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Abstract. In this article, we study a fractional Laplace equation on a compact Riemannian manifold involving a Hardy potential and a nonlinearity with critical exponent,

$$(-\Delta_g)^s u - \mu \frac{u}{d_g(x,x_0)^{2s}} = \lambda f(x)|u|^{p-2}u + k(x)|u|^{2_s^*-2}u \quad \text{in } M,$$

where $n > 2s$, $s \in (0,1)$, $2 < p < 2_s^*$, and $2_s^* = \frac{2n}{n-2s}$ denotes the fractional Sobolev critical exponent. Under suitable conditions on the parameters μ, λ and the smooth positive functions f and k, we employ critical point theory to establish the existence of nontrivial solutions.

1. Introduction

Let (M, g) be a compact Riemannian manifold without boundary. We investigate the existence of nontrivial weak solutions to the following fractional critical elliptic equation with a Hardy potential and combined nonlinearities

$$(-\Delta_g)^s u - \mu \frac{u}{d_g(x,x_0)^{2s}} = \lambda f(x)|u|^{p-2}u + k(x)|u|^{2_s^*-2}u \quad \text{in } M, \tag{1.1}$$ 1

where $0 < s < 1$, $2 < p < 2_s^*$, $\lambda, \mu > 0$, and $f(x), k(x) \in C^\infty(M)$ are given positive smooth functions. Here $d_g(x, x_0)$ denotes the Riemannian distance from x to a fixed point $x_0 \in M$, and the fractional Laplace operator $(-\Delta_g)^s$ is defined by

$$(-\Delta_g)^s u(x) := 2 \lim_{\varepsilon \to 0^+} \int_{M \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{d_g(x,y)^{n+2s}}\, dv_g(y),$$

where $B_\varepsilon(x)$ denotes the geodesic ball of radius ε centered at x, and dv_g denotes the volume element induced by the metric g.

In the Euclidean setting, nonlocal operators, especially the fractional Laplacian $(-\Delta)^s$, have garnered significant attention owing to their theoretical profundity and wide-ranging applications in fields such as physics and finance [?, ?, ?, ?, ?, ?]. Key properties of solutions to such equations have been established in [?, ?, ?, ?]. A paradigmatic and challenging example is the operator $(-\Delta)^s - \mu|x|^{-2s}$, which originates from problems related to stability in relativistic matter [?]. The main analytical challenge stems from the interplay between the nonlocality of $(-\Delta)^s$ and the singularity induced by the Hardy-type potential.

To tackle this class of problems, Fall [?] established existence and nonexistence results for semilinear fractional elliptic equations with Hardy potentials, extending the classical work of Brezis and Nirenberg [?]. Dipierro et al. [?] studied the equation

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = u^{2_s^*-1}, \quad \text{in } \mathbb{R}^n, \tag{1.2}$$ 2

showing that every positive solution U_μ satisfies the estimate

$$\frac{c}{\big(|x|^{1-\gamma_\mu}(1+|x|^{2\gamma_\mu})\big)^{\frac{n-2s}{2}}} \le U_\mu(x) \le \frac{C}{\big(|x|^{1-\gamma_\mu}(1+|x|^{2\gamma_\mu})\big)^{\frac{n-2s}{2}}}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

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where $C_1, C_2 > 0$ are constants, $\gamma_\mu = 1 - \frac{2\alpha_\mu}{n-2s}$ with $\alpha_\mu \in \big(0, \frac{n-2s}{2}\big)$, and the critical parameter α_μ is the unique solution to $\varphi_{s,n}(\alpha_\mu) = \mu$. Here $\varphi_{s,n}$ denotes the strictly increasing function defined by

$$\varphi_{s,n}(\alpha_\mu) = 2^{2s} \frac{\Gamma\big(\frac{\alpha_\mu+2s}{2}\big)\Gamma\big(\frac{n-\alpha_\mu}{2}\big)}{\Gamma\big(\frac{n-\alpha_\mu-2s}{2}\big)\Gamma\big(\frac{\alpha_\mu}{2}\big)} = \mu.$$

Subsequent advancements include the work of Shang et al. [?], who studied the Euclidean counterpart of (??),

$$(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = \lambda f(x) u^{p-1} + k(x) u^{2_s^*-1}, \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

in

and proved the existence of positive solutions under appropriate conditions. Ghoussoub and Shakerian [?] considered equations with double critical exponents, analyzing the non-compactness of Palais-Smale sequences to establish nontrivial weak solutions for

$$\begin{gathered}(-\Delta)^s u - \gamma \frac{u}{|x|^{2s}} = |u|^{2_s^*-2} u + \frac{|u|^{2_s^*(\alpha)-2} u}{|x|^\alpha} \quad \text{in } \mathbb{R}^n, \\ u > 0 \quad \text{in } \mathbb{R}^n,\end{gathered}$$

where $0 \le \alpha < s < 1$, $n > 2s$, $2_s^* = \frac{2n}{n-2s}$ is the fractional Sobolev critical exponent, and $2_s^*(\alpha) = \frac{2(n-\alpha)}{n-2s}$ is the weighted fractional Sobolev critical exponent.

Compared to the Euclidean setting, the analysis of nonlocal problems on Riemannian manifolds remains relatively underdeveloped, primarily due to geometric complexities and the intricate behavior of the Riemannian distance function. Most existing results focus on local operators. For example, Maliki and Terki [?] investigated the existence of multiple solutions to the local elliptic equation

$$-\Delta_g u - \frac{h(x)}{d_g(x,x_0)^2} u = f(x)|u|^{2^*-2} u, \quad \text{in } M,$$

where $2^* = \frac{2n}{n-2}$ is the classical Sobolev critical exponent. Moreover, Ghomari and Maliki [?] established a Struwe-type decomposition for Palais-Smale sequences corresponding to the p-Laplacian equation

$$-\Delta_{p,g} u - \frac{h(x)}{d_g(x,x_0)^\alpha} |u|^{p-2} u = f(x)|u|^{p^*-2} u, \quad \text{in } M,$$

where $p^* = \frac{pn}{n-p}$ denotes the p-Sobolev critical exponent and $0 < \alpha \le p$.

Motivated by the aforementioned works, we investigate the existence of nontrivial weak solutions to (??) on Riemannian manifolds without boundary. To derive our existence results, we impose the following assumptions

- (H1) $k(x) \in C^\infty(M)$ is positive on M and attains its maximum at some point $P \in M$.
- (H2) $f(x) \in C^\infty(M)$ is positive on M and satisfies $f(x) \ge f_0 > 0$ for all $x \in B_r(x_0)$, where $B_r(x_0)$ denotes the geodesic ball of radius r centered at x_0.
- (H3) The parameter μ satisfies $0 < \mu < \mu_0$, where μ_0 is the largest constant such that the operator $\mathcal{L}_\mu$ is coercive.

Let $\mathcal{L}_\mu$ denote the linear operator defined by

$$\mathcal{L}_\mu u = (-\Delta_g)^s u - \mu \frac{u}{d_g(x,x_0)^{2s}}$$

for all $u \in W^{s,2}(M)$. We call $\mathcal{L}_\mu$ is coercive if there exists a constant $\xi > 0$ such that

$$\mathcal{E}_\mu(u) = \iint_{M\times M} \frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}} \, dv_g(x) \, dv_g(y) - \mu \int_M \frac{|u|^2}{d_g(x,x_0)^{2s}} \, dv_g(x) \ge \xi \|u\|^2_{W^{s,2}(M)}, \tag{1.4}$$

coercive

where $\|u\|^2_{W^{s,2}(M)}$ is defined by (??). The energy functional corresponding to (??) is the C^1-map $I : W^{s,2}(M) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\mathcal{E}_\mu(u) - \frac{\lambda}{p} \int_M f(x)|u|^p \, dv_g(x) - \frac{1}{2_s^*} \int_M k(x)|u|^{2_s^*} \, dv_g(x). \tag{1.5}$$

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Weak solutions to (??) are exactly the critical points of I. Our main result reads as follows.

Th2 **Theorem 1.1.** *Let $2 < p < 2_s^*$ and assume that* (H1)–(H3) *hold. Then*

(i) *If $n > \frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2}$, then* (**??**) *has at least one nontrivial weak solution for all $\lambda > 0$.*

(ii) *If $n \leq \frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2}$, then there exists $\lambda_0 > 0$ such that* (**??**) *has at least one nontrivial weak solution for all $\lambda \geq \lambda_0$,*

Remark 1.2. The analysis of (**??**) on a compact manifold poses several key challenges: the critical exponent 2_s^* causes a loss of compactness in the underlying Sobolev embedding, and the interaction between the nonlocal fractional Laplacian $(-\Delta_g)^s$ and the singular Hardy potential introduces further technical complexities. To surmount these obstacles, we first establish an optimal Sobolev inequality (Theorem **??**), which allows us to determine a sharp energy threshold for the Palais-Smale condition. This in turn facilitates the application of critical point theory to prove our main result.

The rest of the paper is organized as follows. Section **??** provides the necessary background on fractional Sobolev spaces over Riemannian manifolds and establishes key embedding theorems. Section **??** develops the variational setting for (**??**) and analyzes the mountain pass geometry of the energy functional I. Section **??** derives energy estimates and proves the existence of nontrivial weak solutions, thereby establishing Theorem **??**.

2. Preliminaries

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This section presents the geometric and analytic foundations for our work. We begin with essential concepts from Riemannian geometry, then define fractional Sobolev spaces on manifolds and establish the key inequalities for our analysis.

2.1. **Riemannian geometry fundamentals.** Let (M, g) be a Riemannian manifold with metric tensor g_{ij} in local coordinates. A fundamental geometric construct is the length of curves. For any C^1-curve $\gamma : [a, b] \to M$ given locally by $\gamma(t) = (x^1(t), \dots, x^n(t))$, the length is

$$L(\gamma) = \int_a^b \sqrt{g_{ij}(\gamma(t))\, \dot{x}^i(t)\dot{x}^j(t)}\, dt,$$

where $\dot{x}^i = dx^i/dt$. This leads to the geodesic distance between points $x, y \in M$, defined as

$$d_g(x, y) := \inf \left\{ L(\gamma) : \gamma \text{ is a piecewise } C^1\text{-curve connecting } x \text{ and } y \right\}.$$

The metric also defines a canonical volume form, which in local coordinates reads

$$dv_g = \sqrt{|g|}\, dx^1 \wedge \cdots \wedge dx^n,$$

with $|g|$ denoting the determinant of the metric tensor.

