{1/2}(\Gamma)\cap L^q(\Gamma))^*$ is continuous, so by (3.21) and Aubin-Lions lemma, there exists a subsequence of $(\{u_{n_j}\}_{j=1}^{\infty},\{v_{n_j}\}_{j=1}^{\infty})$ such that(we still note the subsequence as $(\{u_{n_j}\}_{j=1}^{\infty},\{v_{n_j}\}_{j=1}^{\infty})$)

$$\begin{aligned} u_{n_j}(\cdot,\tau,\omega)&\to u(\cdot,\tau,\omega) \text{ in } L^2((\tau,\tau+T),L^2(\mathcal{O})),\\ v_{n_j}(\cdot,\tau,\omega)&\to v(\cdot,\tau,\omega) \text{ in } L^2((\tau,\tau+T),L^2(\Gamma)).\end{aligned} \tag{3.44}$$

By (3.44), it obviously exists a further subsequence of $(\{u_{n_j}\}_{j=1}^{\infty},\{v_{n_j}\}_{j=1}^{\infty})$ such that

$$\begin{aligned} u_{n_j}(\cdot,\tau,\omega)(x)&\to u(\cdot,\tau,\omega)(x) \text{ for almost all } (t,x)\in(\tau,\tau+T)\times\mathcal{O},\\ v_{n_j}(\cdot,\tau,\omega)(x)&\to v(\cdot,\tau,\omega)(x) \text{ for almost all } (t,x)\in(\tau,\tau+T)\times\Gamma.\end{aligned} \tag{3.45}$$

By (3.45) and the continuity of f,h, we obtain

$$\begin{aligned} f(u_{n_j}(\cdot,\tau,\omega)(x))&\to f(u(\cdot,\tau,\omega)(x)) \text{ for almost all } (t,x)\in(\tau,\tau+T)\times\mathcal{O},\\ h(v_{n_j}(\cdot,\tau,\omega)(x))&\to h(v(\cdot,\tau,\omega)(x))\ textforalmostall(t,x)\in(\tau,\tau+T)\times\Gamma.\end{aligned} \tag{3.46}$$

Then by [18], (3.29) and (3.46), we obtain

$$\begin{aligned} f(u_{n_j}(\cdot,\tau,\omega))&\rightharpoonup f(u(\cdot,\tau,\omega)) \text{ in } L^{p^*}((\tau,\tau+T),L^{p^*}(\mathcal{O})),\\ h(v_{n_j}(\cdot,\tau,\omega))&\rightharpoonup h(v(\cdot,\tau,\omega)) \text{ in } L^{q^*}((\tau,\tau+T),L^{q^*}(\Gamma)).\end{aligned} \tag{3.47}$$

By using (3.32)-(3.43) and (3.47), and taking the limits as $n\to\infty$ in (3.12), we can have that, in the sense of distribution, for all $(\eta,\gamma_0\eta)\in(L^p(\mathcal{O}))\cap H^1(\mathcal{O}))\times(L^q(\Gamma)\cap H^{1/2}(\Gamma))$,

$$\begin{aligned} &\frac{d}{dt}(u,\eta)_{\mathcal{O}}+\frac{d}{dt}(v,\gamma_0\eta)_\Gamma+(A(u),\eta)_{\mathcal{O}}\\ &=(f(u),\eta)_{(L^{p^*}(O),L^p(O))}+(h(v),\gamma_0\eta)_{(L^{q^*}(O),L^q(O))}+(g(t),\eta)_{\mathcal{O}}.\end{aligned} \tag{3.48}$$

Furthermore, by a theorem in [12], we have

$$(u(\cdot,\tau,\omega),v(\cdot,\tau,\omega))\in C([\tau,\tau+T],L^2(\mathcal{O})\times L^2(\Gamma)), \tag{3.49}$$
$$u(\tau,\tau,\omega)=u_0(\omega),\quad v(\tau,\tau,\omega)=v_0(\omega), \tag{3.50}$$
$$u(t_0,\tau,\omega)=\bar{u},\quad v(t_0,\tau,\omega)=\bar{v}, \tag{3.51}$$

$$\begin{aligned} &\frac{d}{dt}\|u(t,\tau,\omega)\|_{\mathcal{O}}^2+\frac{d}{dt}\|v(t,\tau,\omega)\|_\Gamma^2+2\|\nabla u(t,\tau,\omega)\|_{\mathcal{O}}^2\\ &=2\int_{\mathcal{O}}f(u(t,\tau,\omega))u(t,\tau,\omega)dx+2\int_\Gamma h(v(t,\tau,\omega))v(t,\tau,\omega)dS+2(g,u(t,\tau,\omega))_{\mathcal{O}}.\end{aligned} \tag{3.52}$$

for almost all $t\in(\tau,\tau+T)$. By (3.43) and (3.51), we obtain

$$\begin{aligned} u_{n_j}(t_0,\tau,\omega)&\rightharpoonup u(t_0,\tau,\omega) \text{ in } L^2(\mathcal{O}),\\ v_{n_j}(t_0,\tau,\omega)&\rightharpoonup v(t_0,\tau,\omega) \text{ in } L^2(\Gamma).\end{aligned} \tag{3.53}$$

From (3.48)-(3.51), we infer that $(u(\cdot,\tau,\omega),v(\cdot,\tau,\omega))$ is a solution of the deterministic problem (3.1) with initial data $(u_0(\omega),v_0(\omega))$ for a fixed ω. In addition, by (3.53) and the uniqueness of the solution to Definition 3.1, the whole sequence $(u_n(t_0,\tau,\omega),v_n(t_0,\tau,\omega))\rightharpoonup(u(t_0,\tau,\omega),v(t_0,\tau,\omega))$ in $L^2(\mathcal{O})\times L^2(\Gamma)$. Since $t_0\in(\tau,\tau+T]$ is arbitrary, $(u_n(t,\tau,\omega),v_n(t,\tau,\omega))\rightharpoonup(u(t,\tau,\omega),v(t,\tau,\omega))$ for any $t\geq\tau$ and $\omega\in\Omega$. Because of (3.19)-(3.24) and (3.32)-(3.37), we have that for every fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $T>0$,

$$\|u(t,\tau,\omega)\|_{\mathcal{O}}^2\leq C_T,\quad \forall t\in[\tau,\tau+T], \tag{3.54}$$

$$\|v(t,\tau,\omega)\|_\Gamma^2 \le C_T, \quad \forall t\in[\tau,\tau+T], \tag{3.55}$$

$$\int_\tau^{\tau+T} \|\nabla u(s,\tau,\omega)\|_{\mathcal{O}}^2 ds \le \frac{1}{2}C_T, \tag{3.56}$$

$$\int_\tau^{\tau+T} \|u(s,\tau,\omega)\|_{L^p(\mathcal{O})}^p ds \le \frac{1}{2\alpha_1}C_T, \tag{3.57}$$

$$\int_\tau^{\tau+T} \|v(s,\tau,\omega)\|_{L^q(\Gamma)}^q ds \le \frac{1}{2\tilde\alpha_1}C_T. \tag{3.58}$$

