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DISCRETISATION OF THE ERMAKOV-PAINLEVÉ II EQUATION: DIRICHLET AND ROBIN-TYPE BOUNDARY VALUE PROBLEMS

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Abstract. Two-point boundary value problems for a discrete Ermakov-Painle-vé II equation are analysed by means of topological methods. In addition, an alternative variational approach is detailed. Existence of solutions is established for appropriate choice of parameters.

1. Introduction

Ermakov-type nonlinear coupled systems with genesis in the classical work [14] have diverse physical applications in both physics and continuum mechanics [30]. These occur, *inter alia* in nonlinear optics [12, 17, 18, 19, 20, 28, 29, 35], spiralling elliptic soliton analysis [13], Bose-Einstein condensate theory [1], oceanographic warm-core eddy evolution [26], 2+1-dimensional magnetogasdynamics [32] and rotating shallow water system theory [5, 27]. A key feature of these systems is the existence of a so-called *Ermakov invariant*, namely a conserved quantity that underpins their integrability and allows for the derivation of nonlinear superposition principles.

On the other hand, the classical Painlevé equations PI-PVI are well-established for their role in modern soliton theory and integrable systems [8, 10]. In particular, the Painlevé II equation notably has diverse physical applications not only in a solitonic context but also in boundary value problems associated with the Nernst-Planck model of ion transport [23, 24]. In [6], Painlevé II was derived *ab initio* in the analysis of a boundary value problem which determines the electric field distribution in a region occupied by an electrolyte.

More recently, a significant advancement has been the introduction of hybrid *Ermakov-Painlevé* systems, which merge the structure of Ermakov models with the profound integrability properties of the Painlevé equations. These hybrids have extended the applicability of Ermakov systems to new areas, including cold plasma physics, electrodiffusion, and capillarity theory. A distinguished member of the family of the hybrid Ermakov-Painlevé systems is the EP-II equation, namely

$$y''(z) = Ay(z)^3 + Bzy(z) + \frac{C}{y(z)^3}, \quad z \in (0,1). \tag{1.1}$$

which provides a rich framework for studying nonlinear wave propagation and other complex dynamics. Boundary value problems for this equation have been recently studied in [4], however, for numerical simulation and the analysis of discrete physical systems, a discrete analogue is essential. This motivates the study of discretizations that faithfully preserve the core mathematical properties of the original continuous system.

In this paper, we undertake a systematic study of a discrete version of the Ermakov-Painlevé II equation. Specifically, we investigate the finite difference equation

$$\Delta^2 u_{x-1} = au_x^3 + bxu_x + \frac{c}{u_x^3}, \quad x = 1, \dots, N-1, \tag{1.2}$$

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subject to two types of boundary conditions: the Dirichlet condition

$$u_0 = D_0, \quad u_N = D_N, \quad D_0, D_N > 0, \tag{1.3}$$

and the nonlinear Robin condition

$$\Delta u_0 = f_0(u_0), \quad \Delta u_{N-1} = f_N(u_N), \tag{1.4}$$

where f_0, $f_N : (0,+\infty) \to \mathbb{R}$ are given continuous functions. Our objective is to establish the existence of positive solutions for these discrete boundary value problems. It is worth observing that, in the literature on singular second-order equations, the cases $c > 0$ and $c < 0$ are usually referred to as repulsive and attractive, respectively, and their analysis often requires different mathematical tools. For a discrete version of this terminology in difference equations, see [22].

The literature on discretisations of Painlevé-type equations often focuses on the preservation of structural features such as integrability or Bäcklund transformations. Indeed, the derivation of discrete Painlevé equations from Bäcklund and Schlesinger transforms of their continuous counterparts was introduced by Fokas, Grammaticos and Ramani [11] and remains an active area of research (see, e.g., [9, 34]). However, in the present work, our primary goal is not the preservation of such invariants but rather the analysis of boundary value problems for the discrete equation using topological and variational methods. The standard finite difference scheme adopted here, while not designed to preserve the Ermakov invariant, is sufficient for the existence and uniqueness analysis we undertake. Moreover, the convergence of this scheme to the continuous problem can be established independently as in [25]. A systematic study of structure-preserving discretisations of the Ermakov-Painlevé II equation is beyond the scope of this paper and is left for future investigation.

It is important to emphasize that the methods used for the continuous Ermakov-Painlevé II model cannot, in general, be extended to its discretised version. This is particularly true for the repulsive case, which requires the development of an alternative method and leaves an open problem to be considered in a subsequent paper. Our work thus bridges the gap between the well-established theory of continuous Ermakov-Painlevé systems and the requirements of discrete analysis and computation. By providing a rigorous treatment of this discrete model, we aim to open new avenues for its application in numerical and physically discrete contexts.

2. Preliminaries

In this section, we summarize some useful facts concerning second order discrete equations that shall be employed throughout the paper. Some short proofs are included for the reader's convenience.

As usual, we may identify the set of all the functions $u : \{0,\dots,N\} \to \mathbb{R}$ with $\mathbb{R}^{N+1}$ and denote $u(x) := u_x$. The first and second order discrete derivatives of u shall be denoted respectively by

$$\Delta u_x := u_{x+1} - u_x \quad x = 0,\dots,N-1,$$
$$\Delta^2 u_{x-1} := \Delta u_x - \Delta u_{x-1} = u_{x+1} - 2u_x + u_{x-1} \quad x = 1,\dots,N-1.$$

It is noted that the discretisation given by (1.2) follows the standard procedure of a finite difference scheme for the continuous Ermakov-Painlevé II equation (1.1). Indeed, letting $u_x := y\left(\frac{x}{N}\right)$ for $x = 0,\dots,N$ and taking into account that $y''\left(\frac{x}{N}\right) \simeq \frac{1}{N^2}\Delta^2 u_{x-1}$, the discrete equation (1.2) is obtained, with $a = \frac{A}{N^2}$, $b = \frac{B}{N^3}$ and $c = \frac{C}{N^2}$. For a more detailed exposition of this discretisation in a similar context, see e.g. [21].

