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GLOBAL WELL-POSEDNESS FOR 3D GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS IN CRITICAL FOURIER-TRIEBEL-LIZORKIN-MORREY SPACES

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Abstract. This article studies the Cauchy problem of the 3D generalized incompressible magnetohydrodynamic equations in critical Fourier-Triebel-Lizorkin-Morrey spaces. The introduction of the Fourier-Triebel-Lizorkin-Morrey spaces facilitates the estimation of nonlinear terms in the system via Fourier transforms. Moreover, the Fourier-Triebel-Lizorkin-Morrey spaces are strictly larger than the Fourier-Triebel-Lizorkin spaces. When the initial data are sufficiently small, the global well-posedness of solutions to the Cauchy problem for the 3D generalized incompressible magnetohydrodynamic equations is established using the Littlewood-Paley theory and the Banach-Picard contraction principle. Furthermore, we derive Gevrey-class regularity of the solutions in the Fourier-Triebel-Lizorkin-Morrey spaces.

1. Introduction

This article studies the Cauchy problem for the 3D generalized incompressible magnetohydrodynamic (GMHD) equations

$$\begin{gathered}\partial_t u + (-\Delta)^\alpha u + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla P = 0, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}^+,\\ \partial_t b + (-\Delta)^\alpha b + (u\cdot\nabla)b - (b\cdot\nabla)u = 0, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}^+,\\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \quad (x,t)\in\mathbb{R}^3\times\mathbb{R}^+,\end{gathered} \tag{1.1}$$

with boundary conditions

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \tag{1.2}$$

Equation 1.1 describes the macroscopic behavior of incompressible conductive fluids in magnetic fields [34], where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the fluid velocity field, $b(x,t) = (b_1(x,t), b_2(x,t), b_3(x,t))$ denotes the magnetic field, $P(x,t)$ denotes the magnetic pressure. $(-\Delta)^\alpha$ denotes the fractional Laplace operator, which is defined as follows

$$\widehat{(-\Delta)^\alpha f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi),$$

where $0 < \alpha \le 1$ is the fractional dissipation index for the velocity and magnetic fields.

If $\alpha = 1$, (1.1) reduce to the classical magnetohydrodynamic (MHD) equations, which describe the movement of conductive fluids such as plasmas, electrolytes, and liquid metals under the action of electromagnetic fields [23]. Formally, this system is coupled with the Navier-Stokes equations and the electromagnetic Maxwell equations. In 1972, Duvaut and Lions [8] proved the existence and uniqueness of solutions to the 3D MHD equations in the Sobolev space H^s with $s \ge 3$, as well as the global existence of solutions for small initial data. Miao and Yuan investigated the Cauchy problem for the 3D incompressible MHD equations in [26, 27], establishing the global well-posedness of solutions for small initial data in the Besov space $\dot{B}^{\frac{3}{p}-1}_{p,q}(\mathbb{R}^3)$ and $BMO^{-1}(\mathbb{R}^3)$ space, respectively. For more results on the well-posedness of solutions to the Cauchy problem for the MHD equations, see [6, 11, 24, 30, 32, 35, 38].

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If $\alpha = 1$ and $b = 0$, (1.1) become the 3D incompressible Navier-Stokes equations

$$\begin{gathered} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla P = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+ \\ \nabla \cdot u = 0. \end{gathered} \tag{1.3}$$

These equations describe the dynamics of incompressible viscous fluids. In 1934, Leray considered the Cauchy problem for (1.3) in his pioneering work [22] and established the existence of a global weak solution for initial data in $L^2(\mathbb{R}^3)$. In 1950, Hopf [15] obtained the global existence result of weak solutions for the Cauchy problem of (1.3) in a bounded domain. In the Sobolev space $\dot{H}^{\frac{1}{2}}$, Fujita and Kato [19] established local well-posedness of solutions to the large initial data problem and global well-posedness of solutions to the small initial data problem for (1.3). Owing to the scale invariance of (1.3), scholars have obtained well-posedness results of solutions to the small initial data problem of (1.3) in various critical function spaces, including the Lebesgue space L^3 [17], the Besov space $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ [5], the BMO^{-1} space [16], Triebel-Lizorkin spaces [7], and the Lei-Lin space χ^{-1} [20].

Similar to (1.3), (1.1) also possess scaling invariance. In the critical pseudomeasure space $\mathcal{PM}^{\alpha}$, Liu, Zhao, and Cui [25] obtained the existence of global solutions to the small initial data problem for (1.1). In the Lei-Lin space $\chi^{1-2\alpha}$, Ye [36] established the global well-posedness of solutions to (1.1) for small initial data. In the critical Fourier-Besov-Morrey spaces, Barakaa and Toumlilin [9] established global well-posedness of solutions to (1.1) for small initial data. Subsequently, several scholars investigated the well-posedness of fluid dynamics models in mixed Besov spaces and Triebel-Lizorkin spaces [1, 10, 13, 14, 28, 29, 37].

By computing Fourier transforms on the function sequences corresponding to different frequency bands in the Triebel-Lizorkin spaces, we can more easily derive estimates for the nonlinear terms in (1.1). This facilitates the investigation of the well-posedness of solutions to the Cauchy problem (1.1). If the L^p norm is replaced by the Morrey norm, the class of function spaces becomes larger. Accordingly, this paper introduces a new class of mixed-type critical Fourier-Triebel-Lizorkin-Morrey spaces and establishes the well-posedness and Gevrey-class regularity of solutions to the Cauchy problem (1.1). Applying the Leray projection operator $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the first equation of (1.1) yields

$$\begin{gathered} \partial_t u + (-\Delta)^{\alpha} u + \mathbb{P}\nabla \cdot (u \otimes u - b \otimes b) = 0, \\ \partial_t b + (-\Delta)^{\alpha} b + \nabla \cdot (u \otimes b - b \otimes u) = 0, \end{gathered} \tag{1.4}$$

where

$$u \otimes b = (b_1 u, b_2 u, b_3 u), \quad \nabla \cdot (u \otimes b) = \left(\nabla \cdot (b_1 u), \nabla \cdot (b_2 u), \nabla \cdot (b_3 u)\right), \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+.$$

The solution to the Cauchy problem (1.1)-(1.2) can be expressed as

$$\begin{aligned} u(x,t) &= e^{-t(-\Delta)^{\alpha}} u_0 - \int_0^t e^{-(t-\tau)(-\Delta)^{\alpha}} \mathbb{P}\nabla \cdot (u \otimes u - b \otimes b)(x,\tau)d\tau, \\ b(x,t) &= e^{-t(-\Delta)^{\alpha}} b_0 - \int_0^t e^{-(t-\tau)(-\Delta)^{\alpha}} \nabla \cdot (u \otimes b - b \otimes u)(x,\tau)d\tau. \end{aligned} \tag{1.5}$$

This solution is referred to as a mild solution [2] to the Cauchy problem (1.1). By establishing linear and bilinear estimates on (1.5) and applying the Banach-Picard contraction principle, we establish the global well-posedness of solutions to the Cauchy problem (1.1)-(1.2) for small initial data.