Definition 2.1. For any smooth function $u \in C^\infty(M)$, the k-th covariant derivative $\nabla^k u$ has a pointwise squared norm expressed in local coordinates as

$$|\nabla^k u|^2 = g^{i_1 j_1} \cdots g^{i_k j_k} (\nabla^k u)_{i_1 \dots i_k} (\nabla^k u)_{j_1 \dots j_k},$$

where g^{ij} denotes the inverse metric tensor, satisfying the relation $g^{ik} g_{kj} = \delta_j^i$ with the Kronecker delta. In particular, for $k = 1$, we have $(\nabla u)_i = \nabla_i u$ and

$$|\nabla u|^2 = g^{ij} \nabla_i u\, \nabla_j u.$$

Definition 2.2. Let x be a point on M. The exponential map at x is a mapping $\exp_x : V_x \to M$ defined on some neighborhood $V_x \subset T_x M$ of the origin by

$$\exp_x(v) = \gamma_v(1),$$

where γ_v is the unique geodesic satisfying $\gamma_v(0) = x$ and $\gamma_v'(0) = v$.

def:inj_radius **Definition 2.3.** The injectivity radius of (M, g), denoted by $\mathrm{inj}(M)$, quantifies the size of regions where the exponential map remains well-behaved. It is defined as

$$\mathrm{inj}(M) := \inf_{x \in M} \sup \left\{ r > 0 \mid \exp_x : B_r(0) \to M \text{ is a diffeomorphism onto its image} \right\}.$$

For any $0 < r < \mathrm{inj}(M)$ and any $x \in M$, the exponential map $\exp_x$ restricts to a diffeomorphism from the tangent space ball $B_r(0) = \{v \in T_xM : |v|_g < r\}$ onto the geodesic ball $B_r(x) = \{y \in M : d_g(x, y) < r\}$.

2.2. **Fractional Sobolev spaces on Riemannian manifolds.** For $0 < s < 1$, the homogeneous fractional Sobolev space $\dot{H}^s(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the Gagliardo seminorm

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} = \Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy\Big)^{1/2}.$$

Following Guo et al. [**?**], we define the fractional Sobolev space on M as

$$W^{s,2}(M) = \big\{u \in L^2(M) \mid [u]_{s,2} < \infty\big\},$$

equipped with the norm

$$\|u\|_{W^{s,2}(M)} = \Big(\int_M |u(x)|^2\, dv_g(x) + [u]_{s,2}^2\Big)^{1/2} \tag{2.1}$$ eq2.11

where the Gagliardo seminorm is

$$[u]_{s,2} = \Big(\iint_{M\times M} \frac{|u(x) - u(y)|^2}{d_g(x, y)^{n+2s}}\, dv_g(x)\, dv_g(y)\Big)^{1/2}.$$

We define the support of a function u as $\mathrm{supp}(u) = \overline{\{x \in M : u(x) \neq 0\}}$. Since (M, g) is compact and without boundary, we have $W_0^{s,2}(M) = W^{s,2}(M)$, where

$$W_0^{s,2}(M) = \big\{u \in W^{s,2}(M) \mid \mathrm{supp}(u) \text{ is compact}\big\}.$$

Thus $W^{s,2}(M)$ serves as our working space.

lem:2.1 **Lemma 2.4** ([**?**, **?**]). *Let (M, g) be a complete Riemannian manifold. Then $W^{s,2}(M)$ is a uniformly convex and reflexive Banach space.*

lem:2.2 **Lemma 2.5** ([**?**]). *Let (M, g) be a compact Riemannian manifold, and let $\{v_j\}_{j=1}^\infty$ be a bounded sequence in $W^{s,2}(M)$. Then for any $q \in [1, 2_s^*)$, there exists $v \in L^q(M)$ such that $v_j \to v$ strongly in $L^q(M)$.*

lem:2.4 **Lemma 2.6** ([**?**]). *Let (M, g) be a compact Riemannian manifold. Then*

(i) *For any $u \in W^{s,2}(M)$ and $1 \le q \le 2_s^*$, there exists $C = C(n, s) > 0$ such that*

$$\|u\|_{L^q(M)}^2 \le C \iint_{M\times M} \frac{|u(x) - u(y)|^2}{d_g(x, y)^{n+2s}}\, dv_g(x)\, dv_g(y).$$

(ii) *For any $u \in W^{s,2}(M)$, there exists $\widetilde{C} = \widetilde{C}(n, s) > 0$ such that*

$$\iint_{M\times M} \frac{|u(x) - u(y)|^2}{d_g(x, y)^{n+2s}}\, dv_g(x)\, dv_g(y) \le \|u\|_{W^{s,2}(M)}^2 \le \widetilde{C} \iint_{M\times M} \frac{|u(x) - u(y)|^2}{d_g(x, y)^{n+2s}}\, dv_g(x)\, dv_g(y).$$

rem:2.5 **Remark 2.7.** By Lemma **??**, we can define the equivalent norm on $W_0^{s,2}(M)$

$$\|u\| = \Big(\iint_{M\times M} \frac{|u(x) - u(y)|^2}{d_g(x, y)^{n+2s}}\, dv_g(x)\, dv_g(y)\Big)^{1/2},$$

which is equivalent to $\|\cdot\|_{W^{s,2}(M)}$.

The optimal constant in the fractional Sobolev inequality on $\mathbb{R}^n$ is

$$S(n, s, 2)^{-2} = \inf_{u\in\dot{H}^s(\mathbb{R}^n), u\not\equiv 0} \frac{\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy}{\big(\int_{\mathbb{R}^n} |u|^{2_s^*}\, dx\big)^{2/2_s^*}}. \tag{2.2}$$ S

As established in [**?**], this infimum is attained by the family of functions

$$v_\varepsilon(x) = \frac{C\varepsilon^{\frac{n-2s}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{n-2s}{2}}}, \quad \varepsilon > 0.$$

Mukherjee and Tiwari [?] showed that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that for all $u \in W^{s,2}(M)$,

$$\begin{aligned}&\Big(\int_M |u|^{2_s^*}\,dv_g(x)\Big)^{2/2_s^*}\\ &\le (S(n,s,2)^2+\epsilon)\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)+C_\epsilon\int_M |u|^2\,dv_g(x).\end{aligned} \tag{2.3}$$ MSob

The Hardy-weighted fractional Sobolev constant is

$$S_\mu^{-2}(n,s,2)=\inf_{u\in\dot H^s(\mathbb{R}^n),\,u\not\equiv 0}\frac{\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy-\mu\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^{2s}}dx}{\big(\int_{\mathbb{R}^n}|u|^{2_s^*}dx\big)^{2/2_s^*}}. \tag{2.4}$$ Sm

As shown in [?], this infimum is attained and yields solutions to (**??**). By (H3), we claim that there exists $C_\epsilon > 0$ such that for all $u \in W^{s,2}(M)$,

$$\begin{aligned}\Big(\int_M |u|^{2_s^*}\,dv_g\Big)^{2/2_s^*} &\le (S_\mu^2(n,s,2)+\epsilon)\Big[\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)\\ &\quad-\mu\int_M\frac{|u|^2}{d_g(x,x_0)^{2s}}\,dv_g(x)\Big]+C_\epsilon\int_M|u|^2\,dv_g.\end{aligned} \tag{2.5}$$ 5

We will prove this in the appendix. Under the condition of (H3), we define the weighted norm

$$\|u\|_\mu^2=\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)-\mu\int_M\frac{u^2}{d_g(x,x_0)^{2s}}\,dv_g(x),$$

with $x_0 \in M$ fixed and $\mu \in \mathbb{R}$. By coercivity and Remark **??**, this norm satisfies

$$\xi\|u\|^2_{W^{s,2}(M)}\le\|u\|_\mu^2\le C\|u\|^2_{W^{s,2}(M)} \tag{2.6}$$ mu-norm

for some $\xi, C > 0$, making $\|\cdot\|_\mu$ equivalent to $\|\cdot\|_{W^{s,2}(M)}$.

2.1 **Lemma 2.8.** *Let (M,g) be a compact Riemannian manifold, then*

(i) *There exists $C > 0$ such that for all $u \in W^{s,2}(M)$,*
$$\|u\|_{2_s^*}\le C\|u\|_{W^{s,2}(M)};$$
(ii) *For $2 < p < 2_s^*$, there exists $C > 0$ such that for all $u \in W^{s,2}(M)$,*
$$\|u\|_p\le C\|u\|_{W^{s,2}(M)}.$$

The above lemma follow from Lemma **??** and the inequality (**??**).

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3. Mountain pass solution

def:PS **Definition 3.1.** A functional $I \in C^1(W^{s,2}(M),\mathbb{R})$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ if every sequence $\{u_n\} \subset W^{s,2}(M)$ satisfying

$$I(u_n)\to c \quad\text{and}\quad I'(u_n)\to 0 \quad\text{in } (W^{s,2}(M))'$$

as $n \to \infty$ admits a subsequence that converges strongly in $W^{s,2}(M)$.

thm:mountainpass **Lemma 3.2** ([?]). *Let $I \in C^1(W^{s,2}(M),\mathbb{R})$. Suppose there exist constants $\rho > 0$, $\alpha > 0$, and an element $v \in W^{s,2}(M)$ such that $I(u) \ge \alpha$ for all $u \in W^{s,2}(M)$ with $\|u\|_{W^{s,2}(M)} = \rho$, and that $I(v) < 0$ with $\|v\|_{W^{s,2}(M)} > \rho$. Define*

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1],W^{s,2}(M)) : \gamma(0) = 0, \gamma(1) = v\}$. Then there exists a Palais-Smale sequence $\{u_n\} \subset W^{s,2}(M)$ at level c. Moreover, if I satisfies the Palais-Smale condition at level c, then c is a critical value of I.

ountain_geometry **Lemma 3.3.** Let $\lambda > 0$ and $2 < p < 2_s^*$. Suppose hypotheses (H1)-(H3) hold. Then

(i) There exist constants $\rho, \alpha > 0$ such that
$$I(u)\ge\alpha\quad\text{for all } u\in W^{s,2}(M) \text{ with } \|u\|_{W^{s,2}(M)}=\rho.$$

(ii) There exists $v \in W^{s,2}(M) \setminus \{0\}$ such that $\|v\|_{W^{s,2}(M)} > \rho$ and $I(v) < 0$.

Proof. For (i), the coercivity condition from (H3) provides $\xi > 0$ such that

$$\mathcal{E}_\mu(u) \geq \xi \|u\|^2_{W^{s,2}(M)}.$$

Since $k(x) \in C^\infty(M)$ by (H1)and M is compact, $k(x)$ is bounded. Combined with Lemma **??**, there exists $C_1 > 0$ such that

$$\int_M k(x)|u|^{2_s^*}\, dv_g(x) \leq C_1 \|u\|^{2_s^*}_{W^{s,2}(M)}.$$

Similarly, $f(x) \in C^\infty(M)$ by (H2) and Lemma **??** imply the existence of $C_2 > 0$ with

$$\int_M f(x)|u|^p\, dv_g(x) \leq C_2 \|u\|^p_{W^{s,2}(M)}.$$

Then,

$$I(u) \geq \frac{\xi}{2}\|u\|^2_{W^{s,2}(M)} - \frac{\lambda C_2}{p}\|u\|^p_{W^{s,2}(M)} - \frac{C_1}{2_s^*}\|u\|^{2_s^*}_{W^{s,2}(M)}.$$

Since $2_s^* > p > 2$, we may choose a small $\rho > 0$ such that $I(u) \geq \alpha > 0$ whenever $\|u\|_{W^{s,2}(M)} = \rho$, which proves (i). For (ii), let $\phi \in C_c^\infty(M)$ and set $v_t = t\phi$. Then

$$I(v_t) = \frac{t^2}{2}\mathcal{E}_\mu(\phi) - \frac{\lambda t^p}{p}\int_M f(x)\phi^p\, dv_g(x) - \frac{t^{2_s^*}}{2_s^*}\int_M k(x)\phi^{2_s^*}\, dv_g(x).$$

