Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 40, pp. 1–13.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.40

EXISTENCE OF WEAK SOLUTIONS FOR FRACTIONAL $(p_1(x,y), p_2(x,y))$-LAPLACIAN PROBLEMS WITH INDEFINITE WEIGHTS

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Abstract. We study nonlocal elliptic problems driven by the fractional $(p_1(x,y), p_2(x,y))$-Laplacian operator under Dirichlet boundary conditions, where $p_1(\cdot,\cdot)$ and $p_2(\cdot,\cdot)$ are continuous functions defined on a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$. The model includes indefinite weight functions, which may change the sign within the domain. By applying variational methods, we establish the existence of at least one nontrivial weak solution. Our results extend recent contributions in the literature on nonlocal problems with variable exponent operators, and provide new insights into the interaction between fractional order, and sign-changing weights.

1. Introduction

The study of differential equations involving the fractional $p(x,y)$-Laplacian or the more general fractional $(p_1(x,y), p_2(x,y))$-Laplacian operator has gained considerable attention in recent years, owing to their ability to model a wide range of complex and nonlocal phenomena. These operators combine the nonlocal nature of fractional derivatives with the flexibility of variable exponents, giving rise to a rich class of nonlinear problems. Such models are better suited to describing physical processes involving spatial heterogeneity and long-range interactions. Important applications arise in fluid mechanics [23], image processing [1, 20], elasticity theory [8, 22, 26], and porous media flow [3].

In this paper, we study the following problem for the fractional $(p_1(x,y), p_2(x,y))$-Laplacian operator, under homogeneous Dirichlet boundary conditions

$$\begin{gathered}(-\Delta)^s_{p_1(x,.)}u(x) + (-\Delta)^s_{p_2(x,.)}u(x) + V(x)|u|^{q(x)-2}u \\ = \lambda m_1(x)|u(x)|^{r_1(x)-2}u(x) - \mu m_2(x)|u(x)|^{r_2(x)-2}u(x) \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, (For example: $\Omega \subset \mathbb{R}^{\mathbb{N}}$ is a bounded domain of class $C^{2,m}$ for $(m \in (0,1))$). $(-\Delta)^s_{p_i(\cdot)}$ $(0 < s < 1)$ is the fractional $p_i(\cdot)$-Laplacian, i.e.

$$(-\Delta)^s_{p_i(x,\cdot)}u(x) = 2\text{p.v.}\int_\Omega \frac{|u(x)-u(y)|^{p_i(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp_i(x,y)}}dy, \quad x \in \Omega,$$

where p.v. is the principal value, $p_i \in C(\overline{\Omega} \times \overline{\Omega})$, $q, r_i \in C(\overline{\Omega})$ $(i = 1,2)$, V, m_1, and m_2 are three indefinite weight functions, and λ and μ are positive parameters. Let

$$p_{\max}(x,y) := \max\{p_1(x,y), p_2(x,y)\}, \quad \overline{p}_{\max}(x) := \max\{p_1(x,x), p_2(x,x)\},$$

and define

$$(\overline{p}_{\max})^*_s(x) := \frac{N\overline{p}_{\max}(x)}{N - s\overline{p}_{\max}(x)},$$

for all $x, y \in \overline{\Omega}$. Throughout this paper, we assume that

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2020 *Mathematics Subject Classification.* 35A15, 35J60, 35R11, 35P30.
Key words and phrases. Indefinite weight; fractional $(p_1(x), p_2(x))$-Laplacian operator; variable exponent; Ekeland's variational principle; variational method.

Submitted February 12, 2026. Published June 8, 2026.

(H1)

$$2 \leq p_i^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p_i(x,y) \leq p_i(x,y) \leq p_i^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p_i(x,y) < \frac{N}{s} < +\infty,$$

p_i is symmetric, that is, $p_i(x,y) = p_i(y,x)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$, and

$$\bar{p}_{\max}(x) \leq q(x) < \min\Big(\frac{\big(\bar{p}_{\max}\big)^*_s(x)\alpha(x)}{\big(\bar{p}_{\max}\big)^*_s(x) + \alpha(x)}, (\bar{p}_{\max})^*_s(x)\frac{N-s}{N}\Big) < \big(\bar{p}_{\max}\big)^*_s(x)$$

for all $x \in \overline{\Omega}$, where α is given by (H2).

(H2) $1 < \max\{r_1(x), r_2(x)\} < p_{\max}(x,y) < \frac{N}{s} < \min\{\alpha(x), \beta_1(x), \beta_2(x)\}$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. where $\alpha, \beta_1, \beta_2 \in C(\overline{\Omega})$, and $m_i \in L^{\beta_i(x)}$ $(i = 1,2)$, such that $m_1 > 0$ in some subset $\Omega_0 \subset\subset \Omega$ with $meas(\Omega_0) > 0$, and $m_2 \geq 0$ in Ω. Moreover $V \in L^{\alpha(x)}(\Omega)$ satisfies

$$\| V \| < \frac{q^-}{2p^+_{\max} C^{q^-}}$$

where C is the best Sobolev embedding constant from the space $W_0(\Omega)$ into the Lebesgue space $L^{q(\cdot)\alpha'(\cdot)}(\Omega)$.

(H3)

$$\min_{x\in\overline{\Omega}_0} r_1(x) < \min\Big\{\min_{(x,y)\in\overline{\Omega}_0\times\overline{\Omega}_0} p_1(x,y), \min_{(x,y)\in\overline{\Omega}_0\times\overline{\Omega}_0} p_2(x,y), \min_{x\in\overline{\Omega}_0} q(x), \min_{x\in\overline{\Omega}_0} r_2(x)\Big\},$$

where Ω_0 is given by (H2).

When $s = 1$, problem (1.1) reduces to the $p(\cdot)$-Laplacian problem characterized by a nonstandard growth condition. This problem has garnered attention from numerous mathematicians; see, for example [6, 16, 17, 25]. Interest in these types of equations has grown, largely due to their relevance in mathematical modeling of non-Newtonian fluids, particularly electrorheological fluids.

When $V(x) = 1$, the problem reduces to a special case already studied in the literature. In particular, in [9], Chung and Toan studied the problem

$$\begin{gathered}(\Delta)^s_{p_1(x,.)}u(x) + (\Delta)^s_{p_2(x,\cdot)}u(x) + |u|^{q(x)-2}u \\ = \lambda V_1(x)|u(x)|^{r_1(x)-2}u(x) - \mu V_2(x)|u(x)|^{r_2(x)-2}u(x) \quad \text{in } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.2}$$

where λ, μ are two positive parameters. Under the conditions:

(1) p_i is symmetric, $2 \leq p_i(x,y) < \frac{N}{s}$ for all $(x,y) \in \bar{\Omega} \times \bar{\Omega}$ and $\bar{p}_{\max}(x) \leq q(x) < \big(\bar{p}_{\max}\big)^*_s(x)$ for all $x \in \bar{\Omega}$,

(2) $1 < \max\{r_1(x), r_2(x)\} < p_{\max}(x,y) < \frac{N}{s} < \min\{\alpha_1(x), \alpha_2(x)\}$ for all $x, y \in \bar{\Omega}$, where $\alpha_1, \alpha_2 \in C(\bar{\Omega}), V_1 \in L^{\alpha_1(\cdot)}(\Omega)$ such that $V_1 > 0$ in $\Omega_0 \subset\subset \Omega$ with $|\Omega_0| > 0$ and $V_2 \in L^{\alpha_2(\cdot)}(\Omega), V_2 \geq 0$ in Ω,

(3) $\inf_{\overline{\Omega}_0} r_1(x) < \min\big\{\inf_{(x,y)\in\overline{\Omega}_0\times\overline{\Omega}_0} p_1(x,y), \inf_{(x,y)\in\overline{\Omega}_0\times\overline{\Omega}_0} p_2(x,y), \inf_{\overline{\Omega}_0} q(x), \inf_{x\in\overline{\Omega}_0} r_2(x)\big\}$.

