Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 41, pp. 1–20.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.41

LEAST SQUARES ESTIMATION FOR SUB-FRACTIONAL BROWNIAN BRIDGE WITH LINEAR DRIFT

NENGHUI KUANG, HUANTIAN XIE

Abstract. In this article we consider least squares estimation (LSE) for sub-fractional Brownian bridge with linear drift defined by

$$dX_t = \frac{\alpha(k - X_t)}{T - t}dt + dS_t^H, \quad 0 \le t < T, \quad X_0 = x_0,$$

where $\alpha > 0$, $k \in \mathbb{R}$ and $\tau := \alpha k$ are unknown parameters, and S^H is a sub-fractional Brownian with Hurst index $H \in (1/2, 1)$. We prove that the LSE has strong consistency as $t \to T$ depending on the value of α. When it has consistency, we obtain the rate of this convergence. This work extends the results by Kuang and Liu [11] who studied the case $k = 0$ and $x_0 = 0$.

1. Introduction and main results

Let W be a standard Brownian motion and let α be a non-negative real parameter. In recent years, the study of various problems related to the (so-called) α-Wiener bridge, that is, to the solution X to

$$dX_t = -\alpha\frac{X_t}{T - t}dt + dW_t, \quad 0 \le t < T, \quad X_0 = 0, \tag{1.1}$$

has attracted interest. For a motivation and further references, we refer the reader to Barczy and Pap [2, 3], as well as Mansuy [15]. Because (1.1) is linear, it is immediate to solve it explicitly, one then gets the formula

$$X_t = (T - t)^\alpha \int_0^t (T - s)^{-\alpha} dW_s, \quad t \in [0, T),$$

the integral with respect to W being a Wiener integral.

An example of interesting problem related to X is the statistical estimation of α when one observes the whole trajectory of X. A natural candidate is the maximum likelihood estimator (MLE), which can be easily computed for this model, because of the specific form of (1.1), one obtains

$$\hat{\alpha}_t = -\frac{\int_0^t \frac{X_u}{T-u}dX_u}{\int_0^t \frac{X_u^2}{(T-u)^2}du}, \quad 0 \le t < T. \tag{1.2}$$

It is worth noticing that the MLE $\hat{\alpha}_t$ coincides with the LSE, indeed, $\hat{\alpha}_t$ (formally) minimizes

$$\alpha \to \int_0^t \big|\dot{X}_u + \alpha\frac{X_u}{T - u}\big|^2 du.$$

By (1.1) and (1.2), we obtain

$$\alpha - \hat{\alpha}_t = \frac{\int_0^t \frac{X_u}{T-u}dW_u}{\int_0^t \frac{X_u^2}{(T-u)^2}du}. \tag{1.3}$$

It is not very difficult to check that $\hat{\alpha}_t$ is indeed a strongly consistent estimator of α. Zhao and Chen [25] studied large deviation expansion for maximum likelihood estimator of α-Brownian

---

2020 *Mathematics Subject Classification.* 60G15, 60H05, 62F12.
Key words and phrases. Least squares estimation; sub-fractional Brownian bridge; strong consistency.
©2026. This work is licensed under a CC BY 4.0 license.
Submitted September 25, 2025. Published June 11, 2026.

bridge. Zhao, Liu and Chen [24] investigated the large deviation principle for maximum likelihood estimator of α-Brownian bridge.

Es-Sebaiy and Nourdin [7] obtained the asymptotic behavior of the LSE when W in (1.1) is fractional Brownian motion. Han, Shen and Yan [8] studied the case when W in (1.1) is weighted fractional Brownian motion. Kuang and Liu [11] investigated the case when W in (1.1) is sub-fractional Brownian motion. Han et al. [9] considered least squares estimation for fractional Brownian bridge with linear drift, and proved that the LSE had strong consistency as $t \to T$ depending on the value of α. When it had consistency, they obtained the rate of this convergence.

Motivated by all these results, in this paper, we study the asymptotic behavior of the LSE for sub-fractional Brownian bridge with linear drift defined by

$$dX_t = \frac{\alpha(k - X_t)}{T-t}dt + dS_t^H,\ 0 \le t < T,\ X_0 = x_0, \tag{1.4}$$

where $\alpha > 0$, $k \in \mathrm{R}$ and $\tau := \alpha k$ are unknown parameters, and S^H is a sub-fractional Brownian with Hurst index $H \in (\frac{1}{2}, 1)$. We extend the results of Kuang and Liu [11]. For more on sub-fractional Brownian motion, we can see Bojdecki et al. [4], Tudor [20, 21, 22], Yan and Shen [23] Diedhiou et al. [6], Shen and Yan [19], Liu and Yan [14], Shen and Chen [18], Kuang and Xie [12], Kuang and Liu [11], Kuang and Li [10], and Kuang and Xie [13].

By using the least square method, the estimators are formally obtained by minimizing the function

$$L(\alpha, \tau) = \int_0^t \Big|\dot{X}_u - \frac{\tau - \alpha X_u}{T-u}\Big|^2 du.$$

We can obtain the least squares estimators of α, τ and k as follows

$$\hat{\alpha}_t = \frac{\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u}{\frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \big(\int_0^t \frac{X_u}{(T-u)^2}du\big)^2}, \tag{1.5}$$

$$\hat{\tau}_t = \frac{\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{T-u}dX_u\int_0^t \frac{X_u}{(T-u)^2}du}{\frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \big(\int_0^t \frac{X_u}{(T-u)^2}du\big)^2}, \tag{1.6}$$

$$\hat{k}_t = \frac{\hat{\tau}_t}{\hat{\alpha}_t} = \frac{\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{T-u}dX_u\int_0^t \frac{X_u}{(T-u)^2}du}{\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u}, \tag{1.7}$$

for all $0 < t < T$.

Our main results read as follows.

Theorem 1.1. (1) *When $0 < \alpha \le \frac{1}{2}$, as $t \to T$, we have*

$$\hat{\alpha}_t \xrightarrow{a.s.} \alpha. \tag{1.8}$$

When $\frac{1}{2} < \alpha < H$, as $t \to T$, we have

$$\hat{\alpha}_t \xrightarrow{a.s.} \frac{1}{2}\Big(1 + \frac{\frac{(x_0-k)^2}{T}}{\int_0^T \frac{(X_u-k)^2}{(T-u)^2}du}\Big). \tag{1.9}$$

(2) *When $0 < \alpha < H$, as $t \to T$, we have*

$$\hat{k}_t \xrightarrow{a.s.} k. \tag{1.10}$$

(3) *As $t \to T$, we have*

$$\hat{\tau}_t = \hat{\alpha}_t\hat{k}_t \xrightarrow{a.s.} \begin{cases} \alpha k = \tau, & \text{if } 0 < \alpha \le 1/2; \\ \frac{k}{2}\Big(1 + \frac{\frac{(x_0-k)^2}{T}}{\int_0^T \frac{(X_u-k)^2}{(T-u)^2}du}\Big), & \text{if } 1/2 < \alpha < H. \end{cases} \tag{1.11}$$

Since (1.4) is linear, one obtains the solution easily:

$$X_t = k + \frac{x_0 - k}{T^\alpha}(T-t)^\alpha + (T-t)^\alpha\int_0^t (T-u)^{-\alpha}dS_u^H, \quad t \in [0, T), \tag{1.12}$$

where the integral can be understood either in the Young sense, or in the Skorohod sense.

For convenience, we introduce the following two processes related to X: for $t \in [0,T]$,

$$\xi_t = \int_0^t (T-u)^{-\alpha} dS_u^H, \tag{1.13}$$

$$\eta_t = \int_0^t \Big[(T-u)^{\alpha-1} \int_0^u (T-v)^{-\alpha} dS_v^H\Big] dS_u^H = \int_0^t (T-u)^{\alpha-1} \xi_u dS_u^H. \tag{1.14}$$

In particular, we observe that, for $t \in [0,T)$,

$$X_t = k + \frac{x_0-k}{T^\alpha}(T-t)^\alpha + (T-t)^\alpha \xi_t. \tag{1.15}$$

For $t \in [0,T]$, we denote

$$\phi_t = \int_0^t (T-u)^{\alpha-1} dS_u^H, \tag{1.16}$$

$$\psi_t = \int_0^t \frac{dS_u^H}{T-u}. \tag{1.17}$$

Thus we obtain the following theorem.

Theorem 1.2. *Let $N \sim \mathcal{N}(0,1)$ be independent of S^H, and let $\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ denote the usual Beta function.*

(1) *Assume that $\alpha \in (0, 1-H)$. Then, as $t \to T$,*

$$(T-t)^{\alpha-H}(\alpha - \hat{\alpha}_t) \xrightarrow{law} \frac{(1-\alpha)^2(1-2\alpha)}{\alpha^2}\sigma_1 \times \frac{N}{\xi_T + \frac{x_0-k}{T^\alpha}}, \tag{1.18}$$

$$(T-t)^{\alpha-H}(\tau - \hat{\tau}_t) \xrightarrow{law} \frac{k(1-\alpha)^2(1-2\alpha)}{\alpha^2}\sigma_1 \times \frac{N}{\xi_T + \frac{x_0-k}{T^\alpha}}, \tag{1.19}$$

$$(T-t)^{-H}(k - \hat{k}_t) \xrightarrow{law} \frac{(1-\alpha)^2(1-2\alpha)}{\alpha^3}\sigma_2 \times N, \tag{1.20}$$

where

$$\begin{aligned}\sigma_1^2 = \frac{\alpha H(2H-1)}{(1-\alpha)(2-\alpha-2H)}\Big[&\frac{(\alpha+2H-1)\beta(2-\alpha-2H, 2H-1)}{1-H-\alpha} \\ &+ \frac{(1-2H)\beta(2-2H, 2H-1)}{(1-\alpha)(1-H)}\Big],\end{aligned} \tag{1.21}$$

$$\begin{aligned}\sigma_2^2 = \frac{\alpha H(2H-1)}{(1-\alpha)(1-2\alpha)(2-\alpha-2H)}\Big[&\frac{(2H-1)\beta(2-\alpha-2H, 2H-1)}{(1-\alpha)(1-H-\alpha)} \\ &+ \frac{(\alpha+1-2H)\beta(2-2H, 2H-1)}{(1-2\alpha)(1-H)}\Big].\end{aligned} \tag{1.22}$$