Theorem 1.1. *Let* $1 \le q \le \infty$, $1 \le p < \infty$, $\frac{1}{2} < \alpha \le 1$, $\max\{3 - (5 - 4\alpha + \frac{2\alpha}{\rho})p, 0\} < \lambda < 3$, $\frac{2\alpha}{2\alpha-1} < \rho \le \infty$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. *If there exists a constant* $\varepsilon = \varepsilon(p,q,\alpha)$ *such that for* $(u_0, b_0) \in \widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}$ *satisfying* $\|(u_0,b_0)\|_{\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}} < \varepsilon$, *then* (1.1) *has a unique global mild solution*

$$(u,b) \in C\Big([0,\infty), \widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big) \cap \mathcal{L}^{\rho}\Big([0,\infty), \widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\Big) \cap \mathcal{L}^{\rho'}\big([0,\infty), \widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\big), \tag{1.6}$$

and

$$\|(u,b)\|_{\mathcal{L}^{\rho}\left([0,\infty),\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\right)\cap\mathcal{L}^{\rho'}\left([0,\infty),\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\right)} \lesssim \|(u_0,b_0)\|_{\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}}. \tag{1.7}$$

Remark 1.2. This well-posedness result also holds for the classical magnetohydrodynamic equations and the Navier-Stokes equations.

Remark 1.3. The Fourier-Triebel-Lizorkin-Morrey space $\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}(\mathbb{R}^3)$ is a critical space.

Regarding the analytical properties of solutions to (1.1)-(1.2), using the proof method from [2, Theorem 2], we obtain Gevrey class regularity for solutions to (1.1)-(1.2).

Theorem 1.4. *Let $1 \le q \le \infty$, $1 \le p < \infty$, $\frac{1}{2} < \alpha \le 1$, $\max\{3-(5-4\alpha+\frac{2\alpha}{\rho})p, 0\} < \lambda < 3$, $\frac{2\alpha}{2\alpha-1} < \rho \le \infty$, $\frac{1}{\rho}+\frac{1}{\rho'} = 1$. If there exists a constant $\varepsilon' = \varepsilon'(p,q,\alpha)$ such that for $(u_0,b_0) \in \dot{\hat{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}$ satisfying $\|(u_0,b_0)\|_{\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}} < \varepsilon'$, then the solution (u,b) in Theorem 1.1 is analytic in the following sense*

$$\|(e^{\sqrt{t}\Lambda^{\alpha}}u, e^{\sqrt{t}\Lambda^{\alpha}}b)\|_{\mathcal{L}^{\rho}\left([0,\infty),\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\right)\cap\mathcal{L}^{\rho'}\left([0,\infty),\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\right)} \lesssim \|(u_0,b_0)\|_{\dot{\hat{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}},$$

where $e^{\sqrt{t}\Lambda^{\alpha}}$ denotes the Fourier multiplier, whose multiplier symbol is $e^{\sqrt{t}|\xi|^{\alpha}}$.

This article is organized as follows: Section 2 introduces the Littlewood-Paley theory, several function spaces, and provides relevant lemmas. Section 3 establishes linear and bilinear estimates for the mild solution (1.5) in the Fourier-Triebel-Lizorkin-Morrey spaces. Sections 4 and 5 provide the proofs of Theorem 1.1 and 1.4, respectively.

2. PRELIMINARIES

This section introduces Morrey spaces, Fourier-Triebel-Lizorkin spaces, and Fourier-Triebel-Lizorkin-Morrey spaces. The definition of Fourier-Triebel-Lizorkin-Morrey spaces is based on the Littlewood-Paley decomposition method associated with the Fourier transform. Therefore, we first introduce the Fourier transform and Littlewood-Paley theory.

Let $f \in \mathcal{S}$, where $\mathcal{S}$ denotes the class of Schwartz functions.

- The Fourier transform is defined as

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-ix\cdot\xi}f(x)dx, \quad \xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3.$$

- The inverse Fourier formula is

$$\mathcal{F}^{-1}f(x) = \check{f}(x) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{ix\cdot\xi}f(\xi)d\xi, \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3.$$

- Convolution is defined as

$$(f*g)(x) = \int_{\mathbb{R}^3} f(x-y)g(y)dy.$$

- There exists $\chi(\xi) \in \mathcal{S}(\mathbb{R}^3)$ such that $0 \le \chi(\xi) \le 1$ and

$$\chi(\xi) = \begin{cases} 1 & |\xi| \le 3/4, \\ 0 & |\xi| > 4/3. \end{cases}$$

Let $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$, therefore $\operatorname{supp}\varphi(\xi) \subset \{\xi | \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Let $\chi_j(\xi) = \chi(2^{-j}\xi)$, $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, $j \in \mathbb{Z}$. There are non-homogeneous unit decompositions and homogeneous unit decompositions

$$\chi(\xi) + \sum_{j\ge 0}\varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^3,$$

$$\sum_{j\in\mathbb{Z}}\varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^3\backslash\{0\}.$$

For any $u \in L^2(\mathbb{R}^3)$, the following non-homogeneous and homogeneous decompositions hold

$$u = \mathcal{F}^{-1}\left(\chi \mathcal{F} u\right) + \sum_{j \geq 0} \mathcal{F}^{-1}\left(\varphi_j \mathcal{F} u\right) = \mathcal{F}^{-1}\chi * u + \sum_{j \geq 0} \mathcal{F}^{-1}\varphi_j * u,$$

$$u = \sum_{j \in \mathbb{Z}} \mathcal{F}^{-1}\left(\varphi_j \mathcal{F} u\right) = \sum_{j \in \mathbb{Z}} \mathcal{F}^{-1}\varphi_j * u.$$

For a homogeneous decomposition, let $\mathcal{F}^{-1}\left(\varphi_j \mathcal{F} u\right) = \mathcal{F}^{-1}\varphi_j * u = \dot{\Delta}_j u$. In addition, the low-frequency cutoff operator is defined as

$$\dot{S}_j u = \sum_{k \leq j-1} \dot{\Delta}_k u.$$

Based on the definitions of $\dot{\Delta}_j$ and $\dot{S}_j$, it follows that

$$\dot{\Delta}_j \dot{\Delta}_k u = 0, \quad |j-k| \geq 2 \quad \text{and} \quad \dot{\Delta}_j(\dot{S}_{k-1} u \dot{\Delta}_k u) = 0, \quad |j-k| \geq 5.$$

For more details about the Littlewood-Paley theory, please refer to [24, 3]. Next, we introduce Bony's paraproduct decomposition formula.

Definition 2.1 ([4])**.** If $u, v \in \mathcal{S}'$, Bony's paraproduct formula reads

$$uv = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v + \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} v \dot{\Delta}_j u + \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \widetilde{\dot{\Delta}}_j v,$$

where

$$\widetilde{\dot{\Delta}}_j v = \sum_{|j-j'| \leq 1} \dot{\Delta}_{j'} v.$$

The Morrey space $M^{p,\lambda}(\mathbb{R}^3)$ is a generalization of $L^p(\mathbb{R}^3)$ space.

Definition 2.2 ([18, 31])**.** Let $1 \leq p < \infty$ and $0 \leq \lambda < 3$, the Morrey space is defined as

$$M^{p,\lambda}(\mathbb{R}^3) = \Big\{ f \in L^p_{loc}(\mathbb{R}^3) : \|f\|_{M^{p,\lambda}(\mathbb{R}^3)} = \sup_{B(x,r) \subset \mathbb{R}^3} \Big(\frac{1}{|B(x,r)|^{\lambda/3}} \int_{B(x,r)} |f(y)|^p dy \Big)^{1/p} < \infty \Big\},$$

where $B(x,r)$ denotes the ball in $\mathbb{R}^3$ centered at x with radius r. If $\lambda = 0$, $M^{p,\lambda}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$. If $\lambda = 3$, $M^{p,\lambda}(\mathbb{R}^3) = L^\infty(\mathbb{R}^3)$. If $\lambda < 0$ or $\lambda > 3$, $M^{p,\lambda}(\mathbb{R}^3) = \Theta$, Θ is the set of measurable functions on $\mathbb{R}^3$ that are almost everywhere equal to 0.