They showed that there exist two positive critical values $\overline{\lambda}$ and $\underline{\lambda}$ such that, for every $\mu > 0$, the problem (1.2) has a nontrivial weak solution for all $\lambda > \overline{\lambda}$ and $0 < \lambda < \underline{\lambda}$,

Inspired by the work mentioned above and building on the variational framework developed in [9], we apply critical point theory specifically Ekeland's variational principle, to investigate the existence of weak solutions for problem (1.1). We establish the existence of a positive critical value $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, problem (1.1) admits at least one nontrivial weak solution.

Our main result extends and generalizes the findings of Chung and Toan [9]. In particular, while their work assumes the presence of a fixed positive weight, our setting incorporates a sign-changing weight function $V(x)$, which introduces additional mathematical challenges. The presence of such an indefinite potential affects both the geometry of the associated energy functional and the compactness properties required to apply variational methods, making our contribution a nontrivial generalization.

The structure of the paper is as follows. In Section 2, we provide essential definitions and fundamental concepts related to generalized Lebesgue and Sobolev spaces with variable exponents, as well as fractional Sobolev spaces with variable exponents. Section 3 is devoted to the statement and proof of our main result.

2. Preliminaries

In this section, we introduce some definitions and concepts within the framework of generalized function spaces. In particular, we consider the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, the generalized Sobolev spaces $W_0^{1,q(\cdot)}(\Omega)$, and the fractional Sobolev spaces $W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. For further details and background, we refer the reader to [5, 11, 24]. We begin by introducing the set

$$C_+(\overline{\Omega}) = \big\{q \in C(\overline{\Omega}) : q(x) > 1 \text{ for all } x \in \overline{\Omega}\big\}.$$

For $q \in C_+(\overline{\Omega})$, we define

$$q^+ = \max_{x\in\overline{\Omega}} q(x) \quad \text{and} \quad q^- = \min_{x\in\overline{\Omega}} q(x).$$

The *variable exponent Lebesgue space* $L^{q(\cdot)}(\Omega)$ is defined as

$$L^{q(\cdot)}(\Omega) = \big\{u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^{q(x)}\,dx < \infty\big\},$$

and it is endowed with the *Luxemburg norm*

$$\|u\|_{q(\cdot)} = \inf\big\{\mu > 0 : \int_\Omega \big|\frac{u(x)}{\mu}\big|^{q(x)} dx \le 1\big\}.$$

It is well known that $\big(L^{q(\cdot)}(\Omega), \|\cdot\|_{q(\cdot)}\big)$ is a separable and uniformly convex Banach space. This space is referred to as a *generalized Lebesgue space*. Let $q' \in C_+(\overline{\Omega})$ be the *conjugate exponent* of q, defined by

$$\frac{1}{q'(x)} + \frac{1}{q(x)} = 1 \quad \text{for all } x \in \overline{\Omega}.$$

Then, for all $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$, the following Hölder-type inequality holds.

Lemma 2.1 ([11]). *If $u \in L^{q(\cdot)}(\Omega)$ and $v \in L^{q'(\cdot)}(\Omega)$, then*

$$\big|\int_\Omega uv\,dx\big| \leqslant \Big(\frac{1}{q^-} + \frac{1}{q'^-}\Big)\|u\|_{q(\cdot)}\|v\|_{q'(\cdot)} \leqslant 2\|u\|_{q(\cdot)}\|v\|_{q'(\cdot)}.$$

It is known that the embedding $L^{q_2(\cdot)}(\Omega) \hookrightarrow L^{q_1(\cdot)}(\Omega)$ holds for all functions $q_1, q_2 \in C_+(\overline{\Omega})$ such that $q_1(x) \le q_2(x)$ for every $x \in \overline{\Omega}$. Moreover, this embedding is continuous.

Among the fundamental tools in the study of generalized Lebesgue spaces is the *modular* associated with the space $L^{q(\cdot)}(\Omega)$. It is defined as the mapping $\rho_{q(\cdot)} : L^{q(\cdot)}(\Omega) \to \mathbb{R}$ given by

$$\rho_{q(\cdot)}(u) = \int_\Omega |u|^{q(x)}\,dx.$$

Proposition 2.2. [15, 14] *If (u_n), $u \in L^{q(\cdot)}(\Omega)$ and $q^+ < +\infty$ then the following relations hold*

1. $\|u\|_{q(\cdot)} > 1 \implies \|u\|_{q(\cdot)}^{q^-} \le \rho_{q(\cdot)}(u) \le \|u\|_{q(\cdot)}^{q^+}$,
2. $\|u\|_{q(\cdot)} < 1 \implies \|u\|_{q(\cdot)}^{q^+} \le \rho_{q(\cdot)}(u) \le \|u\|_{q(\cdot)}^{q^-}$,
3. $\|u_n - u\|_{q(\cdot)} \to 0 \iff \rho_{q(\cdot)}(u_n - u) \to 0$.

Proposition 2.3 ([12]). *Let γ and q be measurable functions such that $\gamma \in L^\infty\big(\mathbb{R}^N\big)$ and $1 \le \gamma(x)q(x) < \infty$, for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{q(\cdot)}\big(\mathbb{R}^N\big), u \neq 0$. Then*

$$\|u\|_{\gamma(\cdot)q(\cdot)} \le 1 \implies \|u\|_{\gamma(\cdot)q(\cdot)}^{\gamma^+} \le \big\|\|u\|^{\gamma(\cdot)}\big\|_{q(\cdot)} \le \|u\|_{\gamma(\cdot)q(\cdot)}^{\gamma^-},$$

$$\|u\|_{\gamma(\cdot)q(\cdot)} \ge 1 \implies \|u\|_{\gamma(\cdot)q(\cdot)}^{\gamma^-} \le \big\|\|u\|^{\gamma(\cdot)}\big\|_{q(\cdot)} \le \|u\|_{\gamma(\cdot)q(\cdot)}^{\gamma^+}.$$