Although u, Δu and $\Delta^2 u$ belong to different spaces, we shall always employ the notation $\|\cdot\|_2$ to refer to the Euclidean norm, namely

$$\|u\|_2 = \Big(\sum_{x=0}^{N} u_x^2\Big)^{1/2}, \quad \|\Delta u\|_2 = \Big(\sum_{x=0}^{N-1} (\Delta u_x)^2\Big)^{1/2},$$
$$\|\Delta^2 u\|_2 = \Big(\sum_{x=0}^{N-2} (\Delta^2 u_x)^2\Big)^{1/2},$$

Lemma 2.1. *Assume that $u_0 = u_N = 0$ or $\Delta u_0 = \Delta u_{N-1} = 0$, then*

$$\|\Delta u\|_2^2 \le \|u\|_2 \|\Delta^2 u\|_2.$$

Proof. We shall use the summation by parts formula, slightly adapted to the present context. A detailed deduction is included for the reader's convenience:

$$\begin{aligned}\sum_{x=1}^{N-1} \Delta^2 u_{x-1} u_x &= \sum_{x=1}^{N-1} (\Delta u_x - \Delta u_{x-1}) u_x \\ &= \sum_{x=1}^{N-1} \Delta u_x u_x - \sum_{x=0}^{N-2} \Delta u_x u_{x+1} \\ &= \Delta u_{N-1} u_{N-1} - \Delta u_0 u_1 + \sum_{x=1}^{N-2} \Delta u_x (u_x - u_{x+1}) \\ &= \Delta u_{N-1} u_{N-1} - \Delta u_0 u_1 - \sum_{x=1}^{N-2} (\Delta u_x)^2.\end{aligned}$$

Under any of the two assumptions, it is immediately seen that

$$\Delta u_{N-1} u_{N-1} - \Delta u_0 u_1 = -(\Delta u_{N-1})^2 - (\Delta u_0)^2,$$

so the result follows from the Cauchy-Schwarz inequality. □

Lemma 2.2. *Assume that $u_0 = u_N = 0$ and let $\lambda_1 := 4 \sin^2\left(\frac{\pi}{2N}\right)$, then*

1. $\lambda_1 \|u\|_2^2 \le \|\Delta u\|_2^2$;
2. $\|u\|_2 \le \frac{1}{\lambda_1} \|\Delta^2 u\|_2$.

Proof. The first inequality follows from the fact that λ_1 is the first eigenvalue of the problem

$$-\Delta^2 u_{x-1} = \lambda u_x \quad u_0 = u_N = 0,$$

while the second one is deduced from the first one, combined with the preceding lemma. □

Lemma 2.3. *Let $g_x : \mathbb{R} \to \mathbb{R}$ be continuous and bounded for $x = 1, \dots, N-1$. Then the equation*

$$\Delta^2 u_{x-1} - u_x = g_x(u_x) \quad x = 1, \dots, N-1$$

has at least one solution, both for arbitrary Dirichlet conditions (1.3) *and Robin conditions* (1.4) *with $f_0, f_N : \mathbb{R} \to \mathbb{R}$ continuous and bounded.*

Proof. For each $v \in \mathbb{R}^{N+1}$ and both conditions (1.3) and (1.4), define $T(v) := u$ as the unique solution of the linear problem

$$Lu_x := \Delta^2 u_{x-1} - u_x = g_x(v_x) \quad x = 1, \dots, N-1$$

satisfying (1.3) or

$$\Delta u_0 = f_0(v_0), \quad \Delta u_{N-1} = f_N(v_N)$$

respectively. It is verified that $T : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ is well defined and continuous. In fact, in both cases, the problem may be written in matrix form as $\mathcal{A}u = \mathcal{B}$, where $\mathcal{A}$ is a (constant) invertible $(N-1) \times (N-1)$ tridiagonal matrix and $\mathcal{B} \in \mathbb{R}^{N-1}$ is a bounded vector depending continuously on v. This is readily applied to prove that the range of the mapping T is bounded, so by Brouwer's fixed point theorem the existence of at least one fixed point is deduced. □

The preceding lemma allows to adapt to the present context the method of upper and lower solutions for a general equation

$$\Delta^2 u_{x-1} = G_x(u_x) \quad x = 1, \dots, N-1 \tag{2.1}$$

with $G_x : \mathbb{R} \to \mathbb{R}$ continuous for $x = 1, \dots, N-1$. We recall that the method was successfully extended to discrete second order problems under various boundary conditions, see e.g. [2, 3, 7, 15]. However, to the best of our knowledge, no results can be found in the literature for the nonlinear Robin condition (1.4). For convenience, we shall assume without loss of generality that the continuous functions f_0 and f_N, as well as G_x for $x = 1, \dots, N-1$ are defined on the whole line.