(2) *Assume that $\alpha \in (1-H, \frac{1}{2})$. Then, as $t \to T$,*

$$(T-t)^{2\alpha-1}(\alpha - \hat{\alpha}_t) \xrightarrow{law} \frac{(1-\alpha)^2(1-2\alpha)\big(\eta_T + \frac{x_0-k}{T^\alpha}\phi_T\big)}{\alpha^2\big(\xi_T + \frac{x_0-k}{T^\alpha}\big)^2}, \tag{1.23}$$

$$(T-t)^{2\alpha-1}(\tau - \hat{\tau}_t) \xrightarrow{law} \frac{k(1-\alpha)^2(1-2\alpha)\big(\eta_T + \frac{x_0-k}{T^\alpha}\phi_T\big)}{\alpha^2\big(\xi_T + \frac{x_0-k}{T^\alpha}\big)^2}, \tag{1.24}$$

$$(T-t)^{\alpha-1}(k - \hat{k}_t) \xrightarrow{law} \frac{(1-\alpha)(1-2\alpha)\left(\eta_T + \frac{x_0-k}{T^\alpha}\phi_T\right)}{\alpha^3\left(\xi_T + \frac{x_0-k}{T^\alpha}\right)}, \tag{1.25}$$

where ξ_T, η_T and ϕ_T are defined by (1.13), (1.14), *and* (1.16), *respectively.*

(3) *Assume that $\alpha = 1/2$. Then, as $t \to T$,*

$$|\log(T-t)|(\alpha - \hat{\alpha}_t) \xrightarrow{law} \frac{\xi_T\left(\frac{1}{2}\xi_T + \frac{x_0-k}{T^\alpha}\right)}{\big(\xi_T + \frac{x_0-k}{T^\alpha}\big)^2}, \tag{1.26}$$

$$|\log(T-t)|(\tau-\hat{\tau}_t) \xrightarrow{law} \frac{k\xi_T\left(\frac{1}{2}\xi_T+\frac{x_0-k}{T^\alpha}\right)}{\left(\xi_T+\frac{x_0-k}{T^\alpha}\right)^2}, \tag{1.27}$$

$$\frac{|\log(T-t)|}{(T-t)^\alpha}(k-\hat{k}_t) \xrightarrow{law} \frac{4\xi_T\left(\frac{1}{2}\xi_T+\frac{x_0-k}{T^\alpha}\right)}{\xi_T+\frac{x_0-k}{T^\alpha}}, \tag{1.28}$$

where ξ_T is defined by (1.13).

Since we can not prove the existence of the limiting variance of $\frac{\phi_t}{\sqrt{|\log(T-t)|}}$ as $t\to T$, we can not obtain the limiting distribution of the estimators $\hat{\alpha}_t, \hat{\tau}_t$ and $\hat{k}_t$ in the case $\alpha = 1-H$. This will be our work in the future. The rest of this paper is organized as follows. In Section 2, we give some lemmas and the proofs of Theorem 1.1 and 1.2. In the appendix, we list the preliminaries tools that we will need throughout this paper: Malliavin derivative, Young integral, Skorohod integral, and the link between Young and Skorohod integrals.

2. Proofs of Theorems 1.1 and 1.2

First we give some useful lemmas. Lemmas 2.1 and 2.4 come from Kuang and Liu [11, Lemmas 3.2, 3.3, 3.4, 3.6 and 3.7].

Lemma 2.1. *Let $\alpha\in(0,H)$ and ξ_t be defined by* (1.13). *Then $\xi_T := \lim_{t\to T}\xi_t$ exists in L^2. Moreover, for all $\epsilon\in(0,H-\alpha)$, the process $\xi=\{\xi_t\}_{t\in[0,T]}$ admits a modification with $(H-\alpha-\epsilon)$-Hölder continuous paths, still denoted ξ in the sequel. In particular, $\xi_t \xrightarrow{a.s.} \xi_T$ as $t\to T$. Furthermore, as $t\to T$,*

(1) *if $0<\alpha<1/2$, then*
$$(T-t)^{1-2\alpha}\int_0^t \xi_s^2(T-s)^{2\alpha-2}ds \xrightarrow{a.s.} \frac{\xi_T^2}{1-2\alpha}, \tag{2.1}$$
(2) *if $\alpha=1/2$, then*
$$\frac{1}{|\log(T-t)|}\int_0^t \frac{\xi_s^2}{T-s}ds \xrightarrow{a.s.} \xi_T^2, \tag{2.2}$$
(3) *if $1/2<\alpha<H$, then*
$$\int_0^t \xi_s^2(T-s)^{2\alpha-2}ds \xrightarrow{a.s.} \int_0^T \xi_s^2(T-s)^{2\alpha-2}ds, \tag{2.3}$$
with $\int_0^T \xi_s^2(T-s)^{2\alpha-2}ds<\infty$, a.s..

Remark 2.2. If $\alpha\in(1-H,1)$. Then $\phi_T := \lim_{t\to T}\phi_t$ exists in L^2. In particular, $\phi_t \xrightarrow{\text{a.s.}} \phi_T$ as $t\to T$.

Remark 2.3. By the Hölderianity of ξ, if $\alpha\in(0,H)$, we have an analogous result: as $t\to T$,
$$(T-t)^{1-\alpha}\int_0^t \xi_s(T-s)^{\alpha-2}ds \xrightarrow{\text{a.s.}} \frac{\xi_T}{1-\alpha}. \tag{2.4}$$

Lemma 2.4. *Let $\alpha\in(1-H,H)$ and η_t be defined by* (1.14). *Then $\eta_T := \lim_{t\to T}\eta_t$ exists in L^2. Moreover, there exists $\kappa>0$ such that $\eta=\{\eta_t\}_{t\in[0,T]}$ admits a modification with $\kappa-$Hölder continuous paths, still denoted η in the sequel. In particular, $\eta_t \xrightarrow{a.s.} \eta_T$ as $t\to T$.*

Furthermore, for any $t\in[0,T)$, we have
$$\begin{aligned}\eta_t &= \int_0^t (T-u)^{\alpha-1}dS_u^H \times \int_0^t (T-s)^{-\alpha}dS_s^H - \int_0^t \delta S_s^H(T-s)^{-\alpha}\int_0^s \delta S_u^H(T-u)^{\alpha-1}\\ &\quad - \int_0^t ds(T-s)^{-\alpha}\int_0^s du(T-u)^{\alpha-1}\varphi(s,u)\\ &= phi_t\xi_t - \int_0^t \delta S_s^H(T-s)^{-\alpha}\int_0^s \delta S_u^H(T-u)^{\alpha-1} - \int_0^t ds(T-s)^{-\alpha}\int_0^s du(T-u)^{\alpha-1}\varphi(s,u).\end{aligned} \tag{2.5}$$

If $\alpha \in (0, 1-H]$. Then

$$\limsup_{t\to T} \mathbf{E}\Big[\Big(\int_0^t \delta S_u^H (T-u)^{-\alpha} \int_0^u \delta S_v^H (T-v)^{\alpha-1}\Big)^2\Big] < \infty. \tag{2.6}$$

Lemma 2.5. *Let Z be any $\sigma(S^H)$-measurable random variable satisfying $\mathbf{P}(Z<\infty)=1$, and let $N \sim \mathcal{N}(0,1)$ be independent of S^H.*

(1) *If $\alpha \in (0, 1-H)$. Then, as $t \to T$,*

$$\Big(Z, (T-t)^{2-H-\alpha}\Big[\frac{t}{T(T-t)}\phi_t - \frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]\Big) \xrightarrow{law} (Z, \sigma_1 N), \tag{2.7}$$

where σ_1^2 is defined by (1.21).

(2) *If $\alpha \in (0, 1-H)$. Then, as $t \to T$,*

$$\Big(Z, (T-t)^{2-H-2\alpha}\Big[\frac{(T-t)^{\alpha-1}}{1-\alpha}\phi_t - \frac{(T-t)^{2\alpha-1}}{1-2\alpha}\psi_t\Big]\Big) \xrightarrow{law} (Z, \sigma_2 N), \tag{2.8}$$

where σ_2^2 is defined by (1.22).

Proof. For any $d \ge 1, s_1, \dots, s_d \in [0,T)$, we shall prove that, as $t \to T$,

$$\Big(S_{s_1}^H, \dots, S_{s_d}^H, (T-t)^{2-H-\alpha}\Big[\frac{t}{T(T-t)}\phi_t - \frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]\Big) \xrightarrow{\text{law}} \Big(S_{s_1}^H, \dots, S_{s_d}^H, \sigma_1 N\Big), \tag{2.9}$$

which is sufficient to obtain the desired conclusion. Because the left-hand side in the previous convergence is a Gaussian vector (see proofs of Mendy [16, Lemma 4.3], Es-Sebaiy and Nourdin [7, Lemma 7], or Kuang and Liu [11, lemma 3.5]). To obtain (2.9), it is sufficient to check the convergence of its covariance matrix. Let us first compute the limiting variance of $(T-t)^{2-H-\alpha}\big[\frac{t}{T(T-t)}\phi_t - \frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\big]$. We have

$$\begin{aligned}
&\mathbf{E}\Big[\Big((T-t)^{2-H-\alpha}\Big[\frac{t}{T(T-t)}\phi_t - \frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]\Big)^2\Big] \\
&= \frac{t^2}{T^2}\mathbf{E}\Big[\Big((T-t)^{1-H-\alpha}\phi_t\Big)^2\Big] + \frac{1}{(1-\alpha)^2}\mathbf{E}\Big[\Big((T-t)^{1-H}\psi_t\Big)^2\Big] \\
&\quad - \frac{2t}{T(1-\alpha)}\mathbf{E}\Big[(T-t)^{2-2H-\alpha}\phi_t\psi_t\Big].
\end{aligned} \tag{2.10}$$

As $t \to T$, we obtain

$$\begin{aligned}
&\mathbf{E}\Big[\Big((T-t)^{1-H-\alpha}\phi_t\Big)^2\Big] \\
&= (T-t)^{2-2H-2\alpha}\int_0^t ds (T-s)^{\alpha-1}\int_0^t du (T-u)^{\alpha-1}\varphi(s,u) \\
&= (T-t)^{-2H}\int_0^t ds\Big(\frac{T-s}{T-t}\Big)^{\alpha-1}\int_0^t du\Big(\frac{T-u}{T-t}\Big)^{\alpha-1}\varphi(s,u) \\
&= (T-t)^{2-2H}\int_1^{\frac{T}{T-t}} ds s^{\alpha-1}\int_1^{\frac{T}{T-t}} du u^{\alpha-1}\varphi(T-(T-t)s, T-(T-t)u) \\
&= \int_1^{\frac{T}{T-t}} ds s^{\alpha-1}\int_1^{\frac{T}{T-t}} du u^{\alpha-1}\Big[(T-t)^{2-2H}\varphi(T-(T-t)s, T-(T-t)u)\Big] \\
&\to H(2H-1)\int_1^{\infty} ds s^{\alpha-1}\int_1^{\infty} du u^{\alpha-1}|s-u|^{2H-2},
\end{aligned}$$

where we use that

$$\begin{aligned}
&\lim_{t\to T}(T-t)^{2-2H}\varphi(T-(T-t)s, T-(T-t)u)\\
&=\lim_{t\to T}(T-t)^{2-2H}H(2H-1)\Big\{(T-t)^{2H-2}|s-u|^{2H-2}-[2T-(T-t)(s+u)]^{2H-2}\Big\}\\
&=H(2H-1)|s-u|^{2H-2}-\lim_{t\to T}H(2H-1)(T-t)^{2-2H}\left[2T-(T-t)(s+u)\right]^{2H-2}\\
&=H(2H-1)|s-u|^{2H-2}.
\end{aligned} \tag{2.11}$$