In particular, if $\gamma(\cdot)=\gamma$ *is constant, then*

$$\big\| \|u\|^{\gamma} \big\|_{q(\cdot)} = \|u\|^{\gamma}_{\gamma q(\cdot)}.$$

Let $q \in C_+(\overline{\Omega})$. The *variable exponent Sobolev space* $W^{1,q(\cdot)}(\Omega)$ consists of all functions $u \in L^{q(\cdot)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and satisfies $\nabla u \in [L^{q(\cdot)}(\Omega)]^N$. That is,

$$W^{1,q(\cdot)}(\Omega) = \big\{u \in L^{q(\cdot)}(\Omega) : |\nabla u| \in L^{q(\cdot)}(\Omega)\big\}.$$

This space can be equipped with the norm

$$\|u\|_{1,q(\cdot)} := \inf\Big\{\lambda > 0 : \int_\Omega \big[\big|\frac{\nabla u(x)}{\lambda}\big|^{q(x)} + \big|\frac{u(x)}{\lambda}\big|^{q(x)}\big] dx \le 1\Big\},$$

or equivalently,

$$\|u\|_{1,q(\cdot)} = \|u\|_{q(\cdot)} + \|\nabla u\|_{q(\cdot)}.$$

We denote by $W_0^{1,q(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,q(\cdot)}(\Omega)$. Both spaces $W^{1,q(\cdot)}(\Omega)$ and $W_0^{1,q(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.

Moreover, the *Poincaré inequality* holds in $W_0^{1,q(\cdot)}(\Omega)$, namely:

$$\|u\|_{1,q(\cdot)} \le C\|\nabla u\|_{q(\cdot)}, \quad \text{for all } u \in W_0^{1,q(\cdot)}(\Omega),$$

which implies that the norms $\|u\|_{1,q(\cdot)}$ and $\|\nabla u\|_{q(\cdot)}$ are equivalent on $W_0^{1,q(\cdot)}(\Omega)$.

Let $r \in C_+(\overline{\Omega})$ be such that $r(x) < q^*(x)$ for all $x \in \overline{\Omega}$, where $q^*(x)$ denotes the *Sobolev critical exponent*, defined by:

$$q^*(x) = \begin{cases} \frac{Nq(x)}{N-q(x)} & \text{if } q(x) < N, \\ +\infty & \text{if } q(x) \ge N. \end{cases}$$

Then, the embedding $W^{1,q(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega)$ is continuous and compact.

In what follows, we introduce some properties of fractional Sobolev spaces with variable exponents. For a more detailed exposition, we refer the reader to [5, 18]. We define the *fractional Sobolev space* with variable exponents using the Gagliardo approach as

$$W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) = \Big\{u \in L^{q(\cdot)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy < \infty \text{ for some } \lambda > 0\Big\},$$

where $q : \overline{\Omega} \to (1,\infty)$ is continuous and satisfies:

$$1 < q^- := \min_{x\in\overline{\Omega}} q(x) \le q(x) \le \max_{x\in\overline{\Omega}} q(x) =: q^+ < \infty.$$

The associated *Gagliardo seminorm* is

$$[u]_{s,p(\cdot,\cdot)} := \inf\Big\{\lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}\, dx\, dy < 1\Big\}.$$

If we equip the space $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ with the norm

$$\|u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)} := [u]_{s,p(\cdot,\cdot)} + \|u\|_{q(\cdot)}, \tag{2.1}$$

then $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ becomes a Banach space.

Definition 2.4 ([4])**.** Let $p : \overline{\Omega}\times\overline{\Omega} \to (1,+\infty)$ be a continuous variable exponent and $s \in (0,1)$. For any $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$, we define the modular $\rho_{p(\cdot,\cdot)} : W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) \to \mathbb{R}$ by

$$\rho_{p(\cdot,\cdot)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dx\, dy + \int_\Omega |u(x)|^{q(x)} dx,$$

and

$$\|u\|_{\rho_{p(\cdot,\cdot)}} = \inf\big\{\lambda > 0 : \rho_{p(\cdot,\cdot)}\Big(\frac{u}{\lambda}\Big) \leqslant 1\big\}.$$

Remark 2.5. (i) $\|\cdot\|_{\rho_{p(\cdot,\cdot)}}$ is a norm on $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ which is equivalent to the norm $\|\cdot\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)}$. (For the proof of this norm-modular equivalence, one can use the same techniques as in [21, Proposition 2.1])

(ii) $\rho_{p(\cdot,\cdot)}$ also checks the results of Proposition 2.2.

Lemma 2.6 ([26]). *Let $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ be a continuous variable exponent and $s \in (0, 1)$. For any $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$, we have*

$$1 \leqslant [u]_{s,p(\cdot,\cdot)} \implies [u]^{p^-}_{s,p(\cdot,\cdot)} \leqslant \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}}\,dx\,dy \leqslant [u]^{p^+}_{s,p(\cdot,\cdot)},$$

$$[u]_{s,p(\cdot,\cdot)} \leqslant 1 \implies [u]^{p^+}_{s,p(\cdot,\cdot)} \leqslant \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}}\,dx\,dy \leqslant [u]^{p^-}_{s,p(\cdot,\cdot)}.$$

In the following theorem, we establish a result on the compact embedding into Lebesgue spaces with variable exponents. This result was demonstrated in [10] under the condition $q(x) > \overline{p}(x) = p(x, x)$ for all $x \in \overline{\Omega}$, and it was further refined by Azroul et al. [4], in the case where $q(x) = \overline{p}(x)$ for all $x \in \overline{\Omega}$.

Theorem 2.7 ([4]). *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $s \in (0, 1)$. Let $p \in C(\overline{\Omega} \times \overline{\Omega})$ with $sp(x, y) < N$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ let (H1) be satisfied. Assume that $\gamma : \overline{\Omega} \to (1, \infty)$ is a continuous function such that*

$$\bar{p}^*_s(x) := \frac{N\bar{p}(x)}{N - s\bar{p}(x)} > \gamma(x) \geq \gamma^- = \min_{x\in\overline{\Omega}} \gamma(x) > 1 \quad \text{for all } x \in \overline{\Omega}.$$

Then, there exists a positive constant $C = C(N, s, p, q, \gamma, \Omega)$ such that for every $u \in W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$, it holds that

$$\|u\|_{\gamma(\cdot)} \leq C\|u\|_{W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)}.$$

Thus, the space $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ is continuously embedded in $L^{\gamma(\cdot)}(\Omega)$, $1 < \gamma(x) < \bar{p}^*_s(x)$ for all $x \in \overline{\Omega}$. Moreover, this embedding is compact.