As $t \to \infty$, $I(v_t) \to -\infty$. Hence, for t sufficiently large, we have $\|v_t\|_{W^{s,2}(M)} > \rho$ and $I(v_t) < 0$, which establishes (ii). □

Lemma 3.4. *If $\{u_m\} \subset W^{s,2}(M)$ is a Palais-Smale sequence for I at level c, then it is bounded.*

Proof. Let $\{u_m\} \subset W^{s,2}(M)$ be a Palais-Smale sequence for I at level c. Then

$$I(u_m) = \frac{1}{2}\mathcal{E}_\mu(u_m) - \frac{\lambda}{p}\int_M f(x)|u_m|^p\, dv_g(x) - \frac{1}{2_s^*}\int_M k(x)|u_m|^{2_s^*}\, dv_g(x) = c + o(1), \tag{3.1}$$

eq:1

$$\langle I'(u_m), u_m\rangle = \mathcal{E}_\mu(u_m) - \lambda\int_M f(x)|u_m|^p\, dv_g(x) - \int_M k(x)|u_m|^{2_s^*}\, dv_g(x) = o(\|u_m\|_{W^{s,2}(M)}). \tag{3.2}$$

eq:2

Combining (**??**) and (**??**), we obtain

$$\Big(1 - \frac{2}{p}\Big)\lambda\int_M f(x)|u_m|^p\, dv_g(x) + \Big(1 - \frac{2}{2_s^*}\Big)\int_M k(x)|u_m|^{2_s^*}\, dv_g(x) = o(\|u_m\|_{W^{s,2}(M)}) + 2c + o(1).$$

Since $p > 2$ and $2_s^* > 2$, the coefficients $1 - \frac{2}{p}$ and $1 - \frac{2}{2_s^*}$ are positive. Moreover, with $\lambda > 0$ and $f(x), k(x)$ strictly positive by (H1) and (H2), all terms on the left-hand side are nonnegative. Consequently, both integrals are bounded, and there exists $C > 0$ such that

$$\int_M f(x)|u_m|^p\, dv_g(x) \leq C \quad \text{and} \quad \int_M k(x)|u_m|^{2_s^*}\, dv_g(x) \leq C.$$

By the coercivity of $\mathcal{L}_\mu$ from (H3), there exists $\xi > 0$ such that $\mathcal{E}_\mu(u_m) \geq \xi\|u_m\|^2_{W^{s,2}(M)}$. Substituting this into (**??**) yields

$$\xi\|u_m\|^2_{W^{s,2}(M)} \leq \lambda\int_M f(x)|u_m|^p\, dv_g(x) + \int_M k(x)|u_m|^{2_s^*}\, dv_g(x) + o(\|u_m\|_{W^{s,2}(M)}).$$

Thus,

$$\xi\|u_m\|^2_{W^{s,2}(M)} \leq \lambda C + C + o(\|u_m\|_{W^{s,2}(M)}),$$

which implies that the sequence $\{u_m\}$ is bounded in $W^{s,2}(M)$. □

tive_convergence **Lemma 3.5.** *Suppose $\{u_m\} \subseteq W^{s,2}(M)$ is a Palais-Smale sequence for the functional I at level c that converges weakly to $u \in W^{s,2}(M)$. Then*

$$\langle I'(u_m), u\rangle \to \langle I'(u), u\rangle \quad \text{as } m \to \infty.$$

Proof. Assume $u_m \rightharpoonup u$ weakly in $W^{s,2}(M)$. Then

$$\begin{aligned}\langle I'(u_m), u\rangle = &\iint_{M\times M} \frac{(u_m(x)-u_m(y))(u(x)-u(y))}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y) - \mu\int_M \frac{u_m(x)u(x)}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x)\\ &- \lambda\int_M f(x)|u_m(x)|^{p-2}u_m(x)u(x)\mathrm{d}v_g(x) - \int_M k(x)|u_m(x)|^{2_s^*-2}u_m(x)u(x)\mathrm{d}v_g(x).\end{aligned}$$

Since $\frac{(u_m(x)-u_m(y))(u(x)-u(y))}{d_g(x,y)^{n+2s}}$ and $|u_m|^{2_s^*-2}u_m u$ is bounded in $L^1(M\times M)$ and $L^{\frac{2_s^*}{2_s^*-1}}(M)$, they converge weakly in $(L^1(M\times M))'$ and $(L^{\frac{2_s^*}{2_s^*-1}}(M))'$ respectively. Thus

$$\iint_{M\times M} \frac{(u_m(x)-u_m(y))(u(x)-u(y))}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y) \to \iint_{M\times M} \frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y),$$

and

$$\int_M k(x)|u_m(x)|^{2_s^*-2}u_m(x)u(x)\mathrm{d}v_g(x) \to \int_M k(x)|u(x)|^{2_s^*}\mathrm{d}v_g(x).$$

By weak convergence $u_m \rightharpoonup u$, we have

$$\mu\int_M \frac{u_m(x)u(x)}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x) \to \mu\int_M \frac{u^2(x)}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x),$$

and

$$\lambda\int_M f(x)|u_m(x)|^{p-2}u_m(x)u(x)\mathrm{d}v_g(x) \to \lambda\int_M f(x)|u(x)|^p\mathrm{d}v_g(x).$$

Combining these, we obtain

$$\begin{aligned}\langle I'(u_m), u\rangle \to \langle I'(u), u\rangle = &\iint_{M\times M} \frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y) - \mu\int_M \frac{u^2(x)}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x)\\ &- \lambda\int_M f(x)|u(x)|^p\mathrm{d}v_g(x) - \int_M k(x)|u(x)|^{2_s^*}\mathrm{d}v_g(x).\end{aligned}$$

□

le_decomposition

Lemma 3.6. *Let $\{u_m\} \subset W^{s,2}(M)$ be a Palais-Smale sequence for I at level c, with $u_m \rightharpoonup u$ weakly in $W^{s,2}(M)$. Set $v_m = u_m - u$. Then $\{v_m\}$ is a Palais-Smale sequence for I at level $c - I(u)$, satisfying*

$$I(v_m) \to c - I(u) \quad \text{and} \quad \|I'(v_m)\|_{(W^{s,2}(M))'} \to 0 \quad \text{as } m \to \infty,$$

and

$$\begin{aligned}&\iint_{M\times M} \frac{|u_m(x)-u_m(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y)\\ &= \iint_{M\times M} \frac{|v_m(x)-v_m(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y) + \iint_{M\times M} \frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y) + o(1),\end{aligned}$$

$$\int_M \frac{\mu u_m^2}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x) = \int_M \frac{\mu v_m^2}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x) + \int_M \frac{\mu u^2}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x) + o(1),$$

$$\int_M \lambda f(x)|u_m|^p\mathrm{d}v_g(x) = \int_M \lambda f(x)|v_m|^p\mathrm{d}v_g(x) + \int_M \lambda f(x)|u|^p\mathrm{d}v_g(x) + o(1),$$

$$\int_M k(x)|u_m|^{2_s^*}\mathrm{d}v_g(x) = \int_M k(x)|v_m|^{2_s^*}\mathrm{d}v_g(x) + \int_M k(x)|u|^{2_s^*}\mathrm{d}v_g(x) + o(1).$$

Proof. Let $u_m = v_m + u$ with $u_m \rightharpoonup u$ weakly in $W^{s,2}(M)$, which implies $v_m \rightharpoonup 0$ weakly in $W^{s,2}(M)$.

$$\begin{aligned}&\iint_{M\times M} \frac{|u_m(x)-u_m(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y)\\ &= \iint_{M\times M} \frac{|v_m(x)-v_m(y)+u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y)\end{aligned}$$

$$
\begin{aligned}
&= \iint_{M\times M} \frac{|v_m(x)-v_m(y)|^2}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y) + \iint_{M\times M} \frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y)\\
&\quad + 2\iint_{M\times M} \frac{(v_m(x)-v_m(y))(u(x)-u(y))}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y).
\end{aligned}
$$

As $m\to\infty$, we have

$$
\begin{aligned}
&\iint_{M\times M} \frac{|u_m(x)-u_m(y)|^2}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y)\\
&= \iint_{M\times M} \frac{|v_m(x)-v_m(y)|^2}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y) + \iint_{M\times M} \frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y) + o(1).
\end{aligned}
$$

By the Brezis-Lieb Lemma[**?**], we have

$$
\begin{gathered}
\int_M \frac{\mu u_m^2}{d_g(x,x_0)^{2s}} \mathrm{d}v_g(x) = \int_M \frac{\mu v_m^2}{d_g(x,x_0)^{2s}} \mathrm{d}v_g(x) + \int_M \frac{\mu u^2}{d_g(x,x_0)^{2s}} \mathrm{d}v_g(x) + o(1),\\
\int_M \lambda f(x)|u_m|^p \mathrm{d}v_g(x) = \int_M \lambda f(x)|v_m|^p \mathrm{d}v_g(x) + \int_M \lambda f(x)|u|^p \mathrm{d}v_g(x) + o(1),\\
\int_M k(x)|u_m|^{2_s^*} \mathrm{d}v_g(x) = \int_M k(x)|v_m|^{2_s^*} \mathrm{d}v_g(x) + \int_M k(x)|u|^{2_s^*} \mathrm{d}v_g(x) + o(1).
\end{gathered}
$$

Substituting into $I(u_m)$ yields

$$
I(u_m) = I(v_m) + I(u) + o(1).
$$

Since $I(u_m)\to c$, it follows that $I(v_m)\to c - I(u)$. Then

$$
\langle I'(u_m),\phi\rangle = \langle I'(v_m),\phi\rangle + \langle I'(u),\phi\rangle + o(1).
$$

As $\{u_m\}$ is a Palais-Smale sequence and $u_m \rightharpoonup u$, then $\langle I'(u_m),\phi\rangle \to 0$ and $\langle I'(u),\phi\rangle = 0$. Therefore $\langle I'(v_m),\phi\rangle \to 0$ for all ϕ, and hence $\|I'(v_m)\|_{(W^{s,2}(M))'} \to 0$. □

lem:5.1 **Lemma 3.7.** *Let $\{u_m\}\subset W^{s,2}(M)$ be a Palais-Smale sequence for the functional I at level c that converges weakly to $u\in W^{s,2}(M)$. Setting $v_m = u_m - u$, if*

$$
c < \frac{s}{n} S_\mu(n,s,2)^{-n/s} \Big(\max_{x\in M} k(x)\Big)^{-\frac{n-2s}{2s}},
$$

then $\{v_m\}$ converges strongly to 0 in $W^{s,2}(M)$, and thus u_m converges strongly to u in $W^{s,2}(M)$.

