Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 36, pp. 1-13.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.36

CONVERGENCE OF FORMAL POWER SERIES SOLUTIONS FOR SOME REGULAR GEVREY-TYPE DIFFERENTIAL EQUATIONS

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Abstract. In this article we focus on the formal solutions for some regular Gevrey-type differential equation, with convergence of such solutions being the main point of interest. The attained results are a generalization of the classical result on the convergence of formal solutions to differential equations. The technique applied rests upon a classical argument based on a precise writing of the formal solution and a dilatation, in which several technical results on general properties of Gevrey-type sequences and Gevrey-type differential operators are applied.

1. Introduction

The aim of this study is to state conditions, under which formal solutions to certain families of functional equations involving Gevrey-type differential operators are convergent formal power series. In utilizing Gevrey-type differential operators we not only generalize the classical derivative, but also comprise other realizations of great interest in applications such as Jackson's q-derivative. Caputo's fractional derivative is also closely related to operators under study in this paper. We refer to Section 2.2 for the definition of the Gevrey-type differential operator and some of its applications.

Another noteworthy aspect of the present approach is the fact that we have elected to slightly depart from the usual framework in which such operators and differential equations are usually studied. In most cases, moment functions are used instead of Gevrey-type sequences. Those functions are in turn closely connected to the so-called kernel functions and only defined through them. As a consequence, the choice of possible sequences m to be used in the operator defined in Section 2 would be more restrictive. For more details on the approach depending on kernel functions we refer the reader to [2] as well as [19, 20, 15] and references therein. Although that framework is very interesting in its own right, as mentioned previously, we have abandoned it in current study, the reason being its various limitations. Using Gevrey-type sequences instead enables us to cover a broader variety of equations with potentially interesting practical applications, such as q-difference differential equations utilizing previously mentioned q-derivatives.

Let us consider positive integers $p \geq 1$, $n_1, \dots, n_p \geq 1$ and a function F analytic on some neighborhood of $\alpha = (0, \alpha_0, (\alpha_{\ell,1}, \dots, \alpha_{\ell,n_\ell})_{1\leq \ell \leq p}) \in \mathbb{C}^{n_1+\dots+n_p+2}$. We shall study convergence of the formal solutions to the equation

$$F(x, u, (\partial_{m_\ell} u, \dots, \partial_{m_\ell}^{n_\ell} u)_{1\leq \ell\leq p}) = 0, \tag{1.1}$$

Here, $m_1, \dots, m_p$ are assumed to be regular Gevrey-type sequences, and $\partial_{m_1}, \dots, \partial_{m_p}$ denote their respective Gevrey-type differential operators. Under the assumption that the formal solution $\varphi(x) \in \mathbb{C}[[x]]$ to (1.1) exists, and moreover satisfies conditions $\varphi(0) = \alpha_0$ and $\partial_{m_\ell}^q \varphi(0) = \alpha_{\ell,q}$, for all $\ell = 1, \dots, p$ and $q = 1, \dots, n_\ell$ (see Assumption 3.1), we provide conditions on F, under which φ is convergent (see Theorem 3.4).

2020 *Mathematics Subject Classification.* 34A34, 34A25, 34M25.
Key words and phrases. Power series; convergence; formal solution; Gevrey-type differential equation; Gevrey-type sequence.
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Submitted February 21, 2026. Published May 12, 2026.

Rate of growth of the coefficients to formal solutions to functional equations, with particular focus on convergence of such solutions, has recently become a topic of increasing interest among the scientific community. The first result in this direction is due to Maillet [17] and it gives Gevrey upper bounds for the coefficients of the formal solution of (1.1) in the case when $p = 1$ and ∂_{m_1} is reduced to the usual derivative, and where F determines a nonlinear algebraic ordinary differential equation. Indeed, results determining upper estimates on growth of the coefficients of formal solutions to functional equations are nowadays known as Maillet-type results. In 1989, Malgrange [18] provides more accurate results on the Gevrey bound for the rate growth of the coefficients of a formal solution to the situation considered in Maillet's study. His approach is based on the slopes of the Newton polygon associated with the equation. The particularization of that result in the case of convergence of the formal solution on some neighborhood of the origin is recalled in Proposition 3.2. In fact, the technique used in the present work heavily rests upon the one developed in that previous work. A different proof was provided in [10], based on the majorant method, estimating the radius of convergence of the solution. More recently, Gontsov and Goryuchkina study in [9, 11] convergence results of generalized formal solutions to an ordinary differential equation in the form $\sum_{j\geq 0} c_j x^{s_j}$, where $s_j \in \mathbb{C}$, and also in the form of Dulac series [12] and Puiseux series [7]. The q-analogue counterpart of convergence and Maillet-type results has been achieved in works such as [4, 13, 6]; also in [14] when dealing with difference equations. Maillet-type results have also been considered when dealing with the formal solutions to partial differential equations. See [16], and the references therein, and the reference [8] for a deeper insight on the topic.

The technique used to achieve the main result of the present work, Theorem 3.4, rests upon the structure of the main equation (1.1) and a precise calculation of the formal solution. The convergence follows from a dilatation argument. On the way, several technical results regarding general properties of Gevrey-type sequences and Gevrey-type differential operators are used. Such results are displayed in Section 2 alongside main definitions and known facts. In Section 3 the main problem is stated and then illustrated by several examples, in which the formal solution of an equation is guaranteed to be convergent on some neighborhood of the origin. Some technical results are gathered in a final Appendix for the sake of clarity.

2. Gevrey-type sequences

2.1. Definitions and basic properties.

Definition 2.1. A sequence $m = (m(j))_{j\geq 0}$ of positive real numbers is called a *Gevrey-type sequence of order* $s \geq 0$ (in short, an *s-Gevrey-type sequence*) if there exist four positive constants $a, A, c, C > 0$ such that the following estimate holds for all $j \geq 0$:

$$ca^j\Gamma(1 + sj) \leq m(j) \leq CA^j\Gamma(1 + sj).$$

Example 2.2. Some examples of Gevrey-type sequences:

