

EXISTENCE OF ENTROPY SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEM IN A TWO-COMPONENT DOMAIN

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ABSTRACT. In this article, we study the degenerated quasilinear elliptic equations

$$\begin{aligned} -\operatorname{div}(\mathcal{B}(x, v_1, Dv_1)) + g(x, v_1) &= f(x) && \text{in } \Omega_1, \\ -\operatorname{div}(\mathcal{B}(x, v_2, Dv_2)) + g(x, v_2) &= f(x) && \text{in } \Omega_2, \\ v_1 &= 0 && \text{on } \partial\Omega, \\ \mathcal{B}(x, v_1, Dv_1) \cdot \nu_1 &= \mathcal{B}(x, v_2, Dv_2) \cdot \nu_1 && \text{on } \Gamma, \\ \mathcal{B}(x, v_1, Dv_1) \cdot \nu_1 &= -h(x)|v_1 - v_2|^{p-2}(v_1 - v_2) && \text{on } \Gamma, \end{aligned}$$

where Ω is a connected bounded open set of \mathbb{R}^N ($N \geq 2$), and can be decomposed as $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where Ω_2 is an open subset included in Ω , and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$ and $\Gamma = \partial\Omega_2$, with $2 + \lambda - \frac{1}{N} < p < N$, and also we assume that $g(\cdot, s)$ satisfying some growth condition. We will show the existence of entropy solutions for this class of equation and we conclude some regularity results.

1. INTRODUCTION

The study of entropy solutions in the case of one-component domains has garnered significant interest among researchers. Bénilan et al. [6] established the existence, uniqueness, and regularity properties of entropy solutions for quasilinear elliptic equations, for the prototype problem

$$\begin{aligned} A(u) &= f(x, u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $A(u) = -\operatorname{div}(\mathcal{B}(x, u)Du)$ and $f(x, s)$ is a Caratheodory function. Boccardo et al. [9] proved the existence of entropy solutions for the elliptic problem (1.1), where the operator $Au = -\operatorname{div} \mathcal{B}(x, u, Du)$ and the term $f(x, u) = f - \operatorname{div} \phi(u)$, with $f \in L^1(\Omega)$ and $\phi(\cdot) \in C^0(\mathbb{R}, \mathbb{R}^N)$. For additional related literature, we refer to [2, 10, 19]. For non-coercive principal operators, Boccardo et al. [11] further investigated the existence and regularity of solutions to problem (1.1), when $f(x, u) = f(x)$ (independent of u) and $f \in L^m(\Omega)$ for some $m \geq 1$. Additional related results can be found in [11, 16, 19, 32, 39].

The concept of entropy solutions was first introduced by Bénilan et al. [6]. The equivalent notion of renormalized solutions was developed by P.L. Lions and Di Perna [20] in their analysis of the Boltzmann equation. This notion was later adapted to elliptic problems with L^1 -data in [12] (see also [34, 35]) and extended to elliptic problems involving bounded measure data in [18]. For parabolic equations with L^1 -data, renormalized solutions were adapted by Blanchard and Murat [7] and further introduced by Blanchard et al. [8].

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In this paper, we are interested in the study of nonlinear elliptic equations of the type

$$\begin{aligned} -\operatorname{div}(\mathcal{B}(x, v_1, Dv_1)) + g(x, v_1) &= f(x) \quad \text{in } \Omega_1, \\ -\operatorname{div}(\mathcal{B}(x, v_2, Dv_2)) + g(x, v_2) &= f(x) \quad \text{in } \Omega_2, \\ v_1 &= 0 \quad \text{on } \partial\Omega, \\ \mathcal{B}(x, v_1, Dv_1) \cdot \nu_1 &= \mathcal{B}(x, v_2, Dv_2) \cdot \nu_1 \quad \text{on } \Gamma, \\ \mathcal{B}(x, v_1, Dv_1) \cdot \nu_1 &= -h(x)|v_1 - v_2|^{p-2}(v_1 - v_2) \quad \text{on } \Gamma, \end{aligned} \tag{1.2}$$

Here, Ω is a connected open bounded subset of \mathbb{R}^N ($N \geq 2$). It is defined as $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where Ω_2 be an open set such that $\overline{\Omega_2}$ is a set included in Ω , with a Lipschitz boundary and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$. We denote by ν_i the vector unit outward normal to Ω_i for $i = 1, 2$. This study extends previous investigations into quasilinear elliptic equations with mixed boundary conditions. Fulgencio and Guibé [26] established the existence of renormalized solutions for systems governed by matrix fields $B(x, v_i)$ where $i = 1, 2$. Subsequent contributions, such as [27], strengthened these findings by proving the uniqueness of solution and expanded the scope to homogenization analysis [21]. Complementary results can be found in [24, 25, 37, 38].

In a related direction, Guibé and Oropeza [28] examined some quasilinear elliptic equations with nonlinear Robin boundary condition

$$\begin{aligned} -\operatorname{div}(A(x, u)\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_0, \\ B(x, u)\nabla u \cdot \vec{n} + \gamma(x)h(u) &= g \quad \text{on } \Gamma_1, \end{aligned} \tag{1.3}$$

Their analysis focused on a domain Ω where solutions satisfy nonlinear flux conditions on a boundary Γ_1 , coupling the solution v and its normal derivative, alongside Dirichlet conditions on another segment Γ_0 . Using approximation methods and a priori bounds, they established both existence and uniqueness. Such problems have attracted substantial interest, with studies such as [4, 5] addressing linear Robin conditions in perforated domains. Moreover, the authors of [14] have resolved quasilinear problems with nonlinear Robin constraints using Schauder's fixed-point theorem, adapting uniqueness arguments from [1, 15]. Cioranescu et al. proved the existence and uniqueness of solution for analogous systems by using Minty-Browder theorem. For additional results in this area, we refer the reader to [4, 22, 23, 29, 30].

In [13], the authors established the existence of renormalized solution to problem (1.2) for $1 < p < N$ and under the standard coercivity condition. The lower order term required to satisfy the coercivity condition $g(x, s)\operatorname{sign}(s) \geq |s|^\delta$ for $\delta > \min\{1, p' - 1\}$ which plays a key role in controlling the interface term on Γ and ensuring its integrability. The present paper focuses on the existence of entropy solutions to problem (1.2) under the assumption $2 + \lambda - \frac{1}{N} < p < N$, with a degenerated coercivity condition and $g(x, s)$ satisfying only a sign and some growth condition.