Definition 2.4. We shall say that β is an upper solution for (2.1)-(1.3) or (2.1)-(1.4) if

$$\Delta^2 \beta_{x-1} \le G_x(\beta_x) \quad x = 1, \dots, N-1$$

and

$$\beta_0 \ge D_0, \quad \beta_N \ge D_N$$

or

$$\Delta \beta_0 \le f_0(\beta_0), \quad \Delta \beta_{N-1} \ge f_N(\beta_N)$$

respectively. A lower solution α is defined analogously, with all the inequalities reversed.

Lemma 2.5. *Assume there exist α and β as before such that $\alpha_x \le \beta_x$ for all $x = 0, \dots, N$. Then the respective problem* (2.1)-(1.3) *or* (2.1)-(1.4) *has at least one solution u such that $\alpha_x \le u_x \le \beta_x$ for all $x = 0, \dots, N$.*

Proof. Define as usual the (continuous) truncation function

$$\mathcal{T}_x(u) := \begin{cases} \beta_x & \text{if } u \ge \beta_x, \\ u & \text{if } \alpha < u < \beta_x, \\ \alpha_x & \text{if } u \le \alpha_x, \end{cases}$$

and, using the previous lemma, set u as a solution of the problem

$$\Delta^2 u_{x-1} - u_x = G_x(\mathcal{T}_x(u_x)) - \mathcal{T}_x(u_x) \quad x = 1, \dots, N-1$$

under the condition (1.3) or

$$\Delta u_0 = f_0(\mathcal{T}_0(u_0)), \quad \Delta u_{N-1} = f_N(\mathcal{T}_N(u_N))$$

respectively. Suppose for example that $w := u - \beta$ achieves a maximum at some $x = 1, \dots, N-1$, then

$$\Delta^2 w_{x-1} = w_{x+1} - w_x + w_{x-1} - w_x \le 0,$$

that is, $\Delta^2 u_{x-1} \le \Delta^2 \beta_{x-1}$. If $u_x > \beta_x$, then

$$\Delta^2 u_{x-1} > G_x(\beta_x) \ge \Delta^2 \beta_{x-1},$$

a contradiction. In the same way, it is verified that $u - \alpha$ cannot achieve a negative minimum at $x = 1, \dots, N-1$. In the Dirichlet case, it is also clear that $\alpha_x \le u_x \le \beta_x$ for $x = 0$ and $x = N$; thus, u lies between α and β for all x and solves the original problem. For the Robin conditions, suppose that $u - \beta$ does not achieve a maximum at $x = 1, \dots, N-1$, then a strict absolute maximum is achieved either at $x = 0$ or $x = N$. In the first case, if moreover $u_0 > \beta_0$, then $u_0 - \beta_0 > u_1 - \beta_1$, that is

$$\Delta \beta_0 \le f(\beta_0) = \Delta u_0 < \Delta \beta_0,$$

a contradiction. The proof is analogous for $x = N$, so we are able to conclude that $\beta \ge u$. Similarly, it is verified that $\alpha \le u$, whence u is a solution of the original problem. □

3. ATTRACTIVE CASE

Throughout this section, we shall assume that the singularity is attractive, that is, $c < 0$.

3.1. **General results.** Applying the lemmas of the previous section, we shall establish our results. Recall that the parameters a, b, c depend on N as defined in Section 2, a fact that will have significance in some of the conditions described below.

Theorem 3.1. *Let $a > 0 > c$. Then (1.2)-(1.3) admits at least one positive solution for arbitrary $D_0, D_N > 0$ and (1.2)-(1.4) admits at least one positive solution, provided that*

$$f_0(\alpha), f_N(\beta) \le 0 \le f_N(\alpha), f_0(\beta)$$

for some $\alpha > 0$ small enough and some $\beta > \alpha$ large enough. Furthermore, let

$$M := -\frac{9}{N-1}\Big(\frac{a^2c}{4}\Big)^{1/3} > 0$$

and set

$$\hat{b} := \begin{cases} \frac{b}{N-1} & \text{if } b \ge 0 \\ b & \text{otherwise.} \end{cases}$$

Then there are no other positive solutions if

$$\hat{b} + M > -\frac{4}{N-1}\sin^2\Big(\frac{\pi}{2N}\Big) \tag{3.1}$$

for the Dirichlet condition, and

$$\hat{b} + M > 0 \tag{3.2}$$

in the Robin case, provided that f_0 is nondecreasing and f_N is nonincreasing.

Proof. Define $G_x(t) := at^3 + bxt + \frac{c}{t^3}$, then taking constants $\alpha > 0$ sufficiently small and $\beta > 0$ sufficiently large it is seen that $G_x(\alpha) < 0 < G_x(\beta)$ for all $x = 0, \dots, N$. In particular, α and β may be chosen in such a way that all the conditions of Lemma 2.5 are satisfied, so the existence of at least one solution follows. Now assume that u and v are positive solutions and set $w := u - v$. Firstly observe that, if $s, t > 0$ then

$$\Big[a(s^3 - t^3) + c\Big(\frac{1}{s^3} - \frac{1}{t^3}\Big)\Big](s-t) = \psi(s,t)(s-t)^2,$$

where $\psi(s,t) := \big(a - \frac{c}{s^3t^3}\big)\,(s^2 + st + t^2)$ and, by computing the critical points of ψ, it is readily seen that $\psi(s,t) \ge -9\big(\frac{a^2c}{4}\big)^{1/3}$ for all $s, t > 0$. Thus,

$$\sum_{x=1}^{N-1} \Delta^2 w_{x-1} w_x = \sum_{x=1}^{N-1} [G_x(u_x) - G_x(v_x)](u_x - v_x) \ge (N-1)M \sum_{x=1}^{N-1} w_x^2 + \sum_{x=1}^{N-1} xbw_x^2,$$

whence

$$\begin{aligned} -\|\Delta w\|_2^2 + w_N\Delta w_{N-1} - w_0\Delta w_0 &\ge (N-1)M \sum_{x=1}^{N-1} w_x^2 + \sum_{x=1}^{N-1} xbw_x^2 \\ &\ge (N-1)(M + \hat{b}) \sum_{x=1}^{N-1} w_x^2, \quad \text{since } 1 \le x \le N-1. \end{aligned}$$