Let $W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$. Theorem 2.7 remains true if $W^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ is replaced by $W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$. Specifically, the embedding $W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega) \hookrightarrow L^{\gamma(\cdot)}(\Omega)$, where $1 < \gamma(x) < \bar{p}^*_s(x)$ for all $x \in \overline{\Omega}$, is continuous and compact. For any $u \in W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$, the following inequality holds,

$$\|u\|_{\gamma(\cdot)} \leq C\|u\|_{W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)}. \tag{2.2}$$

Moreover, from (2.2) if $1 < q^- \leq q(x) \leq q^+ < \bar{p}^*_s(x)$ for all $x \in \overline{\Omega}$, then $W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)$ is a separable reflexive Banach space with the norm (see [26])

$$\|u\|_{W_0^{s,q(\cdot),p(\cdot,\cdot)}(\Omega)} = [u]_{s,p(\cdot,\cdot)}. \tag{2.3}$$

3. MAIN RESULTS

In this section, we present our main results. We follow the notation introduced in Sections 1 and 2. Under assumption (H1), the fractional Sobolev spaces $W^{s,q(\cdot),p_1(\cdot,\cdot)}(\Omega)$ and $W^{s,q(\cdot),p_2(\cdot,\cdot)}(\Omega)$ are defined as in Section 2. To simplify the notation, we set

$$W := W^{s,q(\cdot),p_{\max}(\cdot,\cdot)}(\Omega).$$

We denote by W_0 the closure of $C_0^\infty(\Omega)$ in the space W. It is straightforward to verify that both W and W_0 are reflexive Banach spaces when equipped with the norms

$$\|u\|_W = [u]_{s,p_{\max}(\cdot,\cdot)} + |u|_{q(\cdot)}, \quad \text{and} \quad \|u\|_{W_0} = [u]_{s,p_{\max}(\cdot,\cdot)},$$

respectively, see (2.1) and (2.3).

Furthermore, we observe that the space W is continuously embedded in each of the fractional Sobolev spaces $W^{s,q(\cdot),p_i(\cdot,\cdot)}(\Omega)$ for $i = 1, 2$, and similarly, W_0 is continuously embedded in the corresponding zero-boundary version $W_0^{s,q(\cdot),p_i(\cdot,\cdot)}(\Omega)$. These embeddings will play a key role in the variational framework used to prove the existence of weak solutions.

Definition 3.1. Let $u \in W_0$, we say that u is a weak solution of problem (1.1), if it holds that

$$\begin{aligned}
&\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p_1(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy \\
&+ \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p_2(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy \\
&+ \int_\Omega V(x)|u(x)|^{q(x)-2}u(x)\varphi(x)\,dx - \lambda\int_\Omega m_1(x)|u(x)|^{r_1(x)-2}u(x)\varphi(x)\,dx \\
&+ \mu\int_\Omega m_2(x)|u(x)|^{r_2(x)-2}u(x)\varphi(x)\,dx = 0
\end{aligned} \tag{3.1}$$

for all $\varphi \in W_0$.

Definition 3.2. For $u \in W_0$, let the energy functional $E_{\lambda,\mu}$ corresponding to the problem (1.1), defined by $E_{\lambda,\mu} : W_0 \to \mathbb{R}$,

$$\begin{aligned}
E_{\lambda,\mu}(u) &= \int_{\Omega\times\Omega} \frac{1}{p_1(x,y)} \frac{|u(x)-u(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy + \int_{\Omega\times\Omega} \frac{1}{p_2(x,y)} \frac{|u(x)-u(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy \\
&\quad + \int_\Omega \frac{V(x)}{q(x)}|u(x)|^{q(x)}\,dx - \lambda\int_\Omega \frac{m_1(x)}{r_1(x)}|u(x)|^{r_1(x)}\,dx + \mu\int_\Omega \frac{m_2(x)}{r_2(x)}|u(x)|^{r_2(x)}\,dx
\end{aligned}$$

for all $\lambda > 0$ and $\mu > 0$.

Standard arguments similar to those used in [4, 5] demonstrate that $E_{\lambda,\mu} \in C^1(W_0, \mathbb{R})$ and for all $u, \varphi \in W_0$, its derivative is given by

$$\begin{aligned}
\langle E'_{\lambda,\mu}(u), \varphi\rangle &= \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p_1(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy \\
&\quad + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p_2(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy \\
&\quad + \int_\Omega V(x)|u(x)|^{q(x)-2}u(x)\varphi(x)\,dx - \lambda\int_\Omega m_1(x)|u(x)|^{r_1(x)-2}u(x)\varphi(x)\,dx \\
&\quad + \mu\int_\Omega m_2(x)|u(x)|^{r_2(x)-2}u(x)\varphi(x)\,dx
\end{aligned}$$

3.1. **Some lemmas.** In this section, we present lemmas showing that the functional $E_{\lambda,\mu}$ satisfies the geometric conditions imposed by the Mountain Pass Theorem (see [2]), under suitable conditions on the weights V, m_1 and m_2. These conditions are essential for proving the existence of a nontrivial solution.

The following lemma shows that the functional $E_{\lambda,\mu}$ satisfies the first geometrical condition of the Mountain Pass Theorem.

Lemma 3.3. *Assume that the conditions* (H1) *and* (H2) *hold. Then there exist* $\lambda^* > 0$ *such that, for any* $\lambda \in (0, \lambda^*)$*, there exist* $\delta \in (0,1)$ *and a constant* $a > 0$ *such that,* $E_{\lambda,\mu}(u) \geq a > 0$ *for all* $u \in W_0$ *with* $\|u\|_{W_0} = \delta$.

Proof. Let $\sigma(x) = q(x)\alpha'(x)$ by conditions (H1) and (H2), we have

$$\sigma(x) < \big(\bar{p}_{\max}\big)^*_s(x)\frac{N-s}{N} \times \frac{N}{N-s},$$

then, $\sigma(x) < (\bar{p}_{\max})^*_s(x)$ for all $x \in \overline{\Omega}$. Therefore, under (H1) and (H2), the embedding $W_0 \hookrightarrow L^{\sigma(\cdot)}(\Omega)$ is continuous and compact. Then there exists $C > 0$ such that

$$\|u\|_{\sigma(x)} < C\|u\|_{W_0}, \quad \text{for all } u \in W_0. \tag{3.2}$$

We fix $\delta \in (0,1)$ such that $\delta = \|u\|_{W_0} < \min\big\{1, \frac{1}{C}\big\}$, then we deduce that $\|u\|_{\sigma(x)} < 1$. From (H1), lemma 2.1 and proposition 2.3 we infer that for all $u \in W_0$ with $\|u\|_{W_0} = \delta$,

$$\Big|\int_\Omega \frac{V(x)}{q(x)}|u(x)|^{q(x)}\,dx\Big| \leq \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\||u|^{q(x)}\|_{\alpha'(\cdot)} \leq \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\|u\|^{q^-}_{\sigma(\cdot)}. \tag{3.3}$$

Since $r_1(x)\beta_1'(x) < (\overline{p}_{\max})_s^*(x)$ similarly, we have

$$\|u\|_{r_1(\cdot)\beta_1'(\cdot)} < C_1\|u\|_{W_0}, \quad \text{for all } u \in W_0. \tag{3.4}$$

where C_1 is the best Sobolev embedding constant from the space $W_0(\Omega)$ into the Lebesgue space $L^{r_1(\cdot)\beta_1'(\cdot)}(\Omega)$. Moreover

$$\Big|\int_\Omega \frac{m_1(x)}{r_1(x)}|u(x)|^{r_1(x)}\,dx\Big| \le \frac{2}{r_1^-}\|m_1\|_{\beta_1(\cdot)}\|u\|^{r_1^-}_{r_1(\cdot)\beta_1'(\cdot)}. \tag{3.5}$$