Because $(u_0,v_0)\in L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))$, we see from the definition of Bochner integration and (3.54)−(3.58) that

$$\begin{gathered}u(\cdot,\tau,u_0)\in L^\infty_{\rm loc}((\tau,\infty),L^2(\Omega,L^2(\mathcal{O})))\cap L^2_{\rm loc}((\tau,\infty),L^2(\Omega,H^1(\mathcal{O})))\cap L^p_{\rm loc}((\tau,\infty),L^p(\Omega,L^p(\mathcal{O}))),\\ v(\cdot,\tau,v_0)\in L^\infty_{\rm loc}((\tau,\infty),L^2(\Omega,L^2(\Gamma)))\cap L^2_{\rm loc}((\tau,\infty),L^2(\Omega,H^{1/2}(\Gamma)))\cap L^q_{\rm loc}((\tau,\infty),L^q(\Omega,L^q(\Gamma))).\end{gathered} \tag{3.59}$$

For every fixed ω, we know that $(u(\cdot,\tau,\omega),v(\cdot,\tau,\omega))\in C([\tau,\infty),L^2(\mathcal{O})\times L^2(\Gamma))$, so by (3.54), (3.55) and the Lebesgue dominated convergence theorem we obtain

$$(u,v)\in C([\tau,\infty),L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))). \tag{3.60}$$

By (3.48)-(3.51) and (3.59)-(3.60), (u,v) is the solution to system (3.1) in the sense of Definition 3.2. In the final part, we need to prove the uniqueness of the solution.

4. Uniqueness of the solution. Assume that (u_1,v_1) and (u_2,v_2) are both solutions of problem (3.1) and $(\tilde u,\tilde v)=(u_1,v_1)-(u_2,v_2)$. Then we obtain

$$\begin{gathered}\frac{d\tilde u}{dt}+A\tilde u=f(u_1)-f(u_2),\\ \frac{d\tilde v}{dt}+\frac{\partial\tilde v}{\partial\vec n}=h(v_1)-h(v_2).\end{gathered} \tag{3.61}$$

By (3.7), we obtain

$$\frac12\frac{d}{dt}\|u_n\|_{\mathcal{O}}^2+\frac12\frac{d}{dt}\|v_n\|_\Gamma^2+\|\nabla u_n\|_{\mathcal{O}}^2\le l\|\tilde u\|_{\mathcal{O}}^2+m\|\tilde v\|_\Gamma^2\le\rho(\|\tilde u\|_{\mathcal{O}}^2+\|\tilde v\|_\Gamma^2). \tag{3.62}$$

where $\rho=\max\{l,m\}$. By (3.62), we have

$$\frac{d}{dt}\|(\tilde u,\tilde v)\|_{\mathcal{O}\times\Gamma}^2\le 2\rho\|(\tilde u,\tilde v)\|_{\mathcal{O}\times\Gamma}^2,$$

then using Gronwall lemma, for $t\ge\tau$, we obtain

$$\|(\tilde u(t,\tau,\omega),\tilde v(t,\tau,\omega))\|_{\mathcal{O}\times\Gamma}^2\le e^{2\rho(t-\tau)}\|(\tilde u(\tau,\tau,\omega),\tilde v(\tau,\tau,\omega))\|_{\mathcal{O}\times\Gamma}^2. \tag{3.63}$$

Therefore,

$$\begin{aligned}&E\left(\|(u_1(t,\tau,\cdot),v_1(t,\tau,\cdot))-(u_2(t,\tau,\cdot),v_2(t,\tau,\cdot))\|_{\mathcal{O}\times\Gamma}^2\right)\\ &\le e^{2\rho(t-\tau)}E\left(\|(u_1(\tau,\tau,\cdot),v_1(\tau,\tau,\cdot))-(u_2(\tau,\tau,\cdot),v_2(\tau,\tau,\cdot))\|_{\mathcal{O}\times\Gamma}^2\right).\end{aligned} \tag{3.64}$$

This implies the uniqueness of the solution.

Finally, we obtain the energy equation (3.11) by taking expectation on the both sides of (3.1). □

Now, we need to define a random dynamical system based on the solution operators of problem (3.1). Let a mapping $\phi:\mathbb{R}^+\times\mathbb{R}\times(L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma)))\to L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))$ satisfy

$$\phi(t,\tau,(u_0,v_0))=(u(t+\tau,\tau,u_0),v(t+\tau,\tau,v_0)),$$

where $(u_0,v_0)\in L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))$, $t\ge0$, $\tau\in\mathbb{R}$, and (u,v) is the solution of problem (3.1) with initial data (u_0,v_0). Moreover, we can easily find that for every $(u_0,v_0)\in L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))$, $t,s\ge0$, and $\tau\in\mathbb{R}$,

$$\phi(t+s,\tau,(u_0,v_0))=\phi(t,s+\tau,(\phi(s,\tau,u_0),\phi(s,\tau,v_0))).$$

So, ϕ is actually a continuous mean random dynamical system on $L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$. From [23], we know the Sobolev embedding

$$H^1(\mathcal{O}) \hookrightarrow L^2(\mathcal{O}), \quad H^1(\mathcal{O}) \hookrightarrow L^2(\Gamma),$$

implies that

$$\|\nabla u\|_{\mathcal{O}}^2 \geq \varrho_1 \|u\|_{\mathcal{O}}^2, \quad \forall u \in H^1(\mathcal{O}), \tag{3.65}$$

$$\|\nabla u\|_{\mathcal{O}}^2 \geq \varrho_2 \|v\|_{\Gamma}^2, \quad \forall u \in H^1(\mathcal{O}). \tag{3.66}$$

where ϱ_1 and ϱ_2 are the embedding constants. We assume $\lambda = min\{\varrho_1, \varrho_2\}$. Let U be a bounded subset of $L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$ and denote by

$$\|U\|_{L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))} = \sup_{(u,v)\in U} (\|u\|^2_{L^2(\Omega, L^2(\mathcal{O}))} + \|v\|^2_{L^2(\Omega, L^2(\Gamma))}). \tag{3.67}$$

We denote a family of nonempty bounded subset of $L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$ by D, i.e.

$$D = \{D(\tau) \subseteq L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma)) : D(\tau) \neq \emptyset \text{ and } D(\tau) \text{ is bounded for each } \tau \in \mathbb{R}\}.$$

Meanwhile, we assume $\mu = \frac{5}{8}\lambda$, where λ is the same positive constant as in (3.65). Then D satisfies

$$\lim_{\tau \to -\infty} e^{\mu\tau} \|D(\tau)\|^2_{L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))} = 0. \tag{3.68}$$

Next by $\mathscr{D}_0$ we denote the collection of all families of nonempty bounded subsets of $L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$, i.e.

$$\begin{gathered} D = \{D(\tau) \subseteq L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma)) : D(\tau) \neq \emptyset \text{ bounded}, \tau \in \mathbb{R}\} \\ \mathscr{D}_0 = \{D \text{ that satisfy (3.68)}\}. \end{gathered} \tag{3.69}$$

For the non-autonmous external g, we assume that

$$\int_{-\infty}^{\tau} e^{\mu s} \|g(s)\|_{\mathcal{O}}^2 ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{3.70}$$

3.2. **Existence of weak mean random attractors for deterministic equations.** We do two steps to prove the existence of the weak mean random attractors of problem (3.1). Firstly, we need the uniform estimate of the solution. Then we construct a $\mathscr{D}_0$-pullback absorbing set in $L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$.