Lemma 2.3 (Hölder inequality [18, 31])**.** *Let* $1 \leq p_1, p_2, p_3 < \infty$ *and* $0 \leq \lambda_1, \lambda_2, \lambda_3 < 3$. *If* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, *then*

$$\|fg\|_{M^{p_3,\lambda_3}(\mathbb{R}^3)} \leq \|f\|_{M^{p_2,\lambda_2}(\mathbb{R}^3)} \|g\|_{M^{p_1,\lambda_1}(\mathbb{R}^3)}.$$

Lemma 2.4 (Young inequality [18, 31])**.** *Let* $1 \leq p < \infty$, $0 \leq \lambda < 3$, *then*

$$\|f * g\|_{M^{p,\lambda}(\mathbb{R}^3)} \leq \|f\|_{L^1(\mathbb{R}^3)} \|g\|_{M^{p,\lambda}(\mathbb{R}^3)}$$

for all $f \in L^1(\mathbb{R}^3)$ *and* $g \in M^{p,\lambda}(\mathbb{R}^3)$.

Lemma 2.5 (Bernstein inequality [12])**.** *Let* $1 \leq q \leq p < \infty$, $0 \leq \lambda_1, \lambda_2 < 3$, $\frac{3-\lambda_1}{p} \leq \frac{3-\lambda_2}{q}$ *and* γ *be a multi-index. If* $\operatorname{supp} \widehat{f} \subset \{\xi : |\xi| \leq A 2^j\}$ *and* $A > 0$ *is a constant, then there exists a constant* $C > 0$ *such that*

$$\|(i\xi)^\gamma \widehat{f}\|_{M^{q,\lambda_2}} \leq C 2^{j|\gamma| + j(\frac{3-\lambda_2}{q} - \frac{3-\lambda_1}{p})} \|\widehat{f}\|_{M^{p,\lambda_1}}.$$

Definition 2.6. Let $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. The homogeneous Fourier-Triebel-Lizorkin space is

$$\dot{\widehat{F}}^s_{p,q}(\mathbb{R}^3) = \{u \in \mathcal{S}' \backslash \mathcal{P} : \|u\|_{\dot{\widehat{F}}^s_{p,q}(\mathbb{R}^3)} < \infty\}$$

is a Banach space with the norm

$$\|u\|_{\widehat{\dot{F}}^{s}_{p,q}(\mathbb{R}^3)} = \begin{cases} \|\Big(\sum_{j\in\mathbb{Z}} 2^{jqs}|\widehat{\dot{\Delta}_j u}|^q\Big)^{1/q}\|_{L^p(\mathbb{R}^3)} & 1 \le q < \infty, \\ \|\sup_{j\in\mathbb{Z}} 2^{js}|\widehat{\dot{\Delta}_j u}|\|_{L^p(\mathbb{R}^3)} & q = \infty, \end{cases}$$

where $\mathcal{P}$ is the set of polynomials.

Definition 2.7. Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q \le \infty$, $0 \le \lambda < 3$. The homogeneous Triebel-Lizorkin-Morrey space

$$\dot{F}^{s}_{p,q,\lambda}(\mathbb{R}^3) = \{u \in \mathcal{S}'\backslash\mathcal{P} : \|u\|_{\dot{F}^{s}_{p,q,\lambda}(\mathbb{R}^3)} < \infty\}$$

is a Banach space with the norm

$$\|u\|_{\dot{F}^{s}_{p,q,\lambda}(\mathbb{R}^3)} = \begin{cases} \|\Big(\sum_{j\in\mathbb{Z}} 2^{jqs}|\dot{\Delta}_j u|^q\Big)^{1/q}\|_{M^{p,\lambda}(\mathbb{R}^3)} & 1 \le q < \infty, \\ \|\sup_{j\in\mathbb{Z}} 2^{js}|\dot{\Delta}_j u|\|_{M^{p,\lambda}(\mathbb{R}^3)} & q = \infty, \end{cases}$$

where $\mathcal{P}$ is the set of polynomials.

Definition 2.8. Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le q \le \infty$, $0 \le \lambda < 3$. The homogeneous Fourier-Triebel-Lizorkin-Morrey space

$$\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^3) = \{u \in \mathcal{S}'\backslash\mathcal{P} : \|u\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^3)} < \infty\}$$

is a Banach space with the norm

$$\|u\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^3)} = \begin{cases} \|\Big(\sum_{j\in\mathbb{Z}} 2^{jqs}|\widehat{\dot{\Delta}_j u}|^q\Big)^{1/q}\|_{M^{p,\lambda}(\mathbb{R}^3)} & 1 \le q < \infty, \\ \|\sup_{j\in\mathbb{Z}} 2^{js}|\widehat{\dot{\Delta}_j u}|\|_{M^{p,\lambda}(\mathbb{R}^3)} & q = \infty, \end{cases}.$$

where $\mathcal{P}$ is the set of polynomials.

Remark 2.9. If $s = -1$, $p = 1$, $q = 1$, $\lambda = 0$, $\widehat{\dot{F}}^{-1}_{1,1,0}(\mathbb{R}^3) = \chi^{-1}(\mathbb{R}^3)$, where $\chi^{-1}(\mathbb{R}^3)$ is Lei-Lin space [20].

Remark 2.10. If $s = \beta$, $q = \infty$, $\lambda = 3$, $\widehat{\dot{F}}^{\beta}_{p,\infty,3}(\mathbb{R}^3) = \mathcal{PM}^{\beta}(\mathbb{R}^3)$, where $\mathcal{PM}^{\beta}(\mathbb{R}^3)$ is critical pseudomeasure space [25].

Lemma 2.11 (Interpolation Inequalities)**.** *Let* $s_1, s_2 \in \mathbb{R}$, $1 \le p_1, p_2 < \infty$, $1 \le q_1, q_2 \le \infty$, $0 \le \lambda_1, \lambda_2 < 3$. *If* $0 \le \theta \le 1$, $s = (1-\theta)s_1 + \theta s_2$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$, $1 \le p < \infty$, $1 \le q \le \infty$, $\frac{\lambda}{p} = \frac{\lambda_1(1-\theta)}{p_1} + \frac{\lambda_2\theta}{p_2}$, *then*

$$\|u\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^3)} \le \|u\|^{1-\theta}_{\widehat{\dot{F}}^{s_1}_{p_1,q_1,\lambda_1}(\mathbb{R}^3)} \|u\|^{\theta}_{\widehat{\dot{F}}^{s_2}_{p_2,q_2,\lambda_2}(\mathbb{R}^3)}.$$

Proof.