On the other hand, it is easy to verify that the following inequality holds,

$$\frac{|u(x)-u(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}} + \frac{|u(x)-u(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}} \ge \frac{|u(x)-u(y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_{\max}(x,y)}}\,. \tag{3.6}$$

By relations (3.2), (3.3), (3.4), (3.5), (3.6) and the conditions (H1) and (H2), we deduce that for any $u \in W_0$ with $\|u\|_{W_0} = \delta \in (0,1)$,

$$\begin{aligned}
E_{\lambda,\mu}(u) &= \int_{\Omega\times\Omega} \frac{1}{p_1(x,y)}\frac{|u(x)-u(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy + \int_{\Omega\times\Omega} \frac{1}{p_2(x,y)}\frac{|u(x)-u(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy \\
&\quad + \int_\Omega \frac{V(x)}{q(x)}|u(x)|^{q(x)}\,dx - \lambda\int_\Omega \frac{m_1(x)}{r_1(x)}|u(x)|^{r_1(x)}\,dx + \mu\int_\Omega \frac{m_2(x)}{r_2(x)}|u(x)|^{r_2(x)}\,dx \\
&\ge \int_{\Omega\times\Omega} \frac{1}{p_{\max}(x,y)}\frac{|u(x)-u(y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_{\max}(x,y)}}\,dx\,dy - \int_\Omega \frac{|V(x)|}{q(x)}|u(x)|^{q(x)}\,dx \\
&\quad - \lambda\int_\Omega \frac{|m_1(x)|}{r_1(x)}|u(x)|^{r_1(x)}\,dx \\
&\ge \frac{1}{p_{\max}^+}\|u\|_{W_0}^{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\|u\|^{q^-}_{\sigma(\cdot)} - \frac{2\lambda}{r_1^-}\|m_1\|_{\beta_1(\cdot)}\|u\|^{r_1^-}_{r_1(\cdot)\beta_1'(\cdot)} \\
&\ge \frac{1}{p_{\max}^+}\|u\|_{W_0}^{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\|u\|^{q^-}_{W_0}C^{q^-} - \frac{2\lambda}{r_1^-}\|m_1\|_{\beta(\cdot)}C_1^{r_1^-}\,\|u\|^{r_1^-}_{W_0} \\
&\ge \frac{1}{p_{\max}^+}\delta^{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\delta^{q^-}\,C^{q^-} - \frac{2\lambda}{r_1^-}\|m_1\|_{\beta(\cdot)}C_1^{r_1^-}\,\delta^{r_1^-} \\
&\ge \delta^{r_1^-}\Big(\frac{1}{p_{\max}^+}\delta^{p_{\max}^+-r_1^-} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\delta^{q^- - r_1^-}C^{q^-} - \frac{2\lambda}{r_1^-}\|m_1\|_{\beta(\cdot)}C_1^{r_1^-}\Big)
\end{aligned} \tag{3.7}$$

By the inequality above, we can choose λ^* such that

$$0 < \frac{1}{p_{\max}^+}\delta^{p_{\max}^+-r_1^-} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\delta^{q^- - r_1^-}C^{q^-} - \frac{2\lambda}{r_1^-}\|m_1\|_{\beta(\cdot)}C_1^{r_1^-},$$

i.e.

$$\lambda < \Big(\frac{1}{p_{\max}^+}\delta^{p_{\max}^+-q^-} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}C^{q^-}\Big)\frac{\delta^{q^- - r_1^-}r_1^-}{2\|m_1\|_{\beta(\cdot)}C_1^{r_1^-}}.$$

If $p_{\max}^+ > q^-$ and $\delta \in (0,1)$ then $\delta^{p_{\max}^+-q^-} < 1$. Hence

$$\lambda < \Big(\frac{1}{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}C^{q^-}\Big)\frac{\delta^{q^- - r_1^-}r_1^-}{2\|m_1\|_{\beta(\cdot)}C_1^{r_1^-}}.$$

By condition (H2) we have $\frac{1}{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}C^{q^-} > 0$. Therefore, we choose

$$\lambda^* = \Big(\frac{1}{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}C^{q^-}\Big)\frac{\delta^{q^- - r_1^-}r_1^-}{2\|m_1\|_{\beta(\cdot)}C_1^{r_1^-}} > 0. \tag{3.8}$$

Otherwise, if $p_{\max}^+ < q^-$ then $\delta^{p_{\max}^+-q^-} > 1$. Hence

$$\frac{1}{p_{\max}^+}\delta^{p_{\max}^+-q^-} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}C^{q^-} > \frac{1}{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}C^{q^-} > 0.$$

Then, we choose

$$\lambda^* = \Big(\frac{1}{p_{\max}^+}\delta^{p_{\max}^+ - q^-} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}C^{q^-}\Big)\frac{\delta^{q^- - r_1^-}r_1^-}{2\|m_1\|_{\beta(\cdot)}C_1^{r_1^-}}. \tag{3.9}$$

Then for all $\lambda \in (0, \lambda^*)$ and all $u \in W_0$ with $\|u\|_{W_0} = \delta$ there exists $a > 0$ such that

$$E_{\lambda,\mu}(u) \ge a > 0.$$

The proof is complete. □

The following lemma shows that the functional $E_{\lambda,\mu}$ satisfies the second geometrical condition of the mountain pass theorem.

Lemma 3.4. *Assume that conditions* (H1)–(H3) *hold. Then for any $\lambda > 0$ and $\mu > 0$ there exists $\rho_0 \in W_0$ such that $\rho_0 > 0$, and $E_{\lambda,\mu}(t\rho_0) < 0$ for all $t > 0$ small enough.*

Proof. We set

$$\begin{gathered} p_{1,0}^- = \min_{(x,y)\in\overline{\Omega}_0\times\overline{\Omega}_0} p_1(x,y), \quad p_{2,0}^- = \min_{(x,y)\in\overline{\Omega}_0\times\overline{\Omega}_0} p_2(x,y), \quad r_{1,0}^- = \min_{x\in\overline{\Omega}_0} r_1(x), \\ r_{2,0}^- = \min_{x\in\overline{\Omega}_0} r_2(x), \quad r_{1,0}^+ = \max_{x\in\overline{\Omega}_0} r_1(x), \quad q_0^- = \min_{x\in\overline{\Omega}_0} q(x), \quad \zeta_0 = \min\left\{p_{1,0}^-, p_{2,0}^-, q_0^-, r_{2,0}^-\right\}. \end{gathered}$$