- the sequence $(\Gamma(1 + sj))_{j\geq 0}$ and, more generally, the sequence $(\Gamma(\alpha + sj))_{j\geq 0}$ with $\alpha > 0$ are s-Gevrey-type sequences for any $s \geq 0$;
- the sequence $((kj)!^s)_{j\geq 0}$ with $k \in \mathbb{N}^*$ is a ks-Gevrey-type sequence for any $s \geq 0$ (to check this fact it suffices to apply Stirling's Formula);
- let $q \in]0, 1[$. The sequence $([j]_q!)_{j\geq 0}$ defined by

$$[j]_q! = \begin{cases} 1 & \text{if } j = 0 \\ [j]_q[j-1]_q \dots [1]_q & \text{if } j \geq 1 \end{cases} \quad \text{with } [j]_q = \frac{1 - q^j}{1 - q}$$

  is a 0-Gevrey-type sequence. Indeed, we have direct inequalities

$$1 \leq [j]_q! \leq \frac{1}{(1 - q)^j} \quad \text{for all } j \geq 0.$$

- let $q \in]0, 1[$. The q-analogue sequence $(\Gamma_q(1 + sj))_{j\geq 0}$ of the sequence $(\Gamma(1 + sj))_{j\geq 0}$ defined by

$$\Gamma_q(1 + sj) = \frac{(q; q)_\infty}{(q^{1+sj}; q)_\infty}(1 - q)^{-sj},$$

with
$$(a;q)_\infty = \prod i = 0^\infty (1-aq^i) \quad \text{for any } a \in]0,1[$$
is a 0-Gevrey-type sequence for any $s \geq 0$. We have indeed
$$(q;q)_\infty \leq (q^{1+sj};q)_\infty \leq 1 \quad \text{for all } j \geq 0;$$
hence,
$$(q;q)_\infty (1-q)^{-sj} \leq \Gamma_q(1+sj) \leq (1-q)^{-sj} \quad \text{for all } j \geq 0. \tag{2.1}$$

Remark 2.3.
- An s-Gevrey-type sequence will be simply called a *Gevrey-type sequence* if we want to omit its order s.
- All Gevrey-type sequences cover the classic set of moment sequences defined by Balser in [3] and used in the theory of the summability, but are also much more general, as shown in Example 2.2 above. Indeed, a 0-Gevrey-type sequence is not a moment sequence.

Given an s-Gevrey-type sequence m, it is clear that Definition 2.1 and Stirling's Formula induce estimates of the form
$$c'a'^j(j+1)^s \leq \frac{m(j+1)}{m(j)} \leq C'A'^j(j+1)^s \quad \text{for all } j \geq 0 \tag{2.2}$$
with convenient positive constants $a', A', c', C' > 0$ independent of j. The following definition introduces an important subset of s-Gevrey-type sequences for which we impose strong restrictive conditions in previous estimates (2.2).

Definition 2.4. A sequence $m = (m(j))_{j\geq 0}$ of positive real numbers is called a *regular Gevrey-type sequence of order* $s \geq 0$ (in short, an *s-regular Gevrey-type sequence*) if there exist two positive constants $c, C > 0$ such that the following estimate holds for all $j \geq 0$:
$$c(j+1)^s \leq \frac{m(j+1)}{m(j)} \leq C(j+1)^s.$$

Example 2.5.
- The sequence $((kj)!^s)_{j\geq 0}$ with $k \in \mathbb{N}^*$ is a ks-regular Gevrey-type sequence for any $s \geq 0$.
- Since Stirling's Formula implies the equivalence
$$\frac{\Gamma(\alpha + s(j+1))}{\Gamma(\alpha + sj)} \underset{j\to+\infty}{\sim} s^s(j+1)^s, \quad s > 0$$
the sequence $(k^j\Gamma(\alpha + sj))_{j\geq 0}$ with $k, \alpha > 0$ is an s-regular Gevrey-type sequence for any $s \geq 0$.
- The sequence $([j]_q!)_{j\geq 0}$ with $q \in]0,1[$ is a 0-regular Gevrey-type sequence. We have indeed
$$1 \leq \frac{[j+1]_q!}{[j]_q!} = [j+1]_q \leq \frac{1}{1-q} \quad \text{for all } j \geq 0.$$
- The sequence $(\Gamma_q(1+sj))_{j\geq 0}$ with $q \in]0,1[$ is a 0-regular Gevrey-type sequence for any $s \geq 0$. Indeed, due to inequalities (2.1), we have
$$(q;q)_\infty(1-q)^{-s} \leq \frac{\Gamma_q(1+s(j+1))}{\Gamma_q(1+sj)} \leq \frac{(1-q)^{-s}}{(q;q)_\infty} \quad \text{for all } j \geq 0.$$

Remark 2.6. It is clear that an s-regular Gevrey-type sequence is an s-Gevrey-type sequence. The converse is, of course, not true. For instance, let us consider the sequence $m = (m(j))_{j\geq 0}$ with
$$m(j) = \lfloor \tfrac{j}{2} \rfloor ! = \begin{cases} \Gamma\left(1 + \frac{j}{2}\right) & \text{if } j \text{ even} \\ \Gamma\left(\frac{1}{2} + \frac{j}{2}\right) & \text{if } j \text{ odd} \end{cases}.$$
From Stirling's Formula, it is clear that m is a 1/2-Gevrey-type sequence. But, it is not a regular Gevrey-type sequence since we have the identity
$$\frac{m(2j+1)}{m(2j)} = 1$$

for any $j \geq 0$, which is contradictory with the fact that this quotient should tend towards $+\infty$ when j goes to infinity.

The following lemma introduces a regular Gevrey-type sequence which will play a fundamental role in remaining parts of the paper.

Lemma 2.7. *Let $m = (m(j))_{j\geq 0}$ be an s-regular Gevrey-type sequence. Then, for any $k \geq 0$, the sequence $\widetilde{m}_k$ defined by*

$$\widetilde{m}_k(j) = m(j+k) \quad \text{for all } j \geq 0$$

is also an s-regular Gevrey-type sequence.

Proof. Let us consider two positive constants $c, C > 0$ such that the following estimate holds for all $j \geq 0$,

$$c(j+1)^s \leq \frac{m(j+1)}{m(j)} \leq C(j+1)^s.$$

Then, for all $j \geq 0$, we obtain

$$c(j+k+1)^s \leq \frac{\widetilde{m}_k(j+1)}{\widetilde{m}_k(j)} \leq C(j+k+1)^s;$$

hence,

$$c(j+1)^s \leq \frac{\widetilde{m}_k(j+1)}{\widetilde{m}_k(j)} \leq C'(j+1)^s$$

with $C' = C(k+1)^s$, which completes the proof. □

Observe the previous result also holds when substituting Gevrey-type sequences by Gevrey sequences.