Proof. From Lemmas **??** and **??**, we have

$$
\langle I'(v_m), v_m\rangle = \langle I'(u_m), u_m\rangle - \langle I'(u), u\rangle + o(1) = o(1).
$$

This implies

$$
\begin{aligned}
&\iint_{M\times M} \frac{|v_m(x)-v_m(y)|^2}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y) - \int_M \frac{\mu v_m^2(x)}{d_g(x,x_0)^{2s}} \mathrm{d}v_g(x)\\
&= \lambda\int_M f(x)|v_m(x)|^p \mathrm{d}v_g(x) + \int_M k(x)|v_m(x)|^{2_s^*} \mathrm{d}v_g(x) + o(1).
\end{aligned} \tag{3.3}
$$

50

We claim that $v_m\to 0$ in $W^{s,2}(M)$. Assume to the contrary that $v_m \not\to 0$. Since $u_m \rightharpoonup u$ and $v_m = u_m - u$, we have $v_m \rightharpoonup 0$. Then

$$
\lambda\int_M f(x)|v_m(x)|^p \mathrm{d}v_g(x) \to 0.
$$

We set

$$
l = \lim_{m\to\infty}\Big(\iint_{M\times M} \frac{|v_m(x)-v_m(y)|^2}{d_g(x,y)^{n+2s}} \mathrm{d}v_g(x)\mathrm{d}v_g(y) - \int_M \frac{\mu v_m^2(x)}{d_g(x,x_0)^{2s}} \mathrm{d}v_g(x)\Big).
$$

From (**??**), we have

$$
\lim_{m\to\infty}\int_M k(x)|v_m(x)|^{2_s^*} \mathrm{d}v_g(x) = l.
$$

Applying (**??**) yields

$$\begin{aligned}&\Big(\int_M k(x)|v_m(x)|^{2_s^*}\mathrm{d}v_g(x)\Big)^{2/2_s^*}\\ &\le \Big(\max_{x\in M} k(x)\Big)^{2/2_s^*}\Big(S_\mu(n,s,2)^2+\varepsilon\Big)\Big(\iint_{M\times M}\frac{|v_m(x)-v_m(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y)\\ &\quad -\int_M \frac{\mu v_m^2(x)}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x)\Big)+C_\epsilon\int_M |v_m(x)|^2\mathrm{d}v_g(x).\end{aligned}$$

Taking the limit as $m\to\infty$ yields

$$l^{2/2_s^*}\le \Big(\max_{x\in M} k(x)\Big)^{2/2_s^*}\left(S_\mu(n,s,2)^2+\varepsilon\right)l.$$

By the arbitrariness of $\varepsilon>0$, we conclude that

$$l\ge \Big(\max_{x\in M} k(x)\Big)^{-\frac{n-2s}{2s}} S_\mu(n,s,2)^{-n/s}. \tag{3.4}$$ eq:6

Since $\langle I'(u),u\rangle=0$, we obtain

$$\begin{aligned}&\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y)-\mu\int_M\frac{u^2(x)}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x)\\ &=\lambda\int_M f(x)|u(x)|^p\mathrm{d}v_g(x)+\int_M k(x)|u(x)|^{2_s^*}\mathrm{d}v_g(x).\end{aligned}\tag{3.5}$$ eq:7

Substituting (**??**) into (**??**), we obtain

$$I(u)=\Big(\frac12-\frac1p\Big)\lambda\int_M f(x)|u(x)|^p\mathrm{d}v_g(x)+\Big(\frac12-\frac{1}{2_s^*}\Big)\int_M k(x)|u(x)|^{2_s^*}\mathrm{d}v_g(x)\ge 0.$$

By Lemma **??**, we have

$$\begin{aligned}&\frac12\iint_{M\times M}\frac{|v_m(x)-v_m(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y)-\frac12\int_M\frac{\mu v_m^2(x)}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x)\\ &-\frac{1}{2_s^*}\int_M k(x)|v_m(x)|^{2_s^*}\mathrm{d}v_g(x)-\frac{\lambda}{p}\int_M f(x)|v_m(x)|^p\mathrm{d}v_g(x)+I(u)\to c.\end{aligned}$$

Taking the limit as $m\to\infty$, we obtain

$$\frac12 l-\frac{1}{2_s^*}l+I(u)=c.$$

Since $I(u)\ge0$, it follows that

$$\Big(\frac12-\frac{1}{2_s^*}\Big)l\le c.$$

Then

$$\frac{s}{n}l\le c.$$

Combining this with (**??**) and taking the limit as $\varepsilon\to0^+$, we derive

$$\frac{s}{n}S_\mu(n,s,2)^{-n/s}\Big(\max_{x\in M}k(x)\Big)^{-\frac{n-2s}{2s}}\le c.$$

This contradicts the assumption on c, so $v_m\to0$ strongly in $W^{s,2}(M)$. □

prop:1 **Proposition 3.8.** *Let $2<p<2_s^*$, $\lambda>0$, with hypotheses (H2) satisfied. For any nonnegative function $v\in W^{s,2}(M)$ with $\|v\|_{W^{s,2}(M)}\ne0$, then*

$$\sup_{t\ge0}I(tv)\le\frac{s}{n}\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)}\Big)^{\frac{n}{2s}}-\frac{\lambda t_v^p}{p}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x),$$

where $I(tv)$ attains its maximum at $t=t_v>0$, and

$$\mathcal{E}_\mu(v)=\iint_{M\times M}\frac{|v(x)-v(y)|^2}{d_g(x,y)^{n+2s}}\mathrm{d}v_g(x)\mathrm{d}v_g(y)-\mu\int_M\frac{v(x)^2}{d_g(x,x_0)^{2s}}\mathrm{d}v_g(x),$$

$$\chi(v)=\Big(\int_M k(x)|v(x)|^{2_s^*}\mathrm{d}v_g(x)\Big)^{2/2_s^*}.$$

Proof. Define a functional for $t\geq 0$ as

$$I(tv)=\frac{1}{2}t^2\mathcal{E}_\mu(v)-\frac{\lambda}{p}\int_M f(x)|tv(x)|^p\mathrm{d}v_g(x)-\frac{1}{2_s^*}\int_M k(x)|tv(x)|^{2_s^*}\mathrm{d}v_g(x).$$

Since $2_s^*>2$, $I(tv)\to-\infty$ as $t\to\infty$, so there exists $t_v>0$ such that

$$I(t_v v)=\sup_{t\geq 0}I(tv).$$

Let $g(t)=I(tv)$. Then

$$g(t)=\frac{t^2}{2}\mathcal{E}_\mu(v)-\frac{t^{2_s^*}}{2_s^*}\chi(v)^{2_s^*/2}-\frac{\lambda t^p}{p}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x),$$

with

$$g'(t)=t\mathcal{E}_\mu(v)-t^{2_s^*-1}\chi(v)^{2_s^*/2}-\lambda t^{p-1}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x).$$

At $t=t_v$, $g'(t_v)=0$ implies

$$t_v\mathcal{E}_\mu(v)=t_v^{2_s^*-1}\chi(v)^{2_s^*/2}+\lambda t_v^{p-1}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x).$$

Thus,

$$\mathcal{E}_\mu(v)=t_v^{2_s^*-2}\chi(v)^{2_s^*/2}+\lambda t_v^{p-2}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x)\geq t_v^{2_s^*-2}\chi(v)^{2_s^*/2},$$

and therefore

$$t_v\leq\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{2_s^*/2}}\Big)^{\frac{1}{2_s^*-2}}=\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{2_s^*/2}}\Big)^{\frac{n-2s}{4s}}.$$

Consider the auxiliary function

$$f(t)=\frac{t^2}{2}\mathcal{E}_\mu(v)-\frac{t^{2_s^*}}{2_s^*}\chi(v)^{\frac{2_s^*}{2}}.$$

Its derivative is

$$f'(t)=t\mathcal{E}_\mu(v)-t^{2_s^*-1}\chi(v)^{\frac{2_s^*}{2}}.$$

Let $t_0=\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{1}{2_s^*-2}}$. A direct computation shows that t_0 is a critical point of $f(t)$, and $f(t)$ is increasing on

$$\Big[0,\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{n-2s}{4s}}\Big].$$

Therefore,

$$I(t_v v)\leq\frac{t_0^2}{2}\mathcal{E}_\mu(v)-\frac{t_0^{2_s^*}}{2_s^*}\chi(v)^{\frac{2_s^*}{2}}-\frac{\lambda t_v^p}{p}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x). \tag{3.6}$$ eq:8

Direct computations show that

$$t_0^2=\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{2}{2_s^*-2}}=\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{n-2s}{2s}},\quad t_0^{2_s^*}=\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{2_s^*}{2_s^*-2}}=\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{n}{2s}}.$$

Substituting this into (**??**), we have

$$\begin{aligned}I(t_v v)&\leq\frac{1}{2}\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{n-2s}{2s}}\mathcal{E}_\mu(v)-\frac{1}{2_s^*}\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{n}{2s}}\chi(v)^{\frac{2_s^*}{2}}-\frac{\lambda t_v^p}{p}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x)\\&=\Big(\frac{1}{2}-\frac{1}{2_s^*}\Big)\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)^{\frac{2_s^*}{2}}}\Big)^{\frac{n}{2s}}\chi(v)^{\frac{2_s^*}{2}}-\frac{\lambda t_v^p}{p}\int_M f(x)|v(x)|^p\mathrm{d}v_g(x).\end{aligned}$$

Using the definition of $\chi(v)$, we obtain

$$I(t_v v) \le \frac{s}{n}\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)}\Big)^{\frac{n}{2s}} - \frac{\lambda t_v^p}{p}\int_M f(x)|v(x)|^p \mathrm{d}v_g(x).$$

Since $t_v \ge c > 0$ and $\lambda > 0$, it follows that $\lambda t_v^p \ge C > 0$, and therefore

$$I(t_v v) \le \frac{s}{n}\Big(\frac{\mathcal{E}_\mu(v)}{\chi(v)}\Big)^{\frac{n}{2s}} - C\int_M f(x)|v(x)|^p \mathrm{d}v_g(x).$$

□

s4

4. Proof of main results

This section establishes crucial energy estimates for the Palais-Smale sequences of the functional I, which are essential for proving the existence of nontrivial solutions to problem (**??**) under critical growth conditions. The analysis relies on the following lemmas.

4.1. **Local geometric expansions and metric comparisons.**

lem:metric_det **Lemma 4.1** ([**?**]). *In normal coordinates about P, the metric determinant expands as*

$$\sqrt{|g|} = 1 - \frac{1}{6}R_{ij}(P)x^i x^j + O(|x|^3),$$

where R_{ij} are components of the Ricci tensor at P.

etric_comparison **Lemma 4.2** ([**?**]). *Let (M,g) be a compact Riemannian manifold. For any $\epsilon > 0$, there exists $\delta \in (0, inj(M))$ such that M has a finite cover by geodesic balls $\{B_\delta(x_i)\}$. In each exponential chart, the distances satisfy*

$$(1-\epsilon)d_E(x,y) \le d_g\left(\exp_{x_i}(x), \exp_{x_i}(y)\right) \le (1+\epsilon)d_E(x,y)$$

for all $x, y \in B_\delta(x_i)$. Moreover, there exists a constant $C > 0$ such that

$$\frac{1}{C}\left|\exp_{x_i}^{-1}(x) - \exp_{x_i}^{-1}(y)\right| \le d_g(x,y) \le C\left|\exp_{x_i}^{-1}(x) - \exp_{x_i}^{-1}(y)\right|$$

for all $x, y \in M$.