From Es-Sebaiy and Nourdin [18], we obtain

$$\int_1^\infty ds s^{\alpha-1}\int_1^\infty du u^{\alpha-1}|s-u|^{2H-2}=\frac{\beta(2-\alpha-2H,2H-1)}{1-H-\alpha}. \tag{2.12}$$

Hence, we obtain

$$\lim_{t\to T}\mathbf{E}\left[\left((T-t)^{1-H-\alpha}\phi_t\right)^2\right]=\frac{H(2H-1)\beta(2-\alpha-2H,2H-1)}{1-H-\alpha}. \tag{2.13}$$

Similarly, by (2.11), for any $t\in[0,T)$, as $t\to T$,

$$\begin{aligned}
\mathbf{E}\Big[\Big((T-t)^{1-H}\psi_t\Big)^2\Big]&=(T-t)^{2-2H}\int_0^t ds\frac{1}{T-s}\int_0^t du\frac{1}{T-u}\varphi(s,u)\\
&=(T-t)^{-2H}\int_0^t ds\frac{T-t}{T-s}\int_0^t du\frac{T-t}{T-u}\varphi(s,u)\\
&=\int_1^{\frac{T}{T-t}}ds\frac{1}{s}\int_1^{\frac{T}{T-t}}du\frac{1}{u}\Big[(T-t)^{2-2H}\varphi(T-(T-t)s,T-(T-t)u)\Big]\\
&\to H(2H-1)\int_1^\infty\int_1^\infty\frac{|s-u|^{2H-2}}{su}\,ds\,du.
\end{aligned}$$

Since

$$\begin{aligned}
\int_1^\infty\int_1^\infty\frac{|s-u|^{2H-2}}{su}\,ds\,du&=\int_1^\infty ds s^{2H-3}\int_1^\infty du\frac{(u-1)^{2H-2}}{u}+\int_1^\infty ds s^{2H-3}\int_{1/s}^1 du\frac{(1-u)^{2H-2}}{u}\\
&=\frac{\beta(2-2H,2H-1)}{2(1-H)}+\int_0^1 du\frac{(1-u)^{2H-2}}{u}\int_{1/u}^\infty ds s^{2H-3}\\
&=\frac{\beta(2-2H,2H-1)}{1-H}.
\end{aligned}$$

We have

$$\lim_{t\to T}\mathbf{E}\Big[\Big((T-t)^{1-H}\psi_t\Big)^2\Big]=\frac{H(2H-1)\beta(2-2H,2H-1)}{1-H}. \tag{2.14}$$

By (2.11), as $t\to T$, we obtain

$$\begin{aligned}
\mathbf{E}\Big[(T-t)^{2-2H-\alpha}\phi_t\psi_t\Big]&=(T-t)^{2-2H-\alpha}\int_0^t\int_0^t(T-s)^{\alpha-1}\frac{1}{T-u}\varphi(s,u)\,ds\,du\\
&=(T-t)^{-2H}\int_0^t\int_0^t\Big(\frac{T-s}{T-t}\Big)^{\alpha-1}\frac{T-t}{T-u}\varphi(s,u)\,ds\,du\\
&=\int_1^{\frac{T}{T-t}}\int_1^{\frac{T}{T-t}}\frac{s^{\alpha-1}}{u}\big[(T-t)^{2-2H}\varphi(T-(T-t)s,T-(T-t)u)\big]\,ds\,du\\
&\to H(2H-1)\int_1^\infty\int_1^\infty\frac{s^{\alpha-1}}{u}|s-u|^{2H-2}\,ds\,du,
\end{aligned}$$

with

$$\int_1^\infty\int_1^\infty\frac{s^{\alpha-1}}{u}|s-u|^{2H-2}\,ds\,du=\frac{\beta(2-\alpha-2H,2H-1)+\beta(2-2H,2H-1)}{2-\alpha-2H}.$$

Therefore,

$$\lim_{t\to T}\mathbf{E}\Big[(T-t)^{2-2H-\alpha}\phi_t\psi_t\Big]=\frac{H(2H-1)[\beta(2-\alpha-2H,2H-1)+\beta(2-2H,2H-1)]}{2-\alpha-2H}. \quad (2.15)$$

By (2.10), (2.13), (2.14) and (2.15), we obtain

$$\lim_{t\to T}\mathbf{E}\Big[\Big((T-t)^{2-H-\alpha}\Big[\frac{t}{T(T-t)}\phi_t-\frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]\Big)^2\Big]=\sigma_1^2,$$

where σ_1^2 is defined by (1.21).

On the other hand, for any $v<t<T$, as $t\to T$, we have

$$\begin{aligned}
&\mathbf{E}\Big\{S_v^H\times(T-t)^{2-H-\alpha}\Big[\frac{t}{T(T-t)}\phi_t-\frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]\Big\}\\
&=\frac{t(T-t)^{1-H-\alpha}}{T}\int_0^t du(T-u)^{\alpha-1}\int_0^v ds\varphi(s,u)-\frac{(T-t)^{1-H}}{1-\alpha}\int_0^t\frac{du}{T-u}\int_0^v ds\varphi(s,u)\\
&=\frac{tH(2H-1)(T-t)^{1-H-\alpha}}{T}\int_0^t du(T-u)^{\alpha-1}\int_0^v ds\,\big[|s-u|^{2H-2}-(s+u)^{2H-2}\big]\\
&\quad-\frac{H(2H-1)(T-t)^{1-H}}{1-\alpha}\int_0^t\frac{du}{T-u}\int_0^v ds\,\big[|s-u|^{2H-2}-(s+u)^{2H-2}\big]\\
&=\frac{tH(T-t)^{1-H-\alpha}}{T}\int_0^t(T-u)^{\alpha-1}\big[u^{2H-1}+sign(v-u)\times|v-u|^{2H-1}\big]du\\
&\quad-\frac{tH(T-t)^{1-H-\alpha}}{T}\int_0^t(T-u)^{\alpha-1}\big[(u+v)^{2H-1}-u^{2H-1}\big]du\\
&\quad-\frac{H(T-t)^{1-H}}{1-\alpha}\int_0^t\frac{1}{T-u}\big[u^{2H-1}+sign(v-u)\times|v-u|^{2H-1}\big]du\\
&\quad+\frac{H(T-t)^{1-H}}{1-\alpha}\int_0^t\frac{1}{T-u}\big[(u+v)^{2H-1}-u^{2H-1}\big]du\to 0.
\end{aligned}$$

Hence we have proveed (2.9), and so (2.7) holds. Similarly, we can obtain the proof of (2.8), therefore, we complete the proof of Lemma 2.5. □

According to the proof, as $t\to T$, we also have

$$\big(Z,(T-t)^{1-H}\psi_t\big)\xrightarrow{\text{law}}\Big(Z,\sqrt{\frac{H(2H-1)\beta(2-2H,2H-1)}{1-H}}N\Big). \quad (2.16)$$

Lemma 2.6. (1) *Let X_t be given by (1.15). If $0<\alpha<H$, then, as $t\to T$,*

$$\frac{X_t-k}{(T-t)^\alpha}\xrightarrow{a.s.}\xi_T+\frac{x_0-k}{T^\alpha}, \quad (2.17)$$

$$(T-t)^{1-\alpha}\int_0^t\frac{X_u-k}{(T-u)^2}du\xrightarrow{a.s.}\frac{1}{1-\alpha}\Big(\xi_T+\frac{x_0-k}{T^\alpha}\Big). \quad (2.18)$$

(2) *If $0<\alpha<\frac12$, then, as $t\to T$,*

$$(T-t)^{1-2\alpha}\int_0^t\frac{(X_u-k)^2}{(T-u)^2}du\xrightarrow{a.s.}\frac{1}{1-2\alpha}\Big(\xi_T+\frac{x_0-k}{T^\alpha}\Big)^2. \quad (2.19)$$

(3) *If $\alpha=\frac12$, then, as $t\to T$,*

$$\frac{1}{|\log(T-t)|}\int_0^t\frac{(X_u-k)^2}{(T-u)^2}du\xrightarrow{a.s.}\Big(\xi_T+\frac{x_0-k}{T^\alpha}\Big)^2. \quad (2.20)$$

(4) *If $\frac12<\alpha<H$, then, as $t\to T$,*

$$\int_0^t\frac{(X_u-k)^2}{(T-u)^2}du\xrightarrow{a.s.}\int_0^T\frac{(X_u-k)^2}{(T-u)^2}du<\infty. \quad (2.21)$$

Proof. By (1.15), we obtain

$$\frac{X_t-k}{(T-t)^\alpha}=\xi_t+\frac{x_0-k}{T^\alpha},$$

$$\begin{aligned}(T-t)^{1-\alpha}\int_0^t\frac{X_u-k}{(T-u)^2}du&=\frac{1}{1-\alpha}\frac{x_0-k}{T^\alpha}+\frac{1}{\alpha-1}\frac{x_0-k}{T}(T-t)^{1-\alpha}\\&\quad+(T-t)^{1-\alpha}\int_0^t\xi_s(T-s)^{\alpha-2}ds,\end{aligned}\tag{2.22}$$

$$\int_0^t\frac{(X_u-k)^2}{(T-u)^2}du=\int_0^t(T-u)^{2\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)^2du.\tag{2.23}$$

Thus, (2.17)-(2.21) can be obtained by Lemma 2.1 and Remark 2.3. □

Lemma 2.7. *Let X_t,ϕ_t and ψ_t be given by (1.15), (1.16) and (1.17), respectively.*

(1) If $\alpha\in(0,\frac12]$, then, for $0<t<T$, we have

$$\begin{aligned}&\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u}{(T-u)^2}du-\frac{t}{T(T-t)}\int_0^t\frac{X_u}{T-u}dX_u\\&=\alpha\Big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\Big(\int_0^t\frac{X_u}{(T-u)^2}du\Big)^2\Big]-\frac{t}{T(T-t)}\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\Big)\\&\quad+\psi_t\int_0^t\frac{X_u-k}{(T-u)^2}du,\end{aligned}\tag{2.24}$$

and

$$\begin{aligned}&\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u^2}{(T-u)^2}du-\int_0^t\frac{X_u}{T-u}dX_u\int_0^t\frac{X_u}{(T-u)^2}du\\&=\tau\Big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\Big(\int_0^t\frac{X_u}{(T-u)^2}du\Big)^2\Big]-k\frac{t}{T(T-t)}\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\Big)-Q_t,\end{aligned}\tag{2.25}$$

where

$$\begin{aligned}Q_t=&\int_0^t(T-u)^{\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)du\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t-k\psi_t\Big)\\&-\psi_t\int_0^t(T-u)^{2\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)^2du.\end{aligned}\tag{2.26}$$