2.2. Gevrey-type differential operator.

Definition 2.8. Let m be a Gevrey-type sequence. We call the linear operator $\partial_m : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ defined by

$$\partial_m\Big(\sum_{j\geq 0} u_j x^j\Big) = \sum_{j\geq 0} \frac{m(j+1)}{m(j)} u_{j+1} x^j$$

a *Gevrey-type differential operator associated with m.*

Example 2.9. When the sequence m is defined by $m(j) = j!$, then the operator ∂_m coincides with the standard derivation operator $\partial = d/dx$ with respect to x. More generally, if $m(j) = (pj)! = \Gamma(1+pj)$ for some positive integer $p \geq 2$, then ∂_m is related to the p-th derivation operator ∂^p with respect to x by the relation

$$\partial^p\Big(\sum_{j\geq 0} u_j x^{jp}\Big) = \sum_{j\geq 0} \frac{\Gamma(1+p(j+1))}{\Gamma(1+pj)} u_{j+1} x^{jp} = \partial_m\Big(\sum_{j\geq 0} u_j x^j\Big)(x^p).$$

This can be generalized to the choice $m(j) = \Gamma(1+sj)$ for some non-integer $s > 0$, by considering the Caputo fractional derivation operator ∂^s of order s with respect to x introduced by Caputo in his 1967 paper [5].

Example 2.10. When the sequence m is defined by $m(j) = [j]_q!$ with $q \in]0,1[$, then the operator ∂_m coincides with the q-difference operator D_q with respect to x,

$$D_q u(x) = \frac{u(qx) - u(x)}{(q-1)x}.$$

Indeed, since

$$D_q(x^j) = \begin{cases} 0 & \text{if } j = 0 \\ [j]_q x^{j-1} & \text{if } j \geq 1 \end{cases}$$

we have

$$D_q\Big(\sum_{j\geq 0} u_j x^j\Big) = \sum_{j\geq 0} [j+1]_q u_{j+1} x^j = \sum_{j\geq 0} \frac{[j+1]_q!}{[j]_q!} u_{j+1} x^j = \partial_m\Big(\sum_{j\geq 0} u_j x^j\Big).$$

As for the standard derivation operator, the operator ∂_m can be related to the Caputo fractional q-difference operator D_q^s of non-integer order $s > 0$ [1, Section 5.2] by choosing the q-analogue $m(j) = \Gamma_q(1+sj)$ of $\Gamma(1+sj)$.

Definition 2.8 can be naturally extended to functions analytic at the origin of $\mathbb{C}$ by means of their representation in the form of an infinite series. Moreover, from inequalities (2.2), one can easily prove the following result.

Lemma 2.11. *Let m be a Gevrey-type sequence and u a function analytic at the origin of $\mathbb{C}$. Then, $\partial_m u$ also defines a function analytic at the origin of $\mathbb{C}$.*

The following proposition presents an important property for Gevrey-type differential operators.

Proposition 2.12. *Let m be a Gevrey-type sequence. Then, for any formal series*

$$u(x) = \sum_{j\geq 0} u_j x^j \in \mathbb{C}[[x]]$$

and any positive integers $k \geq q \geq 1$, the following identity holds:

$$\partial_m^q(x^k u)(x) = x^k \partial_{\widetilde{m}_k}^q u(x) + \sum_{j=k-q}^{k-1} \frac{m(j+q)}{m(j)} u_{j-k+q} x^j \tag{2.3}$$

with $\widetilde{m}_k$ being the Gevrey-type sequence defined by $\widetilde{m}_k(j) = m(j+k)$ (see Lemma 2.7). In particular, if $u(x) \in x^q\mathbb{C}[[x]]$, then $\partial_m^q(x^k u)(x) = x^k \partial_{\widetilde{m}_k}^q u(x)$.

Proof. It is sufficient to prove identity (2.3). Let $k \geq q \geq 1$. Since

$$x^k u(x) = \sum_{j\geq 0} v_j x^j \quad \text{with } v_j = \begin{cases} 0 & \text{if } j < k \\ u_{j-k} & \text{if } j \geq k \end{cases},$$

we first obtain

$$\partial_m^q(x^k u)(x) = \sum_{j\geq 0} \frac{m(j+q)}{m(j)} v_{j+q} x^j = \sum_{j\geq k-q} \frac{m(j+q)}{m(j)} u_{j-k+q} x^j.$$

On the other hand,

$$x^k \partial_{\widetilde{m}_k}^q u(x) = x^k \sum_{j\geq 0} \frac{\widetilde{m}_k(j+q)}{\widetilde{m}_k(j)} u_{j+q} x^j = \sum_{j\geq k} \frac{m(j+q)}{m(j)} u_{j-k+q} x^j,$$

and identity (2.3) follows, which completes the proof. □

3. Regular Gevrey-type differential equations

In this section, we are interested in equations of the form

$$F(x, u, (\partial_{m_\ell} u, \dots, \partial_{m_\ell}^{n_\ell} u)_{1\leq \ell \leq p}) = 0, \tag{3.1}$$

where the following conditions are met:

- $p \geq 1$ and $n_1, \dots, n_p \geq 1$ are positive integers;
- $F(x, y_0, (y_{\ell,1}, \dots, y_{\ell,n_\ell})_{1\leq \ell\leq p}) \not\equiv 0$ is an analytic function in $n_1 + \dots + n_p + 2$ variables $(x, y_0, (y_{\ell,1}, \dots, y_{\ell,n_\ell})_{1\leq\ell\leq p})$ in a neighborhood of a point $\alpha = (0, \alpha_0, (\alpha_{\ell,1}, \dots, \alpha_{\ell,n_\ell})_{1\leq\ell\leq p}) \in \mathbb{C}^{n_1+\dots+n_p+2}$;
- $m_1, \dots, m_p$ are regular Gevrey-type sequences of respective order $s_1, \dots, s_p \geq 0$ satisfying $s_p > 0$ and $s_p n_p \geq s_\ell n_\ell$ for all $\ell = 1, \dots, p$.

Observe that in the case where all the sequences m_ℓ are defined as the sequence $(j!)_{j\geq 0}$, then (3.1) is reduced to a nonlinear differential equation

$$F(x, u, \partial u, \dots, \partial^n u) = 0, \tag{3.2}$$

with a convenient positive integer $n \geq 1$ (see Example 2.9).

Assumption 3.1. Assume that (3.1) admits a formal solution $\varphi(x) \in \mathbb{C}[[x]]$ satisfying $\varphi(0) = \alpha_0$ and $\partial_{m_\ell}^q \varphi(0) = \alpha_{\ell,q}$ for all $\ell = 1, \dots, p$ and all $q = 1, \dots, n_\ell$.

Under this assumption, we address the question

Under what conditions is the formal solution φ convergent?

In the particular case of (3.2), the response to this question was given by Malgrange in [18] by considering the Newton polygon of the linearized differential operator of F along φ.