To establish the existence of solutions to the fractional equation on M via variational methods, we construct a test function that concentrates near a point $P \in M$. Let $(x^1, \dots, x^n)$ denote normal coordinates centered at P. Select $\delta > 0$ sufficiently small such that $B_\delta(P) \subset M$ and choose $\theta > 1$ with $\theta\delta \in (0, \mathrm{inj}(M))$. Define a cutoff function $\eta \in C_0^\infty(B_{\theta\delta}(P))$ by

$$\eta(x) = \begin{cases} 1 & \text{if } x \in B_\delta(P), \\ 0 & \text{if } x \in M \setminus B_{\theta\delta}(P), \end{cases}$$

with $0 \le \eta(x) \le 1$. Let U be the positive radial minimizer for the fractional Sobolev inequality (**??**), satisfying

$$\Big(\int_{\mathbb{R}^n} |U|^{2_s^*} dx\Big)^{2/2_s^*} = S_\mu(n,s,2)^2\Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \mu\int_{\mathbb{R}^n}\frac{|U|^2}{|x|^{2s}}dx\Big). \qquad (4.1)$$ 9

Setting $K := S_\mu^{-2}(n,s,2)$, testing the following equation

$$(-\Delta)^s U - \mu\frac{U}{|x|^{2s}} = U^{2_s^*-1},$$

then we have

$$\|U\|_\mu^2 = \|U\|_{2_s^*}^{2_s^*} = K^{\frac{n}{2s}}.$$

For $0 < \epsilon \ll \theta\delta$, we define an exponential chart $u_\epsilon(x) = \eta(x)U_\epsilon\big(|x|\big)$, where

$$U_\varepsilon(x) = \varepsilon^{-\frac{n-2s}{2}}U\big(\frac{x}{\varepsilon}\big), \quad x \in \mathbb{R}^n.$$

4.3 **Lemma 4.3.** *Let $h:\mathbb{R}\to\mathbb{R}$ be an increasing function and define*

$$H(t)=\int_0^t \sqrt{h'(\tau)}d\tau, \quad t\in\mathbb{R}.$$

Then for every $a,b\in\mathbb{R}$,

$$(a-b)(h(a)-h(b))\geq |H(a)-H(b)|^2.$$

Proof. Assume $a\geq b$ without loss of generality. Since h is increasing, $h'\geq 0$ almost everywhere. Consider

$$H(a)-H(b)=\int_b^a \sqrt{h'(\tau)}d\tau.$$

By the Cauchy-Schwarz inequality

$$\Big[\int_b^a \sqrt{h'(\tau)}d\tau\Big]^2\leq \Big[\int_b^a 1^2 d\tau\Big]\Big[\int_b^a \Big(\sqrt{h'(\tau)}\Big)^2 d\tau\Big]=(a-b)\int_b^a h'(\tau)d\tau.$$

Since

$$\int_b^a h'(\tau)d\tau=h(a)-h(b),$$

we obtain $|H(a)-H(b)|^2\leq (a-b)(h(a)-h(b))$ which completes the proof. □

lem:decay **Lemma 4.4.** *Let U be a minimizer of (**??**). Then there exist constants $c_1,c_2>0$ and $\theta>1$ such that for all $r>1$, the function $U(r)$ satisfies*

$$\frac{c_1}{\big[r^{1-\gamma_\mu}\left(1+r^{2\gamma_\mu}\right)\big]^{\frac{n-2s}{2}}}\leq U(r)\leq \frac{c_2}{\big[r^{1-\gamma_\mu}\left(1+r^{2\gamma_\mu}\right)\big]^{\frac{n-2s}{2}}}, \tag{4.2}$$ 10

where $\gamma_\mu=1-\frac{2\alpha_\mu}{n-2s}$ and $\alpha_\mu\in\big(0,\frac{n-2s}{2}\big)$. Moreover, $U(r)$ satisfies

$$\frac{U(\theta r)}{U(r)}\leq \frac{1}{2}. \tag{4.3}$$ 11

Proof. Inequality (**??**) follows from the asymptotic analysis of solutions to (**??**), thus

$$U(\theta r)\leq \frac{c_2}{\big[(\theta r)^{1-\gamma_\mu}\left(1+(\theta r)^{2\gamma_\mu}\right)\big]^{\frac{n-2s}{2}}},\quad U(r)\geq \frac{c_1}{\big[r^{1-\gamma_\mu}\left(1+r^{2\gamma_\mu}\right)\big]^{\frac{n-2s}{2}}}.$$

Then

$$\frac{U(\theta r)}{U(r)}\leq \frac{c_2}{c_1}\Big(\frac{r^{1-\gamma_\mu}\left(1+r^{2\gamma_\mu}\right)}{(\theta r)^{1-\gamma_\mu}\left(1+(\theta r)^{2\gamma_\mu}\right)}\Big)^{\frac{n-2s}{2}},$$

which simplifies to

$$\frac{U(\theta r)}{U(r)}\leq C\Big(\frac{1+r^{2\gamma_\mu}}{\theta^{1-\gamma_\mu}\left(1+(\theta r)^{2\gamma_\mu}\right)}\Big)^{\frac{n-2s}{2}}.$$

For large $\theta>1$,

$$\frac{U(\theta r)}{U(r)}\leq C\Big(\frac{r^{2\gamma_\mu}}{\theta^{1-\gamma_\mu}(\theta r)^{2\gamma_\mu}}\Big)^{\frac{n-2s}{2}}=C\theta^{-\frac{(1+\gamma_\mu)(n-2s)}{2}}.$$

Choosing θ sufficiently large such that $C\theta^{-\frac{(1+\gamma_\mu)(n-2s)}{2}}\leq\frac{1}{2}$. □

lemma4.4 **Lemma 4.5.** *Let (M,g) be a compact Riemannian manifold and $P\in M$. For sufficiently small $\varepsilon>0$, we have the following estimates*

$$[u_\varepsilon]_{s,2}^2\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy+O\Big(\epsilon^{\frac{(n-2s)(1+\gamma_\mu)}{2}}\Big), \tag{4.4}$$ 12

$$\int_M \frac{u_\varepsilon^2(x)}{d_g(x,x_0)^{2s}}\,dv_g(x)\geq \int_{\mathbb{R}^n}\frac{U^2(x)}{|x|^{2s}}dx-\begin{cases} C\epsilon^{\gamma_\mu(n-2s)} & \text{if } \gamma_\mu(n-2s)<2,\\ C\epsilon^2|\ln\epsilon| & \text{if } \gamma_\mu(n-2s)=2,\\ C\epsilon^2 & \text{if } \gamma_\mu(n-2s)>2,\end{cases} \tag{4.5}$$ 13

$$\int_M k(x)|u_\epsilon(x)|^{2_s^*}\,dv_g(x)=\begin{cases} \max k(x)\|U\|_{2_s^*}^{2_s^*}\left[1+O(\varepsilon^{n\gamma_\mu})\right] & \text{if } n\gamma_\mu<2,\\ \max k(x)\|U\|_{2_s^*}^{2_s^*}\left[1+O(\varepsilon^2|\ln\varepsilon|)\right] & \text{if } n\gamma_\mu=2,\\ \max k(x)\|U\|_{2_s^*}^{2_s^*}\left[1+O(\varepsilon^2)\right] & \text{if } n\gamma_\mu>2,\end{cases} \tag{4.6}$$ 14

$$\int_M f(x)|u_\varepsilon|^p\,dv_g \geq \begin{cases} C\varepsilon^{\frac{\gamma_\mu(n-2s)p}{2}} - C\varepsilon^{n-\frac{(n-2s)p}{2}} & \text{if } n < \frac{2sp(1+\gamma_\mu)}{p(1+\gamma_\mu)-2}, \\ C\varepsilon^{n-\frac{(n-2s)p}{2}}|\ln\varepsilon| & \text{if } n = \frac{2sp(1+\gamma_\mu)}{p(1+\gamma_\mu)-2}, \\ -C\varepsilon^{\frac{\gamma_\mu(n-2s)p}{2}} + C\varepsilon^{n-\frac{(n-2s)p}{2}} & \text{if } n > \frac{2sp(1+\gamma_\mu)}{p(1+\gamma_\mu)-2}. \end{cases} \tag{4.7}$$ 15

Proof. As the explicit form of U remains undetermined, we adapt the test function framework to suit the manifold context. We define

$$m_\epsilon := \frac{U_\epsilon(\delta)}{U_\epsilon(\delta) - U_\epsilon(\theta\delta)}$$

and consider the non-increasing function

$$g_\epsilon(t) := \begin{cases} 2\big(m_\epsilon - 1\big)U_\epsilon(\delta) & \text{if } 0 \leq t \leq U_\epsilon(\theta\delta), \\ m_\epsilon^2\big(U_\epsilon(\theta\delta) - t\big) + 2\big(m_\epsilon - 1\big)U_\epsilon(\delta) & \text{if } U_\epsilon(\theta\delta) \leq t \leq U_\epsilon(\delta), \\ \big(m_\epsilon - 1\big)U_\epsilon(\delta) - t & \text{if } t \geq U_\epsilon(\delta). \end{cases}$$

Because of the absolute continuity of g_ϵ, we define the nondecreasing function

$$G_\epsilon(t) := \int_0^t \sqrt{-g_\epsilon'(r)}\,dr = \begin{cases} 0 & \text{if } 0 \leq t \leq U_\epsilon(\theta\delta), \\ m_\epsilon\big(t - U_\epsilon(\theta\delta)\big) & \text{if } U_\epsilon(\theta\delta) \leq t \leq U_\epsilon(\delta), \\ t & \text{if } t \geq U_\epsilon(\delta). \end{cases}$$

Consider the radially symmetric non-increasing function $u_\epsilon(x) := G_\epsilon\big(U_\epsilon(|x|)\big)$ which satisfies

$$u_\epsilon(x) = \begin{cases} U_\epsilon(|x|) & \text{if } |x| \leq \delta, \\ 0 & \text{if } |x| \geq \theta\delta. \end{cases}$$

Let $\exp_P : T_PM \to M$ denote the exponential map at $P \in M$. For some $r > 0$, the restriction $\exp_P|_{B_r} : B_r \to B_r(P)$ is a diffeomorphism. Define the scaled function on $B_{\theta\delta}(P)$ as

$$U_\epsilon(x) = \epsilon^{-\frac{n-2s}{2}} U\Big(\frac{1}{\epsilon}\exp_P^{-1}(x)\Big), \quad \text{for } x \in B_{\theta\delta}(P).$$

We define

$$u_\epsilon(x) = \eta\big(d_g(P,x)\big)U_\epsilon(x),$$

where $d_g(P,x)$ denotes the geodesic distance from P to x. Split the integral as

$$\begin{aligned} &[u_\epsilon]_{s,2}^2 \\ &= \iint_{M\times M} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y) \\ &= \iint_{B_\delta(P)\times B_\delta(P)} \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y) + 2\iint_D \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y) \\ &\quad + 2\iint_E \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y) + \iint_{B_\delta^C(P)\times B_\delta^C(P)} \frac{|u_\epsilon(x) - u_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y), \end{aligned} \tag{4.8}$$ 13b

where

$$\begin{aligned} D &= \big\{(x,y) \in M\times M \mid x \in B_\delta(P),\ y \in B_\delta^C(P),\ \text{and } d_g(x,y) > \delta/2\big\}, \\ E &= \big\{(x,y) \in M\times M \mid x \in B_\delta(P),\ y \in B_\delta^C(P),\ \text{and } d_g(x,y) \leq \delta/2\big\}. \end{aligned}$$