In the classical one-component setting, the notion of entropy and renormalized solutions are known to be equivalent. However, for the two-components setting, where the interface condition involves the jump term $h(x)|v_1 - v_2|^{p-2}(v_1 - v_2)$. The difficulty is that the regularity guaranteed by the renormalized framework (see [13, Definition 2]) is not sufficient to ensure that this term is integrable in the entropy formulation. To address this issue, we introduce the notion of entropy solution and exploit Lemma 2.5 (see section 2), which provides additional regularity for the solution in Lorentz spaces. Specifically, under the condition $2 - \frac{1}{N} + \lambda < p < N$, this lemma yields for $i = 1, 2$,

$$|v_i|^{p-2-\lambda}v_i \in L^{\frac{N}{N-p}, \infty}(\Omega_i) \quad \text{and} \quad |Dv_i|^{p-2-\lambda}Dv_i \in L^{\frac{N}{N-1}, \infty}(\Omega_i),$$

which in turn guaranties that the jump term

$$|v_1 - v_2|^{p-2}(v_1 - v_2) \in L^1(\Gamma),$$

is well-defined using the Lorentz space regularity which making sense the notion of entropy solution for problem (1.2).

This paper is organized as follows. Section 2 presents the fundamental assumptions on the Carathéodory functions $\mathcal{B}(x, s, \xi)$ and $g(x, s)$ that guarantee the existence of at least one entropy

solution. This section also recalls the essential technical lemmas required for establishing our main result. Section 3 presents the main contribution of this paper, together with the definition of entropy solution for the problem (1.2). The last section is devoted to proving the existence of entropy solutions for the degenerated quasilinear elliptic problem (1.2), and conclude some regularity results.

2. ESSENTIAL ASSUMPTIONS

Consider a connected, bounded, open domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) with Lipschitz-continuous boundary $\partial\Omega$. The domain Ω can be partitioned into two open sets Ω_1 and Ω_2 , separated by an interface Γ , such that

- Ω_2 is an open subset of Ω with a Lipschitz boundary $\Gamma = \partial\Omega_2$. The closure of Ω_2 lies entirely within Ω (i.e., $\overline{\Omega_2} \subset \Omega$).
- Ω_1 is defined as $\Omega \setminus \overline{\Omega_2}$ representing the remaining part of Ω outside $\overline{\Omega_2}$.

Thus, the domain decomposes as $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$, where Γ is the internal interface between Ω_1 and Ω_2 . Let p satisfy

$$2 + \lambda - \frac{1}{N} < p < N, \tag{2.1}$$

where $0 < \lambda < N - 2 + \frac{1}{N}$ and let $p' = \frac{p}{p-1}$ denote its Hölder conjugate exponent, and $p^* = \frac{Np}{N-p}$ the Sobolev conjugate exponent. Let v be a measurable function defined on $\Omega \setminus \Gamma$. The restriction of v to each subdomain Ω_i is denoted by $v_i = v|_{\Omega_i}$ for $i = 1, 2$.

The principal aim of this study is to demonstrate the existence of an entropy solution for the class of degenerated quasilinear elliptic problems modeled by the prototype problem

$$\begin{aligned} -\operatorname{div}(\mathcal{B}(x, v_1, Dv_1)) + g(x, v_1) &= f(x) && \text{in } \Omega_1, \\ -\operatorname{div}(\mathcal{B}(x, v_2, Dv_2)) + g(x, v_2) &= f(x) && \text{in } \Omega_2, \\ v_1 &= 0 && \text{on } \partial\Omega, \\ \mathcal{B}(x, v_1, Dv_1) \cdot \nu_1 &= \mathcal{B}(x, v_2, Dv_2) \cdot \nu_1 && \text{on } \Gamma, \\ \mathcal{B}(x, v_1, Dv_1) \cdot \nu_1 &= -h(x)|v_1 - v_2|^{p-2}(v_1 - v_2) && \text{on } \Gamma, \end{aligned} \tag{2.2}$$

where $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory functions, which satisfy the following conditions:

- (i) There exists $\Lambda \in L^{p'}(\Omega)$, and $\beta > 0$ such that

$$|\mathcal{B}(x, s, \zeta)| \leq \beta(\Lambda(x) + |s|^{p-1} + |\zeta|^{p-1}), \tag{2.3}$$

- (ii) The Carathéodory function \mathcal{B} is strictly monotone in ζ ; that is, for all $\zeta \neq \zeta'$,

$$(\mathcal{B}(x, s, \zeta) - \mathcal{B}(x, s, \zeta'))(\zeta - \zeta') > 0, \tag{2.4}$$

- (iii) There exists a positive constant $b_0 > 0$ such that

$$\mathcal{B}(x, s, \zeta)\zeta \geq b(|s|)|\zeta|^p \quad \text{and} \quad b(|s|) \geq \frac{b_0}{(1 + |s|)^\lambda}, \tag{2.5}$$

where $0 \leq \lambda < \min\{\frac{p-1}{N}, N - 2 + \frac{1}{N}\}$, and the decreasing function $b(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

The function $g(\cdot, \cdot) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is also of Carathéodory type and satisfies only the growth condition:

$$g(x, s) \operatorname{sign}(s) \geq 0 \quad \text{and} \quad \sup_{\{|s| \leq k\}} |g(x, s)| \in L^1(\Omega) \quad \text{for all } k > 0. \tag{2.6}$$

Finally we assume that

$$f \in L^1(\Omega \setminus \Gamma), \quad \text{and} \quad h(\cdot) \in L^\infty(\Gamma) \quad \text{with} \quad 0 < h_0 \leq h(x) \text{ a.e. on } \Gamma. \tag{2.7}$$

Let us begin by reviewing essential definitions and a key proposition required to establish our principal theorems. We first define the space

$$V_1 = \{v \in W^{1,p}(\Omega_1) : v = 0 \text{ on } \partial\Omega\} \quad \text{with} \quad \|v\|_{V_1} := \|Dv\|_{L^p(\Omega_1)},$$

and

$$V := \{v \equiv (v_1, v_2) : v_1 \in V_1 \text{ and } v_2 \in W^{1,p}(\Omega_2)\},$$

equipped with the norm

$$\|v\|_V^p := \|Dv_1\|_{L^p(\Omega_1)}^p + \|Dv_2\|_{L^p(\Omega_2)}^p + \|v_1 - v_2\|_{L^p(\Gamma)}^p. \quad (2.8)$$

Remark 2.1 (see [29]). The norm (2.8) is equivalent to the norm of $V_1 \times W^{1,p}(\Omega_2)$, i.e., there exist two positive constants c_1 and c_2 such that

$$c_1 \|v\|_V \leq \|v\|_{V_1 \times W^{1,p}(\Omega_2)} \leq c_2 \|v\|_V \quad \text{for all } v \in V.$$

Now, we recall some definition and key properties of Lorentz spaces, which we will use in the following. For more details, we refer the reader to [36].