1. Uniform estimate for the solution.

Lemma 3.4. *Suppose* (3.2)-(3.7) *and* (3.70) *hold. Then for every* $\tau \in \mathbb{R}$ *and* $D = \{D(t)\}_{t\in\mathbb{R}} \in \mathscr{D}_0$, *there exists* $T = T(\tau, D) > 0$ *such that for all* $t \geq T$,

$$\begin{gathered} E\left(\|u(\tau, \tau - t, u_0)\|_{\mathcal{O}}^2 + \|v(\tau, \tau - t, v_0)\|_{\Gamma}^2\right) \leq C_1 + C_2 e^{-\mu\tau} \int_{-\infty}^{\tau} e^{\mu s} \|g(s)\|_{\mathcal{O}}^2 ds, \\ \int_{\tau - t}^{\tau} e^{\mu s} E\left(\|\nabla u(s, \tau - t, u_0)\|_{\mathcal{O}}^2\right) ds \leq C_1 e^{\mu\tau} + C_2 \int_{-\infty}^{\tau} e^{\mu s} \|g(s)\|_{\mathcal{O}}^2 ds, \\ \int_{\tau - t}^{\tau} e^{\mu s} E\left(\|u(s, \tau - t, u_0)\|_{L^p(\mathcal{O})}^p\right) ds \leq C_1 e^{\mu\tau} + C_2 \int_{-\infty}^{\tau} e^{\mu s} \|g(s)\|_{\mathcal{O}}^2 ds, \\ \int_{\tau - t}^{\tau} e^{\mu s} E\left(\|v(s, \tau - t, v_0)\|_{L^q(\Gamma)}^q\right) ds \leq C_1 e^{\mu\tau} + C_2 \int_{-\infty}^{\tau} e^{\mu s} \|g(s)\|_{\mathcal{O}}^2 ds, \end{gathered}$$

where $(u_0, v_0) \in D(\tau - t)$, *and* C_1 *and* C_2 *are positive constants independent of* τ *and* D.

Proof. Using (3.11) and (3.65), we have

$$
\begin{aligned}
&\frac{d}{ds}E\left(\|u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)+\frac{d}{ds}E\left(\|v(s,\tau-t,v_0)\|_{\Gamma}^2\right)+\frac{7}{8}\lambda E\left(\|u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)\\
&+\frac{5}{8}\lambda E\left(\|v(s,\tau-t,v_0)\|_{\Gamma}^2\right)+\frac{1}{2}E\left(\|\nabla u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)\\
&\le 2E\Big(\int_{\mathcal{O}} f(u(s,\tau-t,u_0))u(s,\tau-t,u_0)dx\Big)\\
&+2E\Big(\int_{\Gamma} h(v(s,\tau-t,v_0))v(s,\tau-t,v_0)dS\Big)+2E\left(g(t),u(s,\tau-t,u_0)\right)_{\mathcal{O}}.
\end{aligned}\tag{3.71}
$$

Using Young inequality, we obtain

$$
2E\left(g(s),u(s,\tau-t,u_0)\right)_{\mathcal{O}}\le\frac{1}{4}\lambda E\left(\|u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)+\frac{4}{\lambda}\|g\|_{\mathcal{O}}^2.\tag{3.72}
$$

By (3.2) and (3.4) we obtain

$$
2E\left(\int_{\mathcal{O}} f(u(s,\tau-t,u_0))u(s,\tau-t,u_0)dx\right)\le-2\alpha_1E\int_{\mathcal{O}}|u(s,\tau-t,u_0)|^pdx+2\beta|\mathcal{O}|,\tag{3.73}
$$

$$
2E\left(\int_{\Gamma} h(v(s,\tau-t,v_0))v(s,\tau-t,v_0)dx\right)\le-2\tilde{\alpha}_1E\int_{\Gamma}|v(s,\tau-t,v_0)|^qdS+2\tilde{\beta}|\Gamma|.\tag{3.74}
$$

Combining (3.71)-(3.74), we obtain

$$
\begin{aligned}
&\frac{d}{ds}E\left(\|u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)+\frac{d}{ds}E\left(\|v(s,\tau-t,v_0)\|_{\Gamma}^2\right)+\mu E\left(\|u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)\\
&+\mu E\left(\|v(s,\tau-t,v_0)\|_{\Gamma}^2\right)+\frac{1}{2}E\left(\|\nabla u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)\\
&+2\alpha_1E\int_{\mathcal{O}}|u(s,\tau-t,u_0)|^pdx+2\tilde{\alpha}_1E\int_{\Gamma}|v(s,\tau-t,v_0)|^qdS\\
&\le 2\beta|\mathcal{O}|+2\tilde{\beta}|\Gamma|+\frac{5}{2\mu}\|g(s)\|_{\mathcal{O}}^2,
\end{aligned}\tag{3.75}
$$

where $\mu=\frac{5}{8}\lambda$. Then multiplying (3.75) by $e^{\mu s}$ and integrating on $(\tau-t,\tau)$ with $t\ge 0$, we have

$$
\begin{aligned}
&E\left(\|u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)+E\left(\|v(s,\tau-t,v_0)\|_{\Gamma}^2\right)\\
&+2\alpha_1e^{-\mu\tau}\int_{\tau-t}^{\tau}e^{\mu s}E\Big(\|u(s,\tau-t,u_0)\|_{L^p(\mathcal{O})}^p\Big)ds\\
&+2\tilde{\alpha}_1e^{-\mu\tau}\int_{\tau-t}^{\tau}e^{\mu s}E\Big(\|v(s,\tau-t,v_0)\|_{L^q(\Gamma)}^q\Big)ds\\
&+\frac{1}{2}e^{-\mu\tau}\int_{\tau-t}^{\tau}e^{\mu s}E\left(\|\nabla u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)ds\\
&\le e^{-\mu\tau}e^{\mu(\tau-t)}E\left(\|(u_0,v_0)\|_{\mathcal{O}\times\Gamma}^2\right)+\frac{2\beta|\mathcal{O}|+2\tilde{\beta}|\Gamma|}{\mu}+\frac{5}{2\mu}e^{-\mu\tau}\int_{\tau-t}^{\tau}e^{\mu s}\|g(s)\|_{\mathcal{O}}^2ds.
\end{aligned}\tag{3.76}
$$

Because $(u_0,v_0)\in D(\tau-t)$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathscr{D}_0$, we obtain

$$
\begin{aligned}
e^{-\mu\tau}e^{\mu(\tau-t)}E\left(\|(u_0,v_0)\|_{\mathcal{O}\times\Gamma}^2\right)&\le e^{-\mu\tau}e^{\mu(\tau-t)}\Big(\|D(\tau-t)\|_{L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\mathcal{O}))}^2\Big)\\
&\to 0\quad\text{as } t\to\infty.
\end{aligned}\tag{3.77}
$$

There exists $T'=T'(\tau,D)>0$ such that for all $t>T'$,

$$
e^{-\mu\tau}e^{\mu(\tau-t)}E\left(\|(u_0,v_0)\|_{\mathcal{O}\times\Gamma}^2\right)\le 1.\tag{3.78}
$$