By condition (H3) we have $r_{1,0}^- < \zeta_0$. Let $\varepsilon > 0$ be such that $r_{1,0}^- + \varepsilon \le \zeta_0$. Since $r_1 \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset\subset \Omega_0$ such that

$$|r_1(x) - r_{1,0}^-| \le \varepsilon \quad \text{for all } x \in \Omega_1.$$

Thus $r_1(x) \le r_{1,0}^- + \varepsilon \le \zeta_0$ for all $x \in \Omega_1$. Let $\rho_0 \in C_0^\infty(\Omega)$ be such that $\operatorname{supp}\big(\rho_0\big) \subset \Omega_1 \subset\subset \Omega_0, \rho_0 = 1$ in a subset $\Omega_1' \subset$ *supp* (ρ_0), $0 \le \rho_0 \le 1$ in Ω_1. Therefore, for any $t \in (0,1)$ we have

$$\begin{aligned} &E_{\lambda,\mu}(t\rho_0) \\ &= \int_{\Omega\times\Omega}\frac{1}{p_1(x,y)}\frac{|t\rho_0(x) - t\rho_0(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy + \int_{\Omega\times\Omega}\frac{1}{p_2(x,y)}\frac{|t\rho_0(x) - t\rho_0(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy \\ &\quad + \int_\Omega \frac{V(x)}{q(x)}|t\rho_0(x)|^{q(x)}\,dx - \lambda\int_\Omega \frac{m_1(x)}{r_1(x)}|t\rho_0(x)|^{r_1(x)}\,dx + \mu\int_\Omega\frac{m_2(x)}{r_2(x)}|t\rho_0(x)|^{r_2(x)}\,dx \\ &\le \frac{t^{p_{1,0}^-}}{p_{1,0}^-}\int_{\Omega_0\times\Omega_0}\frac{|\rho_0(x) - \rho_0(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy + \frac{t^{p_{2,0}^-}}{p_{2,0}^-}\int_{\Omega_0\times\Omega_0}\frac{|\rho_0(x) - \rho_0(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy \\ &\quad + \frac{t^{q_0^-}}{q_0^-}\int_{\Omega_0}|V(x)||\rho_0(x)|^{q(x)}\,dx - \frac{\lambda t^{r_{1,0}^- + \varepsilon}}{r_{1,0}^+}\int_{\Omega_1} m_1(x)|\rho_0(x)|^{r_1(x)}\,dx \\ &\quad + \frac{\mu t^{r_{2,0}^-}}{r_{2,0}^-}\int_{\Omega_0} m_2(x)|\rho_0(x)|^{r_2(x)}\,dx \\ &\le t^{\zeta_0}\Big[A(\Omega_0) + \frac{1}{q_0^-}\int_{\Omega_0}|V(x)||\rho_0(x)|^{q(x)}\,dx + \frac{\mu}{r_{2,0}^-}\int_{\Omega_0} m_2(x)|\rho_0(x)|^{r_2(x)}\,dx\Big] \\ &\quad - \frac{\lambda t^{r_{1,0}^- + \varepsilon}}{r_{1,0}^+}\int_{\Omega_1} m_1(x)|\rho_0(x)|^{r_1(x)}\,dx, \end{aligned}$$

such that

$$A(\Omega_0) = \frac{1}{p_{1,0}^-}\int_{\Omega_0\times\Omega_0}\frac{|\rho_0(x) - \rho_0(y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy + \frac{1}{p_{2,0}^-}\int_{\Omega_0\times\Omega_0}\frac{|\rho_0(x) - \rho_0(y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy.$$

Then $E_{\lambda,\mu}(t\rho_0) < 0$ for all $0 < t < \theta^{\varpi}$, where

$$\varpi = \frac{1}{\zeta_0 - r_{1,0}^- - \varepsilon} \quad \text{and} \quad 0 < \theta < \min\{1, \theta_0\},$$

and $\theta_0 > 0$ is given by

$$\theta_0 = \frac{\lambda \int_{\Omega_1} m_1(x)|\rho_0(x)|^{r_1(x)}\,dx}{r_{1,0}^+\Big(A(\Omega_0) + \frac{1}{q_0^-}\int_{\Omega_0}|V(x)||\rho_0(x)|^{q(x)}\,dx + \frac{\mu}{r_{2,0}^-}\int_{\Omega_0} m_2(x)|\rho_0(x)|^{r_2(x)}\,dx\Big)}.$$

Now, we point out that

$$A(\Omega_0) + \frac{1}{q_0^-}\int_{\Omega_0}|V(x)||\rho_0(x)|^{q(x)}\,dx + \frac{\mu}{r_{2,0}^-}\int_{\Omega_0} m_2(x)|\rho_0(x)|^{r_2(x)}\,dx > 0.$$

Indeed, if it is not true then

$$A(\Omega_0) + \frac{1}{q_0^-}\int_{\Omega_0}|V(x)||\rho_0(x)|^{q(x)}\,dx + \frac{\mu}{r_{2,0}^-}\int_{\Omega_0} m_2(x)|\rho_0(x)|^{r_2(x)}\,dx = 0,$$

thus $\|\rho_0\|_{W_0} = 0$, hence $\rho_0 = 0$ in Ω_0. This is a contradiction. The proof is complete. □

3.2. **Existence result.**

Theorem 3.5. *Assume that conditions* (H1)–(H3) *are satisfied. Then, for every* $\mu > 0$ *there exists a constant* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$, *problem* (1.1) *has a nontrivial weak solution.*

Proof. We use the same techniques as in [4] and [9]. Let $\lambda^* > 0$ be defined as in (3.8) or (3.9) and $\lambda \in (0, \lambda^*)$, $\mu > 0$. By Lemma (3.3) it follows that

$$\inf_{\partial B_\delta(0)} E_{\lambda,\mu} > 0 \tag{3.10}$$

where $\partial B_\delta(0) = \{u \in B_\delta(0) : \|u\|_{W_0} = \delta\}$ and $B_\delta(0)$ is the ball centered at the origin and of radius δ.

On the other hand, by Lemma (3.4), there exists $\rho_0 \in W_0$ such that $E_{\lambda,\mu}(t\rho_0) < 0$ for all $t > 0$ small enough. Moreover, by inequality (3.7) we have

$$E_{\lambda,\mu}(u) \ge \frac{1}{p_{\max}^+}\|u\|_{W_0}^{p_{\max}^+} - \frac{2}{q^-}\|V\|_{\alpha(\cdot)}\|u\|_{W_0}^{q^-}C^{q^-} - \frac{2\lambda}{r_1^-}\|m_1\|_{\beta(\cdot)}C^{r_1^-}\|u\|_{W_0}^{r_1^-} \tag{3.11}$$

for all $u \in B_\delta(0)$. It follows that

$$-\infty < \overline{c} := \inf_{\overline{B_\delta(0)}} E_{\lambda,\mu} < 0. \tag{3.12}$$

By (3.10) and (3.12), we have

$$0 < \inf_{\partial B_\delta(0)} E_{\lambda,\mu} - \inf_{B_\delta(0)} E_{\lambda,\mu},$$

hence, we can suppose that

$$0 < \tau \le \inf_{\partial B_\delta(0)} E_{\lambda,\mu} - \inf_{B_\delta(0)} E_{\lambda,\mu}.$$