Let us denote by f^* the decreasing rearrangement of f , i.e.

$$f^*(t) = \inf\{s \geq 0 : \text{meas}\{x \in \Omega : |f(x)| > s\} < t\}, \quad t \in [0, \text{meas}(\Omega)]. \quad (2.9)$$

Moreover, for $1 < q < \infty$ and $1 < s \leq \infty$, we denote

$$\|f\|_{L^{q,s}(\Omega)} = \begin{cases} \left(\int_0^{\text{meas}(\Omega)} [f^*(t)t^{\frac{1}{q}}]^s \frac{dt}{t} \right)^{1/s} & \text{if } s < \infty, \\ \sup_{t>0} t [\text{meas}\{x \in \Omega : |f(x)| > t\}]^{1/q} & \text{if } s = \infty. \end{cases} \quad (2.10)$$

The Lorentz space $L^{q,s}(\Omega)$ is the space of Lebesgue measurable functions such that

$$\|f\|_{L^{q,s}(\Omega)} < +\infty.$$

These spaces are equipped with the norm specified in equation (2.10). They serve as interpolation spaces between Lebesgue spaces, in the sense that for every $1 < s < r < \infty$, we have

$$L^{r,1}(\Omega) \subset L^{r,r}(\Omega) = L^r(\Omega) \subset L^{r,\infty}(\Omega) \subset L^{s,1}(\Omega). \quad (2.11)$$

Lemma 2.2. (see. [29]) Let $h > 0$, assuming that (2.3)-(2.5) hold, and let $(v_n := (v_{n,1}, v_{n,2}))_n$ be a sequence in V and let $v = (v_1, v_2)$ be such that $v_n \rightharpoonup v$ weakly in V , and

$$\sum_{i=1}^2 \int_{\Omega_i} \left(\mathcal{B}(x, T_h(v_{n,i}), Dv_{n,i}) - \mathcal{B}(x, T_h(v_{n,i}), Dv_i) \right) (Dv_{n,i} - Dv_i) dx \longrightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (2.12)$$

then $v_n \rightarrow v$ strongly in V for a subsequence.

Definition 2.3. Let $k > 0$, we consider the truncation function $T_k(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ ks/|s| & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T} := \{v := (v_1, v_2) : \Omega \setminus \Gamma \rightarrow \mathbb{R} \text{ measurable, such that } T_k(v) \in V \text{ for all } k > 0\}.$$

Proposition 2.4 (see [6], [26], [28]). Let $v := (v_1, v_2) \in \mathcal{T}$. For $i = 1, 2$, we have:

- (1) There exists a unique measurable function $u_i : \Omega_i \rightarrow \mathbb{R}^N$ such that for all $k \geq 1$,

$$DT_k(v_i) = u_i \chi_{\{|v_i| < k\}} \quad \text{a.e. in } \Omega_i, \quad (2.13)$$

where $\chi_{\{|v_i| < k\}}$ denotes the characteristic function of $\{x \in \Omega_i : |v_i(x)| < k\}$. We denote by u_i the gradient of v_i , i.e., $u_i = Dv_i$.

- (2) If $\sup_{k \geq 1} \frac{1}{k} \|T_k(v)\|_V^p < \infty$, then there exists a unique measurable function $w_i : \Gamma \mapsto \mathbb{R}$, $\Omega_1 = \Omega \setminus \Omega_2$ for $i = 1, 2$ such that for all $k \geq 1$, we have

$$\gamma_i(T_k(v_i)) = T_k(w_i) \quad \text{almost everywhere on } \Gamma,$$

where $\gamma_i : W^{1,p}(\Omega_i) \mapsto L^p(\Gamma)$ is the trace operator, we define the function w_i as the trace of v_i on Γ and we set

$$\gamma_i(v_i) = w_i.$$

Lemma 2.5. *Let $v := (v_1, v_2) \in \mathcal{T}$, and we assume that p satisfies (2.1), such that*

$$\|T_k(v)\|_V^p \leq Mk^{1+\lambda}, \tag{2.14}$$

where $M > 0$, then

$$\left\| |v_i|^{p-1-\lambda} \right\|_{L^{\frac{N}{N-p}, \infty}(\Omega_i)} \leq C(N, p)M \quad \text{for } i = 1, 2, \tag{2.15}$$

and

$$\left\| |Dv_i|^{p-1-\lambda} \right\|_{L^{\frac{N}{N-1}, \infty}(\Omega_i)} \leq C(N, p)(1 + M^{\frac{N+\lambda}{N}} + M^{\frac{N-p+\lambda}{N-p}}) \quad \text{for } i = 1, 2. \tag{2.16}$$

where $C(N, p)$ is a positive constant depending only on N and p .

Proof. We first establish the validity of (2.15). Applying the Gagliardo-Nirenberg-Sobolev inequality, Proposition 2.1, and relation (2.14), we obtain:

$$\begin{aligned} k^{p^*} \text{meas}\{x \in \Omega_i : |v_i| > k\} &\leq \int_{\Omega_i} |T_k(v_i)|^{p^*} dx \\ &\leq C_0(N, p) \|T_k(v)\|_{V_1 \times W^{1,p}(\Omega_2)}^{p^*} \\ &\leq C_1(N, p) \|T_k(v)\|_V^{p^*} \\ &\leq C_2(N, p) (Mk^{1+\lambda})^{p^*/p}. \end{aligned} \tag{2.17}$$

Let $h = k^{p-1-\lambda}$, we obtain

$$h^{\frac{p^*}{p-1-\lambda}} \text{meas}\{x \in \Omega_i : |v_i|^{p-1-\lambda} > h\} \leq C_2(N, p) (Mh^{\frac{1+\lambda}{p-1-\lambda}})^{p^*/p}.$$

Then

$$\text{meas}\{x \in \Omega_i : |v_i|^{p-1-\lambda} > h\} \leq C_3(N, p) (Mh^{\frac{1+\lambda}{p}-1})^{\frac{p^*}{p-1-\lambda}} = C_3(N, p) (Mh^{-1})^{p^*/p}.$$

Which implies that, for all $h > 0$,

$$h \text{meas}\{x \in \Omega_i : |v_i|^{p-1-\lambda} > h\}^{\frac{p}{p^*}} \leq C_4(N, p)M.$$

