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NONLINEAR PARABOLIC EQUATIONS WITH LAPLACIAN-LIKE OPERATORS AND YOUNG MEASURES

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Abstract. This article explores the existence of weak solutions to a parabolic problem governed by the τ-Laplacian-like operator $-\Delta_\tau^\ell \delta$ and a nonlinear source term η div $\phi(y,t,\delta)$. Under suitable growth conditions on the nonlinear function ϕ, we ensure that the weak formulation of the problem is well- posed, leading to the existence result. This result is obtained through the application of Galerkin's approximation technique to build approximate solutions, as well as the Young measures theory, which provides a framework for handling the complexities introduced by the nonlinearity.

1. Introduction and main results

Equations involving τ-Laplacian-like operators, which generalize the τ-Laplacian are used to model a wide range of nonlinear phenomena in diverse applications [1, 3]. Consequently, the investigation of these problems and their extensive generalizations has drawn the interest of numerous mathematicians in various fields, including applied mathematics, engineering, and in various occurrences in physics, such as image processing, phase transitions, Non-Newtonian fluid mechanics and elasticity theory. Readers who wish to explore further are encouraged to refer to [2, 7, 27, 28, 30, 32] for more details.

In recent years, many scholars have made significant progress on the following τ-Laplacian-like operators $-\Delta_\tau^\ell$, specified by

$$\Delta_\tau^\ell \delta := \operatorname{div}\Big(|\nabla\delta|^{\tau-2}\nabla\delta + \frac{|\nabla\delta|^{2\tau-2}\nabla\delta}{\sqrt{1+|\nabla\delta|^{2\tau}}}\Big).$$

The study of such τ-Laplacian-like operators has gained prominence due to their relevance in modeling capillarity phenomena, which describe a liquid's capacity to move through small areas without assistance from outside sources, or even against the force of gravity. For instance, Chen and Luo in [7] investigated the eigenvalue problem associated with a generalized model of capillarity controlled by the τ-Laplacian-like operator, as detailed below.

$$\begin{gathered}
-\operatorname{div}\big((1+\frac{|\nabla\delta|^\tau}{\sqrt{1+|\nabla\delta|^{2\tau}}})|\nabla\delta|^{\tau-2}\nabla\delta\big) = \sigma\big(|\delta|^{p-2}\delta + |\delta|^{q-2}\delta\big) \quad \text{in } \mathscr{S}, \\
\delta = 0 \quad \text{on } \partial\mathscr{S}.
\end{gathered}$$

Furthermore, drawing on techniques from the analysis of nonlinear elliptic boundary value problems involving the τ-Laplacian-like operator, Wei et al. [29] analyzed a class of nonlinear generalized capillarity equations subject to Neumann boundary conditions. For the system

$$\begin{gathered}
-\operatorname{div}\big((1+\frac{|\nabla\delta|^\tau}{\sqrt{1+|\nabla\delta|^{2\tau}}})|\nabla\delta|^{\tau-2}\nabla\delta\big) + \sigma\big(|\delta|^{p-2}\delta + |\delta|^{q-2}\delta\big) + \mathfrak{g}(y,\delta(y)) = \mathfrak{h}(y) \quad \text{in } \mathscr{S}, \\
-\big\langle \eta', \big(1+\frac{|\nabla\delta|^\tau}{\sqrt{1+|\nabla\delta|^{2\tau}}}\big)|\nabla\delta|^{\tau-2}\nabla\delta\big\rangle \in \gamma_y(\delta(y)) \quad \text{on } Q,
\end{gathered}$$

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Calvert and Gupta' [6] applied the perturbation outcomes for the ranges with respect to m-accretive mappings.

This work contributes to the study of nonlinear problems involving τ-Laplacian-like operators within the analytical framework of Young measures. This approach has proven to be a powerful and flexible tool for the analysis of nonlinear partial differential equations exhibiting nonstandard growth, lack of compactness, and oscillatory phenomena.

Recently, a number of related problems combining fractional operators, double-phase structures, and variable exponent frameworks have been extensively investigated in this setting. For instance, existence results for Dirichlet problems involving fractional $p(z)$-Laplacian operators were established in [24], while fractional double-phase elliptic equations with critical Hardy potentials were studied in [22]. In addition, parabolic problems driven by fractional $q(z)$-Laplacian operators were analyzed in [8].

Further contributions in this research direction include a Dirichlet boundary value problem involving double-phase $(\eta(s), \zeta(s))$-Laplacian-like operators [18], elliptic double-phase problems with logarithmic growth and nonlinear boundary conditions [26], as well as nonlinear elliptic and parabolic models involving logarithmic double-phase operators [20, 21, 23, 25]. These results highlight the richness and effectiveness of analytical approaches based on Young measures, variable exponent techniques, and Musielak–Orlicz spaces.

In addition, several recent works have addressed problems involving $p(x)$-Laplacian-like operators under various boundary conditions. In particular, the existence of weak solutions for Neumann boundary value problems was established in [9]. Moreover, Kirchhoff-type problems with Dirichlet boundary conditions were investigated in [11], where existence results were obtained by means of topological degree theory. Related results on $p(x)$-Kirchhoff-type problems with Neumann boundary conditions were obtained in [10]. These contributions further emphasize the growing interest in nonlinear operators with nonstandard growth and highlight the effectiveness of topological degree methods and variable exponent techniques.