On the one hand, in the Dirichlet case, it is seen that $w_0 = w_N = 0$, so the previous inequality, combined with the assumption (3.1) yields

$$-\|\Delta w\|_2^2 \ge (N-1)(M + \hat{b})\|w\|_2^2 > -\lambda_1\|w\|_2^2,$$

and the proof follows from Lemma 2.2.

On the other hand, under the Robin condition, the monotonicity assumptions on f_0 and f_N imply

$$w_N\Delta w_{N-1} \le 0 \le w_0\Delta w_0;$$

thus, the assumption $M + \hat{b} > 0$ directly implies that

$$\|\Delta w\|^2 = \sum_{x=1}^{N-1} w_x^2 = 0.$$

We conclude that w is constant and $w_x = 0$ for $x = 1, \dots, N-1$ which, in turn, implies $w_0 = w_N = 0$. $\square$

Remark 3.2. It is interesting to compare the previous result with the corresponding one for the continuous equation (1.1); particularly, the uniqueness conditions (3.1) and (3.2) which, letting $N \to \infty$, yield

$$\min\{B, 0\} - 9\Big(\frac{A^2C}{4}\Big)^{1/3} \geq -\pi^2$$

and $B \geq 0$ respectively. The latter two inequalities can be easily obtained from (1.1) by direct computation and, in the Dirichlet case, the condition given in [4] is thus improved.

Next, we shall consider the case $a \leq 0$. In the previous paper [4], a somewhat sharp sufficient condition was obtained by considering the solutions of the autonomous problem

$$u''(x) = au(x)^3 - b^- u(x) + \frac{c}{u(x)^3}, \quad 0 < x < 1$$

as upper solutions of the original continuous version of (1.2). However, the bounds given in [4] strongly relied on the fact that the autonomous equation can be integrated when multiplied by $u'(x)$, a procedure that cannot be imitated here, due to the failure of the chain rule. In contrast with that, the discrete problem has an advantage with respect to the continuous one, for which a constant upper solution $\beta > 0$ does not exist when $a \leq 0$, since the inequality

$$bx\beta \geq -\Big(a\beta^3 + \frac{c}{\beta^3}\Big) > 0$$

needs to be satisfied for all $x \in (0, 1)$. As a consequence, the following result provides sufficient conditions for the existence of a constant upper solution for (1.2) that depend on N and do not survive in the limit $N \to \infty$ when $a < 0$.

Theorem 3.3. *Assume $c < 0 < b$, then:*

1. *For $a = 0$, the Dirichlet problem* (1.2)-(1.3) *has a unique positive solution. If furthermore*

$$f_0(\alpha) \leq 0 \leq f_N(\alpha), \quad f_0(\beta) \geq 0 \geq f_N(\beta)$$

for some sufficiently small $\alpha > 0$ and some sufficiently large $\beta > \alpha$, then the nonlinear Robin problem (1.2)-(1.4) *has also a positive solution, which is unique when f_0 is nondecreasing and f_N is nonincreasing.*

2. *For $a < 0$, assume that*

$$4b^3 \geq -27ca^2, \tag{3.3}$$

and define

$$\beta = \sqrt{\frac{-2b}{3a}}. \tag{3.4}$$

Then (1.2) *has at least one positive solution satisfying* (1.3) *or* (1.4)*, provided that*

$$\beta \geq D_0, D_N$$

and

$$f_0(\alpha) \leq 0 \leq f_N(\alpha), \quad f_0(\beta) \geq 0 \geq f_N(\beta)$$

for some sufficiently small $\alpha > 0$, respectively.

Proof. The proof for the case $a = 0$ follows exactly as in the preceding result. For $a < 0$, let G_x be defined as in the previous proof, then clearly $G_x \geq G_1$, so it suffices to verify that $G_1(\beta) \geq 0$. In turn, this is equivalent to inequality (3.3), so the result follows. $\square$

Remark 3.4. According to the preceding comment regarding the continuous model (1.1), it is observed, in the second case, that (3.3) cannot be satisfied when N is large.

It is observed, in the latter proof, that the constant $\beta = \beta(b)$ is computed as the (unique) positive value in which the absolute maximum of the function $\varphi(z) := az^6 + bz^4$ with $z \geq 0$ is achieved. This shows that, if the inequality in (3.3) is strict, then the conditions may be relaxed by considering the maximal interval $\mathcal{I}_c \subset (0, +\infty)$ such that $\varphi(z) \geq -c$ for $z \in I_c$. With this in mind, let us set the notation M_c for the upper endpoint of $\mathcal{I}_c$.

Corollary 3.5. *For $a, c < 0 < b$, assume that (3.3) holds and let $\mathcal{I}_c$ and M_c be defined as before. Then*

1. *Problem (1.2)-(1.3) has at least one positive solution for arbitrary $D_0, D_N \in (0, M_c]$.*
2. *Problem (1.2)-(1.4) has at least one positive solution, provided that*
$$f_0(\alpha) \leq 0 \leq f_N(\alpha), \quad f_0(\beta) \geq 0 \geq f_N(\beta)$$
for some sufficiently small $\alpha > 0$ and some $\beta \in \mathcal{I}_c$.