Thus

$$\left\| |v_i|^{p-1-\lambda} \right\|_{L^{\frac{N}{N-p}, \infty}(\Omega_i)} = \sup_{h>0} \left(h \text{meas}\{x \in \Omega_i : |v_i|^{p-1-\lambda} > h\}^{p/p^*} \right) \leq C_4(N, p)M.$$

Now, we will show (2.16). Thanks to (2.14), for all $\delta > 0$ which yields

$$\begin{aligned} \delta^p \text{meas}\{x \in \Omega_i : |Dv_i| > \delta, |v_i| \leq k\} &\leq \int_{\{|v_i| \leq k\}} |Dv_i|^p dx \\ &\leq \|T_k(v)\|_{V_1 \times W^{1,p}(\Omega_2)}^p \\ &\leq C_5 \|T_k(v)\|_V^p \\ &\leq C_5 M k^{1+\lambda}. \end{aligned}$$

Let $\mu = \delta^{p-1-\lambda}$, we obtain

$$\mu^{\frac{p}{p-1-\lambda}} \text{meas}\{x \in \Omega_i : |Dv_i|^{p-1-\lambda} > \mu, |v_i| \leq k\} \leq C_5 M k^{1+\lambda}. \tag{2.18}$$

By combining (2.17) and (2.18), yield that

$$\begin{aligned} &\text{meas}\{x \in \Omega_i : |Dv_i|^{p-1-\lambda} > \mu\} \\ &\leq \text{meas}\{x \in \Omega_i : |Dv_i|^{p-1-\lambda} > \mu, |v_i| \leq k\} + \text{meas}\{x \in \Omega_i : |v_i| > k\} \\ &\leq \frac{C_5 M k^{1+\lambda}}{\mu^{\frac{p}{p-1-\lambda}}} + \frac{C_2(N, p) (Mk^{1+\lambda})^{p^*/p}}{k^{p^*}}. \end{aligned} \tag{2.19}$$

Since $M > 0$ and $\mu > 0$, by taking $k = M^{\frac{1}{N-1}} \mu^{\frac{N-p}{(N-1)(p-1-\lambda)}}$ in (2.19), we arrive at

$$\text{meas}\{x \in \Omega_i : |Dv_i|^{p-1-\lambda} > \mu\} \leq C_5 M^{\frac{N+\lambda}{N-1}} \mu^{\frac{-N}{N-1} - \frac{\lambda p}{(N-1)(p-1-\lambda)}} + C_2(N, p) M^{\frac{N(N-p+\lambda)}{(N-1)(N-p)}} \mu^{\frac{-N}{N-1}}.$$

It follows that for all $\mu \geq 1$, we have

$$\begin{aligned} \mu \operatorname{meas}\{x \in \Omega_i : |Dv_i|^{p-1-\lambda} > \mu\}^{\frac{N-1}{N}} &\leq C_6 M^{\frac{N+\lambda}{N}} \mu^{-\frac{\lambda p}{N(p-1-\lambda)}} + C_7(N, p) M^{\frac{N-p+\lambda}{N-p}} \\ &\leq C_8(N, p) (M^{\frac{N+\lambda}{N}} + M^{\frac{N-p+\lambda}{N-p}}). \end{aligned}$$

Having in mind that

$$\mu \operatorname{meas}\{x \in \Omega_i : |Dv_i|^{p-1-\lambda} > \mu\}^{\frac{N-1}{N}} \leq \operatorname{meas}\{\Omega_i\}^{\frac{N-1}{N}} \quad \text{for all } 0 < \mu \leq 1.$$

We conclude that

$$\begin{aligned} \left\| |Dv_i|^{p-1-\lambda} \right\|_{L^{\frac{N}{N-1}, \infty}(\Omega_i)} &= \sup_{\mu > 0} \mu \operatorname{meas}\{x \in \Omega_i : |Dv_i|^{p-1-\lambda} > \mu\}^{\frac{N-1}{N}} \\ &\leq C_9(N, p) (1 + M^{\frac{N+\lambda}{N}} + M^{\frac{N-p+\lambda}{N-p}}). \end{aligned} \quad \square$$

3. MAIN RESULTS

Definition 3.1. A measurable function v is called an entropy solution of the degenerated quasi-linear elliptic problem (2.2), if $T_k(v) \in V$,

$$\|T_k(v)\|_V^p \leq Ck^{1+\lambda} \quad \text{for all } k > 0, \quad (3.1)$$

and

$$g(x, v_i) \in L^1(\Omega_i) \quad \text{for } i = 1, 2; \quad (3.2)$$

Moreover v satisfies the inequality

$$\begin{aligned} &\int_{\Omega_1} \mathcal{B}(x, v_1, Dv_1) \cdot DT_k(v_1 - \varphi_1) \, dx + \int_{\Omega_2} \mathcal{B}(x, v_2, Dv_2) \cdot DT_k(v_2 - \varphi_2) \, dx \\ &+ \int_{\Omega_1} g(x, v_1) T_k(v_1 - \varphi_1) \, dx + \int_{\Omega_2} g(x, v_2) T_k(v_2 - \varphi_2) \, dx \\ &+ \int_{\Gamma} h(x) |v_1 - v_2|^{p-2} (v_1 - v_2) (T_k(v_1 - \varphi_1) - T_k(v_2 - \varphi_2)) \, d\sigma \\ &\leq \int_{\Omega} f T_k(v - \varphi) \, dx, \end{aligned} \quad (3.3)$$

for every $\varphi = (\varphi_1, \varphi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$.

Remark 3.2. Under the conditions (3.1)–(3.2), and the fact that $T_k(v) \in V$ for any $k > 0$, all the terms in (3.3) are defined. Indeed

- For each $i = 1, 2$, it follows that

$$DT_k(v_i - \varphi_i) = (DT_M(v_i) - D\varphi_i) \chi_{\{|v_i - \varphi_i| \leq k\}} \quad \text{a.e. in } \Omega_i,$$

where $M = k + \|\varphi\|_\infty$. Using assumption (2.3), and the fact that $T_M(v) \in V$, we conclude that for all $\varphi = (\varphi_1, \varphi_2) \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$, we have

$$\begin{aligned} &\left| \int_{\Omega_i} \mathcal{B}(x, v_i, Dv_i) \cdot DT_k(v_i - \varphi_i) \, dx \right| \\ &= \left| \int_{\Omega_i} \mathcal{B}(x, T_M(v_i), DT_M(v_i)) \cdot (DT_M(v_i) - D\varphi_i) \chi_{\{|v_i - \varphi_i| \leq k\}} \, dx \right| \\ &\leq \beta \left(\int_{\Omega_i} (K + M^{p-1}) |DT_M(v_i)| + |DT_M(v_i)|^p \right. \\ &\quad \left. + (K + M^{p-1} + |DT_M(v_i)|^{p-1}) |D\varphi| \, dx \right) < \infty. \end{aligned} \quad (3.4)$$