$$\begin{aligned} \|u\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^3)} &= \big\|\|2^{j(1-\theta)s_1}|\widehat{\dot{\Delta}_j u}|^{1-\theta}2^{j\theta s_2}|\widehat{\dot{\Delta}_j u}|^{\theta}\|_{l^q}\big\|_{M^{p,\lambda}} \\ &\le \big\|\|2^{js_1}|\widehat{\dot{\Delta}_j u}|\|^{1-\theta}_{l^{q_1}}\|2^{js_2}|\widehat{\dot{\Delta}_j u}|\|^{\theta}_{l^{q_2}}\big\|_{M^{p,\lambda}} \\ &= \sup_{B(x,r)} r^{-\lambda/p}\big\|\|2^{js_1}|\widehat{\dot{\Delta}_j u}|\|^{1-\theta}_{l^{q_1}}\|2^{js_2}|\widehat{\dot{\Delta}_j u}|\|^{\theta}_{l^{q_2}}\big\|_{L^p(B(x,r))} \\ &\le \sup_{B(x,r)} r^{-\frac{\lambda_1(1-\theta)}{p_1}}\big\|\|2^{js_1}|\widehat{\dot{\Delta}_j u}|\|_{l^{q_1}}\big\|^{1-\theta}_{L^{p_1}(B(x,r))} r^{-\frac{\lambda_2\theta}{p_2}}\big\|\|2^{js_2}|\widehat{\dot{\Delta}_j u}|\|_{l^{q_2}}\big\|^{\theta}_{L^{p_2}(B(x,r))} \\ &\le \|u\|^{1-\theta}_{\widehat{\dot{F}}^{s_1}_{p_1,q_1,\lambda_1}(\mathbb{R}^3)}\|u\|^{\theta}_{\widehat{\dot{F}}^{s_2}_{p_2,q_2,\lambda_2}(\mathbb{R}^3)}. \end{aligned}$$

□

Lemma 2.12. *Operator* $\partial_x^\beta : \widehat{\dot{F}}^{s+|\beta|}_{p,q,\lambda}(\mathbb{R}^3) \to \widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^3)$ *is a bounded operator and*

$$\begin{aligned}
\|\Lambda^\alpha f\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^n)} &\lesssim \|f\|_{\widehat{\dot{F}}^{s+\alpha}_{p,q,\lambda}(\mathbb{R}^3)},\\
\|\nabla\cdot f\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^n)} &\lesssim \|f\|_{\widehat{\dot{F}}^{s+1}_{p,q,\lambda}(\mathbb{R}^3)},\\
\|\Delta f\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^n)} &\lesssim \|f\|_{\widehat{\dot{F}}^{s+2}_{p,q,\lambda}(\mathbb{R}^3)}.
\end{aligned}$$

Proof. If for all $j \in \mathbb{Z}$, $\operatorname{supp}\varphi_j(\xi) \subset \big\{\xi | 2^j \frac{3}{4} \le |\xi| \le 2^j \frac{8}{3}\big\}$, then

$$\begin{aligned}
\|\partial_x^\beta f\|_{\widehat{\dot{F}}^{s}_{p,q,\lambda}(\mathbb{R}^3)} &= \Big\|\big\|2^{js}|\varphi_j \widehat{\partial_x^\beta f}|\big\|_{l^q(j\in\mathbb{Z})}\Big\|_{M^{p,\lambda}(\mathbb{R}^3)}\\
&\le \Big\|\big\|2^{js}|\varphi_j |\xi|^{|\beta|}\widehat{f}|\big\|_{l^q(j\in\mathbb{Z})}\Big\|_{M^{p,\lambda}(\mathbb{R}^3)}\\
&\lesssim \Big\|\big\|2^{js}2^{j|\beta|}|\varphi_j \widehat{f}|\big\|_{l^q(j\in\mathbb{Z})}\Big\|_{M^{p,\lambda}(\mathbb{R}^3)}\\
&\lesssim \|f\|_{\widehat{\dot{F}}^{s+|\beta|}_{p,q,\lambda}(\mathbb{R}^3)}.
\end{aligned}$$

□

Definition 2.13. Let $s \in \mathbb{R}$, $1 \le p < \infty$, $1 \le \rho, q \le \infty$, $0 \le \lambda < 3$, and $I = [0,T)$, $T \in (0,\infty]$. The Chemin-Lerner type Fourier-Triebel-Lizorkin-Morrey space

$$\mathcal{L}^\rho(I, \widehat{\dot{F}}^{s}_{p,q,\lambda}) = \{u \in \mathcal{S}'\backslash\mathcal{P} : \|u\|_{\mathcal{L}^\rho(I,\widehat{\dot{F}}^{s}_{p,q,\lambda})} < \infty\}$$

is a Banach space with the norm

$$\|u\|_{\mathcal{L}^\rho(I,\widehat{\dot{F}}^{s}_{p,q,\lambda})} = \begin{cases} \Big\|\Big(\sum_{j\in\mathbb{Z}} 2^{jqs}\|\widehat{\dot{\Delta}_j u}\|^q_{L^\rho(I)}\Big)^{1/q}\Big\|_{M^{p,\lambda}(\mathbb{R}^3)} & 1 \le q < \infty,\\ \big\|\sup_{j\in\mathbb{Z}} 2^{js}\|\widehat{\dot{\Delta}_j u}\|_{L^\rho(I)}\big\|_{M^{p,\lambda}(\mathbb{R}^3)} & q = \infty, \end{cases}$$

where $\mathcal{P}$ is the set of polynomials.

Lemma 2.14 (Banach-Picard contraction principle [21])**.** *X is a Banach space, $\|\cdot\|_X$ denotes its norm. $B : X \times X \mapsto X$ is a bounded bilinear operator satisfying*

$$\|B(u,v)\|_X \le \eta\|u\|_X\|v\|_X$$

for all $u, v \in X$ and a constant $\eta > 0$. If $y \in X$ satisfy $\|y\|_X < \varepsilon < \frac{1}{4\eta}$, the equation $x = y + B(x,x)$ has a unique solution satisfying $\|x\| \le 2\varepsilon$. Furthermore, the solution depends continuously on y. In other words, if $\|y'\|_X < \varepsilon$, x' be a solution of $x' = y' + B(x',x')$, then

$$\|x - x'\|_X \le \frac{1}{1-4\varepsilon\eta}\|y - y'\|_X.$$

Lemma 2.15 ([33])**.** *If $0 < s \le t < \infty$ and $0 \le \alpha \le 1$, then*

$$t|x|^\alpha - \frac{1}{2}(t^2 - s^2)|x|^{2\alpha} - s|x-y|^\alpha - s|y|^\alpha \le \frac{1}{2}$$

holds for $\forall x, y \in \mathbb{R}^3$.

3. LINEAR AND BILINEAR ESTIMATES IN THE FOURIER-TRIEBEL-LIZORKIN-MORREY SPACES

Lemma 3.1. *If $1 \le p < \infty$, $1 \le q \le \infty$, $0 \le \lambda < 3$, $s \in \mathbb{R}$, $1 \le r \le \infty$ and $\frac{1}{2} < \alpha \le 1$. Let $I = [0,T)$, $0 < T \le \infty$, then*

$$\|(e^{-t(-\Delta)^\alpha}u_0, e^{-t(-\Delta)^\alpha}b_0)\|_{\mathcal{L}^r(I,\widehat{\dot{F}}^{s}_{p,q,\lambda})} \lesssim \|(u_0,b_0)\|_{\widehat{\dot{F}}^{s-\frac{2\alpha}{r}}_{p,q,\lambda}}. \tag{3.1}$$