(2) If $\alpha\in(0,1-H]$, for $0<t<T$, we also have

$$\begin{aligned}&\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u}{(T-u)^2}du-\frac{t}{T(T-t)}\int_0^t\frac{X_u}{T-u}dX_u\\&=\alpha\Big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\Big(\int_0^t\frac{X_u}{(T-u)^2}du\Big)^2\Big]\\&\quad-\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\Big[\frac{t}{T(T-t)}\phi_t-\frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]-A_t,\end{aligned}\tag{2.27}$$

and

$$\begin{aligned}&\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u^2}{(T-u)^2}du-\int_0^t\frac{X_u}{T-u}dX_u\int_0^t\frac{X_u}{(T-u)^2}du\\&=\tau\Big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\Big(\int_0^t\frac{X_u}{(T-u)^2}du\Big)^2\Big]\\&\quad-k\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\Big[\frac{t}{T(T-t)}\phi_t-\frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]-kA_t-B_t,\end{aligned}\tag{2.28}$$

where ξ_t is given by (1.13), and A_t, B_t are defined by

$$A_t=\frac{\psi_t^2}{1-\alpha}+\frac{x_0-k}{T}\frac{\psi_t}{1-\alpha}-\frac{t}{T(T-t)}(\phi_t\xi_t-\eta_t),\tag{2.29}$$

$$B_t = \Big(\xi_t + \frac{x_0-k}{T^\alpha}\Big)^2\Big[\frac{(T-t)^{\alpha-1}}{1-\alpha}\phi_t - \frac{(T-t)^{2\alpha-1}}{1-2\alpha}\psi_t\Big] + R_t, \tag{2.30}$$

with

$$\begin{aligned} R_t &= \Big(\xi_t + \frac{x_0-k}{T^\alpha}\Big)\phi_t\Big[\frac{\psi_t}{(1-\alpha)(1-2\alpha)} - \frac{1}{1-\alpha}\frac{x_0-k}{T}\Big] + \frac{\psi_t}{1-2\alpha}\frac{(x_0-k)^2}{T} \\ &\quad - \Big[\int_0^t (T-u)^{\alpha-2}\Big(\xi_u + \frac{x_0-k}{T^\alpha}\Big)du + \frac{2\psi_t}{1-2\alpha}\Big](\phi_t\xi_t - \eta_t). \end{aligned} \tag{2.31}$$

Proof. By (1.4), we obtain

$$\begin{aligned} &\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u \\ &= \alpha\Big[\frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \Big(\int_0^t \frac{X_u}{(T-u)^2}du\Big)^2\Big] \\ &\quad + \psi_t\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dS_u^H. \end{aligned} \tag{2.32}$$

By (1.15), we obtain

$$\int_0^t \frac{X_u}{T-u}dS_u^H = k\psi_t + \frac{x_0-k}{T^\alpha}\phi_t + \eta_t, \tag{2.33}$$

$$\int_0^t \frac{X_u}{(T-u)^2}du = \frac{kt}{T(T-t)} + \int_0^t (T-u)^{\alpha-2}\Big(\xi_u + \frac{x_0-k}{T^\alpha}\Big)du. \tag{2.34}$$

From (2.34), we obtain

$$\int_0^t \frac{X_u-k}{(T-u)^2}du = \int_0^t \frac{X_u}{(T-u)^2}du - \frac{kt}{T(T-t)} = \int_0^t (T-u)^{\alpha-2}\Big(\xi_u + \frac{x_0-k}{T^\alpha}\Big)du. \tag{2.35}$$

Hence (2.24) holds from (2.32)-(2.35). From (3.5), we have

$$\int_0^t (T-u)^{\alpha-2}\Big(\xi_u + \frac{x_0-k}{T^\alpha}\Big)du = \Big(\xi_t + \frac{x_0-k}{T^\alpha}\Big)\frac{(T-t)^{\alpha-1}}{1-\alpha} - \frac{\psi_t}{1-\alpha} - \frac{1}{1-\alpha}\frac{x_0-k}{T}. \tag{2.36}$$

Thus (2.27) holds from (2.24), (2.35) and (2.36).

Similarly, we obtain

$$\begin{aligned} &\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{T-u}dX_u\int_0^t \frac{X_u}{(T-u)^2}du \\ &= \tau\Big[\frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \Big(\int_0^t \frac{X_u}{(T-u)^2}du\Big)^2\Big] \\ &\quad + \psi_t\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{(T-u)^2}du\int_0^t \frac{X_u}{T-u}dS_u^H. \end{aligned} \tag{2.37}$$

By (1.12), we have

$$\begin{aligned} \int_0^t \frac{X_u^2}{(T-u)^2}du &= \frac{k^2t}{T(T-t)} + 2k\int_0^t (T-u)^{\alpha-2}\Big(\xi_u + \frac{x_0-k}{T^\alpha}\Big)du \\ &\quad + \int_0^t (T-u)^{2\alpha-2}\Big(\xi_u + \frac{x_0-k}{T^\alpha}\Big)^2du. \end{aligned} \tag{2.38}$$

Hence (2.25) holds from (2.37), (2.38), (2.34), and (2.33).

By (3.5), we obtain

$$\begin{aligned} &\frac{1-2\alpha}{2}\int_0^t (T-u)^{2\alpha-2}\Big(\xi_u + \frac{x_0-k}{T^\alpha}\Big)^2du \\ &= \frac{(T-t)^{2\alpha-1}}{2}\Big(\xi_t + \frac{x_0-k}{T^\alpha}\Big)^2 - \eta_t - \frac{x_0-k}{T^\alpha}\phi_t - \frac{(x_0-k)^2}{2T}. \end{aligned} \tag{2.39}$$

We can obtain (2.28) by (2.37), (2.38), (2.34), (2.33), (2.36), and (2.39) after some calculations. So the proof is complete. □

Remark 2.8. By (1.13) and (1.16), when $\alpha = \frac{1}{2}$, we have $\xi_t = \phi_t$. Thus, we have $\eta_t = \frac{1}{2}\xi_t^2$ by (2.39) when $\alpha = 1/2$.

2.1. **Proof of Theorem 1.1.** The proof is done in 5 steps. (1) prove (1.8) holds as $0 < \alpha < \frac{1}{2}$. (2) prove (1.8) holds as $\alpha = \frac{1}{2}$. (3) prove (1.9) holds as $\frac{1}{2} < \alpha < H$. (4) prove (1.10) holds as $0 < \alpha < H$. (5) prove (1.11) holds by (1.8), (1.9) and (1.10).

Note that

$$\begin{aligned} &\frac{t}{T(T-t)}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \Big(\int_0^t \frac{X_u-k}{(T-u)^2}du\Big)^2 \\ &= \frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \Big(\int_0^t \frac{X_u}{(T-u)^2}du\Big)^2. \end{aligned}$$

By (1.5), we have

$$\begin{aligned} \hat{\alpha}_t &= \frac{\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u}{\frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \big(\int_0^t \frac{X_u}{(T-u)^2}du\big)^2} \\ &= \frac{\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u}{\frac{t}{T(T-t)}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \big(\int_0^t \frac{X_u-k}{(T-u)^2}du\big)^2}. \end{aligned} \tag{2.40}$$

By (3.5), for any $t \in [0, T)$, we obtain

$$\int_0^t \frac{dX_u}{T-u} = \frac{X_t}{T-t} - \frac{x_0}{T} - \int_0^t \frac{X_u}{(T-u)^2}du = \frac{X_t-k}{T-t} - \int_0^t \frac{X_u-k}{(T-u)^2}du - \frac{x_0-k}{T}, \tag{2.41}$$

and

$$\begin{aligned} \int_0^t \frac{X_u}{T-u}dX_u &= \frac{1}{2}\Big[\frac{X_t^2}{T-t} - \frac{x_0^2}{T} - \int_0^t \frac{X_u^2}{(T-u)^2}du\Big] \\ &= \frac{1}{2}\Big[\frac{(X_t-k)^2}{T-t} - \int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \frac{(x_0-k)^2}{T}\Big] + k\int_0^t \frac{dX_u}{T-u}. \end{aligned} \tag{2.42}$$

Then, we obtain

$$\begin{aligned} &\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u \\ &= \Big[\frac{X_t-k}{T-t} - \int_0^t \frac{X_u-k}{(T-u)^2}du - \frac{x_0-k}{T}\Big]\int_0^t \frac{X_u-k}{(T-u)^2}du \\ &\quad - \frac{t}{2T(T-t)}\Big[\frac{(X_t-k)^2}{T-t} - \int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \frac{(x_0-k)^2}{T}\Big]. \end{aligned}$$

Step 1: If $0 < \alpha < \frac{1}{2}$, then by (2.17), (2.18), and (2.19). As $t \to T$, we have

$$\begin{aligned} &(T-t)^{2-2\alpha}\Big[\frac{t}{T(T-t)}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \Big(\int_0^t \frac{X_u-k}{(T-u)^2}du\Big)^2\Big] \\ &= \frac{t}{T}(T-t)^{1-2\alpha}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \Big[(T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du\Big]^2 \\ &\xrightarrow{\text{a.s.}} \frac{\alpha^2}{(1-2\alpha)(1-\alpha)^2}\Big(\xi_T + \frac{x_0-k}{T^\alpha}\Big)^2 \end{aligned} \tag{2.43}$$

and

$$\begin{aligned}
&(T-t)^{2-2\alpha}\Big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u\Big]\\
&= \Big[\frac{X_t-k}{(T-t)^\alpha} - (T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du - (T-t)^{1-\alpha}\frac{x_0-k}{T}\Big](T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du\\
&\quad - \frac{t}{2T}\Big[\frac{(X_t-k)^2}{(T-t)^{2\alpha}} - (T-t)^{1-2\alpha}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - (T-t)^{1-2\alpha}\frac{(x_0-k)^2}{T}\Big]\\
&\xrightarrow{\text{a.s.}} \frac{\alpha^3}{(1-2\alpha)(1-\alpha)^2}\Big(\xi_T + \frac{x_0-k}{T^\alpha}\Big)^2.
\end{aligned} \tag{2.44}$$

By (2.40), (2.43), and (2.44), we obtain $\hat{\alpha}_t \xrightarrow{\text{a.s.}} \alpha$ as $t \to T$, Thus (1.8) holds.