- In view of Lemma 2.5, the assumption (3.1), and the continuous embedding

$$L^{\frac{N}{N-1}, \infty}(\Omega_i) \hookrightarrow L^s(\Omega_i) \quad \text{for all } 1 < s < \frac{N}{N-1},$$

we conclude that

$$v_i \in W^{1,q}(\Omega_i) \quad \text{for all } 1 \leq q < \frac{N(p-1-\lambda)}{N-1}. \tag{3.5}$$

In light of the condition $p > 2 + \lambda - \frac{1}{N}$, and the continuity of the trace operators

$$\gamma_i : W^{1,q}(\Omega_i) \rightarrow L^q(\Gamma) \quad \text{for } i = 1, 2,$$

it follows that

$$\gamma_i(v_i) \in L^q(\Gamma) \quad \text{for all } 1 \leq q < \frac{N(p-1-\lambda)}{N-1},$$

Therefore,

$$|v_1 - v_2|^{p-1} \in L^1(\Gamma) \quad \text{for } p > 2 + \lambda - \frac{1}{N}.$$

Hence

$$\begin{aligned} & \left| \int_{\Gamma} h(x) |v_1 - v_2|^{p-2} (v_1 - v_2) (T_k(v_1 - \varphi_1) - T_k(v_2 - \varphi_2)) \, d\sigma \right| \\ & \leq 2 \|h(\cdot)\|_{\infty} k \int_{\Gamma} |v_1 - v_2|^{p-1} \, d\sigma < \infty. \end{aligned} \tag{3.6}$$

- Thanks to (3.2) and the assumption $f(x) \in L^1(\Omega \setminus \Gamma)$ it follows that

$$g(x, v_i) T_k(v_i - \varphi_i) \in L^1(\Omega_i) \quad \text{and} \quad f(x) T_k(v - \varphi) \in L^1(\Omega \setminus \Gamma).$$

Theorem 3.3. *Assume that $2 + \lambda - \frac{1}{N} < p < N$, and that conditions (2.3)-(2.7) hold. Then the degenerated quasilinear elliptic problem (2.2) has at least one entropy solution v . Moreover, $v = (v_1, v_2) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ for all $1 \leq q < \frac{N(p-1-\lambda)}{N-1}$.*

4. EXISTENCE OF ENTROPY SOLUTIONS

This section is divided into three subsections:

- Subsection 4.1 formulates the approximate problem;
- Subsection 4.2 establishes crucial a priori estimates and develops the limiting arguments;
- Subsection 4.3 completes the proof of Theorem 3.3 by combining these results.

4.1. Approximate problems. Let $n > 0$. Define

- (1) $f_n(x) = T_n(f(x)) \in L^\infty(\Omega \setminus \Gamma)$. It easy to see that $f_n(x)$ converges to $f(x)$ strongly in $L^1(\Omega \setminus \Gamma)$ as $n \rightarrow \infty$.
- (2) $g_n(x, r) = T_n(g(x, r))$ for a.e. $x \in \Omega$ and for any $r \in \mathbb{R}$.
- (3) $\mathcal{B}_n(x, r, \vartheta) = \mathcal{B}(x, T_n(r), \vartheta)$ for a.e. $x \in \Omega$ and for all $(r, \vartheta) \in \mathbb{R} \times \mathbb{R}^N$.

Then, the approximate approximate problem is formulated as follows

$$\begin{aligned} & -\operatorname{div}(\mathcal{B}_n(x, v_{n,1}, Dv_{n,1})) + g_n(x, v_{n,1}) = f_n(x) \quad \text{in } \Omega_1, \\ & -\operatorname{div}(\mathcal{B}_n(x, v_{n,2}, Dv_{n,2})) + g_n(x, v_{n,2}) = f_n(x) \quad \text{in } \Omega_2, \\ & v_{n,1} = 0 \quad \text{on } \partial\Omega, \\ & \mathcal{B}_n(x, v_{n,1}, Dv_{n,1}) \cdot \nu_1 = \mathcal{B}_n(x, v_{n,2}, Dv_{n,2}) \cdot \nu_1 \quad \text{on } \Gamma, \\ & \mathcal{B}_n(x, v_{n,1}, Dv_{n,1}) \cdot \nu_1 = -h(x) |v_{n,1} - v_{n,2}|^{p-2} (v_{n,1} - v_{n,2}) \quad \text{on } \Gamma. \end{aligned} \tag{4.1}$$

From Proposition 2.1 and by using [33, Theorem 2.7, p.180], we conclude that there exists at least one weak solution $v_n := (v_{n,1}, v_{n,2}) \in V$, i.e.

$$\begin{aligned} & \int_{\Omega_1} \mathcal{B}(x, T_n(v_{n,1}), Dv_{n,1}) \cdot D\varphi_1 \, dx + \int_{\Omega_2} \mathcal{B}(x, T_n(v_{n,2}), Dv_{n,2}) \cdot D\varphi_2 \, dx \\ & + \int_{\Omega_1} g_n(x, v_{n,1})\varphi_1 \, dx + \int_{\Omega_2} g_n(x, v_{n,2})\varphi_2 \, dx \\ & + \int_{\Gamma} h(x)|v_{n,1} - v_{n,2}|^{p-2}(v_{n,1} - v_{n,2})(\varphi_1 - \varphi_2) \, d\sigma \\ & = \int_{\Omega \setminus \Gamma} f_n(x)\varphi \, dx, \end{aligned} \tag{4.2}$$

for all $\varphi \in V$.