The next corollary is directly deduced from the fact that, as $b \to +\infty$, the value $\beta(b)$ tends to $+\infty$ and the lower endpoint of $\mathcal{I}_c$ tends to 0.

Corollary 3.6. *Assume that $a, c < 0$, then:*

1. *Given arbitrary $D_0, D_N > 0$, there exists $b^* > 0$ such that problem (1.2)-(1.3) has at least one positive solution if $b \geq b^*$.*
2. *There exists $b^* > 0$ such that problem (1.2)-(1.3) has at least one positive solution if $b \geq b^*$, provided that*
$$f_0(\beta) \geq 0 \geq f_N(\beta)$$
for some $\beta > 0$ and
$$f_0(\alpha_n) \leq 0 \leq f_N(\alpha_n)$$
for some sequence $\alpha_n \to 0^+$.

In the same spirit of the latter corollary, it is seen that solutions exist when the negative constant c is close enough to 0.

Corollary 3.7. *Assume that $a < 0 < b$ and define $\beta = \beta(b)$ as before. Then:*

1. *There exists $c_* > 0$ such that (1.2)-(1.3) has at least one positive solution for $-c_* < c < 0$, provided that $\beta \geq D_0, D_N$.*
2. *There exists $c_* > 0$ such that (1.2)-(1.4) has at least one positive solution for $-c_* < c < 0$, provided that*
   (a) *$f_0(\beta) \geq 0 \geq f_N(\beta)$.*
   (b) *$f_0(\alpha_n) \leq 0 \leq f_N(\alpha_n)$ for some sequence $\alpha_n \to 0^+$.*

3.2. **Homogeneous Dirichlet problem.** In this section, we shall consider problem (1.2) under the homogeneous condition
$$u_0 = u_N = 0. \tag{3.5}$$
In contrast with the continuous case, here the singularity at the boundary does not imply that the derivatives are unbounded. In fact, if a sequence of solutions is bounded, then so is the sequence of its derivatives. This suggests us to look for a solution of (1.2)-(3.5) as the limit of a sequence of positive solutions of (1.2)-(1.3) with Dirichlet conditions converging to 0. Such a procedure yields the following result.

Theorem 3.8. *Problem (1.2)-(3.5) has at least one solution with $u_x > 0$ for $x = 1, \dots, N-1$, provided that one of the following conditions holds:*

1. *$a > 0 > c$.*
2. *$a = 0$ and $b > 0 > c$.*
3. *$a, c < 0$ and $4b^3 > -27ca^2$.*

Furthermore, the solution is unique in the second case, and also in the first case under assumption (3.1).

Proof. Fix an arbitrary sequence $r_k \searrow 0$ and a constant $\beta > 0$ such that $G_x(\beta) \geq 0$ for $x = 1, \dots, N-1$. According to Lemma 2.5, β serves as an upper solution of the Dirichlet problem when $D_0, D_N \leq \beta$. Thus, for each $k \in \mathbb{N}$ we may assume that r_k is a lower solution with $r_k < \beta$ for all k and define $u^{(k)}$ as a solution of (1.2) between r_k and β satisfying $u_0^{(k)} = u_N^{(k)} = r_k$. Since $u_x^{(k)} \leq \beta$ for all $x = 0, \dots, N$, taking a subsequence we may assume that $\{u^{(k)}\}$ converges to some u such that $u_0 = u_N = 0$. Furthermore, the identity

$$\Delta^2 u_{x-1}^{(k)} = a(u_x^{(k)})^3 + bxu_x^{(k)} + \frac{c}{(u_x^{(k)})^3} \quad x = 1, \dots, N-1,$$

combined with the fact that the left-hand side term is bounded imply that $u_x > 0$ and u verifies (1.2) for $x = 1, \dots, N-1$. Uniqueness is deduced exactly as before. □

Remark 3.9. Again, condition (3) in the previous theorem depends on N and, not surprisingly, for fixed $A, C < 0$, it does not hold when N is large.

4. REPULSIVE CASE

Here, we deal with the case $c > 0$. We shall consider firstly the Dirichlet problem.

Theorem 4.1. *Assume $c > 0 > a$, then (1.2)-(1.3) with $D_0, D_N \geq 0$ has at least one solution u with $u_x > 0$ for $x = 1, \dots, N-1$.*

Proof. Let us firstly observe that the problem may be also written as

$$u_{x+1} + u_{x-1} = au_x^3 + (2 + bx)u_x + \frac{c}{u_x^3} \quad x = 1, \dots, N-1,$$

where it is already assumed that $u_0 = D_0$ and $u_N = D_N$. This shows that u is a (positive) solution if and only if $P_x(u) = 0$ for all $x = 1, \dots, N-1$, where

$$P_x(u) = au_x^6 + (2 + bx)u_x^4 - (u_{x-1} + u_{x+1})u_x^3 + c.$$

Set

$$\Omega := \{u \in \mathbb{R}^{N-1} : 0 < u_x < R,\ x = 1, \dots, N-1\}$$

for some $R > 0$ to be defined, and the homotopy $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^{N-1}$ given by

$$H(u, \lambda)_x := au_x^6 + \lambda \left[(2 + bx)u_x^4 - (u_{x-1} + u_{x+1})u_x^3\right] + c.$$