Step 2: If $\alpha = \frac{1}{2}$, then by (2.17), (2.18) and (2.20). As $t \to T$, we have

$$\begin{aligned}
&\frac{T-t}{|\log(T-t)|}\Big[\frac{t}{T(T-t)}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \Big(\int_0^t \frac{X_u-k}{(T-u)^2}du\Big)^2\Big]\\
&= \frac{t}{T|\log(T-t)|}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \frac{1}{|\log(T-t)|}\Big[(T-t)^{1/2}\int_0^t \frac{X_u-k}{(T-u)^2}du\Big]^2\\
&\xrightarrow{\text{a.s.}} \Big(\xi_T + \frac{x_0-k}{T^\alpha}\Big)^2,
\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}
&\frac{T-t}{|\log(T-t)|}\Big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u\Big]\\
&= \Big[\frac{1}{|\log(T-t)|}\frac{X_t-k}{(T-t)^{1/2}} - \frac{(T-t)^{1/2}}{|\log(T-t)|}\int_0^t \frac{X_u-k}{(T-u)^2}du\\
&\quad - \frac{(T-t)^{1/2}}{|\log(T-t)|}\frac{x_0-k}{T}\Big](T-t)^{1/2}\int_0^t \frac{X_u-k}{(T-u)^2}du\\
&\quad - \frac{t}{2T}\Big[\frac{1}{|\log(T-t)|}\frac{(X_t-k)^2}{T-t} - \frac{1}{|\log(T-t)|}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \frac{1}{|\log(T-t)|}\frac{(x_0-k)^2}{T}\Big]\\
&\xrightarrow{\text{a.s.}} \frac{1}{2}\Big(\xi_T + \frac{x_0-k}{T^\alpha}\Big)^2.
\end{aligned} \tag{2.46}$$

Then by (2.40), (2.45), and (2.46), we obtain $\hat{\alpha}_t \xrightarrow{\text{a.s.}} \alpha = \frac{1}{2}$ as $t \to T$. Hence (1.8) holds.

Step 3: If $\frac{1}{2} < \alpha < H$, then by (2.17), (2.18), and (2.21). As $t \to T$, we have

$$\begin{aligned}
&(T-t)\Big[\frac{t}{T(T-t)}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \Big(\int_0^t \frac{X_u-k}{(T-u)^2}du\Big)^2\Big]\\
&= \frac{t}{T}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - (T-t)^{2\alpha-1}\Big[(T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du\Big]^2\\
&\xrightarrow{\text{a.s.}} \int_0^T \frac{(X_u-k)^2}{(T-u)^2}du < \infty,
\end{aligned} \tag{2.47}$$

and

$$
\begin{aligned}
&(T-t)\Big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u\Big] \\
&= (T-t)^{2\alpha-1}\Big[\frac{X_t-k}{(T-t)^\alpha} \\
&\quad - (T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du - (T-t)^{1-\alpha}\frac{x_0-k}{T}\Big](T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du \\
&\quad - \frac{t}{2T}\Big[(T-t)^{2\alpha-1}\frac{(X_t-k)^2}{(T-t)^{2\alpha}} - \int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \frac{(x_0-k)^2}{T}\Big] \\
&\xrightarrow{\text{a.s.}} \frac{1}{2}\int_0^T \frac{(X_u-k)^2}{(T-u)^2}du + \frac{(x_0-k)^2}{2T}.
\end{aligned} \tag{2.48}
$$

Thus (1.9) holds by (2.40), (2.47) and (2.48).

Step 4: Now we prove (1.10). By (2.41) and (2.42), we obtain

$$
\begin{aligned}
&\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{T-u}dX_u\int_0^t \frac{X_u}{(T-u)^2}du \\
&= \int_0^t \frac{dX_u}{T-u}\Big[\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du + k\int_0^t \frac{X_u-k}{(T-u)^2}du + k\int_0^t \frac{X_u}{(T-u)^2}du\Big] \\
&\quad - \int_0^t \frac{X_u}{T-u}dX_u\int_0^t \frac{X_u}{(T-u)^2}du \\
&= \int_0^t \frac{dX_u}{T-u}\Big[\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du + k\int_0^t \frac{X_u-k}{(T-u)^2}du\Big] \\
&\quad - \Big[\int_0^t \frac{X_u}{T-u}dX_u - k\int_0^t \frac{dX_u}{T-u}\Big]\int_0^t \frac{X_u}{(T-u)^2}du \\
&= \Big[\frac{X_t-k}{T-t} - \int_0^t \frac{X_u-k}{(T-u)^2}du - \frac{x_0-k}{T}\Big]\Big[\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du + k\int_0^t \frac{X_u-k}{(T-u)^2}du\Big] \\
&\quad - \frac{1}{2}\Big[\frac{(X_t-k)^2}{T-t} - \int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \frac{(x_0-k)^2}{T}\Big]\Big[\int_0^t \frac{X_u-k}{(T-u)^2}du + \frac{kt}{T(T-t)}\Big].
\end{aligned}
$$

Hence, when $0<\alpha<1/2$, by (2.17), (2.18) and (2.19), as $t\to T$, we have

$$
\begin{aligned}
&(T-t)^{2-2\alpha}\Big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{T-u}dX_u\int_0^t \frac{X_u}{(T-u)^2}du\Big] \\
&= \Big[\frac{X_t-k}{(T-t)^\alpha} - (T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du - (T-t)^{1-\alpha}\frac{x_0-k}{T}\Big] \\
&\quad \times \Big[(T-t)^\alpha(T-t)^{1-2\alpha}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du + k(T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du\Big] \\
&\quad - \frac{1}{2}\Big[\frac{(X_t-k)^2}{(T-t)^{2\alpha}} - (T-t)^{1-2\alpha}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - (T-t)^{1-2\alpha}\frac{(x_0-k)^2}{T}\Big] \\
&\quad \times \Big[(T-t)^\alpha(T-t)^{1-\alpha}\int_0^t \frac{X_u-k}{(T-u)^2}du + \frac{kt}{T}\Big] \\
&\xrightarrow{\text{a.s.}} \frac{k\alpha^3}{(1-2\alpha)(1-\alpha)^2}\Big(\xi_T + \frac{x_0-k}{T^\alpha}\Big)^2.
\end{aligned} \tag{2.49}
$$

By (1.7), (2.44), and (2.49), we obtain $\hat{k}_t \xrightarrow{\text{a.s.}} k$ as $t \to T$. When $\alpha = \frac{1}{2}$, by (2.17), (2.18), and (2.20), as $t \to T$, we have

$$\begin{aligned}
&\frac{T-t}{|\log(T-t)|}\Big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{T-u}dX_u \int_0^t \frac{X_u}{(T-u)^2}du\Big] \\
&= \Big[\frac{X_t-k}{(T-t)^{1/2}} - (T-t)^{1/2}\int_0^t \frac{X_u-k}{(T-u)^2}du - (T-t)^{1/2}\frac{x_0-k}{T}\Big] \\
&\quad \times \Big[\frac{(T-t)^{1/2}}{|\log(T-t)|}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du + k\frac{(T-t)^{1/2}}{|\log(T-t)|}\int_0^t \frac{X_u-k}{(T-u)^2}du\Big] \\
&\quad - \frac{1}{2}\Big[\frac{1}{|\log(T-t)|}\frac{(X_t-k)^2}{T-t} - \frac{1}{|\log(T-t)|}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du - \frac{1}{|\log(T-t)|}\frac{(x_0-k)^2}{T}\Big] \\
&\quad \times \Big[(T-t)^{1/2}(T-t)^{1/2}\int_0^t \frac{X_u-k}{(T-u)^2}du + \frac{kt}{T}\Big] \\
&\xrightarrow{\text{a.s.}} \frac{k}{2}\Big(\xi_T + \frac{x_0-k}{T^\alpha}\Big)^2.
\end{aligned} \tag{2.50}$$

By (1.7), (2.46), and (2.50), we obtain $\hat{k}_t \xrightarrow{\text{a.s.}} k$ as $t \to T$. When $\frac{1}{2} < \alpha < H$, similarly, as $t \to T$, we obtaain

$$\begin{aligned}
&(T-t)\Big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u^2}{(T-u)^2}du - \int_0^t \frac{X_u}{T-u}dX_u \int_0^t \frac{X_u}{(T-u)^2}du\Big] \\
&\xrightarrow{\text{a.s.}} \frac{k}{2}\Big[\int_0^T \frac{(X_u-k)^2}{(T-u)^2}du + \frac{(x_0-k)^2}{T}\Big].
\end{aligned} \tag{2.51}$$

By (1.7), (2.48), and (2.51), we obtain $\hat{k}_t \xrightarrow{\text{a.s.}} k$ as $t \to T$. Thus (1.10) holds.

Step 5: It is obvious that (1.11) also holds from (1.8)-(1.10). So we complete the proof of Theorem 1.1.

2.2. **Proof of Theorem 1.2.** We prove the theorem in three steps.

Step 1: we prove the asymptotic properties of estimator $\hat{\alpha}$, namely (1.18), (1.23), and (1.26).

(1) If $0 < \alpha < 1-H$, by (1.5) and (2.27), we have

$$\begin{aligned}
&(T-t)^{\alpha-H}(\alpha - \hat{\alpha}_t) \\
&= (T-t)^{2-H-\alpha}\Big[\frac{t}{T(T-t)}\phi_t - \frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\Big]\frac{\xi_t + \frac{x_0-k}{T^\alpha}}{(T-t)^{2-2\alpha}\Big[\frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \big(\int_0^t \frac{X_u}{(T-u)^2}du\big)^2\Big]} \\
&\quad + \frac{(T-t)^{2-H-\alpha}A_t}{(T-t)^{2-2\alpha}\Big[\frac{t}{T(T-t)}\int_0^t \frac{X_u^2}{(T-u)^2}du - \big(\int_0^t \frac{X_u}{(T-u)^2}du\big)^2\Big]} \\
&:= a_t \times b_t + c_t.
\end{aligned}$$

By (2.7), we obtain that $a_t \xrightarrow{\text{law}} \sigma_1 N$. Combing Lemma 2.1 with (2.43), we have, as $t \to T$,

$$b_t \xrightarrow{\text{a.s.}} \frac{(1-\alpha)^2(1-2\alpha)}{\alpha^2\big(\xi_T + \frac{x_0-k}{T^\alpha}\big)}.$$

Since

$$\begin{aligned}
(T-t)^{2-H-\alpha}A_t &= (T-t)^{H-\alpha}\frac{[(T-t)^{1-H}\psi_t]^2}{1-\alpha} + (T-t)^{1-\alpha}\frac{x_0-k}{T}\frac{(T-t)^{1-H}\psi_t}{1-\alpha} \\
&\quad - (T-t)^{1-H-\alpha}\frac{t}{T}(\phi_t\xi_t - \eta_t) \xrightarrow{\text{law}} 0,
\end{aligned} \tag{2.52}$$

as $t \to T$, where we use (2.29) and (2.16). Therefore (1.18) holds.

(2) If $1-H<\alpha<\frac{1}{2}$, by (1.5) and (2.24), as $t\to T$, we have

$$\begin{aligned}(T-t)^{2\alpha-1}(\alpha-\hat{\alpha}_t) &= \frac{\frac{t}{T}\big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\big)-(T-t)^{\alpha-(1-H)}\big[(T-t)^{1-H}\psi_t\big](T-t)^{1-\alpha}\int_0^t\frac{X_u-k}{(T-u)^2}du}{(T-t)^{2-2\alpha}\big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\big(\int_0^t\frac{X_u}{(T-u)^2}du\big)^2\big]}\\ &\xrightarrow{\text{law}}\frac{(1-\alpha)^2(1-2\alpha)}{\alpha^2}\frac{\eta_T+\frac{x_0-k}{T^\alpha}\phi_T}{\big(\xi_T+\frac{x_0-k}{T^\alpha}\big)^2},\end{aligned}$$

where we use Lemma 2.4, Remark 2.2, (2.16), (2.18), and (2.43). Hence (1.23) holds.