Proof. Since $\operatorname{supp}\varphi_j(\xi) \subset \{\xi | 2^j \frac{3}{4} \le |\xi| \le 2^j \frac{8}{3}\}$, then

$$
\begin{aligned}
&\|e^{-t(-\Delta)^\alpha} u_0\|_{\mathcal{L}^r(I,\widehat{\dot{F}}^{s}_{p,q,\lambda})} + \|e^{-t(-\Delta)^\alpha} b_0\|_{\mathcal{L}^r(I,\widehat{\dot{F}}^{s}_{p,q,\lambda})} \\
&\lesssim \big\| \|2^{js} \|e^{-t2^{2\alpha j}} \widehat{\dot{\Delta}_j u_0}\|_{L^r(I)}\|_{l^q} \big\|_{M^{p,\lambda}} + \big\| \|2^{js} \|e^{-t2^{2\alpha j}} \widehat{\dot{\Delta}_j b_0}\|_{L^r(I)}\|_{l^q} \big\|_{M^{p,\lambda}} \\
&\lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} |\widehat{\dot{\Delta}_j u_0}|^q \Big(\int_0^T e^{-tr2^{2\alpha j}} dt \Big)^{q/r} \Big)^{1/q} \Big\|_{M^{p,\lambda}} \\
&\quad + \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} |\widehat{\dot{\Delta}_j b_0}|^q \Big(\int_0^T e^{-tr2^{2\alpha j}} dt \Big)^{q/r} \Big)^{1/q} \Big\|_{M^{p,\lambda}} \\
&\lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jq(s-\frac{2\alpha}{r})} |\widehat{\dot{\Delta}_j u_0}|^q \Big)^{1/q} \Big\|_{M^{p,\lambda}} + \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jq(s-\frac{2\alpha}{r})} |\widehat{\dot{\Delta}_j b_0}|^q \Big)^{1/q} \Big\|_{M^{p,\lambda}} \\
&\lesssim \|(u_0, b_0)\|_{\widehat{\dot{F}}^{s-\frac{2\alpha}{r}}_{p,q,\lambda}}.
\end{aligned}
$$

□

Lemma 3.2. *If* $1 \le p < \infty$, $1 \le q \le \infty$, $0 \le \lambda < 3$, $s \in \mathbb{R}$, $1 \le r \le \infty$, $\frac{1}{2} < \alpha \le 1$ *and* $1 \le r_1 \le r \le \infty$. *Let* $I = [0,T)$, $0 < T \le \infty$, *then*

$$
\Big\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} f(\tau) d\tau \Big\|_{\mathcal{L}^r(I,\widehat{\dot{F}}^{s}_{p,q,\lambda})} \lesssim \|f\|_{\mathcal{L}^{r_1}\big(I,\widehat{\dot{F}}^{s-2\alpha(1+\frac{1}{r}-\frac{1}{r_1})}_{p,q,\lambda}\big)}. \tag{3.2}
$$

Proof. Utilizing $1 + \frac{1}{r} = \frac{1}{\theta} + \frac{1}{r_1}$ and Young's inequality,

$$
\begin{aligned}
\Big\| \int_0^t e^{-(t-\tau)(-\Delta)^\alpha} f(\tau) d\tau \Big\|_{\mathcal{L}^r(I,\widehat{\dot{F}}^{s}_{p,q,\lambda})} &= \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \Big\| \int_0^t e^{-|\xi|^{2\alpha}(t-\tau)} \widehat{\dot{\Delta}_j f} d\tau \Big\|^q_{L^r(I)} \Big)^{1/q} \Big\|_{M^{p,\lambda}} \\
&\lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|e^{-2^{2\alpha j}t} * \widehat{\dot{\Delta}_j f}\|^q_{L^r(I)} \Big)^{1/q} \Big\|_{M^{p,\lambda}} \\
&\lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|e^{-2^{2\alpha j}t}\|^q_{L^\theta(I)} \|\widehat{\dot{\Delta}_j f}\|^q_{L^{r_1}(I)} \Big)^{1/q} \Big\|_{M^{p,\lambda}} \\
&\lesssim \Big\| \Big(\sum_{j\in\mathbb{Z}} 2^{jq(s-\frac{2\alpha}{\theta})} \|\widehat{\dot{\Delta}_j f}\|^q_{L^{r_1}(I)} \Big)^{1/q} \Big\|_{M^{p,\lambda}} \\
&\lesssim \|f\|_{\mathcal{L}^{r_1}\big(I,\widehat{\dot{F}}^{s-2\alpha(1+\frac{1}{r}-\frac{1}{r_1})}_{p,q,\lambda}\big)}.
\end{aligned}
$$

□

Lemma 3.3. *If* $1 \le p < \infty$, $1 \le q \le \infty$, $\max\{3-(5-2\alpha)p, 0\} < \lambda < 3$, $s \in \mathbb{R}$, $\frac{2\alpha}{2\alpha-1} < \rho \le \infty$, $\frac{1}{2} < \alpha \le 1$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. *Let* $I = [0,T)$, $0 < T \le \infty$, *there exists a constant* C *such that*

$$
\begin{aligned}
\|uv\|_{\mathcal{L}^1\big(I,\widehat{\dot{F}}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\big)} &\le C\|u\|_{\mathcal{L}^\rho\big(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\big)} \|v\|_{\mathcal{L}^{\rho'}\big(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\big)} \\
&\quad + C\|v\|_{\mathcal{L}^\rho\big(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\big)} \|u\|_{\mathcal{L}^{\rho'}\big(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\big)}.
\end{aligned}
$$

Proof. By Bony's paraproduct decomposition formula,

$$
\begin{aligned}
\dot{\Delta}_j(uv) &= \sum_{|k-j|\le 4} \dot{\Delta}_j(\dot{S}_{k-1}u\dot{\Delta}_k v) + \sum_{|k-j|\le 4} \dot{\Delta}_j(\dot{S}_{k-1}v\dot{\Delta}_k u) + \sum_{k\ge j-3} \dot{\Delta}_j(\dot{\Delta}_k u\widetilde{\dot{\Delta}}_k v) \\
&= I_j + II_j + III_j.
\end{aligned}
$$

Then

$$
\|uv\|_{\mathcal{L}^1\big(I;\widehat{\dot{F}}^{5+\frac{\lambda-3}{p}-2\alpha}\big)} \le \Big\| \|2^{j(5+\frac{\lambda-3}{p}-2\alpha)} \|\widehat{I_j}\|_{L^1(I)}\|_{l^q} \Big\|_{M^{p,\lambda}} + \Big\| \|2^{j(5+\frac{\lambda-3}{p}-2\alpha)} \|\widehat{II_j}\|_{L^1(I)}\|_{l^q} \Big\|_{M^{p,\lambda}}
$$

$$+\Big\|\|2^{j(5+\frac{\lambda-3}{p}-2\alpha)}\|\widehat{III}_j\|_{L^1(I)}\|_{l^q}\Big\|_{M^{p,\lambda}}.$$

First we have the estimate

$$\begin{aligned}
\|\widehat{I}_j\|_{L^1(I)} &\le \sum_{|k-j|\le 4}\big\|\varphi_j\cdot\widehat{\dot{S}_{k-1}u\dot{\Delta}_k v}\big\|_{L^1(I)}\\
&\le \sum_{|k-j|\le 4}\sum_{l\le k-2}\big\|\varphi_j\cdot\widehat{\dot{\Delta}_l u\dot{\Delta}_k v}\big\|_{L^1(I)}\\
&\le \sum_{|k-j|\le 4}\sum_{l\le k-2}\big\|\widehat{\dot{\Delta}_l u}*\widehat{\dot{\Delta}_k v}\big\|_{L^1(I)}\\
&\le \sum_{|k-j|\le 4}\sum_{l\le k-2}\Big(\big\|\widehat{\dot{\Delta}_l u}\big\|_{L^\rho(I)}*\big\|\widehat{\dot{\Delta}_k v}\big\|_{L^{\rho'}(I)}\Big)\\
&\le \sum_{|k-j|\le 4}\sum_{l\le k-2}\int_{\mathbb{R}^3}\big\|\widehat{\dot{\Delta}_l u}(\xi-y,\cdot)\big\|_{L^\rho(I)}\big\|\widehat{\dot{\Delta}_k v}(y,\cdot)\big\|_{L^{\rho'}(I)}dy.
\end{aligned}\tag{3.3}$$