Motivated by these developments, we study the existence of weak solutions for a Dirichlet boundary value problem of parabolic type involving a τ-Laplacian-like operator within the framework of Young measures [5, 12, 14].

$$\begin{gathered}\frac{\partial \delta}{\partial t} - \Delta_\tau^\ell \delta = \eta \operatorname{div} \phi(y,t,\delta) \quad \text{in } \mathscr{S}_{\mathcal{T}},\\ \delta(y,t) = 0 \quad \text{on } \partial\mathscr{S}_{\mathcal{T}},\\ \delta(y,0) = \delta_0(y) \quad \text{in } \mathscr{S},\end{gathered} \tag{1.1}$$

where the bounded open set $\mathscr{S} \subset \mathbb{R}^N, N \geq 2$ is defined by a Lipschitz boundary, which is represented as $\partial\mathscr{S}$. $\mathscr{S}_{\mathcal{T}} = \mathscr{S} \times (0, \mathcal{T})$, indicates that a constant time $\mathcal{T} > 0$ is given and $\partial\mathscr{S}_{\mathrm{T}} = \partial\mathscr{S} \times (0, \mathcal{T})$. In addition, $\delta_0 \in L^2(\mathscr{S})$ and η is a real positive parameter. Furthermore, the function ϕ meets the following conditions of continuity and growth:

- (A1) The function $\phi : \mathscr{S}_{\mathcal{T}} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, meaning that it is measurable for $(y,t) \in \mathscr{S}_{\mathcal{T}}$ and continuous for $\delta \in \mathbb{R}$.
- (A2) Thee exists a function $\mathrm{K} \in L^{\tau'}(\mathscr{S}_{\mathcal{T}})$, with $\mathrm{K} \geq 0$ such that for all $\delta \in \mathbb{R}$ and a.e.] $(y,t) \in \mathscr{S}_{\mathcal{T}}$ it holds
$$|\phi(y,t,\delta)| \leq \mathrm{K}(y,t) + |\delta|^{\tau-1}.$$

To provide the primary results, first present the formal definition of a weak solution, since it serves as the basis for our investigation.

Definition 1.1. For problem (1.1), given any $\psi \in \mathcal{S}$, the weak solution $\delta \in \mathcal{S} \cap L^\infty\big((0,\mathcal{T}); L^2(\mathscr{S})\big)$ is provided by

$$\int_0^{\mathcal{T}} \langle \frac{\partial \delta}{\partial t}, \psi \rangle dt + \int_{\mathscr{S}_{\mathcal{T}}} \big(|\nabla \delta|^{\tau-2}\nabla\delta + \frac{|\nabla\delta|^{2\tau-2}\nabla\delta}{\sqrt{1+|\nabla\delta|^{2\tau}}}\big)\nabla\psi \, dy dt = \eta \int_{\mathscr{S}_{\mathcal{T}}} \phi(y,t,\delta)\nabla\psi \, dy \, dt$$

The essential result reads as follows. .

Theorem 1.2. *Let $\psi \in \mathcal{S}$ and* (A1), (A2) *hold. Then system* (1.1) *admits a weak solution.*

Next, we describe the structure to help readers better grasp our work. In the opening section, we outline problem (1.1) and the functions it involves. In the subsequent section, we give preliminaries, notation and definitions of the functional spaces we employ, along with the framework of Young measures. Lastly, we establish the existence of a weak solution to problem (1.1), by means of Galerkin's approximation and Young's measures.

2. Theoretical background in mathematics

First, we go over some crucial Sobolev space properties. For a more detailed discussion, the reader is referred to [16, 31]. Additionally, to further our analysis, we will offer a concise review of the Young measure, a key tool for capturing the weak limits of nonlinear functions, see [5, 12, 14] for more details.

2.1. **Fundamental results on Lebesgue and Sobolev spaces.** Now we present a brief summary of the foundational concepts and intrinsic features of Lebesgue spaces. For $1 \leq \tau < \infty$, The Lebesgue spaces are specified by

$$L^\tau(\mathscr{S}) = \big\{\delta : \mathscr{S} \to \mathbb{R} \text{ is measurable}, \int_{\mathscr{S}} |\delta(y)|^\tau \, dy < +\infty\big\}.$$

The association norm is

$$\|\delta\|_{L^\tau(\mathscr{S})} = \Big(\int_{\mathscr{S}} |\delta|^\tau \, dy\Big)^{1/\tau}.$$

$\big(L^\tau(\mathscr{S}), \|\cdot\|_\tau\big)$ is a reflexive Banach space for $1 < \tau < \infty$, and separable for $1 \leq \tau < \infty$. Additionally, we have the Hölder's inequality

$$\int_{\mathscr{S}} \delta\omega \, dy \leq \|\delta\|_{L^\tau(\mathscr{S})} \|\omega\|_{L^{\tau'}(\mathscr{S})},$$

where the conjugate space of $L^\tau(\mathscr{S})$ is $L^{\tau'}(\mathscr{S})$ with $\frac{1}{\tau} + \frac{1}{\tau'} = 1$. Furthermore, for $\mathfrak{r} \leq \mathfrak{s}$, the embedding $L^{\mathfrak{s}}(\mathscr{S}) \hookrightarrow L^{\mathfrak{r}}(\mathscr{S})$ is continuous.

We define the Sobolev space

$$W^{1,\tau}(\mathscr{S}) = \big\{\delta \in L^\tau(\mathscr{S}) : |\nabla\delta| \in L^\tau(\mathscr{S})\big\},$$

equipped with the norm

$$\|\delta\|_{1,\tau} = \|\delta\|_\tau + \|\nabla\delta\|_\tau.$$

The subspace of $W^{1,\tau}(\mathscr{S})$, consisting of closure of $C_0^\infty(\mathscr{S})$ with regard to the norm $\|\cdot\|$, is denoted by $W_0^{1,\tau}(\mathscr{S})$. Additionally, the dual space of $W_0^{1,\tau}(\mathscr{S})$ is denoted by $W^{-1,\tau'}(\mathscr{S})$.