(3) If $\alpha=\frac{1}{2}$, by (1.5) and (2.24), as $t\to T$, we have

$$\begin{aligned}&|\log(T-t)|(\alpha-\hat{\alpha}_t)\\ &= \frac{\frac{t}{T}\big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\big)-(T-t)^{H-\frac{1}{2}}\big[(T-t)^{1-H}\psi_t\big](T-t)^{1/2}\int_0^t\frac{X_u-k}{(T-u)^2}du}{\frac{T-t}{|\log(T-t)|}\big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\big(\int_0^t\frac{X_u}{(T-u)^2}du\big)^2\big]}\\ &\xrightarrow{\text{law}}\frac{\eta_T+\frac{x_0-k}{T^\alpha}\phi_T}{\big(\xi_T+\frac{x_0-k}{T^\alpha}\big)^2}\\ &= \frac{\xi_T\big(\frac{1}{2}\xi_T+\frac{x_0-k}{T^\alpha}\big)}{\big(\xi_T+\frac{x_0-k}{T^\alpha}\big)^2},\end{aligned}$$

where we use Lemma 2.4, Remark 2.2, (2.16), (2.18), (2.45), and Remark 2.8. Hence (1.26) holds.

Step 2: we prove the asymptotic properties of estimator $\hat{\tau}$, namely (1.19), (1.24), and (1.27).

(1') If $0<\alpha<1-H$, by (1.7) and (2.28), we have

$$\begin{aligned}&(T-t)^{\alpha-H}(\tau-\hat{\tau}_t)\\ &= k\frac{\big(\xi_t+\frac{x_0-k}{T^\alpha}\big)(T-t)^{2-H-\alpha}\big[\frac{t}{T(T-t)}\phi_t-\frac{(T-t)^{\alpha-1}}{1-\alpha}\psi_t\big]}{(T-t)^{2-2\alpha}\big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\Big(\int_0^t\frac{X_u}{(T-u)^2}du\Big)^2\big]}\\ &\quad+\frac{(T-t)^{2-H-\alpha}(kA_t+B_t)}{(T-t)^{2-2\alpha}\big[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\big(\int_0^t\frac{X_u}{(T-u)^2}du\big)^2\big]}.\end{aligned}\tag{2.53}$$

Note that, by (2.30), (2.31), and (2.36),

$$\begin{aligned}(T-t)^{2-H-\alpha}B_t &= \Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)^2(T-t)^{2-H-\alpha}\Big[\frac{(T-t)^{\alpha-1}}{1-\alpha}\phi_t-\frac{(T-t)^{2\alpha-1}}{1-2\alpha}\psi_t\Big]\\ &\quad+(T-t)^{2-H-\alpha}R_t,\end{aligned}\tag{2.54}$$

and

$$\begin{aligned}&(T-t)^{2-H-\alpha}R_t\\ &= \Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\phi_t\Big[\frac{(T-t)^{1-\alpha}(T-t)^{1-H}\psi_t}{(1-\alpha)(1-2\alpha)}-\frac{(T-t)^{2-H-\alpha}}{1-\alpha}\frac{x_0-k}{T}\Big]\\ &\quad+\frac{(T-t)^{1-\alpha}(T-t)^{1-H}\psi_t}{1-2\alpha}\frac{(x_0-k)^2}{T}\\ &\quad= \Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\phi_t\Big[\frac{(T-t)^{1-\alpha}(T-t)^{1-H}\psi_t}{(1-\alpha)(1-2\alpha)}-\frac{(T-t)^{2-H-\alpha}}{1-\alpha}\frac{x_0-k}{T}\Big]\\ &\quad+\frac{(T-t)^{1-\alpha}(T-t)^{1-H}\psi_t}{1-2\alpha}\frac{(x_0-k)^2}{T}\\ &\quad-\Big[\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\frac{(T-t)^{1-H}}{1-\alpha}-\frac{(T-t)^{1-\alpha}(T-t)^{1-H}\psi_t}{1-\alpha}\\ &\quad-\frac{(T-t)^{2-H-\alpha}}{1-\alpha}\frac{x_0-k}{T}+\frac{2(T-t)^{1-\alpha}(T-t)^{1-H}\psi_t}{1-2\alpha}\Big](\phi_t\xi_t-\eta_t)\xrightarrow{\text{law}}0,\end{aligned}\tag{2.55}$$

as $t \to T$, by Lemma 2.1, Remark 2.2, (2.16), (2.5), and (2.6). Hence (1.19) holds from (2.52)-(2.55), Lemma 2.1, (2.7), (2.8) and (2.43).

(2') If $1-H<\alpha<\frac{1}{2}$, by (1.7) and (2.25), we have

$$(T-t)^{2\alpha-1}(\tau-\hat{\tau}_t)=\frac{k\frac{t}{T}\left(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\right)+(T-t)Q_t}{(T-t)^{2-2\alpha}\left[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\left(\int_0^t\frac{X_u}{(T-u)^2}du\right)^2\right]}. \tag{2.56}$$

Note that, by (2.26), (2.36), and (2.39),

$$\begin{aligned}
&(T-t)Q_t\\
&=(T-t)\int_0^t(T-u)^{\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)du\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t-k\psi_t\Big)\\
&\quad-(T-t)\psi_t\int_0^t(T-u)^{2\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)^2du\\
&=(T-t)\Big[\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\frac{(T-t)^{\alpha-1}}{1-\alpha}-\frac{\psi_t}{1-\alpha}-\frac{1}{1-\alpha}\frac{x_0-k}{T}\Big]\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t-k\psi_t\Big)\\
&\quad-(T-t)\psi_t\int_0^t(T-u)^{2\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)^2du\\
&=\Big[\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\frac{(T-t)^{\alpha-(1-H)}}{1-\alpha}-\frac{(T-t)^{2H-1}(T-t)^{1-H}\psi_t}{1-\alpha}-\frac{(T-t)^H}{1-\alpha}\frac{x_0-k}{T}\Big]\\
&\quad\times\Big[(T-t)^{1-H}\eta_t+(T-t)^{1-H}\frac{x_0-k}{T^\alpha}\phi_t-k(T-t)^{1-H}\psi_t\Big]\\
&\quad-(T-t)^{1-H}\psi_t\Big[\frac{(T-t)^{2\alpha-1+H}}{1-2\alpha}\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)^2-\frac{2(T-t)^H\eta_t}{1-2\alpha}\\
&\quad-\frac{2(T-t)^H}{1-2\alpha}\frac{x_0-k}{T^\alpha}\phi_t-\frac{(T-t)^H}{1-2\alpha}\frac{(x_0-k)^2}{T}\Big]\xrightarrow{\text{law}}0,
\end{aligned}\tag{2.57}$$

as $t\to T$, by Lemma 2.1, Lemma 2.4, Remark 2.2 and (2.16). Hence (1.24) holds from (2.56), (2.57), Lemma 2.4, Remark 2.2 and (2.43).

(3') If $\alpha=\frac{1}{2}$, by (1.7) and (2.25), we have, as $t\to T$,

$$\begin{aligned}
|\log(T-t)|(\tau-\hat{\tau}_t)&=\frac{k\frac{t}{T}\left(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\right)+(T-t)Q_t}{\frac{T-t}{|\log(T-t)|}\left[\frac{t}{T(T-t)}\int_0^t\frac{X_u^2}{(T-u)^2}du-\left(\int_0^t\frac{X_u}{(T-u)^2}du\right)^2\right]}\\
&\xrightarrow{\text{law}}\frac{k\left(\eta_T+\frac{x_0-k}{T^\alpha}\phi_T\right)}{\left(\xi_T+\frac{x_0-k}{T^\alpha}\right)^2}\\
&=\frac{k\xi_T\left(\frac{1}{2}\xi_T+\frac{x_0-k}{T^\alpha}\right)}{\left(\xi_T+\frac{x_0-k}{T^\alpha}\right)^2},
\end{aligned}$$

where we used Lemma 2.4, Remark 2.2, (2.57), (2.45), and Remark 2.8. Hence (1.27) holds.

Step 3: We prove the asymptotic properties of estimator $\hat{k}$, namely (1.20), (1.25), and (1.28). It is obvious that

$$k-\hat{k}_t=\frac{1}{\hat{\alpha}_t}\left[\tau-\hat{\tau}_t-k(\alpha-\hat{\alpha}_t)\right].$$

(1") If $0<\alpha<1-H$, by (2.27) and (2.28), we have

$$\begin{aligned}
(T-t)^{-H}(k-\hat{k}_t)&=\frac{\left(\xi_t+\frac{x_0-k}{T^\alpha}\right)^2(T-t)^{2-H-2\alpha}\left[\frac{(T-t)^{\alpha-1}}{1-\alpha}\phi_t-\frac{(T-t)^{2\alpha-1}}{1-2\alpha}\psi_t\right]}{(T-t)^{2-2\alpha}\left[\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u}{(T-u)^2}du-\frac{t}{T(T-t)}\int_0^t\frac{X_u}{T-u}dX_u\right]}\\
&\quad+\frac{(T-t)^{2-H-2\alpha}R_t}{(T-t)^{2-2\alpha}\left[\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u}{(T-u)^2}du-\frac{t}{T(T-t)}\int_0^t\frac{X_u}{T-u}dX_u\right]}.
\end{aligned}\tag{2.58}$$

Note that

$$
\begin{aligned}
&(T-t)^{2-H-2\alpha}R_t\\
&=\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\phi_t\Big[\frac{(T-t)^{1-2\alpha}(T-t)^{1-H}\psi_t}{(1-\alpha)(1-2\alpha)}-\frac{(T-t)^{1-H-\alpha+1-\alpha}}{1-\alpha}\frac{x_0-k}{T}\Big]\\
&\quad+\frac{(T-t)^{1-2\alpha}(T-t)^{1-H}\psi_t}{1-2\alpha}\frac{(x_0-k)^2}{T}\\
&-(T-t)^{2-H-2\alpha}\Big[\int_0^t(T-u)^{\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)du+\frac{2\psi_t}{1-2\alpha}\Big](\phi_t\xi_t-\eta_t)\\
&=\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\phi_t\Big[\frac{(T-t)^{1-2\alpha}(T-t)^{1-H}\psi_t}{(1-\alpha)(1-2\alpha)}-\frac{(T-t)^{1-H-\alpha+1-\alpha}}{1-\alpha}\frac{x_0-k}{T}\Big]\\
&\quad+\frac{(T-t)^{1-2\alpha}(T-t)^{1-H}\psi_t}{1-2\alpha}\frac{(x_0-k)^2}{T}\\
&\quad-\Big[\Big(\xi_t+\frac{x_0-k}{T^\alpha}\Big)\frac{(T-t)^{1-H-\alpha}}{1-\alpha}-\frac{(T-t)^{1-2\alpha}(T-t)^{1-H}\psi_t}{1-\alpha}\\
&\quad-\frac{(T-t)^{1-H-\alpha+1-\alpha}}{1-\alpha}\frac{x_0-k}{T}+\frac{2(T-t)^{1-2\alpha}(T-t)^{1-H}\psi_t}{1-2\alpha}\Big](\phi_t\xi_t-\eta_t)\xrightarrow{\text{law}}0,
\end{aligned}
\tag{2.59}
$$

as $t\to T$, by Lemma 2.1, Remark 2.2, (2.16), (2.5), and (2.6). Thus (1.20) holds from (2.58), Lemma 2.1, (2.8), (2.44) and (2.59).