Proposition 3.2 ([18, Théorème 1.2])**.** *Assume that φ is a non-singular solution of* (3.2)*, that is*

$$\frac{dF}{dy_n}(x,\varphi,\partial\varphi,\dots,\partial^n\varphi)\not\equiv 0.$$

Let L_φ be the linearized differential operator of F along φ defined as

$$L_\varphi=\sum_{q=0}^{n}\frac{dF}{dy_q}(x,\varphi,\partial\varphi,\dots,\partial^n\varphi)\partial^q.$$

Assume that the Newton polygon of L_φ has no positive slope. Then φ is convergent.

The aim of this section is to extend this result to the general Gevrey-type differential equation (3.1). Let us first extend the classical notion of Newton polygon of a linear differential operator [21, 22] to linear Gevrey-type differential operators.

3.1. **Newton polygon of a linear Gevrey-type differential operator.** For any $(a,b)\in\mathbb{R}^2$, we denote by $C(a,b)$ the domain

$$C(a,b)=\{(x,y)\in\mathbb{R}^2 \text{ such that } x\le a \text{ and } y\ge b\}.$$

Let us consider the linear Gevrey-type differential operator

$$L=a_0(x)+\sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell}a_{\ell,q}(x)\partial_{m_\ell}^q$$

with $a_0(x)\in\mathbb{C}[[x]]$, $a_{\ell,q}(x)\in\mathbb{C}[[x]]$ for all $\ell=1,\dots,p$ and all $q=1,\dots,n_\ell$, and $a_{\ell,n_\ell}(x)\not\equiv 0$ for all $\ell=1,\dots,p$.

Definition 3.3. We define the *Newton polygon of L* as the convex hull $\mathcal{N}_L$ of

$$\bigcup_{\ell=1}^{p}\bigcup_{q=1,\,a_{\ell,q}\not\equiv 0}^{n_\ell}C(s_\ell q,v(a_{\ell,q})-q)\quad\text{if } a_0(x)\equiv 0$$

$$C(0,v(a_0))\bigcup\bigcup_{\ell=1}^{p}\bigcup_{q=1,\,a_{\ell,q}\not\equiv 0}^{n_\ell}C(s_\ell q,v(a_{\ell,q})-q)\quad\text{if } a_0(x)\not\equiv 0$$

where $v(a)$ stands for the valuation at 0 of any nonzero formal series $a(x)\in\mathbb{C}[[x]]$.

3.2. **Main result.** We are now able to state the main result of this paper.

Theorem 3.4. *Let $L_{m_1,\dots,m_p,\varphi}$ be the linearized Gevrey-type differential operator of F along φ defined as*

$$L_{m_1,\dots,m_p,\varphi}=\frac{dF}{dy_0}(x,\varphi,(\partial_{m_\ell}\varphi,\dots,\partial_{m_\ell}^{n_\ell}\varphi)_{1\le\ell\le p})+\sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell}\frac{dF}{dy_{\ell,q}}(x,\varphi,(\partial_{m_\ell}\varphi,\dots,\partial_{m_\ell}^{n_\ell}\varphi)_{1\le\ell\le p})\partial_{m_\ell}^q.$$

Let us denote by v_0 (resp. $v_{\ell,q}$ for all $\ell=1,\dots,p$ and all $q=1,\dots,n_\ell$) the valuation at $x=0$ of the formal power series

$$\frac{dF}{dy_0}(x,\varphi,(\partial_{m_\ell}\varphi,\dots,\partial_{m_\ell}^{n_\ell}\varphi)_{1\le\ell\le p})\quad\text{(resp. }\frac{dF}{dy_{\ell,q}}(x,\varphi,(\partial_{m_\ell}\varphi,\dots,\partial_{m_\ell}^{n_\ell}\varphi)_{1\le\ell\le p})\text{)}.$$

Assume that the following three conditions hold:

- (C1) $\frac{dF}{dy_{p,n_p}}(x,\varphi,(\partial_{m_\ell}\varphi,\dots,\partial_{m_\ell}^{n_\ell}\varphi)_{1\le\ell\le p})\not\equiv 0$;
- (C2) $\mathcal{N}_{L_{m_1,\dots,m_p,\varphi}}=C(s_pn_p,v_{p,n_p}-n_p)$, *that is the Newton polygon of $L_{m_1,\dots,m_p,\varphi}$, has no positive slope;*
- (C3) $v_{p,n_p}-n_p<v_{\ell,n_\ell}-n_\ell$ *for any $\ell=1,\dots,p-1$ such that $s_pn_p=s_\ell n_\ell$ (if any exists).*

Then φ is convergent.

Remark 3.5. When $p=1$ and ∂_{m_1} is reduced to the classical derivation operator $\partial = d/dx$ with respect to x, Theorem 3.4 coincides with the result proved by Malgrange in [18] for nonlinear differential equations (see Proposition 3.2).

Before we move on to the proof of Theorem 3.4, let us give two examples illustrating our result. In the first one we study the formal solutions of some equations, and in the second one we focus our attention on a classical problem of functional analysis with the study of some generating functions.

Example 3.6 (Some equations)**.** (1) Let us consider the linear q-difference-differential equation

$$\begin{gathered}(x^2+1)\partial^2 u + 2xD_q u - (3x+1)u = f(x)\\ u(0)=1, \quad \partial u(0) = 1\end{gathered} \tag{3.3}$$

with $q \in]0,1[$ and $f(x) \in \mathbb{C}\{x\}$. If we write the inhomogeneity $f(x)$ in the form $f(x) = \sum_{j\geq 0} f_j x^j$, looking for a formal solution $\varphi(x) \in \mathbb{C}[[x]]$ of (3.3) of the form

$$\varphi(x) = \sum_{j\geq 0} \varphi_j x^j$$

yields the recurrence relation

$$(j+2)(j+1)\varphi_{j+2} = f_j - (j(j-1) - 2[j]_q - 1)\varphi_j + 3\varphi_{j-1}$$

for all $j \geq 0$ (with the classical convention that $\varphi_{-1} = 0$) together with the initial condition $\varphi_0 = \varphi_1 = 1$. Consequently, (3.3) admits a unique formal power series solution. Since the Newton polygon of its associated linear operator is $C(2,-2)$, that admits no positive slope, we deduce from Theorem 3.4 that the formal solution $\varphi(x)$ is convergent (conditions (C1) and (C3) are straightforward).