Proposition 2.1. *Assume that the domain $\mathscr{S}$ has a continuous boundary and for $1 \leq \tau < \infty$, the Poincaré inequality is then obtained for any $\delta \in W^{1,\tau}(\mathscr{S})$, meaning that, one can find a constant $C > 0$ that depends entirely on $\mathscr{S}$ and τ such that*

$$\|\delta\|_\tau \leq C\|\nabla\delta\|_\tau, \quad \textit{for all } \delta \in W_0^{1,\tau}(\mathscr{S}). \tag{2.1}$$

We refer to [31] for broader interpretations of the Poincaré inequality. The following equivalent norm on $W_0^{1,\tau}(\mathscr{S})$ will be applied in this study.

$$\|\delta\| = \|\nabla\delta\|_\tau$$

Proposition 2.2 ([16]). *The spaces $\big(W^{1,\tau}(\mathscr{S}), \|\cdot\|_{1,\tau}\big)$ and $\Big(W_0^{1,\tau}(\mathscr{S}), \|\cdot\|\Big)$ are separable for $1 \leq \tau < \infty$ and reflexive for $1 < \tau < \infty$. Moreover, both spaces are Banach spaces.*

Theorem 2.3. *The following compact embedding holds*

$$W^{1,\tau}(\mathscr{S}) \hookrightarrow L^p(\mathscr{S}),$$

where $\mathscr{S}$ is a bounded open subset of $\mathbb{R}^N$ and $p < \tau^*$, τ^* is defined as

$$\tau^* = \frac{N\tau}{N-\tau}, \quad \textit{for all } \tau < N.$$

Since $\tau < \tau^*$, it follows that for every $y \in \mathscr{S}$, we have

$$W^{1,\tau}(\mathscr{S}) \hookrightarrow\hookrightarrow L^\tau(\mathscr{S}).$$

To demonstrate the existence result, we use the following space to build our framework. Within a specified time interval $0 < \mathcal{T} < \infty$,

$$\mathcal{S} := L^\tau\Big(0, \mathcal{T}; W_0^{1,\tau}(\mathscr{S})\Big)$$

is a Banach space, both separable and reflexive, endowed with the norm

$$\|\delta\|_{\mathcal{S}} = \Big(\int_0^{\mathcal{T}} \|\delta\|^\tau dt\Big)^{1/\tau}.$$

Moreover, according to [17], there exists a continuous embedding $\mathcal{S} \hookrightarrow L^\tau(\mathscr{S}_{\mathcal{T}})$.

2.2. **Basic properties of the Young measure.** Features of Young measures will be vital for establishing the existence result. The space of real-valued continuous functions on $\mathbb{R}^k$ with compact support, endowed with the $\|\cdot\|_\infty$-norm is represented by $\mathcal{C}_0(\mathbb{R}^k)$. Then $\mathcal{M}(\mathbb{R}^k)$ is its dual space and it represents the space of signed Radon measures with finite mass. The following represents the duality pairing between these spaces:

$$\langle \zeta, \mathscr{N} \rangle = \int_{\mathbb{R}^k} \mathscr{N}(\varsigma)\, d\zeta(\varsigma).$$

Definition 2.4 ([12])**.** In $L^\infty(\mathscr{S})$, let $\{z_j\}_{j\geq 1}$ be a bounded sequence. For a.e. $y \in \mathscr{S}$, we obtain a Borel probability measure ζ_y on $\mathbb{R}^k$ along with a subsequence $\{z_m\} \subset \{z_j\}$. Consequently, the following is true for each $\mathscr{N} \in \mathcal{C}(\mathbb{R}^k)$.

$$\mathscr{N}(z_m) \overset{*}{\rightharpoonup} \overline{\mathscr{N}} \quad \text{weakly in } L^\infty(\mathscr{S}).$$

with $\overline{\mathscr{N}}(y) = \langle \zeta_y, \mathscr{N} \rangle = \int_{\mathbb{R}^k} \mathscr{N}(\varsigma)\, d\zeta_y(\varsigma)$ for a.e. $y \in \mathscr{S}$.

The basic theorem about Young measures is expressed in the following Lemma.

Lemma 2.5 ([14])**.** *Let $\mathscr{S} \subset \mathbb{R}^N$ be Lebesgue measurable and a closed subset $\mathcal{Q} \subset \mathbb{R}^k$, take a sequence of Lebesgue measurable functions $z_j : \mathscr{S} \to \mathbb{R}^k, j \in \mathbb{N}$, which achieves for $j \to \infty$, $z_j \to \mathcal{Q}$ in measure, that is, for every open neighborhood $\mathscr{K}$ of $\mathcal{Q}$ in $\mathbb{R}^k$*

$$\lim_{j\to\infty} |\{y \in \mathscr{S} : z_j(y) \notin \mathscr{K}\}| = 0.$$

Consequently, a subsequence z_m and a family of non-negative Radon measures $\{\varsigma_y\}_{y\in\mathscr{S}}$ exist on $\mathbb{R}^k$, so that

(i) *For almost $y \in \mathscr{S}$, $\|\zeta_y\|_{\mathcal{M}(\mathbb{R}^k)} := \int_{\mathbb{R}^k} d\zeta_y(\varsigma) \leq 1$.*
(ii) *$\mathscr{N}(z_m) \overset{*}{\rightharpoonup} \overline{\mathscr{N}}$ weakly in $L^\infty(\mathscr{S})$ for all $\mathscr{N} \in \mathcal{C}_0(\mathbb{R}^k)$, for $\overline{\mathscr{N}}(y) = \langle \zeta_y, \mathscr{N} \rangle$.*
(iii) *For all $\mathsf{R} > 0$, it follows that.*

$$\lim_{\nu\to\infty} \sup_{m\in\mathbb{N}} |\{y \in \mathscr{S} \cap B_{\mathsf{R}}(0) : |z_m(y)| \geq \nu\}| = 0. \tag{2.2}$$

Thus, $\|\zeta_y\| = 1$.