Using the above information, the functional $E_{\lambda,\mu} : \overline{B_\delta(0)} \to \mathbb{R}$, is bounded from below on $\overline{B_\delta(0)}$ and $E_{\lambda,\mu} \in C^1\left(\overline{B_\delta(0)}, \mathbb{R}\right)$. Then applying Ekeland's variational principle [13], there exists $u_\varepsilon \in \overline{B_\delta(0)}$ such that

$$\begin{gathered}\inf_{\overline{B_\delta(0)}} E_{\lambda,\mu} \le E_{\lambda,\mu}(u_\varepsilon) \le \inf_{\overline{B_\delta(0)}} E_{\lambda,\mu} + \varepsilon, \\ E_{\lambda,\mu}(u_\varepsilon) < E_{\lambda,\mu}(u) + \varepsilon\|u - u_\varepsilon\|_{W_0} \quad \text{for all } u \in \overline{B_\delta(0)} \text{ and } u \neq u_\varepsilon.\end{gathered} \tag{3.13}$$

So

$$E_{\lambda,\mu}(u_\varepsilon) \le \inf_{\overline{B_\delta(0)}} E_{\lambda,\mu} + \varepsilon \le \inf_{B_\delta(0)} E_{\lambda,\mu} + \varepsilon < \inf_{\partial B_\delta(0)} E_{\lambda,\mu}.$$

Then we deduce that $u_\varepsilon \in B_\delta(0)$. Now, we consider the function $\mathcal{J}_{\lambda,\mu}^\varepsilon : \overline{B_\delta(0)} \to \mathbb{R}$ defined as

$$\mathcal{J}_{\lambda,\mu}^\varepsilon(u) = E_{\lambda,\mu}(u) + \varepsilon\|u - u_\varepsilon\|_{W_0}.$$

By (3.13), we obtain

$$\mathcal{J}_{\lambda,\mu}^\varepsilon(u_\varepsilon) = E_{\lambda,\mu}(u_\varepsilon) < \mathcal{J}_{\lambda,\mu}^\varepsilon(u) \quad \text{for all } u \neq u_\varepsilon.$$

It follows that u_ε is a minimum point of $\mathcal{J}^{\varepsilon}_{\lambda,\mu}$ on $\overline{B_\delta(0)}$. Then, for any $t>0$ small enough and $v\in B_\delta(0)$,

$$\frac{\mathcal{J}^{\varepsilon}_{\lambda,\mu}(u_\varepsilon+tv)-\mathcal{J}^{\varepsilon}_{\lambda,\mu}(u_\varepsilon)}{t}\geq 0.$$

Then, we obtain that

$$\frac{E_{\lambda,\mu}(u_\varepsilon+tv)-E_{\lambda,\mu}(u_\varepsilon)}{t}+\varepsilon\|v\|_{W_0}\geq 0.$$

Let t tends to 0^+. Then

$$\langle E'_{\lambda,\mu}(u_\varepsilon),v\rangle+\varepsilon\|v\|_{W_0}\geq 0,$$

thus

$$\|E'_{\lambda,\mu}(u_\varepsilon)\|_{W_0^*}\leqslant\varepsilon. \tag{3.14}$$

From this inequality, we deduce that there exists a sequence $\{w_n\}\subset B_\delta(0)$ such that

$$E_{\lambda,\mu}(w_n)\to\bar{c}\quad\text{and}\quad E'_{\lambda,\mu}(w_n)\to 0. \tag{3.15}$$

From (3.11) and (3.15), it is clear that $\{w_n\}$ is bounded in W_0. Thus there exists $w\in W_0$ such that $w_n\rightharpoonup w$ in W_0. By (H1) and (H2), we have that

$$\begin{aligned}
q(x)<\frac{(\overline{p}_{\max})^*_s(x)\alpha(x)}{(\overline{p}_{\max})^*_s(x)+\alpha(x)}
&\iff q(x)\left((\overline{p}_{\max})^*_s(x)+\alpha(x)\right)<(\overline{p}_{\max})^*_s(x)\alpha(x)\\
&\iff \alpha(x)q(x)<(\overline{p}_{\max})^*_s(x)\alpha(x)-(\overline{p}_{\max})^*_s(x)q(x)\\
&\iff \frac{\alpha(x)q(x)}{\alpha(x)-q(x)}<(\overline{p}_{\max})^*_s(x).
\end{aligned}$$

Therefore, $k(x)=\frac{\alpha(x)q(x)}{\alpha(x)-q(x)}<(\overline{p}_{\max})^*_s(x)$ for all $x\in\overline{\Omega}$, so by Theorem 2.7, we deduce that W_0 is compactly embedded in $L^{k(\cdot)}(\Omega)$; then

$$w_n\to w\quad\text{in } L^{k(\cdot)}(\Omega). \tag{3.16}$$

Using Hölder inequality, we have

$$\begin{aligned}
\int_\Omega V(x)|w_n|^{q(x)-2}w_n(w_n-w)\,dx&\leq 2\|V(x)\|_{\alpha(\cdot)}\||w_n|^{q(x)-2}w_n(w_n-w)\|_{\alpha'(\cdot)}\\
&\leq 2\|V(x)\|_{\alpha(\cdot)}\||w_n|^{q(x)-2}w_n\|_{q'(\cdot)}\|w_n-w\|_{k(\cdot)}.
\end{aligned}$$

If $\big\||w_n|^{q(x)-2}w_n\big\|_{q'(\cdot)}\leq 1$, then

$$\int_\Omega V(x)|w_n|^{q(x)-2}w_n(w_n-w)\,dx\leq 2\|V(x)\|_{\alpha(\cdot)}\|w_n-w\|_{k(\cdot)}\to 0.$$

Now if $\big\||w_n|^{q(x)-2}w_n\big\|_{q'(\cdot)}>1$, thus we have $\big\||w_n|^{q(x)-2}w_n\big\|_{q'(\cdot)}\leq\|u_n\|^{q^+}_{q(\cdot)}$. Since $q(x)<(\overline{p}_{\max})^*_s(x)$ for all $x\in\overline{\Omega}$, then we have the compact embedding $W_0\hookrightarrow L^{q(\cdot)}(\Omega)$, and by (3.16), we obtain

$$\lim_{n\to+\infty}\int_\Omega V(x)|w_n|^{q(x)-2}w_n(w_n-w)\,dx=0. \tag{3.17}$$

From (H2) we have $r_i(x)<(\overline{p}_{\max})^*_s(x)$ and $h_i(x)=\frac{\beta_i(x)r_i(x)}{\beta_i(x)-r_i(x)}<(\overline{p}_{\max})^*_s(x)$. Using the same techniques as above, we obtain

$$\lim_{n\to+\infty}\int_\Omega m_1(x)|w_n|^{r_1(x)-2}w_n(w_n-w)\,dx=0, \tag{3.18}$$

$$\lim_{n\to+\infty}\int_\Omega m_2(x)|w_n|^{r_2(x)-2}w_n(w_n-w)\,dx=0. \tag{3.19}$$