It is observed that, if $u_x = 0$ for some x, then $H_\lambda(u)_x := H(u, \lambda)_x = c \neq 0$. Furthermore, taking R sufficiently large it is also clear that also $H_\lambda(u)_x \neq 0$ when $u_x = R$; in other words, we have verified that H_λ does not vanish on $\partial\Omega$ for arbitrary $\lambda \in [0,1]$. From the homotopy invariance of the Brouwer degree, we deduce that

$$\deg(H_1, \Omega, 0) = \deg(H_0, \Omega, 0) = (-1)^{N-1},$$

where the latter equality follows from the fact that $H_0(u)_x = au_x^6 + c$, which has a unique simple positive root $u_* := \sqrt[6]{\frac{-c}{a}}$ and the sign of the Jacobian determinant of H_0 at $(u_*, \dots, u_*)$ coincides with that of a^{N-1}. Hence, the existence property of the Brouwer degree implies that H_1 has at least one root in Ω, which corresponds to a solution of the problem. □

Remark 4.2. (1) The previous proof is also valid when $c < 0 < a$; thus, the existence part of Theorem 3.1 is retrieved, including the homogeneous case treated in Theorem 3.8. In both situations, the fact that $ac < 0$ allows the direct use the Poincaré-Miranda theorem instead of the Brouwer degree. Indeed, it suffices to observe, for $u \in \overline{\Omega}$, that

$$P_x(u) = c \quad \text{if } u_x = 0$$

and

$$P_x(u) \sim aR^6 \quad \text{if } u_x = R,$$

which shows that P_x changes sign at the corresponding faces of Ω. However, the topological degree is more general and may constitute an useful tool when searching for multiple solutions.

(2) It is worth recalling, for the non-homogeneous continuous case (1.1), that the assumption $A < 0 < C$ yields in fact the existence of infinitely many solutions. More precisely, for all $k \in \mathbb{N}$ sufficiently large there exist at least two solutions with exactly k nodal regions with respect to the segment joining the boundary values. Such a result cannot be reproduced in the discrete model for obvious reasons, although it is expected that the number of solutions increases as N gets larger. As a toy model, let $N = 2$, then the problem reduces to find the zeros of the polynomial

$$P_1(u_1) = au_1^6 + (2+b)u_1^4 - (D_0 + D_2)u_1^3 + c,$$

which trivially verifies

$$P_1(0) = c > 0, \quad P_1(u_1) \to -\infty \quad \text{as } u_1 \to +\infty.$$

By computing its derivative (or simply by the Descartes rule), it is easy to conclude that there exist at most 3 positive solutions; moreover, the solution is unique if and only if

$$b + 2 < \frac{9}{4}\Big(\frac{-a(D_0 + D_2)^2}{2}\Big)^{1/3}.$$

Next, we proceed with the Robin case.

Theorem 4.3. *Assume $a < 0 < c$ and*

(1) *$f_0(\eta) \le 0 \le f_N(\eta)$ for some $\eta > 0$ small enough.*
(2) *There exists a sequence $R_n \to +\infty$ such that*

$$\liminf_{n\to\infty} \frac{f_0(R_n)}{R_n} > -1, \quad \limsup_{n\to\infty} \frac{f_N(R_n)}{R_n} < 1.$$

Then (1.2)-(1.4) *has at least one positive solution.*

Proof. We shall follow the outline of the previous proof, now taking into account the Robin condition, which motivates to define

$$P_0(u) := f_0(u_0) + u_0 - u_1, \quad P_N(u) := -f_N(u_N) + u_N - u_{N-1}.$$

Within this context, the polynomials P_x for $x = 1, \dots x_{N-1}$ are defined as before for arbitrary $u \in (0, +\infty)^{N+1}$. It is noted that the homotopy of the preceding proof may be extended by setting

$$H_\lambda(u)_0 := \lambda f_0(u_0) + u_0 - u_1, \quad H_\lambda(u)_N := -\lambda f_N(u_N) + u_N - u_{N-1}.$$

According to the assumptions, we may fix $\varepsilon > 0$ and n_0 such that

$$f_0(R_n) + R_n > \varepsilon R_n, \quad f_N(R_n) - R_n < -\varepsilon R_n$$

for all $n \ge n_0$. Next, fix $R := R_n$ for some sufficiently large $n \ge n_0$ such that if $H_\lambda(u)_x = 0$ for all $x = 1, \dots, N-1$ with $0 < u_y \le R$ for all $y = 0, \dots, N$, then $u_x \le \varepsilon R$ for $x = 1, \dots, N-1$. Suppose now that $H_\lambda(u) = 0$ with $0 < u_x \le R$ for all x and $u_0 = R$, then

$$\lambda f_0(R) + R = u_1 \le \varepsilon R.$$

This implies $f_0(R) < 0$ and, consequently,

$$\lambda f_0(R) + R \ge f_0(R) + R > \varepsilon R,$$

a contradiction. In the same way, it is verified that, if $H_\lambda(u) = 0$ with $0 < u_x \le R$, then $u_N < R$. Now fix $\eta > 0$ such that if $0 < u_x \le R$ for all $x = 0, \dots N$ and $u_j \le \eta$ for some $j = 1, \dots, N-1$, then $H_\lambda(u)_j \ne 0$. Suppose that $H_\lambda(u) = 0$ with $\eta \le u_x < R$ for all x and $u_0 = \eta$, then

$$\lambda f_0(\eta) + \eta = u_1 > \eta.$$

From the hypothesis, η may be chosen in such a way that $f_0(\eta) \le 0$, which yields a contradiction. In the same way, we deduce that $u_N > \eta$, whence the homotopy does not vanish on $\partial\Omega$, where

$$\Omega := \{u \in \mathbb{R}^{N+1} : \eta < u_x < R, \quad x = 0, \dots, N\}.$$