Let $\bar{x} = \exp_P^{-1}(x)$ and $\bar{y} = \exp_P^{-1}(y)$. Define

$$d_{g_\epsilon}(x,y) := d_g(\exp_P(\epsilon\bar{x}), \exp_P(\epsilon\bar{y}))$$

and $g_\epsilon(x) := (\exp_P^* g)(\epsilon\bar{x})$ as the pullback of g via the exponential chart at P. Then

$$\begin{aligned}
&\iint_{B_\delta(P)\times B_\delta(P)} \frac{|U_\epsilon(x)-U_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)\\
&= \epsilon^{-n+2s}\iint_{B_{\delta/\epsilon}\times B_{\delta/\epsilon}} \frac{|U(x)-U(y)|^2}{d_{g_\epsilon}(x,y)^{n+2s}}\cdot \epsilon^{2n}\,dv_{g_\epsilon}(x)\,dv_{g_\epsilon}(y) \qquad (4.9)\\
&= \epsilon^{n+2s}\iint_{B_{\delta/\epsilon}\times B_{\delta/\epsilon}} \frac{|U(x)-U(y)|^2}{d_{g_\epsilon}(x,y)^{n+2s}}\,dv_{g_\epsilon}(x)\,dv_{g_\epsilon}(y).
\end{aligned}$$

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For a fixed $R>0$, set

$$\ell_R = \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(B_R\times B_R)} \frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

By the dominated convergence theorem, $\ell_R\to 0$ as $R\to+\infty$. For sufficiently small $\epsilon>0$ with $R<\delta/\epsilon$

$$\iint_{(B_{\delta/\epsilon}\times B_{\delta/\epsilon})\setminus(B_R\times B_R)} \frac{|U(x)-U(y)|^2}{d_{g_\epsilon}(x,y)^{n+2s}}\,dv_{g_\epsilon}(x)\,dv_{g_\epsilon}(y) \le C\epsilon^{-n-2s}\ell_R.$$

By Lemma **??**, we have

$$\frac{\epsilon}{d_{g_\epsilon}(x,y)} \to \frac{1}{|x-y|} \quad \text{as } \epsilon\to 0$$

Thus, for any fixed $R>0$

$$\lim_{\epsilon\to 0}\epsilon^{n+2s}\iint_{B_R\times B_R} \frac{|U(x)-U(y)|^2}{d_{g_\epsilon}(x,y)^{n+2s}}\,dv_{g_\epsilon}(x)\,dv_{g_\epsilon}(y) = \iint_{B_R\times B_R}\frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

Since

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \iint_{B_R\times B_R}\frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \ell_R,$$

taking the limits $R\to\infty$ and $\epsilon\to 0$ gives

$$\lim_{\epsilon\to 0}\epsilon^{n+2s}\iint_{B_{\delta/\epsilon}\times B_{\delta/\epsilon}} \frac{|U(x)-U(y)|^2}{d_{g_\epsilon}(x,y)^{n+2s}}\,dv_{g_\epsilon}(x)\,dv_{g_\epsilon}(y) \le \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

To estimate the second term in (**??**), applying Lemma **??** yields

$$\begin{aligned}
&\iint_D \frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)\\
&\le C\iint_{\substack{x\in B_\delta\\ y\notin B_\delta\\ |x-y|\ge\delta/2}} \frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&\le C\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u_\epsilon(x)-u_\epsilon(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&= C\big[G_\epsilon(U_\epsilon)\big]_{s,2}^2\\
&\le C\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{(U_\epsilon(x)-U_\epsilon(y))(g_\epsilon(U_\epsilon(x))-g_\epsilon(U_\epsilon(y)))}{|x-y|^{n+2s}}\,dx\,dy\\
&= C\int_{\mathbb{R}^n} U_\epsilon^{2_s^*-1}(g_\epsilon(U_\epsilon)+U_\epsilon)dx + C\mu\int_{\mathbb{R}^n}\frac{U_\epsilon(g_\epsilon(U_\epsilon)+U_\epsilon)}{|x|^{2s}}dx - C\int_{\mathbb{R}^n} U_\epsilon^{2_s^*}dx - C\mu\int_{\mathbb{R}^n}\frac{U_\epsilon^2}{|x|^{2s}}dx\\
&= C\int_{\mathbb{R}^n} U_\epsilon^{2_s^*-1}\big(g_\epsilon(U_\epsilon)+U_\epsilon\big)dx + C\mu\int_{\mathbb{R}^n}\frac{U_\epsilon\big(g_\epsilon(U_\epsilon)+U_\epsilon\big)}{|x|^{2s}}dx - C\big[U_\epsilon\big]_{s,2}^2\\
&\le C\int_{\mathbb{R}^n} U_\epsilon^{2_s^*-1}(g_\epsilon(U_\epsilon)+U_\epsilon)dx + C\mu\int_{\mathbb{R}^n}\frac{U_\epsilon(g_\epsilon(U_\epsilon)+U_\epsilon)}{|x|^{2s}}dx.
\end{aligned}$$

For all $t > 0$

$$g_\epsilon(t) + t \le 2U_\epsilon(\delta)m_\epsilon = 2\epsilon^{-\frac{n-2s}{2}} U\Big(\frac{\delta}{\epsilon}\Big)\Big[1 - \frac{U(\frac{\theta\delta}{\epsilon})}{U(\frac{\delta}{\epsilon})}\Big]^{-1} \le 4\epsilon^{-\frac{n-2s}{2}} U\Big(\frac{\delta}{\epsilon}\Big).$$

Since

$$U(r) \le \frac{c_2}{[r^{1-\gamma_\mu}(1 + r^{2\gamma_\mu})]^{\frac{n-2s}{2}}},$$

we obtain

$$g_\epsilon(t) + t \le 4c\epsilon^{-\frac{n-2s}{2}} \left(\frac{\epsilon}{\delta}\right)^{\frac{(n-2s)(1-\gamma_\mu)}{2}} \cdot \frac{1}{\left(\frac{\delta}{\epsilon}\right)^{\gamma_\mu(n-2s)}} = 4c\frac{\epsilon^{\frac{(n-2s)\gamma_\mu}{2}}}{\delta^{\frac{(n-2s)(\gamma_\mu+1)}{2}}}.$$

The scaling properties of U_ϵ imply

$$\int_{\mathbb{R}^n} U_\varepsilon(x)^{2_s^*-1} dx = \varepsilon^{\frac{n-2s}{2}} \int_{\mathbb{R}^n} U(x)^{2_s^*-1} dx,$$
$$\int_{\mathbb{R}^n} \frac{U_\epsilon(x)}{|x|^{2s}} dx = \epsilon^{\frac{n-2s}{2}} \int_{\mathbb{R}^n} \frac{U(x)}{|x|^{2s}} dx.$$

Since integrals $\int_{\mathbb{R}^n} U(x)dx$ and $\int_{\mathbb{R}^n} \frac{U(x)}{|x|^{2s}}dx$ are finite, we have

$$\iint_D \frac{|U_\epsilon(x) - U_\epsilon(y)|^2}{d_g(x,y)^{n+2s}}\, dv_g(x)\, dv_g(y) = O\left(\epsilon^{\frac{(n-2s)(1+\gamma_\mu)}{2}}\right).$$

The remaining terms in (**??**) are estimated similarly, establishing (**??**). □

Next, we analyze the singular integral

$$\mathcal{H}(\epsilon) = \int_M \frac{u_\epsilon^2(x)}{d_g(x,x_0)^{2s}}\, dv_g(x).$$

Using normal coordinates centered at x_0, we localize the integral to a geodesic ball $B_{\theta\delta}(x_0)$. The metric volume expansion $dv_g = (1 + O(|x|^2))dx$ gives

$$\mathcal{H}(\epsilon) = \int_{|x|\le\theta\delta} \frac{u_\epsilon^2(x)}{|x|^{2s}} \left(1 + O(|x|^2)\right) dx = \int_{|x|\le\theta\delta} \frac{u_\epsilon^2(x)}{|x|^{2s}} dx + \int_{|x|\le\theta\delta} \frac{u_\epsilon^2(x)}{|x|^{2s}} O(|x|^2) dx.$$

The first term satisfies

$$I_1 = \int_{|x|\le\theta\delta} \frac{u_\epsilon^2(x)}{|x|^{2s}} dx = \int_{|x|\le\delta/\epsilon} \frac{U^2(x)}{|x|^{2s}} dx + O(1) = \int_{\mathbb{R}^n} \frac{U^2(x)}{|x|^{2s}} dx - \int_{|x|>\delta/\epsilon} \frac{U^2(x)}{|x|^{2s}} dx + O(1),$$

where $O(1)$ arises from the integral over $\delta < |x| \le \theta\delta$. Using the asymptotic decay of $U(x)$, we obtain

$$\begin{aligned}\int_{|x|>\delta/\epsilon} \frac{U^2(x)}{|x|^{2s}} dx &\le c_2 \int_{|x|>\delta/\epsilon} \frac{1}{\left(|x|^{1-\gamma_\mu}(1 + |x|^{2\gamma_\mu})\right)^{n-2s} |x|^{2s}} dx \\ &= c_2\omega_{n-1} \int_{\delta/\epsilon}^\infty \frac{r^{n-1}}{r^{n-\gamma_\mu(n-2s)}(1 + r^{2\gamma_\mu})^{n-2s}} dr \\ &\le C \int_{\delta/\epsilon}^\infty r^{-1+\gamma_\mu(n-2s)} dr \\ &\le C\epsilon^{\gamma_\mu(n-2s)}.\end{aligned}$$

Thus

$$I_1 \ge \int_{\mathbb{R}^n} \frac{U^2(x)}{|x|^{2s}} dx - C\epsilon^{\gamma_\mu(n-2s)} + O(1). \tag{4.10}$$ 17

For the second integral,

$$\begin{aligned}I_2 &= \int_{|x|\le\theta\delta} \frac{u_\epsilon^2(x)}{|x|^{2s}} O(|x|^2) dx \\ &\ge -C \int_{|x|\le\theta\delta} \frac{u_\epsilon^2(x)}{|x|^{2s}} |x|^2 dx\end{aligned}$$

$$
\begin{aligned}
&= -C\epsilon^2 \int_{|x|\le \theta\delta/\epsilon} \frac{U^2(x)}{|x|^{2s-2}} dx \\
&= -C\epsilon^2 \int_0^{\theta\delta/\epsilon} U^2(r) r^{n+1-2s} dr.
\end{aligned}
$$

By the behavior of $U(r)$,

$$
\begin{aligned}
\epsilon^2 \int_0^{\theta\delta/\epsilon} U^2(r) r^{n+1-2s} dr &= \epsilon^2 \Big[\int_0^1 U^2(r) r^{n+1-2s} dr + \int_1^{\theta\delta/\epsilon} U^2(r) r^{n+1-2s} dr \Big] \\
&\le C\epsilon^2 \Big[\int_1^{\theta\delta/\epsilon} r^{1-\gamma_\mu(n-2s)} dr + O(1) \Big] \\
&\le C\epsilon^2 \Big[\int_1^{\theta\delta/\epsilon} r^{1-\gamma_\mu(n-2s)} dr + O(1) \Big] \\
&\le \begin{cases} C\epsilon^{\gamma_\mu(n-2s)} & \text{if } \gamma_\mu(n-2s) < 2, \\ C\epsilon^2 |\ln\epsilon| & \text{if } \gamma_\mu(n-2s) = 2, \\ C\epsilon^2 & \text{if } \gamma_\mu(n-2s) > 2. \end{cases}
\end{aligned} \tag{4.11}
$$ 18