Given that for almost every $y \in \mathscr{I}$, the sequence $\mathscr{N}(z_m)$ converges weakly to $\overline{\mathscr{N}} = \langle \zeta_y, \mathscr{N} \rangle$ in $L^1(\mathscr{S}')$, for a continuous function $\mathscr{N}$ and any measurable subset $\mathscr{N}' \subset \mathscr{N}$, the sequence $\mathscr{N}(z_m)$ is weakly precompact in $L^1(\mathscr{S}')$.

3. Proof of main results

This section aims to establish the existence of weak solutions to problem (1.1). To demonstrate Theorem 1.2, we first apply Galerkin's method to formulate an approximate solution for problem (1.1). Let $\{\lambda_j\}_{j\geq 1}$ be an orthonormal basis in L^2, such that:

$$\{\lambda\}_{j\geq 1} \subset C_0^\infty(\mathscr{S}) \subset \overline{\cup_{m\geq 1}\mathcal{U}_m}^{C^1(\bar{\mathscr{S}})},$$

in which $\mathcal{U}_m = \operatorname{span}\{\lambda_1, \dots, \lambda_m\}$. We define

$$\delta_m(y,t) = \sum_{j=1}^{m} \gamma_{mj}(t)\lambda_j(y),$$

where $\gamma_{mj} : [0,\mathcal{T}) \to \mathbb{R}^+$ satisfy

$$\begin{aligned} &\int_{\mathscr{S}} \frac{\partial \delta_m}{\partial t}\lambda_j(y)dy + \int_{\mathscr{S}} \big(|\nabla\delta_m|^{\tau-2}\nabla\delta_m + \frac{|\nabla\delta_m|^{2\tau-2}\nabla\delta_m}{\sqrt{1+|\nabla\delta_m|^{2\tau}}}\big)\nabla\lambda_j(y)dy \\ &= \eta\int_{\mathscr{S}} \phi(y,t,\delta_m)\nabla\lambda_j(y)\,dy. \end{aligned} \tag{3.1}$$

Suppose that $\delta_m \in \mathcal{S}$, thus δ_m Meets the requirement $\delta(y,t) = 0$ on $\partial\mathscr{S}_{\mathcal{T}}$. For $\delta(y,0) = \delta_0(y)$ in $\mathscr{S}$, we set $\gamma_{mj}(0) := (\delta_0, \lambda_j)_{L^2(\mathscr{S})}$ in such a way that

$$\delta_m(\cdot,0) = \sum_{j=1}^{m} \gamma_{mj}(0)\lambda_m(\cdot) \to \delta_0 \quad \text{in } L^2(\mathscr{S}) \quad \text{as } m \to \infty.$$

The inner product in $\mathscr{L}^2(\mathscr{S})$ is $(\cdot,\cdot)_{L^2}$. Consider the interval $J = [0,\Gamma_t]$ and take $m \in \mathbb{N}$, $0 < \Gamma_t < \mathcal{T}$. Given that $R > 0$ is sufficiently large, we choose the ball $B_R(0) := B(0,R) \subset \mathbb{R}^m$ to encompass the vectors $(\gamma_{m1}(0), \dots, \gamma_{mm}(0))$.

Now we define the operator $\mathcal{N}$ from $J \times \overline{B_R(0)}$ into $\mathbb{R}^k$ by

$$\begin{aligned} \mathcal{N}(t,\gamma_1,\dots,\gamma_m) = \eta\int_{\mathscr{S}} \phi\left(y,t,\delta_m\right)\nabla\lambda_i(y)dy - \int_{\mathscr{S}} \big(|\nabla\delta_m|^{\tau-2}\nabla\delta_m \\ + \frac{|\nabla\delta_m|^{2\tau-2}\nabla\delta_m}{\sqrt{1+|\nabla\delta_m|^{2\tau}}}\big)\nabla\lambda_j(y)dy; \quad i = 1,\dots,m, \end{aligned}$$

where $\delta_m := \sum_{j=1}^{m} \gamma_{mj}(t)\lambda_j(y)$. $\mathcal{N}$ is a Carathéodory function. Moreover, it is also possible to estimate each component $\mathcal{N}_i$ on $J \times \overline{B_R(0)}$ by

$$\begin{aligned} |\mathcal{N}_i\left(t,\gamma_1,\dots,\gamma_m\right)| &\le \eta\Big(\int_{\mathscr{S}} |\phi\left(y,t,\delta_m\right)|^{\tau'}dy\Big)^{1/\tau'}\Big(\int_{\mathscr{S}} |\nabla\lambda_i|^{\tau}dy\Big)^{1/\tau} \\ &\quad + 2\Big(\int_{\mathscr{S}} |\nabla\delta_m(y,t)|^{\tau}dy\Big)^{\frac{\tau-1}{\tau}}\Big(\int_{\mathscr{S}} |\nabla\lambda_i|^{\tau}dy\Big)^{1/\tau}. \end{aligned} \tag{3.2}$$

By Hölder's inequality, we can express (3.2) as follows

$$|\mathcal{N}_i\left(t,\gamma_1,\dots,\gamma_m\right)| \le \mathcal{L}(R,m)E(t),$$

Uniformly over $J \times \overline{B_R(0)}$, where $\mathcal{L}(R,m)$ is a constant that relies with m and R, while $E(t) \in L^1(J)$ remains independent of both R and m.