At last, by (3.76) and (3.78), we have

$$\begin{aligned}
&E\left(\|u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)+E\left(\|v(s,\tau-t,v_0)\|_{\Gamma}^2\right)\\
&+2\alpha_1 e^{-\mu\tau}\int_{\tau-t}^{\tau} e^{\mu s}E\left(\|u(s,\tau-t,u_0)\|_{L^p(\mathcal{O})}^p\right)ds\\
&+2\tilde{\alpha}_1 e^{-\mu\tau}\int_{\tau-t}^{\tau} e^{\mu s}E\left(\|v(s,\tau-t,v_0)\|_{L^q(\Gamma)}^q\right)ds\\
&+\frac{1}{2}e^{-\mu\tau}\int_{\tau-t}^{\tau} e^{\mu s}E\left(\|\nabla u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)ds\\
&\le 1+\frac{2\beta|\mathcal{O}|+2\tilde{\beta}|\Gamma|}{\mu}+\frac{5}{2\mu}e^{-\mu\tau}\int_{-\infty}^{\tau} e^{\mu s}\|g(s)\|_{\mathcal{O}}^2 ds,
\end{aligned} \tag{3.79}$$

for all $t \ge T'$. The proof is complete. □

2. Construct a $\mathscr{D}_0$-pullback absorbing set.

Lemma 3.5. *Suppose conditions* (3.2)-(3.7) *and* (3.70) *hold. For each $\tau \in \mathbb{R}$, let*

$$K(\tau)=\left\{(u,v)\in L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma)) : E\left(\|(u,v)\|_{\mathcal{O}\times\Gamma}^2\right)\le R(\tau)\right\}. \tag{3.80}$$

where

$$R(\tau)=C_1+C_2 e^{-\mu\tau}\int_{-\infty}^{\tau} e^{\mu s}\|g(s)\|_{\mathcal{O}}^2 ds.$$

Where C_1 and C_2 are the constants as same as in Lemma 3.4. *Moreover, the family $K=\{K(\tau):\tau\in\mathbb{R}\}\in\mathscr{D}_0$ is a weakly compact $\mathscr{D}_0$-pullback absorbing set of ϕ.*

Proof. Since for $\forall\tau\in\mathbb{R}$, $K(\tau)$ is a bounded closed convex subset of $L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))$ by (3.80), hence $K(\tau)$ is weakly compact in $L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))$. Moreover, by Lemma 3.4, we notice that for every $\tau\in\mathbb{R}$ and $D=\{D(t)\}_{t\in\mathbb{R}}\in\mathscr{D}_0$, there exists $T=T(\tau,D)>0$ such that for all $t\ge T$,

$$\big(u(\tau,\tau-t,u_0),v(\tau,\tau-t,v_0)\big)=\phi(t,\tau-t,D(\tau-t))\subseteq K(\tau).$$

Then we need to prove $K(\tau)\subseteq\mathscr{D}_0$, i.e. $K(\tau)$ satisfies (3.68). By (3.80), we have

$$\begin{aligned}
&\lim_{\tau\to-\infty} e^{\mu\tau}\|K(\tau)\|_{L^2(\Omega,L^2(O))\times L^2(\Omega,L^2(\Gamma))}^2=\lim_{\tau\to-\infty} e^{\mu\tau}R(\tau)\\
&=\lim_{\tau\to-\infty} e^{\mu\tau}C_1+\lim_{\tau\to-\infty} C_2\int_{-\infty}^{\tau} e^{\mu s}\|g(s)\|_{\mathcal{O}}^2 ds.
\end{aligned} \tag{3.81}$$

By (3.70) and the arbitrariness of τ, we obtain

$$\int_{-\infty}^{0} e^{\mu s}\ \|g(s)\|_{\mathcal{O}}^2 ds<\infty. \tag{3.82}$$

Then combining (3.81) and (3.82), we obtain

$$\lim_{\tau\to-\infty} e^{\mu\tau}\|K(\tau)\|_{L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))}^2=0. \tag{3.83}$$

□

Next, we prove the existence of the weak $\mathscr{D}_0$-pullback mean random attractor for system ϕ.

Theorem 3.6. *Suppose* (3.2)-(3.7) *and* (3.70) *hold. Then problem* (3.1) *has a unique weak $\mathscr{D}_0$-pullback mean random attractor $\mathscr{A}_0=\{\mathscr{A}_0(\tau):\tau\in\mathbb{R}\}$ in $L^2(\Omega,L^2(\mathcal{O}))\times L^2(\Omega,L^2(\Gamma))$.*

Proof. Now we know that ϕ has a weakly compact $\mathscr{D}_0$-pullback absorbing set $K=\{K(\tau):\tau\in\mathbb{R}\}$. By Theorem 2.6, we can get the existence and uniqueness of the weak $\mathscr{D}_0$-pullback mean random attractor for system ϕ. □

4. Weak mean random attractors for stochastic equations

In this section, we prove the existence of the weak $\bar{\mathscr{D}}_0$-pullback mean random attractors for the stochastic reaction-diffusion equations with dynamical boundary conditions. We firstly define a mean random dynamical system.

4.1. **Generation of mean random dynamical system.** We assume that $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$ is a complete filtered probability space and W is a two-sided cylindrical Q-Wiener process with respect to $\{\mathscr{F}_t\}_{t\in\mathbb{R}}$, in which $Q = I$ on another separable Hilbert space B. Let $L_2(B, H)$ be a Hilbert space of all Hilbert-Schmidt operators defined from B to H.

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ. Consider the following non-autonomous stochastic equation for every $\tau \in \mathbb{R}$ and $t > \tau$:

$$
\begin{aligned}
du - \Delta u dt = (f(u) + g(x,t))dt + \epsilon G(t,u)dW, \quad &\text{in } \mathcal{O} \times \mathbb{R}^+,\\
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \vec{n}} = h(u), \quad &\text{on } \Gamma \times \mathbb{R}^+,\\
u(x,\tau) = u_0(x), \quad &\text{in } \mathcal{O},\\
u(x,\tau) = v_0(x), \quad &\text{on } \Gamma,
\end{aligned}
\tag{4.1}
$$

in which $\epsilon \in (0,1]$ is a constant, $g \in L^2_{\rm loc}(\mathbb{R}, L^2(\mathcal{O}))$. And the stochastic term in (4.1) is understood in the sense of Ito's integration. We assume $f, h : \mathbb{R} \to \mathbb{R}$ is a smooth nonlinear function such that for all $m \in \mathbb{R}$,

$$f'(m) \le \gamma_1, \tag{4.2}$$
$$|f(m)| \le \gamma_2(1 + |m|), \tag{4.3}$$
$$h'(m) \le \tilde{\gamma}_1, \tag{4.4}$$
$$|h(m)| \le \tilde{\gamma}_2(1 + |m|), \tag{4.5}$$

where $\gamma_1 \ge 0$ and $0 < \gamma_2 + \tilde{\gamma_2} < \frac{3}{8}\lambda$. Then we assume $G(t,u) : \mathbb{R} \times H^1(\mathcal{O}) \to L_2(B, L^2(\mathcal{O}))$ satisfies, for some $\gamma_3 \in \mathbb{R}$, for all $t \in \mathbb{R}$, and for all $u_1, u_2 \in H^1(\mathcal{O})$:

$$2\gamma_1\|u_1 - u_2\|^2_{\mathcal{O}\times\Gamma} + \|G(t,u_1) - G(t,u_2)\|^2_{L_2(B,L^2(\mathcal{O}))} \le 2\|\nabla u_1 - \nabla u_2\|^2_{\mathcal{O}} + \gamma_3\|u_1 - u_2\|^2_{\mathcal{O}\times\Gamma}, \tag{4.6}$$

and for some $\gamma_4 > 0$, $\gamma_5 \in \mathbb{R}$ and for all $t \in \mathbb{R}$, where $\theta \in L^1_{\rm loc}(\mathbb{R})$,

$$\gamma_4\|\nabla u\|^2_{\mathcal{O}} + \|G(t,u)\|^2_{L_2(B,L^2(\mathcal{O}))} \le 2\|\nabla u\|^2_{\mathcal{O}} + \gamma_5\|u\|^2_{\mathcal{O}\times\Gamma} + \theta(t). \tag{4.7}$$