Using the generalized Minkowski inequality and Young's inequality, and assuming $\rho>\frac{2\alpha}{2\alpha-1}$, it follows from (3.3) that

$$\begin{aligned}
&\big\|2^{j(5+\frac{\lambda-3}{p}-2\alpha)}\|\widehat{I}_j\|_{L^1(I)}\big\|_{l^q}\\
&=\Big\|\int_{\mathbb{R}^3}\sum_{|k-j|\le 4}\sum_{l\le k-2}2^{j(5+\frac{\lambda-3}{p}-2\alpha)}\big\|\widehat{\dot{\Delta}_l u}(\xi-y,\cdot)\big\|_{L^\rho(I)}\big\|\widehat{\dot{\Delta}_k v}(y,\cdot)\big\|_{L^{\rho'}(I)}dy\Big\|_{l^q}\\
&\le \int_{\mathbb{R}^3}\Big\|\sum_{|k-j|\le 4}\sum_{l\le k-2}2^{j(5+\frac{\lambda-3}{p}-2\alpha)}\big\|\widehat{\dot{\Delta}_l u}(\xi-y,\cdot)\big\|_{L^\rho(I)}\big\|\widehat{\dot{\Delta}_k v}(y,\cdot)\big\|_{L^{\rho'}(I)}\Big\|_{l^q}dy\\
&\le \int_{\mathbb{R}^3}\Big\|\sum_{|k-j|\le 4}\sum_{l\le k-2}2^{j(5+\frac{\lambda-3}{p}-2\alpha)}2^{l(1-\frac{2\alpha}{\rho'})}2^{l(\frac{2\alpha}{\rho'}-1)}\big\|\widehat{\dot{\Delta}_l u}(\xi-y,\cdot)\big\|_{L^\rho(I)}\big\|\widehat{\dot{\Delta}_k v}(y,\cdot)\big\|_{L^{\rho'}(I)}\Big\|_{l^q}dy\\
&\lesssim \int_{\mathbb{R}^3}\Big\|\sum_{|k-j|\le 4}2^{(j-k)(5+\frac{\lambda-3}{p}-2\alpha)}2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot{\Delta}_k v}(y,\cdot)\big\|_{L^{\rho'}(I)}\Big\|_{l^q}\\
&\quad\times\Big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot{\Delta}_l u}(\xi-y,\cdot)\big\|_{L^\rho(I)}\Big\|_{l^\infty}dy\\
&\lesssim \int_{\mathbb{R}^3}\Big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot{\Delta}_k v}(y,\cdot)\big\|_{L^{\rho'}(I)}\Big\|_{l^q(k\in\mathbb{Z})}\Big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot{\Delta}_l u}(\xi-y,\cdot)\big\|_{L^\rho(I)}\Big\|_{l^\infty(l\in\mathbb{Z})}dy\\
&\lesssim \Big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot{\Delta}_k v}\big\|_{L^{\rho'}(I)}\Big\|_{l^q(k\in\mathbb{Z})}*\Big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot{\Delta}_l u}\big\|_{L^\rho(I)}\Big\|_{l^\infty(l\in\mathbb{Z})}.
\end{aligned}$$

By Lemma 2.4, Lemma 2.5 and the inclusion relation $l^q\subset l^\infty$, it follows that

$$\begin{aligned}
&\Big\|\big\|2^{j\left(5+\frac{\lambda-3}{p}-2\alpha\right)}\|\widehat{I}_j\|_{L^1(I)}\big\|_{l^q}\Big\|_{M^{p,\lambda}}\\
&\lesssim \Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot{\Delta}_k v}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}*\big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot{\Delta}_l u}\big\|_{L^\rho(I)}\big\|_{l^\infty(l\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\\
&\lesssim \Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot{\Delta}_k v}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\Big\|\big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot{\Delta}_l u}\big\|_{L^\rho(I)}\big\|_{l^\infty(l\in\mathbb{Z})}\Big\|_{L^1}\\
&\lesssim \Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot{\Delta}_k v}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\Big\|\Big\|2^{l(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot{\Delta}_l u}\big\|_{L^\rho(I)}\Big\|_{l^\infty(l\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\\
&\lesssim \Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot{\Delta}_k v}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\Big\|\big\|2^{l(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot{\Delta}_l u}\big\|_{L^\rho(I)}\big\|_{l^q(l\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\\
&\lesssim \|u\|_{\mathcal{L}^\rho\left(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\right)}\|v\|_{\mathcal{L}^{\rho'}\left(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\right)}.
\end{aligned}\tag{3.4}$$

Similarly,

$$\Big\|\big\|2^{j\left(5+\frac{\lambda-3}{p}-2\alpha\right)}\|\widehat{II_j}\|_{L^1(I)}\big\|_{l^q}\Big\|_{M^{p,\lambda}} \lesssim \|v\|_{\mathcal{L}^{\rho}\left(I,\hat{\dot F}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\right)}\|u\|_{\mathcal{L}^{\rho'}\left(I,\hat{\dot F}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\right)}. \tag{3.5}$$

And

$$\begin{aligned}
\|\widehat{III_j}\|_{L^1(I)} &\leq \sum_{k\geq j-3}\sum_{|k-l|\leq 1}\big\|\widehat{\dot\Delta_k u}*\widehat{\dot\Delta_l v}\big\|_{L^1(I)}\\
&\leq \sum_{k\geq j-3}\sum_{|k-l|\leq 1}\Big(\big\|\widehat{\dot\Delta_k u}\big\|_{L^{\rho'}(I)}*\big\|\widehat{\dot\Delta_l v}\big\|_{L^{\rho}(I)}\Big)\\
&\leq \sum_{k\geq j-3}\sum_{|k-l|\leq 1}\int_{\mathbb{R}^3}\big\|\widehat{\dot\Delta_k u}(\xi-y,\cdot)\big\|_{L^{\rho'}(I)}\big\|\widehat{\dot\Delta_l v}(y,\cdot)\big\|_{L^{\rho}(I)}dy.
\end{aligned} \tag{3.6}$$

Using the generalized Minkowski inequality, Young's inequality, and the conditions $\rho>\frac{2\alpha}{2\alpha-1}$ and $5+\frac{\lambda-3}{p}-2\alpha>0$, it follows that