Ordinary differential equations are applied to the following system as a result of the Carathéodory existence (see [15]).

$$\begin{gathered} \gamma_i'(t) = \mathcal{N}_i\left(t,\gamma_1(t),\dots,\gamma_m(t)\right), \\ \gamma_i(0) = \gamma_{mi}(0). \end{gathered} \tag{3.3}$$

For $i = 1,\dots,m$ there exists a continuous, distributional solution γ_i which depends on m of (3.3) over $[0,t')$ a time interval, with $t' > 0$ possibly depending on m. The associated integral form of Equation (3.3) is expressed as,

$$\gamma_i(t) = \gamma_i(0) + \int_0^t \mathcal{N}_i\left(z,\gamma_1(z),\dots,\gamma_m(z)\right)dz.$$

holds on $[0,t')$. Consequently, $\delta_m(y,t) = \sum_{i=1}^{m} \gamma_{mj}(t)\lambda_j(y)$ is the intended solution of (3.1) with the initial condition $\delta_m(\cdot,0) = \sum_{j=1}^{m} \gamma_{mj}(0)\lambda_j(\cdot) \to \delta_0$ in $L^2(\mathscr{S})$ for $m \to \infty$.

Next, we extend these local solutions to the full time interval $[0,\mathcal{T}]$. This is done by obtaining a priori estimates on the local solutions, which guarantee that the approximate solutions remain bounded and well-behaved over the entire interval after we ensure that these approximations hold globally in time, not just locally. This is accomplished by multiplying equation (3.1)by $\gamma_{mi}(t)$, then summing over $i = 1, \dots, m$, to acquire for $\Gamma_t \in [0, \mathcal{T})$.

$$\int_{\mathscr{S}_{\Gamma_t}} \frac{\partial \delta_m}{\partial t} \delta_m \, dy \, dt + \int_{\mathscr{S}_{\Gamma_t}} \big(|\nabla \delta_m|^\tau + \frac{|\nabla \delta_m|^{2\tau}}{\sqrt{1 + |\nabla \delta_m|^{2\tau}}} \big) \, dy \, dt = \eta \int_{\mathscr{S}_{\Gamma_t}} \phi\,(y, t, \delta_m)\, \nabla \delta_m \, dy \, dt,$$

where $\mathscr{S}_{\Gamma_t} = \mathscr{S} \times (0, \Gamma_t)$. We have

$$\mathbf{M}_1 \equiv \int_{\mathscr{S}_{\Gamma_t}} \frac{\partial \delta_m}{\partial t} \delta_m \, dy \, dt = \frac{1}{2} \|\delta_m(\cdot, \Gamma_t)\|^2_{L^2(\mathscr{S})} - \frac{1}{2} \|\delta_m(\cdot, 0)\|^2_{L^2(\mathscr{S})}.$$

By considering that $\sqrt{1 + |\nabla \delta_m|^{2\tau}} > |\nabla \delta_m|^\tau$ we deduce that

$$\begin{aligned}
|\mathbf{M}_2| &\equiv \Big| \int_{\mathscr{S}_{\Gamma_t}} \big(|\nabla \delta_m|^\tau + \frac{|\nabla \delta_m|^{2\tau}}{\sqrt{1 + |\nabla \delta_m|^{2\tau}}} \big) \, dy \, dt \Big| \\
&= \int_{\mathscr{S}_{\Gamma_t}} |\nabla \delta_m|^\tau \, dy \, dt + \int_{\mathscr{S}_{\Gamma_t}} \frac{|\nabla \delta_m|^{2\tau}}{\sqrt{1 + |\nabla \delta_m|^{2\tau}}} \frac{\sqrt{1 + |\nabla \delta_m|^{2\tau}}}{\sqrt{1 + |\nabla \delta_m|^{2\tau}}} \, dy \, dt \\
&\geqslant \int_{\mathscr{S}_{\Gamma_t}} |\nabla \delta_m|^\tau \, dy \, dt + \int_{\mathscr{S}_{\Gamma_t}} \sqrt{1 + |\nabla \delta_m|^{2\tau}} \Big(1 - \frac{1}{1 + |\nabla \delta_m|^{2\tau}} \Big) \, dy \, dt \\
&\geqslant \int_{\mathscr{S}_{\Gamma_t}} |\nabla \delta_m|^\tau \, dy \, dt + \int_{\mathscr{S}_{\Gamma_t}} \sqrt{1 + |\nabla \delta_m|^{2\tau}} \Big(1 - \frac{1}{1 + \sigma'} \Big) \, dy \, dt \text{ due to } |\nabla \delta_m| \geqslant \sigma' > 0 \\
&\geqslant \int_{\mathscr{S}_{\Gamma_t}} |\nabla \delta_m|^\tau \, dy \, dt + C_1 \int_{\mathscr{S}_{\Gamma_t}} \sqrt{1 + |\nabla \delta_m|^{2\tau}} \, dy \, dt \\
&\geqslant (1 + C_1) \|\delta_m\|^\tau_{\mathcal{S}},
\end{aligned}$$

and by applying conditions $(\mathcal{Y}_1) - (\mathcal{Y}_2)$, we deduce that

$$|\mathbf{M}_3| \equiv \Big| \int_{\mathscr{S}_{\Gamma_t}} \eta \phi\,(y, t, \delta_m)\, \nabla \delta_m \, dy \, dt \Big| \leq \eta C_2 \|\mathrm{K}\|_{\tau'} \|\delta_m\|_{\mathcal{S}} + \eta C_3 \|\delta_m\|^\tau_{\mathcal{S}}.$$

We know that $\delta_m(y, 0) = \Phi_m(y) \to \delta_0$ in $L^2(\mathscr{S})$, thus

$$\int_{\mathscr{S}} \delta_m^2(y, 0) \, dy = \int_{\mathscr{S}} |\Phi_m(y)|^2 \, dy \leq d \quad \text{for all } m \in \mathbb{N}.$$

As a result, we deduce that $\|\delta_m(\cdot, \Gamma_t)\|^2_{L^2(\mathscr{S})} \leq d$ in accordance with the estimates on $\mathbf{M}_\ell$, $\ell = 1, \dots, 3$.