Definition 4.1. Suppose $\tau \in \mathbb{R}$ and $(u_0, v_0) \in L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$. A $L^2(\mathcal{O})$-valued $\{\mathscr{F}_t\}_{t\in\mathbb{R}}$-adapted process $\{(u_t, v_t)\}_{t\in[\tau,\infty)}$ is a solution of problem (4.1) with initial condition (u_0, v_0) if $u \in C([\tau,\infty), L^2(\mathcal{O})) \cap L^2_{\rm loc}((\tau,\infty), H^1(\mathcal{O}))$, $v \in C([\tau,\infty), L^2(\Gamma)) \cap L^2_{\rm loc}((\tau,\infty), H^{\frac{1}{2}}(\Gamma))$ P-a.s., and satisfies for every $t > \tau$ and $(\eta, \gamma_0\eta) \in (H^1(\mathcal{O}), H^{1/2}(\Gamma))$,

$$
\begin{aligned}
&(u(t),\eta)_{\mathcal{O}} + (v(t),\gamma_0\eta)_\Gamma + \int_\tau^t (\nabla u, \nabla\eta)_{\mathcal{O}}\, ds\\
&= (u_0,\eta)_{\mathcal{O}} + (v_0,\gamma_0\eta)_\Gamma + \int_\tau^t\int_{\mathcal{O}} f(u)\eta\, dx\, ds + \int_\tau^t\int_\Gamma h(v)\gamma_0\eta\, dS\, ds\\
&\quad + \int_\tau^t\int_{\mathcal{O}} g(x,s)\eta\, dx\, ds + \epsilon\int_\tau^t (\eta, G(s,u)dW(s)),
\end{aligned}
\tag{4.8}
$$

P-almost everywhere.

From [18], we know that if (4.2)-(4.5) are satisfied, then for every $\tau \in \mathbb{R}$ and $(u_0, v_0) \in L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$, problem (4.1) has a unique solution (u, v) in the sense of Definition 4.1. Moreover, for every $T > 0$,

$$E\Big(\sup_{t\in[\tau,\tau+T]}\Big(\|u(t)\|^2_{\mathcal{O}} + \|v(t)\|^2_\Gamma\Big)\Big) < \infty. \tag{4.9}$$

Because $(u,v) \in C([\tau,\infty), L^2(\mathcal{O})) \times C([\tau,\infty), L^2(\Gamma))$ P-a.s., by (4.9) and the Lebesgue dominated convergence theorem, we can obtain $u \in C([\tau,\infty), L^2(\Omega, L^2(\mathcal{O})))$ and $v \in C([\tau,\infty), L^2(\Omega, L^2(\Gamma)))$. Now we can define a mean random dynamical system ϕ for problem (4.1). Suppose a mapping

$$\phi : \mathbb{R}^+ \times \mathbb{R} \times (L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))) \to (L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))),$$

for $t \geq 0$, $\tau \in \mathbb{R}$, $u_0 \in L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O}))$, $v_0 \in L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$, satisfy

$$\phi(t,\tau,(u_0,v_0)) = (u(t+\tau,\tau,u_0), v(t+\tau,\tau,v_0)),$$

where (u,v) is the solution of system (4.1) with initial data (u_0,v_0). Moreover, since (u,v) is the unique solution, there are

$$\phi(t+s,\tau,(u_0,v_0)) = \phi(t, s+\tau, (\phi(s,\tau,u_0), \phi(s,\tau,u_0))$$

for every $(u_0,v_0) \in L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$, $t, s \geq 0$, and $\tau \in \mathbb{R}$. So, ϕ is a mean random dynamical system on $L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$ over $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in\mathbb{R}}, P)$. We denote D as a family of nonempty bounded subset of $L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$, so

$$D = \big\{D(\tau) \subseteq L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma)) : D(\tau) \neq \emptyset \text{and is bounded for each } \tau \in \mathbb{R}\big\}. \tag{4.10}$$

Meanwhile, D satisfies

$$\lim_{\tau\to-\infty} e^{\mu\tau} \|D(\tau)\|^2_{L^2(\Omega,\mathscr{F}_\tau;L^2(\mathcal{O}))\times L^2(\Omega,\mathscr{F}_\tau;L^2(\Gamma))} = 0, \tag{4.11}$$

where $\mu = \frac{5}{8}\lambda$ and λ is the same positive constant as in (3.65). Then suppose $\bar{\mathscr{D}}_0$ is the collection of all families of nonempty bounded subsets of $L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$,

$$\begin{aligned} \bar{\mathscr{D}}_0 = \big\{D = \big\{&D(\tau) \subseteq L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma)) \\ &: D(\tau) \neq \emptyset \text{ bounded}, \tau \in \mathbb{R}\big\} : D \text{satisfies (4.11)}\big\}. \end{aligned} \tag{4.12}$$

For the function g, we assume:

$$\int_{-\infty}^{\tau} e^{\mu s} \left(\|g(s)\|^2_{\mathcal{O}} + |\theta(s)|\right) ds < \infty, \quad \forall \tau \in \mathbb{R}. \tag{4.13}$$

4.2. **Existence of Weak mean random attractors for stochastic equations.** In this section, to prove the existence of the weak $\bar{\mathscr{D}}_0$-pullback mean random attractors of problem (4.1), we divide into two steps. Firstly, we need a uniform estimate of the solution. Then construct a $\bar{\mathscr{D}}_0$-pullback absorbing set.

1. Obtain a uniform estimate of the solution.

Lemma 4.2. *Let* (4.2)-(4.7) *and* (4.13) *hold. Then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ and for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \bar{\mathscr{D}}_0$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$,*

$$E\left(\|u(\tau,\tau-t,u_0)\|^2_{\mathcal{O}} + \|v(r,\tau-t,v_0)\|^2_{\Gamma}\right) \leq C_3 + C_4 e^{\mu\tau} \int_{-\infty}^{\tau} e^{\mu s} \left(\|g(s)\|^2_{\mathcal{O}} + |\theta(s)|\right) ds,$$

where $(u_0,v_0) \in D(\tau-t)$, and C_3 and C_4 are positive constants independent of τ and D.