(2) Let us now consider the nonlinear Gevrey-type differential equation

$$\begin{gathered}\partial_m^2 u - a(x)(\partial_m u)^2 = f(x)\\ u(0)=\alpha, \quad \partial_m u(0) = \beta\end{gathered} \tag{3.4}$$

with m being a regular Gevrey-type sequence of order $s>0$, $a(x), f(x) \in \mathbb{C}\{x\}$ and $\alpha, \beta \in \mathbb{C}$. After writing these coefficients in the same form as above and looking for a formal solution $\varphi(x) \in \mathbb{C}[[x]]$ on the same type, we obtain the recurrence relation

$$\frac{m(j+2)}{m(j)}\varphi_{j+2} = f_j + \sum_{j_0+j_1+j_2=j} \frac{m(j_1+1)}{m(j_1)}\frac{m(j_2+1)}{m(j_2)} a_{j_0}\varphi_{j_1+1}\varphi_{j_2+1}$$

for all $j\geq 0$ together with the initial conditions $\varphi_0 = \alpha$ and $\varphi_1 = \frac{m(0)\beta}{m(1)}$. Consequently, Eq. (3.4) admits a unique formal power series solution. Now, let us introduce the function F associated with (3.4),

$$F(x,y_0,y_1,y_2) = y_2 - a(x)y_1^2 - f(x).$$

We easily check that condition (C1) of Theorem 3.4 is satisfied. On the other hand, the linearized Gevrey-type differential operator of F along φ is defined as

$$L_{m,\varphi} = \partial_m^2 - 2a(x)\partial_m\varphi\partial_m.$$

Since its Newton polygon is reduced to $C(2s,-2)$; hence, has no positive slope, we deduce that condition (C2) of Theorem 3.4 is satisfied. The last condition (C3) being also satisfied, Theorem 3.4 applies and implies that the formal solution $\varphi(x)$ of Eq. (3.4) is convergent.

Example 3.7 (Generating function)**.** Let us choose a positive real number $s>0$, and let us consider a sequence $(\varphi_j)_{j\geq 0}$ satisfying the recurrence relation

$$(j+1)^{2s}\varphi_{j+1} = 1 + \sum_{j_1+j_2+j_3=j}\Big(\prod_{h=1}^{3}((j_h+1)(j_h+2))^s\varphi_{j_h}\Big)$$

for all $j \geq 0$. It is clear that the generating function

$$\varphi(x) = \sum_{j\geq 0} \varphi_j x^j \in \mathbb{C}[[x]]$$

of $(\varphi_j)_{j\geq 0}$ is a formal solution of the regular Gevrey-type differential equation

$$\begin{gathered} \partial_{m_2} u = \frac{1}{1-x} + \left(\partial_{m_1}^2 (x^2 y)\right)^3 \\ u(0) = \varphi_0 \end{gathered} \tag{3.5}$$

where m_1 and m_2 are respectively defined by $m_1(j) = j!^s$ and $m_2(j) = j!^{2s}$ for all $j \geq 0$. If we then apply Proposition 2.12, equation (3.5) can be rewritten in the form

$$\begin{gathered} F(x, u, \partial_{\widetilde{m}_{1;2}} u, \partial_{\widetilde{m}_{1;2}}^2 u, \partial_{m_2} u) = 0 \\ u(0) = \varphi_0 \end{gathered}$$

with

$$F(x, y_0, y_{1,1}, y_{1,2}, y_{2,1}) = y_{2,1} - \Big(x^2 y_{1,2} + \frac{m(2)}{m(0)}\varphi_0 + \frac{m(3)}{m(1)}\varphi_1 x\Big)^3 - \frac{1}{1-x}$$

and $\widetilde{m}_{1;2}(j) = m_1(j+2)$, for all $j \geq 0$. It is clear that condition (C1) of Theorem 3.4 is satisfied. Now, the linearized Gevrey-type differential operator of F along φ is defined as

$$L_{\widetilde{m}_{1;2}, m_2, \varphi} = \partial_{m_2} - 3x^2 \Big(x^2 \partial_{\widetilde{m}_{1;2}}^2 \varphi + \frac{m(2)}{m(0)}\varphi_0 + \frac{m(3)}{m(1)}\varphi_1 x\Big)^2 \partial_{\widetilde{m}_{1;2}}^2.$$

Its Newton polygon being $C(2s, -1)$, it has no positive slope and condition (C2) of Theorem 3.4 is satisfied. As for condition (C3), it is also satisfied since the valuation of the coefficient of ∂_{m_2} is 0, and the one of the coefficient of $\partial_{\widetilde{m}_{1;2}}^2$ is at least 2. Consequently, Theorem 3.4 apply and implies the convergence of $\varphi(x)$.

Let us now turn to the proof of our main result.

Proof of Theorem 3.4. From conditions (C1) and (C2), we have $0 \leq v_{p,n_p} < +\infty$, $v_{p,n_p} - n_p \leq v_0$ and $v_{p,n_p} - n_p \leq v_{\ell,q} - q$ for all $\ell = 1, \dots, p$ and all $q = 0, \dots, n_\ell$. Hence, we can consider the homogeneous part $a_0 x^{v_{p,n_p} - n_p}$ of degree $v_{p,n_p} - n_p$ of

$$\frac{dF}{dy_0}(x, \varphi, (\partial_{m_\ell}\varphi, \dots, \partial_{m_\ell}^{n_\ell}\varphi)_{1\leq \ell\leq p}),$$

and the homogeneous parts $a_{\ell,q} x^{v_{p,n_p} - n_p + q}$ of degree $v_{p,n_p} - n_p + q$ of

$$\frac{dF}{dy_{\ell,q}}(x, \varphi, (\partial_{m_\ell}\varphi, \dots, \partial_{m_\ell}^{n_\ell}\varphi)_{1\leq \ell\leq p})$$

for all $\ell = 1, \dots, p$ and all $q = 1, \dots, n_\ell$. Obviously, we have $a_{p,n_p} \neq 0$ and $a_0 = 0$ (resp. $a_{\ell,q} = 0$) as soon as $v_{p,n_p} - n_p < 0$ (resp. $v_{p,n_p} - n_p + q < 0$) or as soon as its corresponding formal power series is zero. Moreover, condition (C3) implies $a_{\ell,n_\ell} = 0$ for any $\ell = 1, \dots, p-1$ such that $s_p n_p = s_\ell n_\ell$ if any exists.