We shall proceed by considering:

$$\mathscr{D} := \{ t \in [0, \mathcal{T}) : \text{ a weak solution to (3.3) exists on } [0, t) \}.$$

$\mathscr{D}$ is a non-empty set since it includes at least a local solution. Moreover, as established in [13], $\mathscr{D}$ is both open and closed. As a result, we deduce that $\mathscr{D} = [0, \mathcal{T})$.

Finally, we employ the Young measure, this step is crucial for dealing with the complexities introduced by the fractional operators. The following lemma explains how this method is utilized to identify weak limits.

Lemma 3.1. *There exists a Young measure $\zeta_{(y,t)}$ generated by $\nabla \delta_m$, if $\{\delta_m\}_m$ is bounded in $\mathcal{S} := L^\tau \Big(0, \mathcal{T}; W_0^{1,\tau}(\mathscr{S}) \Big)$. This measure has the subsequent features:*

(1) $\zeta_{(y,t)}$ *is a probability measure, meaning, for almost every $(y, t) \in \mathscr{S}_{\mathcal{T}}$ we have*

$$\big\| \zeta_{(y,t)} \big\|_{\mathcal{M}^{k \times n}} := \int_{\mathcal{M}^{k \times n}} d\zeta_{(y,t)}(\varsigma) = 1.$$

(2) *The weak L^1-limit of $\nabla \delta_m$ is provided by $\big\langle \zeta_{(y,t)}, \mathcal{I}d \big\rangle = \int_{\mathcal{M}^{k \times n}} \varsigma \, d\zeta_{(y,t)}(\varsigma)$.*

(3) *For almost every $(y, t) \in \mathscr{S}_{\mathcal{T}}$, $\zeta_{(y,t)}$ satisfies $\big\langle \zeta_{(y,t)}, \mathcal{I}d \big\rangle = \nabla \delta(y, t)$.*

Proof. (1) We demonstrate that $\{\nabla\delta_m\}$ fulfills the equation (2.2) to verify the first part of Lemma 3.1. Furthermore, for any $R>0$, we know that $\mathscr{S}\cap B_R\subseteq\mathscr{S}$, where $B_R=B(0,R)$ is the ball centered at 0 with radius R. Since $\{\nabla\delta_m\}_m$ is bounded in $L^\tau(\mathscr{S}_\mathcal{T})$, it follows that there exists $\kappa\geq 0$ such that

$$\begin{aligned}\kappa &\geq \int_{\mathscr{S}_\mathcal{T}} |\nabla\delta_m|^\tau\,dy\,dt\\ &\geq \int_{\{(y,t)\in(\mathscr{S}\cap B_R)\times(0,\mathcal{T})\ :\ |\nabla\delta_m(y,t)|\geq\nu\}} |\nabla\delta_m|^\tau\,dy\,dt\\ &\geq \nu^\tau|\{(y,t)\in(\mathscr{S}\cap B_R)\times(0,\mathcal{T}):\ |\nabla\delta_m(y,t)|\geq\nu\}|.\end{aligned}$$

Thus

$$\sup_{m\in\mathbb{N}}|\{(y,t)\in(\mathscr{S}\cap B_R)\times(0,\mathcal{T}):\ |\nabla\delta_m(y,t)|\geq\nu\}|\leq\frac{\kappa}{\nu^\tau}\to 0,\ \text{as}\ \nu\to\infty.$$

According to assertion 3 in Lemma 2.5, we have $\|\zeta_{(y,t)}\|=1$.

(2) The existence of a subsequence (still represented by $\{\delta_m\}_m$) that is weakly convergent in $\mathcal{S}$ is implied by the reflexivity of $\mathcal{S}:=L^\tau\big(0,\mathcal{T};W_0^{1,\tau}(\mathscr{S})\big)$ for $\tau>1$. Thus, the reflexivity of $L^\tau(\mathscr{S}_\mathcal{T})$ implies the existence of a subsequence $\{\nabla\delta_m\}_m$ weakly convergent in $L^\tau(\mathscr{S}_\mathcal{T})$, thus weakly convergent in $L^1(\mathscr{S}_\mathcal{T})$. Considering $\mathscr{N}$ as the identity mapping $\mathcal{I}d$ and applying Lemma 2.5, we obtain

$$\nabla\delta_m\rightharpoonup\langle\zeta_{(y,t)},\mathcal{I}d\rangle=\int_{\mathcal{M}^{k\times n}}\varsigma\,d\zeta_{(y,t)}(\varsigma)\quad\text{weakly in } L^1(\mathscr{S}_\mathcal{T}).$$

(3) Since $\delta_m\rightharpoonup\delta$ in $\mathcal{S}:=L^\tau(0,\mathcal{T};W_0^{1,\tau}(\mathscr{S}))$ and $\delta_m\to\delta$ in $L^\tau(\mathscr{S}_\mathcal{T})$, we have

$$\nabla\delta_m\rightharpoonup\nabla\delta\quad\text{in } L^\tau(\mathscr{S}_\mathcal{T}).$$