Since $O(1)$ is positive, combining estimates (**??**) and (**??**) for sufficiently small $\epsilon > 0$ yields

$$
\begin{aligned}
\mathcal{H}(\epsilon) &\ge \int_{\mathbb{R}^n} \frac{U^2(x)}{|x|^{2s}} dx - C\epsilon^{\gamma_\mu(n-2s)} - \begin{cases} C\epsilon^{\gamma_\mu(n-2s)} & \text{if } \gamma_\mu(n-2s) < 2, \\ C\epsilon^2 |\ln\epsilon| & \text{if } \gamma_\mu(n-2s) = 2, \\ C\epsilon^2 & \text{if } \gamma_\mu(n-2s) > 2. \end{cases} \\
&= \int_{\mathbb{R}^n} \frac{U^2(x)}{|x|^{2s}} dx - \begin{cases} C\epsilon^{\gamma_\mu(n-2s)} & \text{if } \gamma_\mu(n-2s) < 2, \\ C\epsilon^2 |\ln\epsilon| & \text{if } \gamma_\mu(n-2s) = 2, \\ C\epsilon^2 & \text{if } \gamma_\mu(n-2s) > 2. \end{cases}
\end{aligned}
$$

Next, we consider the integral transformation at a maximum point P of $k(x)$. In $B_{\theta\delta}(P)$, Taylor expansion gives $k(x) = k(P) + O(|x|^2)$. Then

$$
\begin{aligned}
\int_M k(x) |u_\varepsilon|^{2_s^*} dv_g &= \int_{|x|\le\theta\delta} k(x) (\eta U_\varepsilon)^{2_s^*} \sqrt{|\mathrm{g}|} dx \\
&= \int_{|x|\le\delta} \big[k(P) + O(|x|^2)\big] U_\varepsilon^{2_s^*}(x) dx + O(1) \\
&= k(P) \int_{|x|\le\delta} U_\varepsilon^{2_s^*}(x) dx + \int_{|x|\le\delta} U_\varepsilon^{2_s^*}(x) O(|x|^2) dx + O(1).
\end{aligned}
$$

For the first integral,

$$
\int_{|x|\le\delta} U_\varepsilon^{2_s^*}(x) dx = \int_{|x|\le\frac{\delta}{\varepsilon}} U^{2_s^*}(x) dx = \|U\|_{2_s^*}^{2_s^*} + O(1).
$$

For the second integral, we have

$$
\begin{aligned}
\int_{|x|\le\delta} U_\varepsilon^{2_s^*}(x) |x|^2 dx &= \varepsilon^2 \int_{|x|\le\delta/\varepsilon} U^{2_s^*}(|x|) |x|^2 dx \\
&= C\varepsilon^2 \Big(\int_0^1 U^{2_s^*}(r) r^{n+1} dr + \int_1^{\frac{\delta}{\varepsilon}} U^{2_s^*}(r) r^{n+1} dr \Big).
\end{aligned} \tag{4.12}
$$ 19

By the properties of $U(r)$, the first integral in (**??**) is constant, so we estimate the second integral over $[1, \frac{\delta}{\varepsilon}]$. Using the decay of $U(r)$,

$$
\int_1^{\frac{\delta}{\varepsilon}} U^{2_s^*}(r) r^{n+1} dr \le C \int_1^{\frac{\delta}{\varepsilon}} r^{n+1} r^{-(1+\gamma_\mu)n} dr = C \int_1^{\frac{\delta}{\varepsilon}} r^{1-n\gamma_\mu} dr.
$$

Thus,

$$\int_{|x|\le\delta} U^{2_s^*}(s)|x|^2\,dx = \begin{cases} O(\varepsilon^{n\gamma_\mu}) & \text{if } n\gamma_\mu < 2,\\ O(\varepsilon^2|\ln\varepsilon|) & \text{if } n\gamma_\mu = 2,\\ O(\varepsilon^2) & \text{if } n\gamma_\mu > 2.\end{cases}$$

Then

$$\int_M k(x)|u_\varepsilon|^{2_s^*}\,dv_g = k(P)\|U\|_{2_s^*}^{2_s^*} + \begin{cases} O(\varepsilon^{n\gamma_\mu}) & \text{if } n\gamma_\mu < 2,\\ O(\varepsilon^2|\ln\varepsilon|) & \text{if } n\gamma_\mu = 2,\\ O(\varepsilon^2) & \text{if } n\gamma_\mu > 2.\end{cases}$$

For the subcritical case, by hypothesis (H2), we have $f(x) > c > 0$ on $B_\delta(x_0)$. Then

$$\begin{aligned}\int_M f(x)|u_\epsilon|^p\,dv_g &\ge C\int_{|x|\le\delta} u_\epsilon^p dx\\ &\ge C\varepsilon^{n-\frac{(n-2s)p}{2}}\int_{1\le|x|\le\delta/\varepsilon}\frac{1}{\left(|x|^{1-\gamma_\mu}(1+|x|^{2\gamma_\mu})\right)^{\frac{(n-2s)p}{2}}}dx\\ &\ge C\varepsilon^{n-\frac{(n-2s)p}{2}}\int_1^{\delta/\varepsilon}\frac{r^{n-1}}{\left(r^{1-\gamma_\mu}(1+r^{2\gamma_\mu})\right)^{\frac{(n-2s)p}{2}}}dr\\ &\ge \begin{cases} C\varepsilon^{\frac{\gamma_\mu(n-2s)p}{2}} - C\varepsilon^{n-\frac{(n-2s)p}{2}} & \text{if } n < \frac{2sp(1+\gamma_\mu)}{p(1+\gamma_\mu)-2},\\ C\varepsilon^{n-\frac{(n-2s)p}{2}}|\ln\varepsilon| & \text{if } n = \frac{2sp(1+\gamma_\mu)}{p(1+\gamma_\mu)-2},\\ -C\varepsilon^{\frac{\gamma_\mu(n-2s)p}{2}} + C\varepsilon^{n-\frac{(n-2s)p}{2}} & \text{if } n > \frac{2sp(1+\gamma_\mu)}{p(1+\gamma_\mu)-2}.\end{cases}\end{aligned}$$

Lemma 4.6. *Suppose $k(x)$ attains its maximum at $P \in M$. For sufficiently small $\varepsilon > 0$,*

$$\frac{\mathcal{E}_\mu(u_\varepsilon)}{\chi(u_\varepsilon)} \le \begin{cases} (\max k)^{-2/2_s^*}S_\mu^{-2}(n,s,2) + O(\varepsilon^{\gamma_\mu(n-2s)}) & \text{if } \gamma_\mu(n-2s) < 2,\\ (\max k)^{-2/2_s^*}S_\mu^{-2}(n,s,2) + O(\varepsilon^2|\ln\varepsilon|) & \text{if } \gamma_\mu(n-2s) = 2,\\ (\max k)^{-2/2_s^*}S_\mu^{-2}(n,s,2) + O(\varepsilon^2) & \text{if } \gamma_\mu(n-2s) > 2.\end{cases}$$

Proof of Theorem **??**. Define the constants

$$\alpha = \frac{\gamma_\mu(n-2s)p}{2},\quad \beta = n - \frac{(n-2s)p}{2},\quad D = \frac{2sp(1+\gamma_\mu)}{p(1+\gamma_\mu)-2},\quad \gamma = \frac{2}{\gamma_\mu} + 2s.$$

Under the condition $2 < p < 2_s^*$, a direct computation shows that $\gamma > D$. By hypothesis (H2), there exists $C > 0$ such that for sufficiently small $\varepsilon > 0$,

$$\int_M f(x)|u_\varepsilon|^p\,dv_g \ge \begin{cases} C\varepsilon^\alpha & \text{if } n < D,\\ C\varepsilon^\beta|\ln\varepsilon| & \text{if } n = D,\\ C\varepsilon^\beta & \text{if } n > D.\end{cases}$$

By Proposition **??**, we obtain

$$\sup_{t\ge0} I(tu_\varepsilon) \le \frac{s}{n}\Big(\frac{\mathcal{E}(u_\varepsilon)}{\chi(u_\varepsilon)}\Big)^{\frac{n}{2s}} - \lambda C\int_M f(x)|u_\varepsilon|^p\,dv_g.$$

We now analyze three cases based on the value of $\gamma_\mu(n-2s)$.

Case 1. $\gamma_\mu(n-2s) < 2$, which implies $n < \gamma$. When $n > D$, we have

$$\sup_{t\ge0} I(tu_\varepsilon) \le \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2) + C\varepsilon^{\gamma_\mu(n-2s)} - \lambda C\varepsilon^\beta.$$

If $\beta < \gamma_\mu(n-2s)$, equivalent to $n > \frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2}$, then

$$\sup_{t\ge0} I(tu_\varepsilon) < \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2).$$

When $n = D$,

$$\sup_{t\ge0} I(tu_\varepsilon) \le \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2) + C\varepsilon^{\gamma_\mu(n-2s)} - \lambda C\varepsilon^\beta|\ln\varepsilon|,$$

which fails to provide the desired estimate. For $n < D$,

$$\sup_{t\geq 0} I(tu_\varepsilon) \leq \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2) + C\varepsilon^{\gamma_\mu(n-2s)} - \lambda C\varepsilon^\alpha.$$

Since $\frac{\gamma_\mu(n-2s)p}{2} - \gamma_\mu(n-2s) > 0$, this estimate is not valid. Thus, for $\frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2} < n < \gamma$, we have

$$\sup_{t\geq 0} I(tu_\varepsilon) < \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2).$$

Case 2. $\gamma_\mu(n-2s) = 2$, which implies $n = \gamma > D$. Then

$$\sup_{t\geq 0} I(tu_\varepsilon) \leq \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2) + C\varepsilon^2 - \lambda C\varepsilon^\beta.$$

Here $\beta = n - \frac{(n-2s)p}{2} \leq n-(n-2s) = 2s < 2$, so

$$\sup_{t\geq 0} I(tu_\varepsilon) < \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2).$$

Case 3. $\gamma_\mu(n-2s) > 2$, which implies $n > \gamma > D$. The analysis is similar to Case 2. Since $\beta < 2$,

$$\sup_{t\geq 0} I(tu_\varepsilon) < \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2).$$

Combining these cases, when $n > \frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2}$, we have

$$\sup_{t\geq 0} I(tu_\varepsilon) < \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2).$$

Now consider $n \leq \frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2}$. Then

$$\sup_{t\geq 0} I(tu_\varepsilon) \leq \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2) + C\varepsilon^{\gamma_\mu(n-2s)} - \lambda C\begin{cases}\varepsilon^\alpha & \text{if } n < D,\\ \varepsilon^\beta|\ln\varepsilon| & \text{if } n = D,\\ \varepsilon^\beta & \text{if } n > D.\end{cases}$$

For sufficiently small $\varepsilon > 0$, choose $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$,

$$\frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2) + C\varepsilon^{\gamma_\mu(n-2s)} < \lambda C\begin{cases}\varepsilon^\alpha & \text{if } n < D,\\ \varepsilon^\beta|\ln\varepsilon| & \text{if } n = D,\\ \varepsilon^\beta & \text{if } n > D.\end{cases}$$

This implies

$$\sup_{t\geq 0} I(tu_\varepsilon) < 0 < \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2).$$

Therefore, for any $\lambda > 0$ when $n > \frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2}$, and for $\lambda > \lambda_0$ when $n \leq \frac{2s(2\gamma_\mu+p)}{p+2\gamma_\mu-2}$, we have

$$\sup_{t\geq 0} I(tu_\varepsilon) < \frac{s}{n}(\max k)^{-\frac{n-2s}{2s}} S_\mu^{\frac{n}{2s}}(n,s,2).$$

By Lemma **??**, there exists a nontrivial solution to (**??**). □

5. Appendix

ThA1 **Theorem 5.1.** *Under hypothesis (H3), for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that the inequality*

$$\Big(\int_M |u|^{2_s^*}\,dv_g(x)\Big)^{2/2_s^*} \leq (S_\mu^2(n,s,2)+\epsilon)\Big[\iint_{M\times M} \frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y) - \mu\int_M \frac{|u|^2}{d_g(x,x_0)^{2s}}\,dv_g(x)\Big] + C_\epsilon\int_M |u|^2\,dv_g(x)$$

holds for all $u \in W^{s,2}(M)$.