(2") If $1-H<\alpha<\frac{1}{2}$, by (2.24) and (2.25), we have

$$
(T-t)^{\alpha-1}(k-\hat{k}_t)=\frac{(T-t)^{1-\alpha}\big(Q_t+k\psi_t\int_0^t\frac{X_u-k}{(T-u)^2}du\big)}{(T-t)^{2-2\alpha}\big[\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u}{(T-u)^2}du-\frac{t}{T(T-t)}\int_0^t\frac{X_u}{T-u}dX_u\big]}.
$$

By (2.26), (2.35), and (2.23), we have

$$
\begin{aligned}
&Q_t+k\psi_t\int_0^t\frac{X_u-k}{(T-u)^2}du\\
&=\int_0^t(T-u)^{\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)du\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t-k\psi_t\Big)\\
&\quad-\psi_t\int_0^t(T-u)^{2\alpha-2}\Big(\xi_u+\frac{x_0-k}{T^\alpha}\Big)^2du+k\psi_t\int_0^t\frac{X_u-k}{(T-u)^2}du\\
&=\int_0^t\frac{X_u-k}{(T-u)^2}du\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t-k\psi_t\Big)-\psi_t\int_0^t\frac{(X_u-k)^2}{(T-u)^2}du+k\psi_t\int_0^t\frac{X_u-k}{(T-u)^2}du\\
&=\Big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\Big)\int_0^t\frac{X_u-k}{(T-u)^2}du-\psi_t\int_0^t\frac{(X_u-k)^2}{(T-u)^2}du.
\end{aligned}
\tag{2.60}
$$

Hence, by Lemma 2.4, Remark 2.2, (2.18), (2.19), (2.16), and (2.44), we obtain, as $t\to T$,

$$
\begin{aligned}
&(T-t)^{\alpha-1}(k-\hat{k}_t)\\
&=\frac{\big(\eta_t+\frac{x_0-k}{T^\alpha}\phi_t\big)(T-t)^{1-\alpha}\int_0^t\frac{X_u-k}{(T-u)^2}du-(T-t)^{\alpha-(1-H)}(T-t)^{1-H}\psi_t(T-t)^{1-2\alpha}\int_0^t\frac{(X_u-k)^2}{(T-u)^2}du}{(T-t)^{2-2\alpha}\big[\int_0^t\frac{dX_u}{T-u}\int_0^t\frac{X_u}{(T-u)^2}du-\frac{t}{T(T-t)}\int_0^t\frac{X_u}{T-u}dX_u\big]}\\
&\xrightarrow{\text{law}}\frac{(1-\alpha)(1-2\alpha)\big(\eta_T+\frac{x_0-k}{T^\alpha}\phi_T\big)}{\alpha^3\big(\xi_T+\frac{x_0-k}{T^\alpha}\big)}.
\end{aligned}
$$

Therefore (1.25) holds.

(3") If $\alpha=1/2$, similarly as $t\to T$, we obtain

$$
\frac{|\log(T-t)|}{(T-t)^{1/2}}(k-\hat{k}_t)
$$

$$
\begin{aligned}
&= \frac{(T-t)^{1/2}\big[\left(\eta_t + \frac{x_0-k}{T^\alpha}\phi_t\right)\int_0^t \frac{X_u-k}{(T-u)^2}du - \psi_t\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du\big]}{\frac{T-t}{|\log(T-t)|}\big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u\big]}\\
&= \Big[\left(\eta_t + \frac{x_0-k}{T^\alpha}\phi_t\right)(T-t)^{1/2}\int_0^t \frac{X_u-k}{(T-u)^2}du\\
&\quad - (T-t)^{H-\frac{1}{2}}|\log(T-t)|(T-t)^{1-H}\psi_t\frac{1}{|\log(T-t)|}\int_0^t \frac{(X_u-k)^2}{(T-u)^2}du\Big]\\
&\quad \div \Big[\frac{T-t}{|\log(T-t)|}\big[\int_0^t \frac{dX_u}{T-u}\int_0^t \frac{X_u}{(T-u)^2}du - \frac{t}{T(T-t)}\int_0^t \frac{X_u}{T-u}dX_u\big]\Big]\\
&\xrightarrow{\text{law}} \frac{4\left(\eta_T + \frac{x_0-k}{T^\alpha}\phi_T\right)}{\xi_T + \frac{x_0-k}{T^\alpha}}\\
&= \frac{4\xi_T\left(\frac{1}{2}\xi_T + \frac{x_0-k}{T^\alpha}\right)}{\xi_T + \frac{x_0-k}{T^\alpha}}.
\end{aligned}
$$

Hence (1.28) holds. The proof of Theorem 1.2 is complete.

3. Appendix

In this section we describe some basic facts on the stochastic calculus with respect to sub-fractional Brownian motion S^H. Some surveys and complete literatures could be found in Alòs and Nualart [1] and Nualart [17].

Fix a time interval $[0,T]$. We denote by $\mathcal{H}_{S^H}$ canonical Hilbert space associated to the sub-fractional Brownian motion S^H. That is, $\mathcal{H}_{S^H}$ is the closure of the linear span ε generated by the indicator function with respect to the scalar product

$$
\langle \mathbf{I}_{[0,t]}, \mathbf{I}_{[0,s]}\rangle = R(t,s) = \mathbf{E}(S_t^H S_s^H) = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |s-t|^{2H}\right].
$$

We know that the covariance of sub-fractional Brownian motion S^H can be written as

$$
R(t,s) = \mathbf{E}(S_t^H S_s^H) = \int_0^t\int_0^s \varphi(u,v)dudv, \tag{3.1}
$$

where $\varphi(u,v) = H(2H-1)\left[|u-v|^{2H-2} - (u+v)^{2H-2}\right]$. Note that for any $H > 1/2$, we have

$$
\varphi(u,v) \le H(2H-1)|u-v|^{2H-2}. \tag{3.2}
$$

Let $C_b^\infty(\mathbb{R}^n,\mathbb{R})$ be the class of infinitely differentiable functions $f:\mathbb{R}^n\to\mathbb{R}$ such that f and its partial derivatives are bounded. We denote by S the class of smooth cylindrical random variables $F = f(S^H(\varphi_1),\dots,S^H(\varphi_n))$, for $\varphi_i \in \mathcal{H}_{S^H}, i = 1,\dots,n$, and $f \in C_b^\infty(\mathbb{R}^n,\mathbb{R})$. The Malliavin derivative operator D of a smooth cylindrical random variables $F = f(S^H(\varphi_1),\dots,S^H(\varphi_n))$ is defined as the $\mathcal{H}_S^H$-valued random variable

$$
D_sF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(S^H(\varphi_1),\dots,S^H(\varphi_n))\varphi_j(s), s \in [0,T].
$$

In particular $D_sS_t^H = \mathbf{I}_{[0,t]}(s)$. As usual, $\mathbf{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$
\|F\|_{1,2}^2 = \mathbf{E}(F^2) + \mathbf{E}[\|DF\|_{\mathcal{H}_{S^H}}^2].
$$

The Skorohod integral δ is the adjoint of the derivative operator D. If a random variable $u \in L^2(\Omega,\mathcal{H}_{S^H})$ belongs to the domain of the Skorohod integral (denoted by $\mathrm{dom}(\delta)$), that is, if it verifies

$$
|\mathbf{E}\langle DF,u\rangle_{\mathcal{H}_{S^H}}| \le c_u\sqrt{\mathbf{E}(F^2)} \quad \text{for all } F \in S.
$$

Then $\delta(u)$ is defined by the duality relationship

$$
\mathbf{E}[\delta(u)F] = \mathbf{E}[\langle DF,u\rangle_{\mathcal{H}_{S^H}}], \quad \text{for every } F \in \mathbf{D}^{1,2}.
$$

We will use the notation

$$\delta(u) = \int_0^T u_s \delta S_s^H, \quad u \in \mathrm{dom}(\delta).$$

It is well-known that $\mathbf{D}^{1,2}(\mathcal{H}_{S^H})$ is included in the domain of δ. Note that $\mathbf{E}(\delta(u)) = 0$ and the variance of $\delta(u)$ is given by

$$\mathbf{E}(\delta^2(u)) = \mathbf{E}\Big(\|u\|^2_{\mathcal{H}_{S^H}}\Big) + \mathbf{E}\left(\langle Du, (Du)^*\rangle_{\mathcal{H}_{S^H}\otimes\mathcal{H}_{S^H}}\right),$$

if $u \in \mathbf{D}^{1,2}(\mathcal{H}_{S^H})$, where $(Du)^*$ is the adjoint of Du in the Hilbert space $\mathcal{H}_{S^H} \otimes \mathcal{H}_{S^H}$. We will use the property

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}_{S^H}},$$

if $F \in \mathbf{D}^{1,2}$ and $u \in \mathrm{dom}(\delta)$ such that $Fu \in \mathrm{dom}(\delta)$. We also need the commutativity relationship between D and δ,

$$D\delta(u) = u + \int_0^1 Du_s \delta S_s^H,$$

if $u \in \mathbf{D}^{1,2}(\mathcal{H}_{S^H})$ and the process $\{Du_s, s \in [0,1]\}$ belongs to the domain of δ.

For every $q \geq 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of S^H, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(S^H(h)), h \in \mathcal{H}_{S^H}, \|h\|_{\mathcal{H}_{S^H}} = 1\}$, where H_q is the qth Hermite polynomial. The mapping $I_q(h^{\otimes q}) = H_q(S^H(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}_{S^H}^{\odot q}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}_{S^H}^{\odot q}} = \sqrt{q!}\|\cdot\|_{\mathcal{H}_{S^H}^{\otimes q}}$ and $\mathcal{H}_q$. Specifically, for all $f, g \in \mathcal{H}_{S^H}^{\odot q}$ and $q \geq 1$, one has

$$\mathbf{E}[I_q(f)I_q(g)] = q!\langle f, g\rangle_{\mathcal{H}_{S^H}^{\otimes q}}. \tag{3.3}$$

On the other hand, it is well-known that any random variable Z belonging to $L^2(\Omega)$ admits the following chaotic expansion:

$$Z = \mathbf{E}[Z] + \sum_{q=1}^{\infty} I_q(f_q), \tag{3.4}$$

where the series converges in $L^2(\Omega)$ and the kernels f_q, belonging to $\mathcal{H}_{S^H}^{\odot q}$, are uniquely determined by Z.