Moreover, $\nabla\delta_m\rightharpoonup\nabla\delta$ in $L^1(\mathscr{S}_\mathcal{T})$ (up to a subsequence). The second assertion in Lemma 3.1 leads to the conclusion that

$$\nabla\delta(y,t)=\langle\zeta_{(y,t)},\mathcal{I}d\rangle\quad\text{for a.e. } (y,t)\in\mathscr{S}_\mathcal{T}.$$

□

Furthermore, by the estimations on $\mathbf{M}_\ell$, $\ell=1,\dots,3$, we can compose

$$\begin{aligned}&\frac{1}{2}\|\delta_m(\cdot,\Gamma_t)\|^2_{L^2(\mathscr{S})}+(1+C_1)\|\delta_m\|^\tau_{\mathcal{S}}\\ &\leq\frac{1}{2}\|\delta_m(\cdot,0)\|^2_{L^2(\mathscr{S})}+\eta C_2\|\mathrm{K}\|_{\tau'}\|\delta_m\|_{\mathcal{S}}+\eta C_3\|\delta_m\|^\tau_{\mathcal{S}}.\end{aligned}\tag{3.4}$$

If $\|\delta_m\|_{\mathcal{S}}$ is unbounded, then $\int_0^\mathcal{T}\|\delta_m\|^\tau dt$ is unbounded, and this contradict (3.4). Thus, $\{\delta_m\}_{m\in\mathbb{N}}$ is bounded in $\mathcal{S}\cap L^\infty\big(0,\mathcal{T};L^2(\mathscr{S})\big)$. For an appropriate subsequence,

$$\delta_m\rightharpoonup\delta\text{ in }\mathcal{S}\quad\text{and}\quad\delta_m\rightharpoonup^*\delta\text{ in }L^\infty\big(0,\mathcal{T};L^2(\mathscr{S})\big).$$

The function $\delta\in\mathcal{S}\cap L^\infty\big(0,\mathcal{T};L^2(\mathscr{S})\big)$ is a candidate to be a weak solution for problem (1.1). By using conditions (A2), we obtain

$$\int_{\mathscr{S}_\mathcal{T}}|\eta\phi(y,t,\delta_m)|^{\tau'}\,dy\,dt\leq C_4\int_{\mathscr{S}_\mathcal{T}}\Big(|\mathrm{K}(y,t)|^{\tau'}+|\delta_m|^\tau\Big)\,dy\,dt.\tag{3.5}$$

Since $\mathrm{K}\in L^{\tau'}(\mathscr{S}_\mathcal{T})$ and $(\delta_m)_m$ is bounded in $\mathcal{S}$, it follows that

$$\eta\phi(y,t,\delta_m)\rightharpoonup\eta\phi(y,t,\delta)\quad\text{in } L^{\tau'}(\mathscr{S}_\mathcal{T}).$$

We are now prepared for proving Theorem 1.2. Let us consider a sequence $\{w_m\}_m$ that is bounded in $\mathcal{S}$. By Lemma 2.5, there exists a Young measure $\zeta_{(y,t)}$ generated by $\nabla\delta_m$ in $L^\tau(\mathscr{S}_\mathcal{T})$ that satisfies the properties established in Lemma 3.1.

First, we establish the convergence $\delta_m\to\delta$ in measure. To this end, consider

$$\mathcal{H}_{m,\varepsilon}=\{(y,t)\in\mathscr{S}_\mathcal{T};|\delta_m(y,t)-\delta(y,t)|\geq\varepsilon\}.$$

Since $\{w_m\}_m$ is bounded in $\mathcal{S}$, it follows that a subsequence, indicated by δ_m, we have $\delta_m \to \delta$ in $L^\tau(\mathscr{S}_\mathcal{T})$. Subsequently, we have

$$\int_{\mathscr{S}_\mathcal{T}} |\delta_m(y,t)-\delta(y,t)|^\tau \, dy\,dt \geq \int_{\mathcal{H}_{m,\varepsilon}} |\delta_m(y,t)-\delta(y,t)|^\tau \, dy\,dt \geq \varepsilon^\tau \, |\mathcal{H}_{m,\varepsilon}| \,.$$

It results that

$$|\mathcal{H}_{m,\varepsilon}| \leq \frac{1}{\varepsilon^\tau} \int_{\mathscr{S}_\mathcal{T}} |\delta_m(y,t)-\delta(y,t)|^\tau \to 0 \quad \text{as } m \to \infty.$$

Thus, on $\mathscr{S}_\mathcal{T}$, the sequence δ_m converges in measure to δ. However, using Lemma 3.1, $\nabla\delta_m$ is bounded in $L^\tau(\mathscr{S}_\mathcal{T})$, leading to the result

$$\nabla\delta_m(y,t) \rightharpoonup \int_{\mathbb{M}^{k\times n}} \varsigma \, d\zeta_{(y,t)}(\varsigma) = \nabla\delta(y,t) \quad \text{weakly in } L^1(\mathscr{S}_\mathcal{T}).$$

In addition, from Hölder's inequality it follows that

$$\int_{\mathscr{S}_\mathcal{T}} |F(\nabla\delta_m)\nabla\psi| \, dy\,dt \leq 2\|\delta_m\|_{\mathcal{S}}^{\tau-1}\|\psi\|_{\mathcal{S}},$$

where $F(\xi) = |\xi|^{\tau-2}\xi + \frac{|\xi|^{2\tau-2}\xi}{\sqrt{1+|\xi|^{2\tau}}}$.