Proof. By Ito's formula, P-a.s., for $r \geq \tau - t$, we obtain,

$$\begin{aligned} &\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}} + \|v(r,\tau-t,v_0)\|^2_{\Gamma} + 2\int_{\tau-t}^{r} \|\nabla u(s,\tau-t,u_0)\|^2_{\mathcal{O}} ds \\ &= \|u_0\|^2_{\mathcal{O}} + \|v_0\|^2_{\Gamma} + 2\int_{\tau-t}^{r} (f(u(s,\tau-t,u_0)), u(s,\tau-t,u_0))_{\mathcal{O}} ds \\ &\quad + 2\int_{\tau-t}^{r} (h(v(s,\tau-t,v_0)), v(s,\tau-t,v_0))_{\Gamma} ds + 2\int_{\tau-t}^{r} (g(s), u(s,\tau-t,u_0))_{\mathcal{O}} ds \\ &\quad + \epsilon^2 \int_{\tau-t}^{r} \|G(s,u(s,\tau-t,u_0))\|^2_{L_2(B,L^2(\mathcal{O}))} ds \\ &\quad + 2\epsilon \int_{\tau-t}^{r} (u(s,\tau-t,u_0), G(s,u(s,\tau-t,u_0)) dW(s))_{\mathcal{O}} ds. \end{aligned} \tag{4.14}$$

Take expectation on the both sides of (4.14) and by the properties of Ito's integration, we obtain, for $r \geq \tau - t$,

$$
\begin{aligned}
&E\left(\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2\right) + E\left(\|v(r,\tau-t,v_0)\|_{\Gamma}^2\right)\\
&+ 2\int_{\tau-t}^{r} E\left(\|\nabla u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right) ds\\
&= E\left(\|u_0\|_{\mathcal{O}}^2 + \|v_0\|_{\Gamma}^2\right) + 2\int_{\tau-t}^{r} E\left(f(u(s,\tau-t,u_0)), u(s,\tau-t,u_0)\right)_{\mathcal{O}} ds\\
&\quad + 2\int_{\tau-t}^{r} E\left(h(v(s,\tau-t,v_0)), v(s,\tau-t,v_0)\right)_{\Gamma} ds\\
&\quad + 2\int_{\tau-t}^{r} E\left(g(s), u(s,\tau-t,u_0)\right)_{\mathcal{O}} ds\\
&\quad + \epsilon^2 \int_{\tau-t}^{r} E\Big(\|G(s,u(s,\tau-t,u_0))\|_{L_2(B,L^2(\mathcal{O}))}^2\Big) ds.
\end{aligned}
\tag{4.15}
$$

Next, take the derivative,

$$
\begin{aligned}
&\frac{d}{dr}E\left(\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2\right) + \frac{d}{dr}E\left(\|v(r,\tau-t,v_0)\|_{\Gamma}^2\right) + 2E\left(\|\nabla u(s,\tau-t,u_0)\|_{\mathcal{O}}^2\right)\\
&= 2E\left(f(u(s,\tau-t,u_0)), u(s,\tau-t,u_0)\right)_{\mathcal{O}}\\
&\quad + 2E\left(h(v(s,\tau-t,v_0)), v(s,\tau-t,v_0)\right)_{\Gamma}\\
&\quad + 2E\left(g(s), u(s,\tau-t,u_0)\right)_{\mathcal{O}} + \epsilon^2 E\Big(\|G(s,u(s,\tau-t,u_0))\|_{L_2(B,L^2(\mathcal{O}))}^2\Big).
\end{aligned}
\tag{4.16}
$$

By (4.2), (4.4) and Young inequality, we obtain

$$
\begin{aligned}
&|\left(f(u(r,\tau-t,u_0)), u(r,\tau-t,u_0)\right)_{\mathcal{O}}|\\
&\leq \gamma_2\Big(\int_{\mathcal{O}} |u(r,\tau-t,u_0)|dx + \|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2\Big)\\
&\leq \frac{1}{8}\Big(\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2\Big)\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2 + \frac{2\gamma_2^2}{\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2}|\mathcal{O}| + \gamma_2\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2.
\end{aligned}
\tag{4.17}
$$

$$
\begin{aligned}
&|\left(h(v(r,\tau-t,v_0)), v(r,\tau-t,v_0)\right)_{\Gamma}|\\
&\leq \tilde{\gamma}_2\left(\int_{\Gamma} |v(r,\tau-t,v_0)|dS + \|v(r,\tau-t,v_0)\|_{\Gamma}^2\right)\\
&\leq \frac{1}{8}\Big(\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2\Big)\|v(r,\tau-t,v_0)\|_{\Gamma}^2 + \frac{2\tilde{\gamma}_2^2}{\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2}|\Gamma| + \tilde{\gamma}_2\|v(r,\tau-t,v_0)\|_{\Gamma}^2.
\end{aligned}
\tag{4.18}
$$

Taking expectation, we have

$$
\begin{aligned}
&2E\left(f(u(r,\tau-t,u_0)), u(r,\tau-t,u_0)\right)_{\mathcal{O}}\\
&\leq \frac{1}{4}\Big(\frac{3}{8}\lambda - \tilde{\gamma}_2 + 7\gamma_2\Big)E\left(\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2\right) + \frac{4\gamma_2^2}{\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2}|\mathcal{O}|,
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
&2E\left(h(v(r,\tau-t,v_0)), v(r,\tau-t,v_0)\right)_{\Gamma}\\
&\leq \frac{1}{4}\Big(\frac{3}{8}\lambda - \gamma_2 + 7\tilde{\gamma}_2\Big)E\left(\|v(r,\tau-t,v_0)\|_{\Gamma}^2\right) + \frac{4\tilde{\gamma}_2^2}{\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2}|\Gamma|.
\end{aligned}
\tag{4.20}
$$

By Young inequality, we obtain

$$
2E\left(g(r), u(r,\tau-t,u_0)\right)_{\mathcal{O}} \leq \frac{1}{4}(\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2)E\left(\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2\right) + \frac{4}{\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2}\|g(r)\|_{\mathcal{O}}^2. \tag{4.21}
$$

We denote

$$
\epsilon_0 = \min\Big\{1, \sqrt{\frac{\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2}{2(\lambda + |\gamma_5|)}}\Big\}. \tag{4.22}
$$

By (4.7), for all $0 < \epsilon \leq \epsilon_0$, we obtain

$$\begin{aligned}
&\epsilon^2 E\Big(\|G(r,u(r,\tau-t,u_0))\|^2_{L_2(B,L^2(\mathcal{O}))}\Big)\\
&\leq 2\epsilon^2 E\left(\|\nabla u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)+\epsilon^2|\gamma_5|E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}\times\Gamma}\right)+\epsilon^2|\theta(r)|\\
&\leq \frac{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}{(\lambda+|\gamma_5|)}E\left(\|\nabla u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)\\
&\quad+\frac{(\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2)|\gamma_5|}{2(\lambda+|\gamma_5|)}E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}\times\Gamma}\right)+|\theta(r)|.
\end{aligned}\tag{4.23}$$

Combining (4.16), (4.19),-(4.21) and (4.23), we have, for almost all $r \geq \tau - t$,

$$\begin{aligned}
&\frac{d}{dr}E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)+\frac{d}{dr}E\left(\|v(r,\tau-t,v_0)\|^2_{\Gamma}\right)\\
&+\Big(2-\frac{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}{\lambda+|\gamma_5|}\Big)E\left(\|\nabla u(s,\tau-t,u_0)\|^2_{\mathcal{O}}\right)\\
&\leq \frac{1}{2}\left(\frac{3}{8}\lambda-\tilde{\gamma}_2+3\gamma_2+\frac{\left(\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2\right)|\gamma_5|}{(\lambda+|\gamma_5|)}\right)E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)\\
&\quad+\frac{1}{2}\Big(\frac{3}{8}\lambda-\gamma_2+3\tilde{\gamma}_2+\frac{\left(\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2\right)|\gamma_5|}{(\lambda+|\gamma_5|)}\Big)E\left(\|v(r,\tau-t,v_0)\|^2_{\Gamma}\right)\\
&\quad+\frac{4\tilde{\gamma}_2^2}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}|\Gamma|+\frac{4\gamma_2^2}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}|\mathcal{O}|+\frac{4}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}\|g(r)\|^2_{\mathcal{O}}+|\theta(r)|.
\end{aligned}\tag{4.24}$$