On the other hand, from (3.15), we have

$$\lim_{n\to+\infty}\langle E'_{\lambda,\mu}(w_n),w_n-w\rangle=0.$$

That is,

$$\begin{aligned}
&\lim_{n\to+\infty}\Big\{\int_{\Omega\times\Omega}\frac{|w_n(x)-w_n(y)|^{p_1(x,y)-2}\,(w_n(x)-w_n(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy\\
&+\int_{\Omega\times\Omega}\frac{|w_n(x)-w_n(y)|^{p_2(x,y)-2}\,(w_n(x)-w_n(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy\\
&+\int_\Omega V(x)|w_n(x)|^{q(x)-2}w_n(x)\big(w_n(x)-w(x)\big)\,dx-\lambda\int_\Omega m_1(x)|w_n(x)|^{r_1(x)-2}w_n(x)\big(w_n(x)-w(x)\big)\,dx\\
&-\mu\int_\Omega m_2(x)|w_n(x)|^{r_2(x)-2}w_n(x)\big(w_n(x)-w(x)\big)\,dx\Big\}=0
\end{aligned}$$

By using the relations (3.17), (3.18) and (3.19) we obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\Big\{\int_{\Omega\times\Omega}\frac{|w_n(x)-w_n(y)|^{p_1(x,y)-2}\,(w_n(x)-w_n(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy\\
&+\int_{\Omega\times\Omega}\frac{|w_n(x)-w_n(y)|^{p_2(x,y)-2}\,(w_n(x)-w_n(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy\Big\}\\
&=0
\end{aligned}\tag{3.20}$$

On the other hand, since $\{w_n\}$ converges weakly to w in W_0, we have $E'_{\lambda,\mu}(w)(w_n-w)\to 0$ as $n\to\infty$ or

$$\begin{aligned}
&\lim_{n\to+\infty}\Big\{\int_{\Omega\times\Omega}\frac{|w(x)-w(y)|^{p_1(x,y)-2}(w(x)-w(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy\\
&+\int_{\Omega\times\Omega}\frac{|w(x)-w(y)|^{p_2(x,y)-2}(w(x)-w(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy\\
&+\int_\Omega V(x)|w|^{q(x)-2}w(w_n-w)\,dx-\lambda\int_\Omega m_1(x)|w|^{r_1(x)-2}w(w_n-w)\,dx\\
&+\mu\int_\Omega m_2(x)|w|^{r_2(x)-2}w(w_n-w)\,dx\Big\}=0,
\end{aligned}$$

which implies by using the same arguments as before that

$$\begin{aligned}
&\lim_{n\to+\infty}\Big\{\int_{\Omega\times\Omega}\frac{|w(x)-w(y)|^{p_1(x,y)-2}(w(x)-w(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy\\
&+\int_{\Omega\times\Omega}\frac{|w(x)-w(y)|^{p_2(x,y)-2}(w(x)-w(y))\,\big(\big(w_n(x)-w(x)\big)-(w_n(y)-w(y))\big)}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy\Big\}\\
&=0.
\end{aligned}\tag{3.21}$$

Let $\Lambda_n(x,y)=w_n(x)-w_n(y)$ and $\Lambda(x,y)=w(x)-w(y)$. By subtracting equation (3.20) from equation (3.21) and using the well-known inequality [9],

$$\big(|\eta_1|^{r-2}\eta_1-|\eta_2|^{r-2}\eta_2\big)\,(\eta_1-\eta_2)\geq\frac{1}{2^r}\,|\eta_1-\eta_2|^r\,,\quad\text{for all } r\geq 2,\ \eta_1,\eta_2\in\mathbb{R}^N,$$

we obtain

$$\begin{aligned}
A_n&=\int_{\Omega\times\Omega}\frac{\Big(|\Lambda_n(x,y)|^{p_1(x,y)-2}\,\Lambda_n(x,y)-|\Lambda(x,y)|^{p_1(x,y)-2}\Lambda(x,y)\Big)\,(\Lambda_n(x,y)-\Lambda(x,y))}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy\\
&\quad+\int_{\Omega\times\Omega}\frac{\Big(|\Lambda_n(x,y)|^{p_2(x,y)-2}\,\Lambda_n(x,y)-|\Lambda(x,y)|^{p_2(x,y)-2}\Lambda(x,y)\Big)\,(\Lambda_n(x,y)-\Lambda(x,y))}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy\\
&\geq\int_{\Omega\times\Omega}\frac{1}{2^{p_1(x,y)}}\,\frac{|\Lambda_n(x,y)-\Lambda(x,y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy+\int_{\Omega\times\Omega}\frac{1}{2^{p_2(x,y)}}\,\frac{|\Lambda_n(x,y)-\Lambda(x,y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2^{p_1^+}} \int_{\Omega\times\Omega} \frac{|\Lambda_n(x,y)-\Lambda(x,y)|^{p_1(x,y)}}{|x-y|^{N+sp_1(x,y)}}\,dx\,dy + \frac{1}{2^{p_2^+}} \int_{\Omega\times\Omega} \frac{|\Lambda_n(x,y)-\Lambda(x,y)|^{p_2(x,y)}}{|x-y|^{N+sp_2(x,y)}}\,dx\,dy \\
&\geq \frac{1}{2^{p_{\max}^+}} \int_{\Omega\times\Omega} \frac{|\Lambda_n(x,y)-\Lambda(x,y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_{\max}(x,y)}}\,dx\,dy \\
&= \frac{1}{2^{p_{\max}^+}} \int_{\Omega\times\Omega} \frac{|(w_n-w)(x)-(w_n-w)(y)|^{p_{\max}(x,y)}}{|x-y|^{N+sp_{\max}(x,y)}}\,dx\,dy \\
&\geq \frac{1}{2^{p_{\max}^+}} \|w_n-w\|_{W_0}^{(p_{\max})^+}.
\end{aligned}
$$

Since $\lim_{n\to+\infty} A_n = 0$, it follows that $\lim_{n\to+\infty} \|w_n - w\|_{W_0} = 0$. Therefore, the sequence (w_n) converges strongly to w in W_0. Since $E_{\lambda,\mu} \in C^1(W_0,\mathbb{R})$, it follows that

$$E'_{\lambda,\mu}(w_n) \to E'_{\lambda,\mu}(w), \quad \text{as } n \to \infty. \tag{3.22}$$

Then by relations (3.22) and (3.15), we obtain

$$E_{\lambda,\mu}(w) = \lim_{n\to+\infty} E_{\lambda,\mu}(w_n) = \bar{c} < 0 \quad \text{and} \quad E'_{\lambda,\mu}(w) = 0.$$

We deduce that w is a nontrivial critical point of $E_{\lambda,\mu}$, then w is a weak solution for problem (1.1). Moreover, since $E_{\lambda,\mu}(w) < 0$ thus, w is a nontrivial weak solution for problem (1.1). Finally, for every $\lambda \in (0,\lambda^*)$, problem (1.1) admits a nontrivial weak solution. The proof of Theorem 3.5 is now complete. □

Acknowledgments. The authors would like to thank the anonymous referees for their valuable comments and constructive suggestions.

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