Therefore, the sequence $(F(\nabla\delta_m)\nabla\psi)$ is equiintegrable; this implies

$$\begin{aligned} &F(\nabla\delta_m)\nabla\psi \\ &\rightharpoonup \Big(\Big|\int_{\mathcal{M}^{k\times n}} \varsigma \, d\zeta_{(y,t)}(\varsigma)\Big|^{\tau-2} \int_{\mathcal{M}^{k\times n}} \varsigma \, d\zeta_{(y,t)}(\varsigma) + \frac{|\int_{\mathcal{M}^{k\times n}} \varsigma \, d\zeta_{(y,t)}(\varsigma)|^{2\tau-2} \int_{\mathcal{M}^{k\times n}} \varsigma \, d\zeta_{(y,t)}(\varsigma)}{\sqrt{1+|\int_{\mathcal{M}^{k\times n}} \varsigma \, d\zeta_{(y,t)}(\varsigma)|^{2\tau}}}\Big)\nabla\psi \\ &= F(\nabla\delta)\nabla\psi \quad \text{weakly in } L^1(\mathscr{S}_\mathcal{T}). \end{aligned}$$

Furthermore, we infer that $F(\nabla\delta_m)\nabla\psi$ is weakly convergent in $L^\tau(\mathscr{S}_\mathcal{T})$ based on the reflexivity of $L^\tau(\mathscr{S}_\mathcal{T})$. Additionally, weakly convergent in $L^1(\mathscr{S}_\mathcal{T})$, since $1 < \tau$ and its weak L^1-limit is determined by $F(\nabla\delta)\nabla\psi$. Consequently

$$\begin{aligned} &\lim_{m\to\infty} \int_{\mathscr{S}_\mathcal{T}} \Big(|\nabla\delta_m|^{\tau-2}\nabla\delta_m + \frac{|\nabla\delta_m|^{2\tau-2}\nabla\delta_m}{\sqrt{1+|\nabla\delta_m|^{2\tau}}}\Big)\nabla\psi \, dy\,dt \\ &= \int_{\mathscr{S}_\mathcal{T}} \Big(|\nabla\delta|^{\tau-2}\nabla\delta + \frac{|\nabla\delta|^{2\tau-2}\nabla\delta}{\sqrt{1+|\nabla\delta|^{2\tau}}}\Big)\nabla\psi \, dy\,dt. \end{aligned}$$

With the aid of Aubin-Simon argument (see [4]), we deduce that $w \in C(0,\mathcal{T};L^2(\mathscr{S}))$ and there is a weak convergence $\delta_m(\cdot,\mathcal{T}) \to \delta(\cdot,\mathcal{T})$ in $L^2(\mathscr{S})$.

Since $\delta_m \to \delta$ in measure for $m \to \infty$, after selecting an appropriate subsequence, it follows that $\delta_m \to \delta$ almost everywhere as $m \to \infty$. Hence, for $\psi \in \mathcal{S}$, from (A1) it follows that

$$\eta\phi\,(y,t,\delta_m)\,\nabla\psi \to \eta\phi\,(y,t,\delta)\,\nabla\psi \quad \text{Almost everywhere.}$$

As $\eta\phi\,(y,t,\delta_m)$ is equiintegrable by (3.5), as a result, using the Vitali Convergence Theorem

$$\eta\phi\,(y,t,\delta_m)\,\nabla\psi \to \eta\phi\,(y,t,\delta)\,\nabla\psi \quad \text{in } L^1(\mathscr{S}_\mathcal{T}).$$

Consequently,

$$\lim_{m\to\infty} \int_{\mathscr{S}_\mathcal{T}} \eta\phi\,(y,t,\delta_m)\,\nabla\psi \, dydt = \int_{\mathscr{S}_\mathcal{T}} \eta\phi\,(y,t,\delta)\,\nabla\psi \, dydt.$$

A test function $\mathcal{O} \in \cup_{m\in\mathbb{N}}\mathcal{U}_m$ and $\beta \in C_0^\infty([0,\mathcal{T}])$ are now used in (3.1) and pass to the limit $m \to \infty$ after integrating over $(0,\mathcal{T})$, this leads to the equation below for every $\mathcal{O} \in \cup_{m\in\mathbb{N}}\mathcal{U}_m$ and $\beta \in C_0^\infty([0,\mathcal{T}])$.

$$\begin{aligned} &\int_{\mathscr{S}_\mathcal{T}} \frac{\partial\delta}{\partial t}\beta(t)\mathcal{O}(y) \, dy\,dt + \int_{\mathscr{S}_\mathcal{T}} \big(|\nabla\delta|^{\tau-2}\nabla\delta + \frac{|\nabla\delta|^{2\tau-2}\nabla\delta}{\sqrt{1+|\nabla\delta|^{2\tau}}}\big)\beta(t)\nabla\mathcal{O}(y) \, dy\,dt \\ &= \int_{\mathscr{S}_\mathcal{T}} \eta\phi\,(y,t,\delta)\,\beta(t)\nabla\mathcal{O}(y) \, dy\,dt. \end{aligned}$$

The density of the linear span of these functions in $\mathcal{S} := L^\tau\big((0,\mathcal{T}); W_0^{1,\tau}(\mathscr{S})\big)$ leads to the conclusion that δ is a weak solution of the main problem.

4. Conclusion

In summary, the existence of weak solutions to a parabolic problem with τ-Laplacian-like operator has been established in this study. Galerkin's approximation method, along with the Young measures theory, was applied to construct approximate solutions and address the complexities introduced by the nonlinearity. These results offer a robust framework for advancing theoretical research. Future work may explore the uniqueness and regularity of solutions, as well as extend the methodology to more complex or higher-dimensional problems.

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