Again, this implies

$$\deg(H_1, \Omega, 0) = \deg(H_0, \Omega, 0) = (-1)^{N+1}$$

because the Jacobian matrix of H_0 at the unique root $(u_*, \dots, u_*)$ is now given by

$$\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 6au_*^5 & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & 6au_*^5 & 0 \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

□

Next, we shall focus on the case $a, c > 0$. In the Dirichlet case, it is immediately seen that solutions cannot exist when c is sufficiently large. The same happens in the Robin case, provided that f_0 and f_N satisfy some appropriate growth conditions. The following result shows that solutions exist in the opposite situation.

Theorem 4.4. *Let $a > 0$, then:*

1. *There exists $c_* > 0$ such that problem* (1.2)-(1.3) *has at least one positive solution for $0 < c < c_*$, in the following cases:*
   - (a) *$D_0 > 0$ or $D_N > 0$.*
   - (b) *$D_0 = D_N = 0$ and $b < -\frac{2}{N-1}$.*
2. *There exists $c_* > 0$ such that problem* (1.2)-(1.4) *has at least one positive solution for $0 < c < c_*$, provided that*

$$f_0(r_0) + r_0 \le 0 \le f_N(r_N) - r_N$$

for some $r_0, r_N > 0$ and

$$\liminf_{n\to\infty} \frac{f_0(R_n)}{R_n} > -1, \quad \limsup_{n\to\infty} \frac{f_N(R_n)}{R_n} < 1$$

for some sequence $R_n \to +\infty$.

Proof. Consider firstly the Dirichlet case with $D_0 > 0$ and define the function

$$Q_x(u) := P_x(u) - c = u_x^3[au_x^3 + (2 + bx)u_x - (u_{x-1} + u_{x+1})].$$

Next, fix $\varepsilon_1 > 0$ such that

$$at^3 + (2 + b)t < D_0 \quad 0 < t \le \varepsilon_1$$

and, inductively, we may set $\varepsilon_2, \dots, \varepsilon_{N-1} > 0$ such that

$$at^3 + (2 + bx)t < \varepsilon_{x-1} \quad 0 < t \le \varepsilon_x.$$

Also, we may fix $R > 0$ such that $aR^3 + (2 + bx)R > 2R$ for $x = 1, \dots, N-1$ and define the open set

$$\Omega := (\varepsilon_1, R) \times \dots \times (\varepsilon_{N-1}, R).$$

It follows from the previous choice of the values ε_x and R that if $u \in \partial\Omega$, then there exists x such that either $u_x = R$ and $Q_x(u) > 0$, or $u_x = \varepsilon_x$ and $Q_x(u) < 0$. Thus, picking an arbitrary point $v \in \Omega$, in both situations it is deduced that $\lambda Q_x(u) + (1 - \lambda)(u_x - v_x) \neq 0$ for $\lambda \in [0, 1]$. In other words, the homotopy $H_\lambda := \lambda Q + (1 - \lambda)(I - v)$ does not vanish, where I denotes the identity map and, in consequence,

$$\deg(Q, \Omega, 0) = \deg(I - v, \Omega, 0) = 1.$$

Because the degree is continuous, we conclude that

$$\deg(P, \Omega, 0) = \deg(Q + (c, \dots, c), \Omega, 0) = 1$$

when c is sufficiently small. The proof is analogous when $D_0 = 0 < D_N$, now starting from an appropriate value $\varepsilon_{N-1} > 0$ such that

$$at^3 + (2 + b(N - 1))t < D_N \quad 0 < t \le \varepsilon_{N-1}$$

and, inductively, choosing values $\varepsilon_{N-2}, \dots, \varepsilon_1 > 0$ such that

$$at^3 + (2 + bx)t < \varepsilon_{x+1} \quad 0 < t \le \varepsilon_x.$$

Finally, if $D_0 = D_N = 0$ and $b < -\frac{2}{N-1}$, it suffices to fix $\varepsilon_{N-1} > 0$ such that

$$at^3 + (2 + b(N-1))t < 0 \quad 0 < t \le \varepsilon_{N-1}$$

and proceed as before.

Concerning the Robin case, set $\varepsilon_0 := r_0$, $\varepsilon_N := r_N$ and ε_x as before for $x = 1, \dots, N-1$. Then, fix a value $R > 0$ as in the proof of Theorem 4.3 and set

$$\tilde{\Omega} := (\varepsilon_0, R) \times \ldots \times (\varepsilon_N, R),$$
$$\tilde{Q}(u); = (f_0(u_0) + u_0 - u_1, Q(u_1, \dots, u_{N-1}), -f_N(u_N) + u_N - u_{N-1}).$$

Again, it is deduced that $\deg(\tilde{Q}, \tilde{\Omega}, 0) = 1$, and the proof follows from the fact that $P = \tilde{Q} + (0, c, \dots, c, 0)$. ☐