4.2. A priori estimates.

Theorem 4.1. *Assume that (2.3)-(2.7) hold. Let $v_n = (v_{n,1}, v_{n,2})$ be a weak solution of the approximate problem (4.1). Then we have the following estimates*

- (i) *There exists a constant $C = C(N, p, \alpha) > 0$ such that*

$$\|T_k(v_n)\|_V^p \leq Ck^{1+\lambda} \quad \text{for all } k > 0.$$

- (ii) *There exists a constant $C = C(N, p, \alpha) > 0$ such that*

$$\text{meas}\{|v_n| > k\} \leq \frac{C}{k^{p-1-\lambda}}.$$

- (iii) *There exists a constant $C = C(N, p, \alpha) > 0$ such that*

$$\text{meas}_{\Gamma}\{|\gamma_i(v_{n,i})| > k\} \leq \frac{C}{k^{p-1-\lambda}} \quad \text{for } i = 1, 2.$$

Proof. (i) By taking $\varphi = (T_k(v_{n,1}), T_k(v_{n,2})) \in V$ as a test function for the approximate problem (4.2), we have

$$I_{1,1} + I_{1,2} + I_{2,1} + I_{2,2} + I_3 = I_4, \tag{4.3}$$

where, for $i = 1, 2$

$$\begin{aligned} I_{1,i} &= \int_{\Omega_i} \mathcal{B}(x, T_n(v_{n,i}), Dv_{n,i}) \cdot DT_k(v_{n,i}) \, dx, \\ I_{2,i} &= \int_{\Omega_i} g_n(x, v_{n,i})T_k(v_{n,i}) \, dx, \\ I_3 &= \int_{\Gamma} h(x)|v_{n,1} - v_{n,2}|^{p-2}(v_{n,1} - v_{n,2})(T_k(v_{n,1}) - T_k(v_{n,2})) \, d\sigma, \\ I_4 &= \int_{\Omega} f_n(x)T_k(v_n) \, dx. \end{aligned}$$

For the first term $I_{1,i}$, using (2.13) and (2.5), we have

$$\begin{aligned} I_{1,i} &= \int_{\Omega_i} \mathcal{B}(x, T_k(v_{n,i}), DT_k(v_{n,i})) \cdot DT_k(v_{n,i}) \, dx \\ &\geq \int_{\Omega_i} b(|T_k(v_{n,i})|)|DT_k(v_{n,i})|^p \, dx \\ &\geq \int_{\Omega_i} \frac{b_0|DT_k(v_{n,i})|^p}{(1 + |T_k(v_{n,i})|)^\lambda} \, dx \\ &\geq \frac{b_0}{(1 + k)^\lambda} \int_{\Omega_i} |DT_k(v_{n,i})|^p \, dx. \end{aligned} \tag{4.4}$$

Concerning the term I_2 , in view of (2.6) it is clear that

$$I_{2,i} = \int_{\Omega_i} g_n(x, v_{n,i}) T_k(v_{n,i}) \, dx \geq 0. \tag{4.5}$$

For the behavior of I_3 , by (2.7) and since $|v_{n,1} - v_{n,2}| \geq |T_k(v_{n,1}) - T_k(v_{n,2})|$ a.e. on Γ , we arrive at

$$I_3 \geq h_0 \int_{\Gamma} |T_k(v_{n,1}) - T_k(v_{n,2})|^p \, d\sigma. \tag{4.6}$$

Now we study the behavior of I_4 . Thanks to (2.6) we have

$$|I_4| = \left| \int_{\Omega \setminus \Gamma} f_n(x) T_k(v_n) \, dx \right| \leq k \int_{\Omega \setminus \Gamma} |f(x)| \, dx. \tag{4.7}$$

It follows that, by putting together (4.3)–(4.7) and using (2.6), there exists a constant $C = C(N, p, \alpha) > 0$ independent of k and n such that

$$\frac{b_0}{(1+k)^\lambda} \sum_{i=1}^2 \int_{\Omega_i} |DT_k(v_{n,i})|^p \, dx + \int_{\Gamma} |T_k(v_{n,1}) - T_k(v_{n,2})|^p \, d\sigma \leq Ck. \tag{4.8}$$

Hence

$$\|T_k(v_n)\|_V^p \leq Ck^{1+\lambda} \quad \text{for } k \geq 1. \tag{4.9}$$

The proofs of (ii) and (iii) follow the same reasoning as in [13, Lemma 2]. \square

Corollary 4.2 ([13, Lemma 3]). *Choosing $k > 0$, and let $v_n = (v_{n,1}, v_{n,2}) \in V$ be a solution of the approximate problem (4.1). Then, there exists a measurable function $v = (v_1, v_2)$ and a subsequence of $(v_n)_n$ still denoted by $(v_n)_n$, which satisfies the following properties*

- (i) $v_{n,i} \rightarrow v_i$ a.e. in Ω_i for $i = 1, 2$;
- (ii) $\gamma_i(v_{n,i}) \rightarrow \gamma_i(v_i)$ a.e. on Γ for $i = 1, 2$;
- (iii) $T_k(v_n) \rightharpoonup T_k(v)$ weakly in V ;
- (iv) $T_k(v_{n,i}) \rightarrow T_k(v_i)$ strongly in $L^p(\Omega_i)$ and a.e. in Ω_i ;
- (v) $\gamma_i(T_k(v_{n,i})) \rightarrow \gamma_i(T_k(v_i))$ strongly in $L^p(\Gamma)$ and a.e. on Γ .

Lemma 4.3. *Under the same assumptions of Theorem 3.3, we have*

$$g_n(x, v_{n,i}) \rightarrow g(x, v_i) \text{ strongly in } L^1(\Omega_i) \quad \text{for } i = 1, 2; \tag{4.10}$$

$$|v_{n,1} - v_{n,2}|^{p-2} (v_{n,1} - v_{n,2}) \rightarrow |v_1 - v_2|^{p-2} (v_1 - v_2) \text{ strongly in } L^1(\Gamma) \tag{4.11}$$

Proof. To establish (4.10), we follow a similar approach to the one outlined in the proof of [3].

Now, we prove that (4.11). According to (4.9), we can apply Lemma 2.5 to deduce that

$$\|v_{n,i}\|_{W^{1,q}(\Omega_i)} \leq C \quad \text{for all } 1 \leq q < \frac{N(p-1-\lambda)}{N-1}. \tag{4.12}$$

This implies that the sequence $(v_{n,i})_n$ is uniformly bounded in $W^{1,q}(\Omega_i)$. In view of (4.12) and the condition $p > 2 + \lambda - \frac{1}{N}$, we have

$$v_{n,i} \rightharpoonup v_i \text{ weakly in } W^{1,q}(\Omega_i) \quad \text{for all } 1 \leq q < \frac{N(p-1-\lambda)}{N-1}.$$

Consequently, since the trace operator $\gamma_i : W^{1,q}(\Omega_i) \rightarrow L^q(\Gamma)$ is compact, there exists a subsequence, still denoted by $(v_{n,i})_n$, such that

$$v_{n,i} \rightarrow v_i \text{ strongly in } L^q(\Gamma) \quad \text{for all } 1 \leq q < \frac{N(p-1-\lambda)}{N-1},$$