$$\begin{aligned}
&\Big\|2^{j\left(5+\frac{\lambda-3}{p}-2\alpha\right)}\|\widehat{III_j}\|_{L^1(I)}\Big\|_{l^q}\\
&=\Big\|\int_{\mathbb{R}^3}\sum_{k\geq j-3}\sum_{|k-l|\leq 1}2^{j(5+\frac{\lambda-3}{p}-2\alpha)}\big\|\widehat{\dot\Delta_k u}(\xi-y,\cdot)\big\|_{L^{\rho'}(I)}\big\|\widehat{\dot\Delta_l v}(y,\cdot)\big\|_{L^{\rho}(I)}dy\Big\|_{l^q}\\
&\leq \int_{\mathbb{R}^3}\Big\|\sum_{k\geq j-3}\sum_{|k-l|\leq 1}2^{j(5+\frac{\lambda-3}{p}-2\alpha)}\big\|\widehat{\dot\Delta_k u}(\xi-y,\cdot)\big\|_{L^{\rho'}(I)}\big\|\widehat{\dot\Delta_l v}(y,\cdot)\big\|_{L^{\rho}(I)}\Big\|_{l^q}dy\\
&\leq \int_{\mathbb{R}^3}\Big\|\sum_{k\geq j-3}\sum_{|k-l|\leq 1}2^{j(5+\frac{\lambda-3}{p}-2\alpha)}2^{l(1-\frac{2\alpha}{\rho'})}2^{l(\frac{2\alpha}{\rho'}-1)}\big\|\widehat{\dot\Delta_k u}(\xi-y,\cdot)\big\|_{L^{\rho'}(I)}\big\|\widehat{\dot\Delta_l v}(y,\cdot)\big\|_{L^{\rho}(I)}\Big\|_{l^q}dy\\
&\leq \int_{\mathbb{R}^3}\Big\|\sum_{k\geq j-3}2^{(j-k)(5+\frac{\lambda-3}{p}-2\alpha)}2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot\Delta_k u}(\xi-y,\cdot)\big\|_{L^{\rho'}(I)}\Big\|_{l^q}\\
&\quad\times\Big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot\Delta_l v}(y,\cdot)\big\|_{L^{\rho}(I)}\Big\|_{l^\infty}dy\\
&\leq \int_{\mathbb{R}^3}\Big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot\Delta_k u}(\xi-y,\cdot)\big\|_{L^{\rho'}(I)}\Big\|_{l^q(k\in\mathbb{Z})}\Big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot\Delta_l v}(y,\cdot)\big\|_{L^{\rho}(I)}\Big\|_{l^\infty(l\in\mathbb{Z})}dy\\
&\leq \Big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot\Delta_k u}\big\|_{L^{\rho'}(I)}\Big\|_{l^q(k\in\mathbb{Z})}*\Big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot\Delta_l v}\big\|_{L^{\rho}(I)}\Big\|_{l^\infty(l\in\mathbb{Z})}.
\end{aligned}$$

By Lemma 2.4, Lemma 2.5 and the inclusion relation $l^q\subset l^\infty$, it follows that

$$\begin{aligned}
&\Big\|\big\|2^{j\left(5+\frac{\lambda-3}{p}-2\alpha\right)}\|\widehat{III_j}\|_{L^1(I)}\big\|_{l^q}\Big\|_{M^{p,\lambda}}\\
&\lesssim\Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot\Delta_k u}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}*\big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot\Delta_l v}\big\|_{L^{\rho}(I)}\big\|_{l^\infty(l\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\\
&\lesssim\Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot\Delta_k u}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\Big\|\big\|2^{l(1-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot\Delta_l v}\big\|_{L^{\rho}(I)}\big\|_{l^\infty(l\in\mathbb{Z})}\Big\|_{L^1}\\
&\lesssim\Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot\Delta_k u}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\Big\|\big\|2^{l(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot\Delta_l v}\big\|_{L^{\rho}(I)}\big\|_{l^\infty(l\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\\
&\lesssim\Big\|\big\|2^{k(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho})}\big\|\widehat{\dot\Delta_k u}\big\|_{L^{\rho'}(I)}\big\|_{l^q(k\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\Big\|\big\|2^{l(4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'})}\big\|\widehat{\dot\Delta_l v}\big\|_{L^{\rho}(I)}\big\|_{l^q(l\in\mathbb{Z})}\Big\|_{M^{p,\lambda}}\\
&\lesssim\|v\|_{\mathcal{L}^{\rho}\left(I,\hat{\dot F}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\right)}\|u\|_{\mathcal{L}^{\rho'}\left(I,\hat{\dot F}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\right)}.
\end{aligned} \tag{3.7}$$

Combining (3.4), (3.5), (3.7), the proof is complete. □

Lemma 3.4. *If* $1\leq p<\infty$, $1\leq q\leq\infty$, $\max\{3-(5-4\alpha+\frac{2\alpha}{\rho})p,0\}<\lambda<3$, $s\in\mathbb{R}$, $1\leq\rho\leq\infty$, $\frac{1}{2}<\alpha\leq 1$, $\frac{1}{\rho}+\frac{1}{\rho'}=1$. *Let* $I=[0,T)$, $0<T\leq\infty$, *then*

$$\|uv\|_{\mathcal{L}^{\rho}\left(I,\hat{\dot F}^{5+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho}}_{p,q,\lambda}\right)}\leq C\|u\|_{\mathcal{L}^{\rho}\left(I,\hat{\dot F}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\right)}\|v\|_{\mathcal{L}^{\infty}\left(I,\hat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)}$$

$$+ C\|v\|_{\mathcal{L}^{\rho}\left(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\right)}\|u\|_{\mathcal{L}^{\infty}\left(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)}.$$

Using the same proof method as above, we obtain the proof of this lemma.

4. Proof of theorem 1.1

In this section, we use Lemma 2.14, the Banach-Picard contraction principle.

Proof. From the mild solution (1.5) of the Cauchy problem (1.1)-1.2, it follows that

$$(u,b) = (e^{-t(-\Delta)^{\alpha}}u_0, e^{-t(-\Delta)^{\alpha}}b_0) + B\left((u,b),(u,b)\right), \tag{4.1}$$

$$B\left((u,b),(u,b)\right) = \left(B_1\left((u,b),(u,b)\right), B_2\left((u,b),(u,b)\right)\right), \tag{4.2}$$

where

$$B_1\left((u,b),(u,b)\right) = -\int_0^t e^{-(t-\tau)(-\Delta)^{\alpha}}\mathbb{P}\operatorname{div}(u\otimes u - b\otimes b)(\tau)d\tau, \tag{4.3}$$

$$B_2\left((u,b),(u,b)\right) = -\int_0^t e^{-(t-\tau)(-\Delta)^{\alpha}}\operatorname{div}(u\otimes b - b\otimes u)(\tau)d\tau. \tag{4.4}$$

First, we need to obtain the linear estimate for $(e^{-t(-\Delta)^{\alpha}}u_0, e^{-t(-\Delta)^{\alpha}}b_0)$ in the work space

$$X = \mathcal{L}^{\rho}\Big([0,\infty), \widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\Big) \cap \mathcal{L}^{\rho'}\Big([0,\infty), \widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}}_{p,q,\lambda}\Big).$$

Let $s = 4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}$, $r=\rho$ and $s = 4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}$, $r=\rho'$ in Lemma 3.1, respectively. Then

$$\|(e^{-t(-\Delta)^{\alpha}}u_0, e^{-t(-\Delta)^{\alpha}}b_0)\|_X \le C_1\|(u_0,b_0)\|_{\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}}. \tag{4.5}$$

Second, we need to obtain a bilinear estimate for (4.2). Let $s = 4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}$, $r=\rho$, $r_1=1$ and $s = 4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho}$, $r=\rho'$, $r_1=1$ in Lemma 3.2, respectively. Using Lemmas 2.12 and 3.3. Based on the boundedness of $\mathbb{P}$ in $\mathcal{L}^1\Big(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)$, we have

$$\begin{aligned}
&\|B\left((u,b),(u,b)\right)\|_X \\
&= \|B_1\left((u,b),(u,b)\right)\|_X + \|B_2\left((u,b),(u,b)\right)\|_X \\
&\lesssim \|\mathbb{P}\operatorname{div}(u\otimes u - b\otimes b)\|_{\mathcal{L}^1\left(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)} + \|\operatorname{div}(u\otimes b - b\otimes u)\|_{\mathcal{L}^1\left(I,\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)} \\
&\lesssim \|u\otimes u\|_{\mathcal{L}^1\left(I,\widehat{\dot{F}}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)} + \|b\otimes b\|_{\mathcal{L}^1\left(I,\widehat{\dot{F}}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)} \\
&+ \|u\otimes b\|_{\mathcal{L}^1\left(I,\widehat{\dot{F}}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)} + \|b\otimes u\|_{\mathcal{L}^1\left(I,\widehat{\dot{F}}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\right)} \\
&\le C_2\left(\|u\|_X^2 + \|b\|_X^2 + 2\|u\|_X\|b\|_X\right) \\
&= C_2\|(u,b)\|_X^2.
\end{aligned} \tag{4.6}$$