Now, for any $k \geq n$ with $n = \max(n_1, \dots, n_p)$, let us write $\varphi(x)$ in the form $\varphi(x) = \varphi_k(x) + x^k \psi_k(x)$ with $\psi_k(x) = O(x^n)$. Then Proposition 2.12 implies

$$\partial_{m_\ell}^q \varphi = \partial_{m_\ell}^q \varphi_k + \partial_{m_\ell}^q (x^k \psi_k) = \partial_{m_\ell}^q \varphi_k + x^k \partial_{\widetilde{m}_{\ell;k}}^q \psi_k$$

for all $\ell = 1,\dots,p$ and all $q = 1,\dots,n_\ell$. Applying next the Taylor expansion with integral remainder of order 2 to $F(x,\varphi,(\partial_{m_\ell}\varphi,\dots,\partial_{m_\ell}^{n_\ell}\varphi)_{1\le\ell\le p})$, we obtain

$$\begin{aligned}
&F(x,\varphi_k,(\partial_{m_\ell}\varphi_k,\dots,\partial_{m_\ell}^{n_\ell}\varphi_k)_{1\le\ell\le p}) + x^k\Big(\frac{dF}{dy_0}(x,\varphi_k,(\partial_{m_\ell}\varphi_k,\dots,\partial_{m_\ell}^{n_\ell}\varphi_k)_{1\le\ell\le p})\psi_k\\
&+\sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell}\frac{dF}{dy_{\ell,q}}(x,\varphi_k,(\partial_{m_\ell}\varphi_k,\dots,\partial_{m_\ell}^{n_\ell}\varphi_k)_{1\le\ell\le p})\partial_{\widetilde{m}_{\ell;k}}^{q}\psi_k\Big)\\
&+x^{2k}\sum_{\ell,\ell'=1}^{p}\sum_{0\le q\le n_\ell,\,0\le q'\le n_{\ell'}}G_{\ell,\ell',q,q'}(\bullet)(\partial_{\widetilde{m}_{\ell;k}}^{q}\psi_k)(\partial_{\widetilde{m}_{\ell',k}}^{q'}\psi_k)=0
\end{aligned}\tag{3.6}$$

with some analytic functions $G_{\ell,\ell',q,q'}(\bullet)$ at the origin of $\mathbb{C}^{2(n_1+\dots+n_p)+3}$ with

$$\bullet = (x,\varphi_k,(\partial_{m_\ell}\varphi_k,\dots,\partial_{m_\ell}^{n_\ell}\varphi_k)_{1\le\ell\le p},x^k\psi_k,(x^k\partial_{\widetilde{m}_{\ell;k}}\psi_k,\dots,x^k\partial_{\widetilde{m}_{\ell;k}}^{n_\ell}\psi_k)_{1\le\ell\le p}).$$

Let us choose three positive integers $k_0,k_1,k_2>0$ such that

(i) $a_0+\sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell}a_{\ell,q}\frac{m_\ell(k')}{m_\ell(k'-q)}\ne 0$ for all $k'\ge k_0$ (the existence of such k_0 is guaranteed by Lemma 4.1 since in the case where $s_pn_p=s_\ell n_\ell$ for some $\ell=1,\dots,p-1$ we have $a_{\ell,n_\ell}=0$ (see remark just above) and the corresponding sum is then summed up to at most $n_\ell-1$ with $s_\ell(n_\ell-1)<s_pn_p$);

(ii) $\frac{dF}{dy_0}(x,\varphi_{k'},(\partial_{m_\ell}\varphi_{k'},\dots,\partial_{m_\ell}^{n_\ell}\varphi_{k'})_{1\le\ell\le p})=O(x^{v_{p,n_p}-n_p})$ for all $k'\ge k_1$.

(iii) $\frac{dF}{dy_{\ell,q}}(x,\varphi_{k'},(\partial_{m_\ell}\varphi_{k'},\dots,\partial_{m_\ell}^{n_\ell}\varphi_{k'})_{1\le\ell\le p}) = O\ (x^{v_{p,n_p}-n_p+q})$ for all $\ell=1,\dots,p$, all $q=1,\dots,n_\ell$ and all $k'\ge k_2$.

Then, for $k=\max(k_0,k_1,k_2,3n-n_p+v_{p,n_p})$, relation (3.6), conditions (ii) and (iii), and identities $\partial_{\widetilde{m}_{\ell,k}}^{q}\psi_k(x)=O(x^{n-q})$ for all $\ell=1,\dots,p$ and all $q=1,\dots,n_\ell$ imply

$$F(x,\varphi_k,(\partial_{m_\ell}\varphi_k,\dots,\partial_{m_\ell}^{n_\ell}\varphi_k)_{1\le\ell\le p})=O(x^{k+v_{p,n_p}-n_p+n}).$$

Let us now observe, on one hand, that the terms $x^k\partial_{\widetilde{m}_{\ell;k}}^{q''}$ appearing in the functions $G_{\ell,\ell',q,q'}$ can be written in the form $x^{k-q''}\times x^{q''}\partial_{\widetilde{m}_{\ell;k}}^{q''}$ with $k-q''\ge 0$ for all $\ell=1,\dots,p$ and all $q''=1,\dots,n_\ell$. On the other hand, the new term $x^{k+n_p-v_{p,n_p}}$ appearing in the place of x^{2k} can be written in the form $x^{k-q-q'+n_p-v_{p,n_p}}\times x^q\times x^{q'}$ with $k-q-q'+n_p-v_{p,n_p}\ge n$ for all $\ell,\ell'=1,\dots,p$, all $q=1,\dots,n_\ell$ and all $q'=1,\dots,n_{\ell'}$. Consequently, dividing by $x^{k+v_{p,n_p}-n_p}$, using the definition of the terms a_0 and $a_{\ell,q}$ for $\ell=1,\dots,p$ and $q=1,\dots,n_\ell$, we receive that ψ_k is a formal solution of an equation of the form

$$\begin{aligned}
&a_0\psi+\sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell}a_{\ell,q}\delta_{\widetilde{m}_{\ell;k}}^{(q)}\psi+xH_{0,0}(x)\psi+x\sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell}H_{\ell,q}(x)\delta_{\widetilde{m}_{\ell;k}}^{(q)}\psi\\
&+x^nH(x,\psi,(\delta_{\widetilde{m}_{\ell,k}}^{(1)}\psi,\dots,\delta_{\widetilde{m}_{\ell;k}}^{(n_\ell)}\psi)_{1\le\ell\le p})=0
\end{aligned}\tag{3.7}$$

with $n_1+\dots+n_p+1$ convenient functions $H_{\ell,q}$ analytic at the origin of $\mathbb{C}$, a convenient function H analytic at the origin of $\mathbb{C}^{n_1+\dots+n_p+2}$ and with $\delta_{\widetilde{m}_{\ell;k}}^{(q)}=x^q\partial_{\widetilde{m}_{\ell;k}}^{q}$ for all $\ell=1,\dots,p$ and all $q=1,\dots,n_\ell$. More precisely, ψ_k is the unique formal power series solution of this equation in $x^n\mathbb{C}[[x]]$. Indeed, looking for a formal solution of (3.7) in the form $\sum_{j\ge n}u_jx^j$, we obtain recurrence relations

$$\Big(a_0+\sum_{\ell=1}^{p}\sum_{q=0}^{n_\ell}a_{\ell,q}\frac{m_\ell(j+k)}{m_\ell(j+k-q)}\Big)u_j+\operatorname{rem}_j(u_{j-1},\dots,u_n)=v_j$$

for all $j\ge n$, where the term $\operatorname{rem}_j(u_{j-1},\dots,u_n)$ only contains terms u_k with $k<j$, and where v_j is a convenient constant entirely determined by H. We conclude that ψ_k is unique by condition (i) above.