By (3.65) and (3.66), it follows that

$$\begin{aligned}
&\frac{d}{dr}E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)+\frac{d}{dr}E\left(\|v(r,\tau-t,v_0)\|^2_{\Gamma}\right)\\
&+\Big(\frac{\gamma_2-\tilde{\gamma}_2}{\lambda}+1-\frac{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}{2(\lambda+|\gamma_5|)}\Big)\lambda E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)\\
&+\Big(1-\frac{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}{2(\lambda+|\gamma_5|)}-\frac{\gamma_2-\tilde{\gamma}_2}{\lambda}\Big)\lambda E\left(\|v(r,\tau-t,v_0)\|^2_{\Gamma}\right)\\
&\leq \frac{1}{2}\Big(\frac{3}{8}\lambda-\tilde{\gamma}_2+3\gamma_2+\frac{\left(\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2\right)|\gamma_5|}{(\lambda+|\gamma_5|)}\Big)E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)\\
&\quad+\frac{1}{2}\Big(\frac{3}{8}\lambda-\gamma_2+3\tilde{\gamma}_2+\frac{\left(\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2\right)|\gamma_5|}{(\lambda+|\gamma_5|)}\Big)E\left(\|v(r,\tau-t,v_0)\|^2_{\Gamma}\right)\\
&\quad+\frac{4\tilde{\gamma}_2^2}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}|\Gamma|+\frac{4\gamma_2^2}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}|\mathcal{O}|+\frac{4}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}\|g(r)\|^2_{\mathcal{O}}+|\theta(r)|.
\end{aligned}\tag{4.25}$$

Obviously, we find that

$$\Big(\frac{\gamma_2-\tilde{\gamma}_2}{\lambda}+1-\frac{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}{2(\lambda+|\gamma_5|)}\Big)\lambda-\frac{1}{2}\Big(\frac{3}{8}\lambda-\tilde{\gamma}_2+3\gamma_2+\frac{\left(\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2\right)|\gamma_5|}{(\lambda+|\gamma_5|)}\Big)=\frac{5}{8}\lambda,\tag{4.26}$$

$$\Big(1-\frac{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}{2(\lambda+|\gamma_5|)}-\frac{\gamma_2-\tilde{\gamma}_2}{\lambda}\Big)\lambda-\frac{1}{2}\Big(\frac{3}{8}\lambda-\gamma_2+3\tilde{\gamma}_2+\frac{\left(\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2\right)|\gamma_5|}{(\lambda+|\gamma_5|)}\Big)=\frac{5}{8}\lambda.\tag{4.27}$$

We assume $\mu = \frac{5}{8}\lambda$. Then it follows from (4.25)–(4.27) that

$$\begin{aligned}
&\frac{d}{dr}E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)+\frac{d}{dr}E\left(\|v(r,\tau-t,v_0)\|^2_{\Gamma}\right)\\
&+\mu E\left(\|u(r,\tau-t,u_0)\|^2_{\mathcal{O}}\right)+\mu E\left(\|v(r,\tau-t,v_0)\|^2_{\Gamma}\right)\\
&\leq \frac{4\tilde{\gamma}_2^2}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}|\Gamma|+\frac{4\gamma_2^2}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}|\mathcal{O}|+\frac{4}{\frac{3}{8}\lambda-\tilde{\gamma}_2-\gamma_2}\|g(r)\|^2_{\mathcal{O}}+|\theta(r)|.
\end{aligned}\tag{4.28}$$

Let $t \geq 0$,we multiplying (4.28) by $e^{\mu r}$ and then integrating on $(\tau - t, \tau)$, so we obtain

$$
\begin{aligned}
&E\left(\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2\right) + E\left(\|v(r,\tau-t,v_0)\|_{\Gamma}^2\right)\\
&\leq e^{-\mu\tau}e^{\mu(\tau-t)}E\left(\|u_0\|_{\mathcal{O}}^2 + \|v_0\|_{\Gamma}^2\right) + \frac{2\tilde{\gamma}_2^2}{\mu\left(\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2\right)}|\Gamma| + \frac{2\gamma_2^2}{\mu\left(\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2\right)}|\mathcal{O}|\\
&\quad + e^{-\mu\tau}\int_{\tau-t}^{\tau} e^{\mu r}\Big(\frac{4}{\frac{3}{8}\lambda - \tilde{\gamma}_2 - \gamma_2}\|g(r)\|_{\mathcal{O}}^2 + |\theta(r)|\Big)dr\\
&\leq e^{-\mu\tau}e^{\mu(\tau-t)}E\left(\|(u_0,v_0)\|_{\mathcal{O}\times\Gamma}^2\right) + c_0 + c_1e^{-\mu\tau}\int_{\tau-t}^{\tau} e^{\mu r}\left(\|g(r)\|_{\mathcal{O}}^2 + |\theta(r)|\right)dr.
\end{aligned}
\tag{4.29}
$$

From $(u_0, v_0) \in D(\tau - t)$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \bar{\mathscr{D}}_0$, we obtain

$$
\begin{aligned}
&e^{-\mu\tau}e^{\mu(\tau-t)}E\left(\|(u_0,v_0)\|_{\mathcal{O}\times\Gamma}^2\right)\\
&\leq e^{-\mu\tau}e^{\mu(\tau-t)}\left(\|D(\tau-t)\|_{L^2(\Omega,\mathscr{F}_\tau;L^2(\mathcal{O}))\times L^2(\Omega,\mathscr{F}_\tau;L^2(\Gamma))}^2\right) \;\to\; 0 \text{ as } t \;\to\; \infty.
\end{aligned}
\tag{4.30}
$$

Then there exists $T'' = T''(\tau, D) > 0$ such that for all $t > T''$,

$$
e^{-\mu\tau}e^{\mu(\tau-t)}E\left(\|(u_0,v_0)\|_{\mathcal{O}\times\Gamma}^2\right) \leq 1. \tag{4.31}
$$

Therefore, by (4.29) and (4.31), we have, for all $t \geq T''$,

$$
\begin{aligned}
&E\left(\|u(r,\tau-t,u_0)\|_{\mathcal{O}}^2\right) + E\left(\|v(r,\tau-t,v_0)\|_{\Gamma}^2\right)\\
&\leq 1 + c_0 + c_1e^{\mu\tau}\int_{-\infty}^{\tau} e^{\mu r}\left(\|g(r)\|_{\mathcal{O}}^2 + |\theta(r)|\right)dr.
\end{aligned}
\tag{4.32}
$$

The proof is complete. □

2. Construct a $\bar{\mathscr{D}}_0$-pullback absorbing set.

Lemma 4.3. *Suppose conditions* (4.2)-(4.5) *and* (4.13) *hold. Then there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$. For each $\tau \in \mathbb{R}$, let*

$$
\bar{K}(\tau) = \left\{(u,v) \in L^2(\Omega,\mathscr{F}_\tau;L^2(\mathcal{O})) \times L^2(\Omega,\mathscr{F}_\tau;L^2(\Gamma)) : E\left(\|(u,v)\|_{\mathcal{O}\times\Gamma}^2\right) \leq \bar{R}(\tau)\right\}. \tag{4.33}
$$

where

$$
\bar{R}(\tau) = C_3 + C_4e^{-\mu\tau}\int_{-\infty}^{\tau} e^{\mu s}\left(\|g(s)\|_{\mathcal{O}}^2 + |\theta(s)|\right)ds,
$$

where C_3 and C_4 are the constants as same as in Lemma 4.2. Moreover, the family $\bar{K} = \{\bar{K}(\tau) : \tau \in \mathbb{R}\} \in \bar{\mathscr{D}}_0$ is a weakly compact $\bar{\mathscr{D}}_0$-pullback absorbing set of ϕ.