Therefore,

$$|v_{n,1} - v_{n,2}|^{p-1} \rightarrow |v_1 - v_2|^{p-1} \text{ strongly in } L^{\bar{q}}(\Gamma) \quad \text{for all } 1 \leq \bar{q} < \frac{p(N-1-\lambda)}{N-1},$$

On the one hand, since the embedding $L^{\bar{q}}(\Gamma) \hookrightarrow L^1(\Gamma)$ is continuous for every $1 \leq \bar{q} < \frac{p(N-1-\lambda)}{N-1}$, we deduce that

$$|v_{n,1} - v_{n,2}|^{p-1} \rightarrow |v_1 - v_2|^{p-1} \text{ strongly in } L^1(\Gamma),$$

Equivalently, we can write

$$|v_{n,1} - v_{n,2}|^{p-2}(v_{n,1} - v_{n,2}) \rightarrow |v_1 - v_2|^{p-2}(v_1 - v_2) \quad \text{strongly in } L^1(\Gamma).$$

□

Lemma 4.4. *Assume that (2.3)-(2.7) hold. Let $v_n = (v_{n,1}, v_{n,2})$ be a weak solution of the approximate problem (4.1), then*

$$T_k(v_n) \rightarrow T_k(v) \quad \text{strongly in } V, \quad (4.13)$$

and

$$Dv_{n,i} \rightarrow Dv_i \quad \text{a.e. in } \Omega_i \quad (\text{for a subsequence}). \quad (4.14)$$

Proof. We set $z_n = v_n - T_h(v_n) + T_k(v_n) - T_k(v)$, by taking $\varphi = T_{2k}(z_n) \in V$ as a test function for the approximate problem (4.2), we obtain

$$\begin{aligned} & \sum_{i=1}^2 \int_{\Omega_i} \mathcal{B}(x, T_n(v_{n,i}), Dv_{n,i}) DT_{2k}(z_{n,i}) dx + \sum_{i=1}^2 \int_{\Omega_i} g_n(x, v_{n,i}) T_{2k}(z_{n,i}) dx \\ & + \int_{\Gamma} h(x) |v_{n,1} - v_{n,2}|^{p-2} (v_{n,1} - v_{n,2}) (T_{2k}(z_{n,1}) - T_{2k}(z_{n,2})) d\sigma \\ & = \sum_{i=1}^2 \int_{\Omega_i} f_n(x) T_{2k}(z_{n,i}) dx. \end{aligned} \quad (4.15)$$

Following the same approach in [31] the first term on the left-hand side of (4.15) can be estimated as follows

$$\begin{aligned} & \int_{\Omega_i} \mathcal{B}(x, T_n(v_{n,i}), Dv_{n,i}) DT_{2k}(z_{n,i}) dx \\ & \geq \int_{\Omega_i} \left(\mathcal{B}(x, T_k(v_{n,i}), DT_k(v_{n,i})) - \mathcal{B}(x, T_k(v_{n,i}), DT_k(v_i)) \right) (DT_k(v_{n,i}) - DT_k(v_i)) dx + \varepsilon(n). \end{aligned} \quad (4.16)$$

where $\lim_{n \rightarrow \infty} \varepsilon(n) = 0$. The convergence of the remaining terms in the equation (4.15) are established using the same method. Firstly, we have $T_{2k}(z_{n,i}) \rightharpoonup T_{2k}(v_i - T_h(v_i))$ weak-* in $L^\infty(\Omega_i)$ and thanks to (4.10), we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega_i} g_n(x, v_{n,i}) T_{2k}(z_{n,i}) dx = \int_{\Omega_i} g(x, v_i) T_{2k}(v_i - T_h(v_i)) dx. \quad (4.17)$$

Observe that $T_{2k}(v_i - T_h(v_i)) \rightarrow 0$ a.e. in Ω_i and

$$|g(x, v_i) T_{2k}(v_i - T_h(v_i))| \leq 2k |g(x, v_i)| \in L^1(\Omega_i).$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$\lim_{h \rightarrow \infty} \int_{\Omega_i} g(x, v_i) T_{2k}(v_i - T_h(v_i)) dx = 0. \quad (4.18)$$

Similarly, we have $T_{2k}(z_{n,i}) \rightharpoonup T_{2k}(v_i - T_h(v_i))$ weak-* in $L^\infty(\Gamma)$, and by using (4.11), we obtain

$$\begin{aligned} & \lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\Gamma} h(x) |v_{n,1} - v_{n,2}|^{p-2} (v_{n,1} - v_{n,2}) T_{2k}(z_{n,i}) d\sigma \\ & = \lim_{h \rightarrow \infty} \int_{\Gamma} h(x) |v_1 - v_2|^{p-2} (v_1 - v_2) T_{2k}(v_i - T_h(v_i)) d\sigma = 0, \end{aligned} \quad (4.19)$$

and

$$\lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=1}^2 \int_{\Omega_i} f_n(x) T_{2k}(z_{n,i}) dx = \lim_{h \rightarrow \infty} \sum_{i=1}^2 \int_{\Omega_i} f(x) T_{2k}(v_i - T_h(v_i)) dx = 0. \quad (4.20)$$

By combining (4.15) and (4.16)-(4.20), we deduce that

$$\begin{aligned} & \sum_{i=1}^2 \int_{\Omega_i} \left(\mathcal{B}(x, T_k(v_{n,i}), DT_k(v_{n,i})) - \mathcal{B}(x, T_k(v_{n,i}), DT_k(v_i)) \right) (DT_k(v_{n,i}) - DT_k(v_i)) dx \\ & \leq \varepsilon(n, h), \end{aligned} \quad (4.21)$$

where $\lim_{h \rightarrow \infty} \lim_{n \rightarrow \infty} \varepsilon(n, h) = 0$. Then, by letting n and h tend to infinity, we arrive at

$$\lim_{n \rightarrow \infty} \sum_{i=1}^2 \int_{\Omega_i} (\mathcal{B}(x, T_k(v_{n,i}), DT_k(v_{n,i})) - \mathcal{B}(x, T_k(v_{n,i}), DT_k(v_i))) \cdot (DT_k(v_{n,i}) - DT_k(v_i)) dx = 0. \quad (4.22)$$

In view of Lemma 2.2, we conclude that

$$T_k(v_n) \longrightarrow T_k(v) \quad \text{strongly in } V \quad \text{and} \quad Dv_n \longrightarrow Dv \quad \text{a.e. in } \Omega \setminus \Gamma$$

4.3. Proof of Theorem 3.3. In this subsection, we prove that v is an entropy solution to the degenerated quasilinear elliptic problem (2.2). First, combining Theorem 4.1 (i) and Corollary 4.2 (iii) we conclude that v satisfies (3.1). Moreover, in view of (4.10), we have v satisfies (3.2).