As a conclusion of this section, let us emphasize that the case $a < 0 < c$ can be analysed by a more careful study of the associated polynomial system and, as mentioned in Remark 4.2, the non-homogeneous Dirichlet problem for (1.1) has infinitely many solutions. In contrast, the discretization involves a system of $N-1$ polynomials with $N-1$ variables and, as is well known, the number of (nondegenerate) positive solutions is always finite. A celebrated -although unsharp- upper bound of this number was given by Khovanskii in [16] and improved in later works; in our case, the system can be reduced to a single equation, so an upper bound is readily obtained from Descartes' rule of signs. However, no general results for lower bounds seem to exist in the literature. Such bounds would be of interest, since it is expected that the number of solutions tends to infinity as $N \to \infty$. Concerning the case $a, c > 0$, existence of solutions of (1.2)-(1.3) is guaranteed when the associated polynomial system has at least one positive root but, again, the literature on this topic is scarce. Some recent results have been obtained (see e.g. [36]), but the conditions do not apply to our specific case. Concerning the case $a, c > 0$, we emphasize that the threshold c_* in Theorem 4.4 depends on N, so the result is therefore of a purely discrete nature. In particular, it is not intended to be carried over to the continuous limit $N \to \infty$.

5. Variational approach

In this section, we shall briefly describe the main aspects of the variational method, adapted to the present boundary value problems. This aims, on the one hand, to give alternative proofs of some of the preceding results and, on the other hand, to propose a useful tool in order to tackle the multiplicity problem suggested in the previous section, with the help of linking-type theorems. This goal shall be pursued in a forthcoming paper.

For the Dirichlet case, define the functional $\mathcal{I} : (0, +\infty)^{N-1} \to \mathbb{R}$ given by

$$\mathcal{I}(u) = \frac{1}{2}\sum_{x=1}^{N}(\Delta u_{x-1})^2 + \frac{a}{4}\sum_{x=1}^{N-1} u_x^4 + \frac{b}{2}\sum_{x=1}^{N-1} x u_x^2 - \frac{c}{2}\sum_{x=1}^{N-1}\frac{1}{u_x^2}, \tag{5.1}$$

where it is assumed that $u_0 = D_0$ and $u_N = D_N$. It is straightforwardly verified that the critical points of $\mathcal{I}$ coincide with the positive solutions of (1.2)-(1.3). Next, for $x = 1, \dots, N-1$ compute

$$\frac{\partial \mathcal{I}}{\partial u_x} = 2u_x - (u_{x-1} + u_{x+1}) + a u_x^3 + b x u_x + \frac{c}{u_x^3}.$$

This shows that u is a critical point of $\mathcal{I}$ if and only if $P_x(u) = 0$ for all $x = 1, \dots, N-1$, where P_x are the polynomials defined in the previous section.

Analogously for condition (1.4), the functional $\mathcal{J} : (0, +\infty)^{N+1} \to \mathbb{R}$ defined by

$$\mathcal{J}(u) := \mathcal{I}(u) + F_0(u_0) - F_N(u_N),$$

with $\mathcal{I}$ defined as in (5.1) and

$$F_0(s) := \int_0^s f_0(t)\, dt, \quad F_N(s) := \int_0^s f_N(t)\, dt.$$

Here, if u is a critical point of $\mathcal{J}$, then taking all the directional derivatives $\frac{\partial \mathcal{J}}{\partial \nu}(u)$ with $\nu_0 = \nu_N = 0$ it is deduced that u satisfies (1.2). Next, for arbitrary $\nu \neq 0$, summation by parts yields

$$0 = \frac{\partial \mathcal{J}}{\partial \nu}(u) = f_0(u_0)\nu_0 - f_N(u_N)\nu_N + \Delta u_{N-1}\nu_N - \Delta u_0 \nu_0,$$

so the Robin condition is verified by taking firstly $\nu_0 = 0 \neq \nu_N$ and next $\nu_0 \neq 0 = \nu_N$. In order to verify that the critical points of $\mathcal{J}$ coincide with the solutions of the problem, let us simply observe that

$$\frac{\partial \mathcal{J}}{\partial u_x} = \frac{\partial \mathcal{I}}{\partial u_x} \quad x = 1, \dots, N-1,$$
$$\frac{\partial \mathcal{J}}{\partial u_0} = f_0(u_0) + u_0 - u_1, \quad \frac{\partial \mathcal{J}}{\partial u_N} = -f_N(u_N) + u_N - u_{N-1}.$$

It is observed that the variational formulation provides an immediate resolution of the cases in which a and c have different signs, since a coercivity argument can be applied. For the Dirichlet case assume, for example, that $a > 0 > c$, then $\mathcal{I}(u) \to +\infty$ as some coordinate of u tends to 0 or to $+\infty$ which, in turn, implies that $\mathcal{I}$ achieves an absolute minimum on $(0, +\infty)^{N-1}$. A similar conclusion follows when $a < 0 < c$ and also for the Robin case, under appropriate assumptions on f_0 and f_N.

6. Discussion and concluding remarks

A discretisation of the Ermakov-Painlevé II equation subject to certain boundary conditions has been carried out using a standard finite difference scheme, and both topological and variational approaches have been applied to analyse the resulting discrete problem.

Although a full numerical study is beyond the scope of this paper, we note that the convergence of solutions of the discrete scheme (1.2) to solutions of the continuous problem (1.1) can be established following the approach in [25] for singular mixed boundary value problems. Moreover, as shown in Remark 3.2, the condition obtained here for uniqueness in the attractive case improves those previously known for the continuous equation (1.1) in the Dirichlet setting, and they reduce to the standard one in the linear Robin context.

The extension of the variational approach to a wider range of discretisation problems (e.g., other singular nonlinearities or boundary conditions) is the subject of current investigation.

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