Proof. Since for all $\tau \in \mathbb{R}$, $\bar{K}(\tau)$ is a bounded closed convex subset of $L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O})) \times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$ by (4.33), hence $K(\tau)$ is weakly compact in $L^2(\Omega, \mathscr{F}_\tau; L^2(\mathcal{O}))\times L^2(\Omega, \mathscr{F}_\tau; L^2(\Gamma))$. Moreover, by Lemma 4.2, we notice that for every $\tau \in \mathbb{R}$ and $D = \{D(t)\}_{t\in\mathbb{R}} \in \bar{\mathscr{D}}_0$, there exists $T = T(\tau, D) > 0$ such that for all $t \geq T$ and $0 < \epsilon \leq \epsilon_0$,

$$
(u(\tau,\tau-t,u_0), v(\tau,\tau-t,v_0)) = \phi(t,\tau-t,D(\tau-t)) \subseteq K(\tau).
$$

Then we use the same method as in Lemma 3.5 to prove $\bar{K}(\tau) \subseteq \bar{\mathscr{D}}_0$. So, $\bar{K}$ is a weakly compact $\bar{\mathscr{D}}_0$-pullback absorbing set of system ϕ. □

Finally, we can easily prove the existence of the weak $\bar{\mathscr{D}}_0$-pullback mean random attractor $\mathscr{A}_0 \in \bar{\mathscr{D}}_0$ for system ϕ by Theorem 2.5.

Theorem 4.4. *Suppose* (4.2)-(4.5) *and* (4.13) *hold. Then problem* (4.1) *has a unique weak $\bar{\mathscr{D}}_0$-pullback mean random attractor $\mathscr{A}_0 = \{\mathscr{A}_0(\tau) : \tau \in \mathbb{R}\}$ in $L^2(\Omega, L^2(\mathcal{O})) \times L^2(\Omega, L^2(\Gamma))$.*

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References

[1] J. Arrieta, P. Quittner, A. Rodríguez-Bernal; *Parabolic problems with nonlinear dynamical boundary conditions and singular initial data*, Differ. Integral Equ. 14 (2001), 1487–1510.
[2] M. Anguiano, P. Marín-Rubio, J. Real; *Pullback attractors for non-autonomous reaction-diffusion equations with dynamical boundary conditions*, J. Math. Ana. Appl. 383 (2011), 608–618.
[3] M. Anguiano, P. Marín-Rubio, J. Real; *Regularity results and exponential growth for pullback attractors of a non-autonomous reaction-diffusion model with dynamical boundary conditions*, Nonlinear Anal. Real World Appl. 20 (2014), 112–125.
[4] J. Below, M. Cuesta, G. Mailly; *Qualitative results for parabolic equations involving the p-laplacian under dynamical boundary conditions*, NWEJM 4 (2018), 59–97.
[5] I. Chueshov, B. Schmalfuss; *Parabolic stochastic partial differential equations with dynamical boundary conditions*, Differ. Integral Equ. 17 (2004), 751–790.
[6] I. Chueshov, B. Schmalfuss; *Qualitative behavior of a class of stochastic parabolic pdes with dynamical boundary conditions*, Discrete Contin. Dyn. Syst. 18 (2007), 315–338.
[7] J. Escher; *Quasilinear parabolic systems with dynamical boundary conditions*, Comm. Partial Differential Equations 18 (1993), 1309–1364.
[8] Z. Fan, C. Zhong; *Attractors for parabolic equations with dynamic boundary conditions*, Nonlinear Anal. 68 (2008), 1723–1732.
[9] A. Gu; *Weak pullback mean random attractors for non-autonomous p-Laplacian equations*, Discrete Contin. Dyn. Syst. Ser. B 26 (2021), 3863–3878.
[10] A. Gu; *Weak pullback mean random attractors for stochastic evolution equations and applications*, Stoch. Dyn. 22 (2022), 2240001.
[11] P. Kloeden, T. Lorenz; *Mean-square random dynamical systems*, J. Differ. Equ. 253 (2012) 1422–1438.
[12] J. Robinson; *Infinite-dimensional dynamical systems*, Cambridge University Press, 2001.
[13] J. Sprekels, H. Wu; *A note on parabolic equation with nonlinear dynamical boundary condition*, Nonlinear Anal., 72 (2010), 3028–3048.
[14] C. Sun, W. Tan; *Non-autonomous reaction-diffusion model with dynamic boundary conditions*, J. Math. Ana. Appl., 443 (2016), 1007–1032.
[15] B. Wang; *Asymptotic behavior of stochastic wave equations with critical exponents on* $\mathbb{R}^3$, Trans. Amer. Math. Soc., 363 (2011), 3639–3663.
[16] B. Wang; *Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems*, J. Differ. Equ., 253 (2012), 1544–1583.
[17] B. Wang; *Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms*, Stoch. Dyn., 14 (2014), 1450009.
[18] B. Wang; *Weak pullback attractors for mean random dynamical systems in bochner spaces*, J. Dyn. Differ. Equ., 31 (2019), 2177–2204.
[19] B. Wang; *Weak pullback attractors for stochastic Navier-Stokes equations with nonlinear diffusion terms*, Proc. Amer. Math. Soc., 147 (2019), 1627–1638.
[20] B. Wang, B. Guo; *Asymptotic behavior of non-autonomous stochastic parabolic equations with nonlinear Laplacian principal part*, Electron. J. Differential Equations, 2013 (2013) No. 191, 1–25.
[21] H. Wu; *Convergence to equilibrium for the semilinear parabolic equation with dynamical boundary condition*, Adv. Math. Sci. Appl. 17 (2007), 67-88 .
[22] L. Yang, M. Yang; *Long-time behavior of reaction-diffusion equations with dynamical boundary condition*, Nonlinear Anal., 74 (2011), 3876–3883.
[23] L. Yang, M. Yang, P. Kloeden; *Pullback attractors for non-autonomous quasi-linear parabolic equations with dynamical boundary conditions*, Discrete Contin. Dyn. Syst. Ser. B, 17 (2012), 2635–2651.
[24] L. Yang, M. Yang; *Long-time behavior of stochastic reaction-diffusion equation with dynamical boundary condition*, Discrete Contin. Dyn. Syst. Ser. B, 22 (2017), 2627–2650.

Sisi Liu
School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, China
Email address: liuss0919@163.com

Mingli Hong
Institute of Disaster Prevention, Sanhe, 065201, China
Email address: hongmingli001@163.com

Lu Yang (corresponding author)
School of Mathematics and Statistics, Lanzhou University, Lanzhou, 730000, China
Email address: yanglu@lzu.edu.cn