To prove that ψ_k is convergent, we use the classical method of dilatation as follows. For all $\lambda \in \mathbb{C}$, let us set $\psi_{(\lambda)}(x) = \psi(\lambda x)$. Since

$$(\delta^{(q)}_{\widetilde{m}_{\ell;k}}\psi)(\lambda x) = \sum_{j\geq n} \frac{m_\ell(j+k)}{m_\ell(j+k-q)}\psi_j(\lambda x)^j = (\delta^{(q)}_{\widetilde{m}_{\ell;k}}\psi_{(\lambda)})(x)$$

for all $\ell = 1,\dots,p$ and all $q = 1,\dots,n_\ell$, we consider, instead of (3.7), the equation

$$\begin{aligned} &a_0\psi_{(\lambda)} + \sum_{\ell=1}^{p}\sum_{q=0}^{n_\ell} a_{\ell,q}\delta^{(q)}_{\widetilde{m}_{\ell;k}}\psi_{(\lambda)} + \lambda x H_{0,0}(\lambda x)\psi_{(\lambda)} + \lambda x \sum_{\ell=1}^{p}\sum_{q=0}^{n_\ell} H_{\ell,q}(\lambda x)\delta^{(q)}_{\widetilde{m}_{\ell;k}}\psi_{(\lambda)} \\ &\quad + \lambda^n x^n H(\lambda x, \psi_{(\lambda)}, (\delta^{(1)}_{\widetilde{m}_{\ell;k}}\psi_{(\lambda)},\dots,\delta^{(n_\ell)}_{\widetilde{m}_{\ell;k}}\psi_{(\lambda)})_{1\leq\ell\leq p}) = 0 \end{aligned} \tag{3.8}$$

for sufficiently small λ. Let us also consider the application

$$\begin{aligned} M(\lambda,\psi) &= a_0\psi + \sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell} a_{\ell,q}\delta^{(q)}_{\widetilde{m}_{\ell;k}}\psi + \lambda x H_{0,0}(\lambda x)\psi \\ &\quad + \lambda x \sum_{\ell=1}^{p}\sum_{q=0}^{n_\ell} H_{\ell,q}(\lambda x)\delta^{(q)}_{\widetilde{m}_{\ell;k}}\psi + \lambda^n x^n H(\lambda x, \psi, (\delta^{(1)}_{\widetilde{m}_{\ell;k}}\psi,\dots,\delta^{(n_\ell)}_{\widetilde{m}_{\ell;k}}\psi)_{1\leq\ell\leq p}), \end{aligned}$$

as well as the Banach spaces $H^r_{n,k}$ (see Section 4.2) defined for any $r \in \mathbb{R}^+$ by

$$H^r_{n,k} = \Big\{\sum_{j\geq n} f_j x^j \text{ such that } \sum_{j\geq n}(j+k)^r|f_j| < +\infty\Big\}.$$

According to Lemmas 4.4 and 4.5, it is clear that M is well defined and $\mathcal{C}^\infty$ in a neighborhood of $(0,0) \in \mathbb{C}\times H^{s_p n_p}_{n,k}$ with values in $H^0_{n,k}$, and satisfies $M(0,0) = 0$. Moreover, its partial derivative in direction $H^{s_p n_p}_{n,k}$ at $(0,0)$ is the linear map

$$\frac{dM}{d\psi}(0,0) = a_0 + \sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell} a_{\ell,q}\delta^{(q)}_{\widetilde{m}_{\ell,k}} : H^{s_p n_p}_{n,k} \to H^0_{n,k}.$$

Since for any $\sum_{j\geq n} g_j x^j \in H^0_{n,k}$ the equation

$$\Big(a_0 + \sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell} a_{\ell,q}\delta^{(q)}_{\widetilde{m}_{\ell,k}}\Big)\Big(\sum_{j\geq n} f_j x^j\Big) = \sum_{j\geq n} g_j x^j$$

implies the identities

$$\Big(a_0 + \sum_{\ell=1}^{p}\sum_{q=1}^{n_\ell} a_{\ell,q}\frac{m_\ell(j+k)}{m_\ell(j+k-q)}\Big) f_j = g_j$$

for all $j \geq n$, we deduce from condition (i) above that the application $dM/d\psi(0,0)$ is invertible. Consequently, the Implicit Function Theorem applies and implies that for all small enough $\lambda \in \mathbb{C}$, there exists a unique $\mathcal{C}^\infty$ function $\psi_{(\lambda)} \in H^{s_p n_p}_{n,k}$ such that $M(\lambda, \psi_{(\lambda)}) = 0$. We conclude thanks to the unicity of the formal solution ψ_k. □

4. Appendix: technical results

4.1. **A fundamental inequality.**

Lemma 4.1. *Let $p \geq 1$ and $\mu_1,\dots,\mu_p \geq 1$ be positive integers, and suppose that $m_1,\dots,m_p$ are regular Gevrey-type sequences of orders $s_1,\dots,s_p \geq 0$, respectively. Let $a_0 \in \mathbb{C}$ and $a_{\ell,1},\dots,a_{\ell,\mu_\ell} \in \mathbb{C}$ with $a_{\ell,\mu_\ell} \neq 0$ for all $\ell = 1,\dots,p$. Assume that $s_p > 0$ and $s_p\mu_p > s_\ell\mu_\ell$ for all $\ell = 1,\dots,p-1$. Then there exists a nonnegative integer $j_0 \geq \max(\mu_1,\dots,\mu_p)$ such that*

$$a_0 + \sum_{\ell=1}^{p}\sum_{q=1}^{\mu_\ell} a_{\ell,q}\frac{m_\ell(j)}{m_\ell(j-q)} \neq 0 \quad \textit{for all } j \geq j_0.$$

Proof. From Definition 2.4 it follows that for $\ell = 1, \dots, p$, there exist, positive constants $c_\ell, C_\ell > 0$ such that the following inequality holds for all $j \geq 0$:

$$c_\ell (j+1)^{s_\ell} \leq \frac{m_\ell(j+1)}{m_\ell(j)} \leq C_\ell (j+1)^{s_\ell}.$$