It remains to show that v satisfies (3.3). Indeed, let $\varphi \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$ and $M = k + \|\varphi\|_\infty$ with $k > 0$. By taking $v = T_k(v_n - \varphi) \in V$ as a test function for the variational formulation (4.2), we obtain

$$\begin{aligned} & \sum_{i=1}^2 \int_{\Omega_i} \mathcal{B}(x, T_n(v_{n,i}), Dv_{n,i}) \cdot DT_k(v_{n,i} - \varphi_i) dx + \sum_{i=1}^2 \int_{\Omega_i} g_n(x, v_{n,i}) T_k(v_{n,i} - \varphi_i) dx \\ & + \int_{\Gamma} h(x) |v_{n,1} - v_{n,2}|^{p-2} (v_{n,1} - v_{n,2}) T_k(v_{n,1} - \varphi_1) d\sigma \\ & - \int_{\Gamma} h(x) |v_{n,1} - v_{n,2}|^{p-2} (v_{n,1} - v_{n,2}) T_k(v_{n,2} - \varphi_2) d\sigma \\ & = \int_{\Omega \setminus \Gamma} f_n(x) T_k(v_n - \varphi) dx. \end{aligned} \quad (4.23)$$

Firstly, we have $\{|v_{n,i} - \varphi_i| \leq k\} \subset \{|v_{n,i}| \leq M\}$, then the first term on the left-hand side of (4.23) can be written as

$$\begin{aligned} & \int_{\Omega_i} \mathcal{B}(x, T_n(v_{n,i}), Dv_{n,i}) \cdot DT_k(v_{n,i} - \varphi_i) \chi_{\{|v_{n,i} - \varphi_i| \leq k\}} dx \\ & = \int_{\Omega} \mathcal{B}(x, T_M(v_{n,i}), DT_M(v_{n,i})) \cdot (DT_M(v_{n,i}) - D\varphi_i) \chi_{\{|v_{n,i} - \varphi_i| \leq k\}} dx \\ & = \int_{\Omega} (\mathcal{B}(x, T_M(v_{n,i}), DT_M(v_{n,i})) - \mathcal{B}(x, T_M(v_{n,i}), D\varphi_i)) \cdot (DT_M(v_{n,i}) - D\varphi_i) \chi_{\{|v_{n,i} - \varphi_i| \leq k\}} dx \\ & \quad + \int_{\Omega} \mathcal{B}(x, T_M(v_{n,i}), D\varphi_i) \cdot (DT_M(v_{n,i}) - D\varphi_i) \chi_{\{|v_{n,i} - \varphi_i| \leq k\}} dx. \end{aligned}$$

In view of Fatou's Lemma, we obtain

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\Omega_i} \mathcal{B}(x, T_n(v_{n,i}), Dv_{n,i}) \cdot DT_k(v_{n,i} - \varphi_i) \chi_{\{|v_{n,i} - \varphi_i| \leq k\}} dx \\ & \geq \int_{\Omega_i} (\mathcal{B}(x, T_M(v_i), DT_M(v_i)) - \mathcal{B}(x, T_M(v_i), D\varphi_i)) \cdot (DT_M(v_i) - D\varphi_i) \chi_{\{|v_i - \varphi_i| \leq k\}} dx \\ & \quad + \lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{B}(x, T_M(v_{n,i}), D\varphi_i) \cdot (DT_M(v_{n,i}) - D\varphi_i) \chi_{\{|v_{n,i} - \varphi_i| \leq k\}} dx \\ & = \int_{\Omega_i} \mathcal{B}(x, T_M(v_i), DT_M(v_i)) \cdot (DT_M(v_i) - D\varphi_i) \chi_{\{|v_i - \varphi_i| \leq k\}} dx \\ & = \int_{\Omega_i} \mathcal{B}(x, v_i, Dv_i) \cdot DT_k(v_i - D\varphi_i) dx. \end{aligned} \quad (4.24)$$

For the remaining terms, we note that $T_k(v_{n,i} - \varphi_i) \rightharpoonup T_k(v_i - \varphi_i)$ weak-* in $L^\infty(\Omega_i)$ and weak-* in $L^\infty(\Gamma)$. Having in mind that (4.10) and (4.11), we conclude that

$$\int_{\Omega_i} g_n(x, v_{n,i}) T_k(v_{n,i} - \varphi_i) dx \rightarrow \int_{\Omega_i} g(x, v_i) T_k(v_i - \varphi_i) dx \quad \text{as } n \rightarrow \infty, \quad (4.25)$$

and

$$\begin{aligned} & \int_{\Gamma} h(x) |v_{n,1} - v_{n,2}|^{p-2} (v_{n,1} - v_{n,2}) T_k(v_{n,i} - \varphi_i) d\sigma \\ & \rightarrow \int_{\Gamma} h(x) |v_1 - v_2|^{p-2} (v_1 - v_2) T_k(v_i - \varphi_i) d\sigma \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4.26)$$

Moreover, we have $f_n(x) \rightarrow f(x)$ strongly in $L^1(\Omega \setminus \Gamma)$, then

$$\int_{\Omega \setminus \Gamma} f_n(x) T_k(v_n - \varphi) dx \rightarrow \int_{\Omega \setminus \Gamma} f(x) T_k(v - \varphi) dx \quad \text{as } n \rightarrow \infty. \quad (4.27)$$

Consequently, by combining (4.23)-(4.27), we deduce that

$$\begin{aligned} & \int_{\Omega_1} \mathcal{B}(x, v_1, Dv_1) \cdot DT_k(v_1 - \varphi_1) dx + \int_{\Omega_2} \mathcal{B}(x, v_2, Dv_2) \cdot DT_k(v_2 - \varphi_2) dx \\ & + \int_{\Omega_1} g(x, v_1) T_k(v_1 - \varphi_1) dx + \int_{\Omega_2} g(x, v_2) T_k(v_2 - \varphi_2) dx \\ & + \int_{\Gamma} h(x) |v_1 - v_2|^{p-2} (v_1 - v_2) (T_k(v_1 - \varphi_1) - T_k(v_2 - \varphi_2)) d\sigma \\ & \leq \int_{\Omega \setminus \Gamma} f(x) T_k(v - \varphi) dx, \end{aligned}$$

for every $\varphi \in V \cap (L^\infty(\Omega_1) \times L^\infty(\Omega_2))$, which complete the proof of the Theorem 3.3.

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