Proof. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that $\delta < \mathrm{inj}(M)$. Cover M by geodesic balls $\{B_\delta(x_i)\}_{i=1}^m$ with subordinate partition of unity $\{\eta_i\}_{i=1}^m$. Jensen's inequality gives

$$\|u\|_{2_s^*}^2 \le \frac{1}{m}\sum_{i=1}^m \|\eta_i u\|_{2_s^*}^2.$$

We define $v_i = (\eta_i u)\circ \exp_{x_i}$, which implies

$$\int_M |\eta_i u|^{2_s^*}\, dv_g \le (1+\varepsilon)\int_{\mathbb{R}^n} |v_i|^{2_s^*} dx.$$

The critical Sobolev inequality in $\mathbb{R}^n$ gives

$$\|\eta_i u\|_{2_s^*}^2 \le (1+\varepsilon) S_\mu(n,s,2)^2\Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \mu\int_{\mathbb{R}^n}\frac{v_i^2}{|x|^{2s}}dx\Big). \tag{5.1}$$ eq19

For sufficiently small $\delta_1 > 0$, we decompose

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &= \iint_{|x-y|\ge\delta_1} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \iint_{|x-y|<\delta_1} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

The first term satisfies

$$\begin{aligned}\iint_{|x-y|\ge\delta_1} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy &\le 2\int_{\mathbb{R}^n}|v_i(x)|^2\Big(\int_{|z|\ge\delta_1}\frac{1}{|z|^{n+2s}}dz\Big)dx\\ &\le C\delta_1^{-2s}\int_M |\eta_i u|^2\,dv_g.\end{aligned} \tag{5.2}$$ eq20

Let $U_i = \exp_{x_i}^{-1}(B_\delta(x_i))$. Since $v_i(x)=0$ for $x\notin U_i$, we obtain

$$\begin{aligned}&\iint_{|x-y|<\delta_1} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &= 2\iint_{\substack{|x-y|<\delta_1\\ x\in U_i,\, y\notin U_i}} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \iint_{\substack{|x-y|<\delta_1\\ x,y\in U_i}} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.\end{aligned}$$

Let $K_i = \exp_{x_i}^{-1}(\mathrm{supp}(\eta_i))$. There exists $a > 0$ such that $|x-y|\ge a$ for all $x\in K_i$, $y\in\mathbb{R}^n\setminus U_i$, and all i. Then

$$\iint_{\substack{|x-y|<\delta_1\\ x\in U_i, y\notin U_i}} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le C\int_{K_i}|v_i(x)|^2dx \le C\int_M |\eta_i u|^2\,dv_g. \tag{5.3}$$ eq21

Since

$$v_i(x)-v_i(y) = \eta_i(\exp_{x_i}(x))\left[u(\exp_{x_i}(x)) - u(\exp_{x_i}(y))\right] + u(\exp_{x_i}(y))\left[\eta_i(\exp_{x_i}(x)) - \eta_i(\exp_{x_i}(y))\right],$$

applying the inequality $(a+b)^2 \le (1+\varepsilon)a^2 + C_\varepsilon b^2$ yields

$$\iint_{\substack{|x-y|<\delta_1\\ x,y\in U_i}} \frac{|v_i(x)-v_i(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \le (1+\varepsilon)I + C_\varepsilon II,$$

where

$$\begin{aligned}I &= \iint_{\substack{|x-y|<\delta_1\\ x,y\in U_i}} |\eta_i(\exp_{x_i}(x))|^2 \frac{|u(\exp_{x_i}(x)) - u(\exp_{x_i}(y))|^2}{|x-y|^{n+2s}}\,dx\,dy,\\ II &= \iint_{\substack{|x-y|<\delta_1\\ x,y\in U_i}} \frac{|\eta_i(\exp_{x_i}(x)) - \eta_i(\exp_{x_i}(y))|^2}{|x-y|^{n+2s}} |u(\exp_{x_i}(y))|^2\,dx\,dy.\end{aligned}$$

Since $(\eta_i u)\circ\exp_{x_i}$ is Lipschitz with a uniform constant, we have

$$II \le C\int_{B_\delta(x_i)} |u|^2\,dv_g. \tag{5.4}$$ eq22

For I, we have

$$\begin{aligned} I &= \iint_{\substack{|x-y|<\delta_1\\ x,y\in U_i}} |\eta_i(\exp_{x_i}(x))|^2 \frac{|u(\exp_{x_i}(x))-u(\exp_{x_i}(y))|^2}{|x-y|^{n+2s}}\,dx\,dy\\ &= \iint_{\substack{|\exp_{x_i}^{-1}(x)-\exp_{x_i}^{-1}(y)|<\delta_1\\ x,y\in B_\delta(x_i)}} \eta_i(x)^2 \frac{|u(x)-u(y)|^2}{|\exp_{x_i}^{-1}(x)-\exp_{x_i}^{-1}(y)|^{n+2s}}\,dv_g(x)\,dv_g(y)\\ &\le \iint_{x,y\in B_\delta(x_i)} \eta_i(x)^2\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y). \end{aligned} \tag{5.5}$$

eq23

The Hardy potential term satisfies

$$\begin{aligned} \int_{\mathbb{R}^n}\frac{v_i^2}{|x|^{2s}}dx &= \int_{x\in U_i}\frac{v_i^2}{|x|^{2s}}\,dx+\int_{x\notin U_i}\frac{v_i^2}{|x|^{2s}}\,dx\\ &\ge \int_{x\in U_i}\frac{v_i^2}{|x|^{2s}}\,dx\\ &= \int_{x\in U_i}\frac{\eta_i^2(\exp_{x_i}(x))u^2(\exp_{x_i}(x))}{|x|^{2s}}\,dx\\ &= \int_{x\in B_\delta(x_i)}\frac{\eta_i^2(x)u^2(x)}{|\exp_{x_i}^{-1}(x)-\exp_{x_i}^{-1}(x_0)|^{2s}}\,dv_g(x)\\ &\ge \int_{B_\delta(x_i)}\frac{\eta_i^2u^2}{d_g(x,x_0)^{2s}}\,dv_g(x). \end{aligned} \tag{5.6}$$

eq24

Combining estimates (**??**) to (**??**), we obtain

$$\begin{aligned} &\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)-\mu\int_M\frac{u^2}{d_g(x,x_0)^{2s}}\,dv_g\\ &\le \sum_{i=1}^m\Big[C\int_M|\eta_i u|^2\,dv_g+C\int_M|u|^2\,dv_g\\ &\quad+\iint_{x,y\in B_\delta(x_i)}\eta_i^2\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)-\mu\int_{B_\delta(x_i)}\frac{\eta_i^2u^2}{d_g(x,x_i)^{2s}}\,dv_g\Big]\\ &\le C\int_M|u|^2\,dv_g+\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)-\mu\int_M\frac{u^2}{d_g(x,x_0)^{2s}}\,dv_g. \end{aligned} \tag{5.7}$$

eq25

Therefore,

$$\begin{aligned} \|u\|_{2_s^*}^2 \le \big(S_\mu(n,s,2)^2+\varepsilon\big)\Big(&\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)\\ &-\mu\int_M\frac{u^2}{d_g(x,x_0)^{2s}}\,dv_g\Big)+C\int_M|u|^2\,dv_g. \end{aligned}$$

To establish sharpness, assume there exist constants $c_1,c_2>0$ such that for all $u\in W^{s,2}(M)$

$$\begin{aligned} &\Big(\int_M|u|^{2_s^*}\,dv_g\Big)^{2/2_s^*}\\ &\le c_1\Big(\iint_{M\times M}\frac{|u(x)-u(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y)-\mu\int_M\frac{u^2}{d_g(x,x_0)^{2s}}\,dv_g\Big)+c_2\int_M|u|^2\,dv_g. \end{aligned} \tag{5.8}$$

eq27

Let $\eta:[0,\infty)\to[0,1]$ be a smooth cutoff function with $\mathrm{supp}(\eta)\subset[0,\theta\delta]$ and $\eta\equiv1$ on $[0,\delta]$, where $\theta\delta<\mathrm{inj}(M)$. For $x_0\in M$, define the test function

$$u_\varepsilon(x)=\eta(d_g(x,x_0))U_\varepsilon(x),\quad\text{where}\quad U_\varepsilon(x)=\varepsilon^{-\frac{n-2s}{2}}U\Big(\frac{\exp_{x_0}^{-1}(x)}{\varepsilon}\Big).$$

From (**??**), we have

$$\begin{aligned} &\Big(\int_M |u_\varepsilon(x)|^{2_s^*}\,dv_g\Big)^{2/2_s^*} \\ &\le c_1\Big(\iint_{M\times M} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{d_g(x,y)^{n+2s}}\,dv_g(x)\,dv_g(y) - \mu\int_M \frac{u_\varepsilon^2}{d_g(x,x_0)^{2s}}\,dv_g\Big) + c_2\int_M |u_\varepsilon|^2\,dv_g. \end{aligned} \tag{5.9}$$ eq28

As $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\int_M |u_\varepsilon|^{2_s^*}\,dv_g = \int_{\mathbb{R}^n} |U|^{2_s^*}dx. \tag{5.10}$$ eq29

Substituting this into (**??**) and applying Lemma **??**, we obtain in the limit as $\varepsilon \to 0$,

$$\Big(\int_{\mathbb{R}^n} |U|^{2_s^*}dx\Big)^{2/2_s^*} \le c_1\Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|U(x)-U(y)|^2}{|x-y|^{n+2s}}\,dx\,dy - \mu\int_{\mathbb{R}^n} \frac{U^2}{|x|^{2s}}dx\Big). \tag{5.11}$$ eq30

Since U is an extremizer for (**??**), the optimal constant equals $S_\mu(n,s,2)^2$, hence

$$c_1 \ge S_\mu(n,s,2)^2.$$

This proves the sharpness of $S_\mu(n,s,2)^2$. □

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