By applying the triangular inequality, we obtain

$$\begin{aligned} &|a_0 + \sum_{\ell=1}^{p} \sum_{q=1}^{\mu_\ell} a_{\ell,q} \frac{m_\ell(j)}{m_\ell(j-q)}| \\ &\geq |a_{p,\mu_p}| \frac{m_p(j)}{m_p(j-\mu_p)} - \sum_{q=1}^{\mu_p - 1} |a_{p,q}| \frac{m_p(j)}{m_p(j-q)} - \sum_{\ell=1}^{p-1} \sum_{q=1}^{\mu_\ell} |a_{\ell,q}| \frac{m_\ell(j)}{m_\ell(j-q)} - |a_0| \end{aligned}$$

for all $j \geq \max(\mu_1, \dots, \mu_p)$. Let us now observe that

$$c_\ell^q (j \dots (j-q+1))^{s_\ell} \leq \frac{m_\ell(j)}{m_\ell(j-q)} \leq C_\ell^q (j \dots (j-q+1))^{s_\ell}$$

for all $\ell = 1, \dots, p$, all $q = 1, \dots, \mu_\ell$ and all $j \geq \max(\mu_1, \dots, \mu_p)$. Consequently,

$$\begin{aligned} &|a_0 + \sum_{\ell=1}^{p} \sum_{q=1}^{\mu_\ell} a_{\ell,q} \frac{m_\ell(j)}{m_\ell(j-q)}| \\ &\geq |a_{p,\mu_p}| c_p^{\mu_p} (j \dots (j-\mu_p+1))^{s_p} - \sum_{q=1}^{\mu_p-1} |a_{p,q}| C_p^q (j \dots (j-q+1))^{s_p} \\ &- \sum_{\ell=1}^{p-1} \sum_{q=1}^{\mu_\ell} |a_{\ell,q}| C_\ell^q (j \dots (j-q+1))^{s_\ell} - |a_0| \end{aligned}$$

for all $j \geq \max(\mu_1, \dots, \mu_p)$, and the result follows since the right hand-side tends to $+\infty$ when $j \to +\infty$ (it is indeed equivalent to $|a_{p,\mu_p}| c_p^{\mu_p} j^{s_p \mu_p}$ at $+\infty$). ☐

Remark 4.2. The assumption "$s_p > 0$" is fundamental in the proof of Lemma 4.1. Without it, the last equivalence fails and we can not reach the conclusion. Moreover, observe that the previous result is not true for classical Gevrey series of some positive orders.

4.2. **Banach spaces $H^r_{n,k}$.** For any integers $k \geq 0$ and $n \geq 1$, and any nonnegative real number $r \in \mathbb{R}^+$, let us set $H^r_{n,k}$ the $\mathbb{C}$-vector space defined by

$$H^r_{n,k} = \Big\{ f = \sum_{j \geq n} f_j x^j \text{ such that } \sum_{j \geq n} (j+k)^r |f_j| < +\infty \Big\}.$$

For any $f = \sum_{j \geq n} f_j x^j \in H^r_{n,k}$, we set $\|f\|_{n,k,r} = \sum_{j \geq n} (j+k)^r |f_j|$.

The proof of the next result can be followed from the classical weighted ℓ^1 spaces, so we omit it.

Lemma 4.3. *Let $k \geq 0$, $n \geq 1$ and $r \in \mathbb{R}^+$. Then*

1. *The map $\| \cdot \|_{n,k,r} : H^r_{n,k} \longrightarrow \mathbb{R}^+$ is a norm on $H^r_{n,k}$.*
2. *The normed vector space $(H^r_{n,k}, \| \cdot \|_{n,k,r})$ is a Banach space.*

Lemma 4.4. *Let $k \geq 0$, $n \geq 1$ and $r \in \mathbb{R}^+$. Let m be a regular Gevrey-type sequence of order $s \geq 0$, and let $q \geq 1$ be a positive integer such that $q \leq n$ and $qs \leq r$. Then $\delta^{(q)}_{\widetilde{m}_k}$ is a $\mathcal{C}^\infty$ linear map from $H^r_{n,k}$ to $H^0_{n,k}$. Moreover, its first differential is constant equal to $\delta^{(q)}_{\widetilde{m}_k}$.*

Proof. It is sufficient to prove that the linear map $\delta^{(q)}_{\widetilde{m}_k}$ is continuous, since any continuous linear map between two Banach spaces is automatically $\mathcal{C}^\infty$ with a constant first differential equal to itself, and the other differentials equal to 0 [23, page 71]. To do that, let us choose $f \in H^r_{n,k}$. By assumption we have

$$\delta^{(q)}_{\widetilde{m}_k} f = \sum_{j \geq n} \frac{m(j+k)}{m(j+k-q)} f_j x^j.$$

Let us now use the fact that m is an s-regular Gevrey-type sequence. From Definition 2.4 follows existence of a positive constant $C > 0$ such that

$$\frac{m(j+1)}{m(j)} \leq C(j+1)^s \quad \text{for all } j \geq 0.$$

Hence,

$$\frac{m(j+k)}{m(j+k-q)} \leq C^q (j+k)^{qs} \leq C^q (j+k)^r \quad \text{for all } j \geq n.$$

Consequently, $\delta^{(q)}_{\widetilde{m}_k} f \in H^0_{n,k}$ and the following inequality concludes the proof,

$$\|\delta^{(q)}_{\widetilde{m}_k} f\|_{n,k,0} = \sum_{j \geq n} \frac{m(j+k)}{m(j+k-q)} |f_j| \leq C^q \sum_{j \geq n} (j+k)^r |f_j| = C^q \|f\|_{n,k,r}.$$

□

Note that he previous result remains valid when considering a Gevrey sequence of order $s \geq 0$.

Lemma 4.5. *Let $k \geq 0$ and $n \geq 1$. Then, the multiplication is a $\mathcal{C}^\infty$ bilinear map from $H^0_{n,k} \times H^0_{n,k}$ to $H^0_{n,k}$.*

Proof. As for the linear map $\delta^{(q)}_{\widetilde{m}_k}$, it is sufficient to prove that the multiplication is continuous [23, page 71]. To do that, let us choose $f, g \in H^0_{n,k}$. Thanks to the Cauchy product and the triangular inequality, it is clear that $fg \in H^0_{n,k}$ with

$$\begin{aligned} \|fg\|_{n,k,0} &= \sum_{j \geq n} \Big| \sum_{h+\ell=j} f_h g_\ell \Big| \\ &\leq \sum_{j \geq n} \Big(\sum_{h+\ell=j} |f_h||g_\ell| \Big) \\ &= \Big(\sum_{j \geq n} |f_j| \Big) \Big(\sum_{j \geq n} |g_j| \Big) \\ &= \|f\|_{n,k,0} \|g\|_{n,k,0}. \end{aligned}$$

This completes the proof. □

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