Let $f, g : [0,T] \to \mathbb{R}$ be Hölder continuous functions of order $\alpha \in (0,1)$ and $\beta \in (0,1)$ with $\alpha+\beta > 1$. Young in 1936 proved that the Riemman-Stiltjes (so-called Young integral) $\int_0^T f(s)dg(s)$ exists. Moreover, if $\alpha = \beta \in (\frac{1}{2}, 1)$ and $\psi : \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^1, the integrals $\int_0^{\bullet} \frac{\partial\psi}{\partial f}(f(u), g(u))df(u)$ and $\int_0^{\bullet} \frac{\partial\psi}{\partial g}(f(u), g(u))dg(u)$ exist in the Young sense and the following change variables holds,

$$\psi(f(t), g(t)) = \psi(f(0), g(0)) + \int_0^t \frac{\partial\psi}{\partial f}(f(u), g(u))df(u) + \int_0^t \frac{\partial\psi}{\partial g}(f(u), g(u))dg(u), \tag{3.5}$$

for $0 \leq t \leq T$. As a consequence, if $\frac{1}{2} < H < 1$ and $(u_t, t \in [0,T])$ is a process with Hölder paths of order $\alpha > 1 - H$, the integral $\int_0^T u_s dS_s^H$ is well-defined as a Young integral. Suppose moreover that for any $t \in [0,T], u_t \in \mathbf{D}^{1,2}$, and

$$\mathbf{P}\Big(\int_0^T \int_0^T |D_s u_t| \left[|t-s|^{2H-2} - (t+s)^{2H-2}\right] ds\, dt < \infty\Big) = 1. \tag{3.6}$$

Then, by (3.2) and the same argument as in Alòs and Nualart [17] or see the proof of Cai, Wang and Xiao [5, Lemma 2.7], we have

$$\int_0^T u_s dS_s^H = \int_0^T u_s \delta S_s^H + \int_0^T \int_0^T D_t u_s \varphi(t,s)\, dt\, ds. \tag{3.7}$$

In fact, let $u_t^{\epsilon} = \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} u_s ds$. Using integration by parts, we have

$$F\delta(u) = \delta(Fu) + \langle DF, u\rangle_{\mathcal{H}_{S^H}},$$

if $F \in \mathbf{D}^{1,2}$ and $u \in \mathrm{dom}(\delta)$ such that $Fu \in \mathrm{dom}(\delta)$. When u satisfied the condition (3.6), we obtain

$$\begin{aligned}
\int_0^T u_s \frac{S^H_{s+\epsilon} - S^H_{s-\epsilon}}{2\epsilon} ds &= \int_0^T u_s \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} dS^H_v \, ds \\
&= \int_0^T u_s \delta\Big(\frac{1}{2\epsilon} \mathbf{I}_{[s-\epsilon, s+\epsilon]}\Big) ds \\
&= \int_0^T \delta\Big(u_s \frac{1}{2\epsilon} \mathbf{I}_{[s-\epsilon, s+\epsilon]}\Big) ds + \frac{1}{2\epsilon} \int_0^T \langle Du_s, \mathbf{I}_{[s-\epsilon, s+\epsilon]} \rangle_{\mathcal{H}_{S^H}} ds \\
&= \delta(u^\epsilon) + \frac{1}{2\epsilon} \int_0^T \langle Du_s, \mathbf{I}_{[s-\epsilon, s+\epsilon]} \rangle_{\mathcal{H}_{S^H}} ds \qquad (3.8) \\
&= \delta(u^\epsilon) + \frac{1}{2\epsilon} \int_0^T \Big[\int_0^T \int_0^T D_t u_s \mathbf{I}_{[s-\epsilon, s+\epsilon]}(r) \varphi(t, r) \, dt \, dr \Big] ds \\
&= \delta(u^\epsilon) + \int_0^T \Big[\int_0^T \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} D_t u_s \varphi(t, r) \, dt \, dr \Big] ds \\
&= \delta(u^\epsilon) + \int_0^T \int_0^T D_t u_s \Big[\frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} \varphi(t, r) dr \Big] \, dt \, ds,
\end{aligned}$$

since

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{s-\epsilon}^{s+\epsilon} \varphi(t, r) dr = \lim_{\epsilon \to 0} \frac{\varphi(t, s+\epsilon) + \varphi(t, s-\epsilon)}{2} = \varphi(t, s).$$

Thus (3.7) can be easily obtained by taking the limit $\epsilon \to 0$ on both sides of (3.8). In particular, when u is a non-random Hölder continuous function of order $\alpha > 1 - H$, we obtain

$$\int_0^t u_s dS^H_s = \int_0^t u_s \delta S^H_s. \qquad (3.9)$$

This article extends the results of Han et al. [8] for fractional Brownian bridge with linear drift. The differences between our paper and Han et al. [9] lie in: (1) We prove the existence of the limiting variance of $(T-t)^{2-H-\alpha} \Big[\frac{t}{T(T-t)} \phi_t - \frac{(T-t)^{\alpha-1}}{1-\alpha} \psi_t \Big]$ by above (2.11). (2) We give the proof of (3.7).

Acknowledgements. The authors wish to thank the anonymous referees for their careful reading and the comments that improved this paper. Nenghui Kuang was supported by the Natural Science Foundation of Hunan Province under Grant 2021JJ30233, and by the Scientific Research Project of the Education Department of Hunan Province under Grant 24A0345. Huantian Xie was partly supported by the NSF of China (No. 12271233), by the Improving innovation ability of enterprizes in the Shandong province (No. 2023TSGC0466), and by the NSF of Shandong Province (No. ZR2019YQ05, 2019KJI003).

References

[1] Alòs, A.; Nualart, D.; Stochastic calculus with respect to fractional Brownian motion. *Stochastics and Stochastics Reports*, 2003, 75(3): 129-152.

[2] Barczy, M.; Pap, G.; α-Wiener bridges: singularity of induced measures and sample path properties. *Stoch. Anal. Appl.*, 2010, 28(3): 447-466.

[3] Barczy. M.; Pap, G.; Explicit formulas for Laplace transforms of certain functionals of some time inhomogeneous diffusions. *J. Math. Anal. Appl.*, 2011, 380(2): 405-424.

[4] Bojdecki. T.; Gorostiza, L.; Talarczyk. A.; Sub-fractional Brownian motion and its relation to occupation times. *Statist. Probab. Lett.*, 2004, 69 (4): 405-419.

[5] Cai, C.; Wang, Q.; Xiao, W.; Mixed sub-fractional Brownian motion and drift estimation of related Ornstein-Uhlenbeck process. *Commun. Math. Stat.*, 2023, 11:229-255.

[6] Diedhiou, A.; Manga. C.; Mendy, I.; Parametric estimation for SDEs with additive subfractional Brownian motion. *J. Numer. Math. Stoch.*, 2011, 3(1):37-45.

[7] Es-Sebaiy, K.; Nourdin, I.; Parameter estimation for α-fractional bridges. *Malliavin Calculus and Stochastic Analysis*, 2013, 34: 385-412.

[8] Han, J.; Shen, G.; Yan, L.; Least squares estimation for α-weighted fractional Brownian bridge. *Appl. Math. J. Chinese Univ. Ser. A*, 2015, 30(4): 432-444.
[9] Han, J.; Sun, Y.; Yan, L.; Least squares estimation for fractional Brownian bridge with linear drift. *Comm. Statist. Theory Methods*, (26 Feb 2025), DOI: 10.1080/03610926.2025.2461615
[10] Kuang, N.; Li, Y.; Berry-Esséen bounds and almost sure CLT for the quadratic variation of the sub-bifractional Brownian motion. *Comm. Statist. Simulation Comput.*, 2022, 51(8): 4257-4275.
[11] Kuang, N.; Liu, B.; Least squares estimator for α-sub-fractional bridges. *Statist. Papers*, 2018, 59(3): 893-912.
[12] Kuang, N.; Xie, H.; Maximum likelihood estimator for the sub-fractional Brownian motion approximated by a random walk. *Ann. Inst. Statist. Math.*, 2015, 67(1): 75-91.
[13] Kuang, N.; Xie, H.; Least squares type estimators for the drift parameters in the sub-bifractional Vasicek processes. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.,* 2023,26(2):2350004.
[14] Liu, J.; Yan, L.; Remarks on asymptotic behavior of weighted quadratic variation of subfractional Brownian motion. *J. Korean Statist. Soc.*, 2012 41(2):177-187.
[15] Mansuy, R.; On a one-parameter generalization of the Brownian bridge and associated quadratic functionals. *J. Theoret. Probab.*, 2004, 17(4): 1021-1029.
[16] Mendy, I.; Parametric estimation for sub-fractional Ornstein-Uhlenbeck process. *J. Statist. Plann. Inference*, 2013, 143(4): 663-674.
[17] Nualart, D.; *The Malliavin calculus and related topics.* 2nd ed. Springer-Verlag, Berlin. 2006
[18] Shen, G.; Chen, C.; Stochastic integration with respect to the sub-fractional Brownian motion with $0 \leq H \leq 1/2$. *Statist. Probab. Lett.*, 2012, 82(2): 240-251.
[19] Shen, G.; Yan, L.; Remarks on an integral functional driven by sub-fractional Brownian motion. *J. Korean Statist. Soc.*, 2011, 40(3): 337-346.
[20] Tudor, C.; Some properties of the sub-fractional Brownian motion. *Stochastics*, 2007, 79 (5): 431-448.
[21] Tudor, C.; On the Wiener integral with respect to a sub-fractional Brownian motion on an interval. *J. Math. Anal. Appl.*, 2009, 351(1):456-468.
[22] Tudor, C.; Berry-Esseen bounds and almost sure CLT for the quadratic variation of the subfractional Brownian motion. *J. Math. Anal. Appl.*, 2011, 375(2):667-676.
[23] Yan, L.; Shen, G.; On the collision local time of sub-fractional Brownian motions. *Statist. Probab. Lett.*, 2010, 80(5):296-308.
[24] Zhao, S.; Liu, Q.; Chen, T.; On the large deviation principle for maximum likelihood estimator of α-Brownian bridge. *Statist. Probab. Lett.*, 2018, 138: 143-150.
[25] Zhao, S.; Chen, T.; Large deviation expansion for maximum-likelihood estimator of α-Brownian bridge. *Comm. Statist. Theory Methods*, 2017, 46(15): 7313-7326.

Nenghui Kuang (corresponding author)
School of Mathematics and Statistics, Hunan University of Science and Technology, Xiangtan, Hunan 411201, China
Email address: knh1552@163.com

Huantian Xie
School of Mathematics and Statistics, Linyi University, Linyi, Shandong 276005, China
Email address: xiehuantian@lyu.edu.cn