If $\|(u_0,b_0)\|_{\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}} \le \frac{1}{4C_1C_2}$, it follows from (4.5) that

$$\|(e^{-t(-\Delta)^{\alpha}}u_0, e^{-t(-\Delta)^{\alpha}}b_0)\|_X \le C_1\|(u_0,b_0)\|_{\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}} \le \frac{1}{4C_2}. \tag{4.7}$$

Combining (4.6), (4.7) and Lemma 2.14, (1.1) has a unique global mild solution which satisfies

$$\|(u,b)\|_X \le 2C_1\|(u_0,b_0)\|_{\widehat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}}. \tag{4.8}$$

To prove $(u,b)\in C\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)$. Let us first prove that $(u,b)\in \mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)$. It follows from 4.8 that

$$\begin{aligned}\|(u,b)\|_{\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)} &\le \|(e^{-t(-\Delta)^\alpha}u_0,e^{-t(-\Delta)^\alpha}b_0)\|_{\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\\ &\quad+\|B\left((u,b),(u,b)\right)\|_{\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}.\end{aligned}\tag{4.9}$$

Let $r=\infty$, $s=4+\frac{\lambda-3}{p}-2\alpha$ in Lemma 3.1, then

$$\|(e^{-t(-\Delta)^\alpha}u_0,e^{-t(-\Delta)^\alpha}b_0)\|_{\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\lesssim\|(u_0,b_0)\|_{\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}}.\tag{4.10}$$

Let $r=\infty$, $s=4+\frac{\lambda-3}{p}-2\alpha$, $r_1=1$ in Lemma 3.2, applying Lemma 2.12, Lemma 3.3 and the boundedness of $\mathbb{P}$ in $\mathcal{L}^1\Big(I,\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)$, we have

$$\begin{aligned}&\|B\left((u,b),(u,b)\right)\|_{\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\\ &\lesssim\|\mathbb{P}\operatorname{div}(u\otimes u-b\otimes b)\|_{\mathcal{L}^1\Big(I,\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}+\|\operatorname{div}(u\otimes b-b\otimes u)\|_{\mathcal{L}^1\Big(I,\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\\ &\lesssim\|u\otimes u\|_{\mathcal{L}^1\Big(I,\widehat{\dot F}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}+\|b\otimes b\|_{\mathcal{L}^1\Big(I,\widehat{\dot F}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\\ &\quad+\|u\otimes b\|_{\mathcal{L}^1\Big(I,\widehat{\dot F}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}+\|b\otimes u\|_{\mathcal{L}^1\Big(I,\widehat{\dot F}^{5+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\\ &\lesssim\left(\|u\|_X^2+\|b\|_X^2+2\|u\|_X\|b\|_X\right)\\ &\lesssim\|(u,b)\|_X^2.\end{aligned}\tag{4.11}$$

Combining estimates of 4.9, 4.10 and 4.11,

$$\|(u,b)\|_{\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\lesssim\|(u_0,b_0)\|_{\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}}+\|(u,b)\|_X^2,\tag{4.12}$$

that is

$$(u,b)\in\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big).\tag{4.13}$$

Second, we prove the continuity of u with respect to time t, the continuity of b follows similarly. For $0\le t_1<t_2<\infty$,

$$\begin{aligned}\|u(t_2)-u(t_1)\|_{\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}}&\le\Big\|\Big(\sum_{j\le N}2^{jq(4+\frac{\lambda-3}{p}-2\alpha)}\big|\widehat{\dot\Delta_j u}(t_2)-\widehat{\dot\Delta_j u}(t_1)\big|^q\Big)^{1/q}\Big\|_{M^{p,\lambda}}\\ &\quad+\Big\|\Big(\sum_{j>N}2^{jq(4+\frac{\lambda-3}{p}-2\alpha)}\big|\widehat{\dot\Delta_j u}(t_2)-\widehat{\dot\Delta_j u}(t_1)\big|^q\Big)^{1/q}\Big\|_{M^{p,\lambda}}.\end{aligned}$$

Since $u\in\mathcal{L}^\infty\Big([0,\infty),\widehat{\dot F}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)$, for all $\varepsilon>0$, there exists N such that

$$\Big\|\Big(\sum_{j>N}2^{jq(4+\frac{\lambda-3}{p}-2\alpha)}\big|\widehat{\dot\Delta_j u}(t_2)-\widehat{\dot\Delta_j u}(t_1)\big|^q\Big)^{1/q}\Big\|_{M^{p,\lambda}}<\frac{\varepsilon}{2}.\tag{4.14}$$

For $\|\big(\sum_{j\le N}2^{jq(4+\frac{\lambda-3}{p}-2\alpha)}|\widehat{\dot\Delta_j u}(t_2)-\widehat{\dot\Delta_j u}(t_1)|^q\big)^{1/q}\|_{M^{p,\lambda}}$, using Hölder's inequality and Lemma 2.12, and Lemma 3.4, we have

$$\begin{aligned}&\Big\|\Big(\sum_{j\le N}2^{jq(4+\frac{\lambda-3}{p}-2\alpha)}\big|\widehat{\dot\Delta_j u}(t_2)-\widehat{\dot\Delta_j u}(t_1)\big|^q\Big)^{1/q}\Big\|_{M^{p,\lambda}}\\ &\lesssim|t_2-t_1|^{1/\rho'}\Big\|\Big(\sum_{j\le N}2^{jq(4+\frac{\lambda-3}{p}-2\alpha)}\|\partial_t\widehat{\dot\Delta_j u}\|^q_{L^\rho[0,\infty)}\Big)^{1/q}\Big\|_{M^{p,\lambda}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim |t_2-t_1|^{1/\rho'}\Big\|\Big(\sum_{j\le N}2^{jq(4+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho})}2^{j(2\alpha-\frac{2\alpha}{\rho})q}\big\|\partial_t\widehat{\dot{\Delta}_j u}\big\|^q_{L^\rho[0,\infty)}\Big)^{1/q}\Big\|_{M^{p,\lambda}}\\
&\lesssim |t_2-t_1|^{1/\rho'}2^{N(2\alpha-\frac{2\alpha}{\rho})}\Big\|\Big(\sum_{j\le N}2^{jq(4+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho})}\big\|\partial_t\widehat{\dot{\Delta}_j u}\big\|^q_{L^\rho[0,\infty)}\Big)^{1/q}\Big\|_{M^{p,\lambda}}\\
&\lesssim |t_2-t_1|^{1/\rho'}\,\|\partial_t u\|_{\mathcal{L}^\rho\Big([0,\infty),\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho}}_{p,q,\lambda}\Big)}\\
&\lesssim |t_2-t_1|^{1/\rho'}\Big(\|(-\Delta)^\alpha u\|_{\mathcal{L}^\rho\Big([0,\infty),\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho}}_{p,q,\lambda}\Big)}+\|\mathbb{P}\operatorname{div}(u\otimes u-b\otimes b)\|_{\mathcal{L}^\rho\Big([0,\infty),\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho}}_{p,q,\lambda}\Big)}\Big)\\
&\lesssim |t_2-t_1|^{1/\rho'}\Big(\|u\|_{\mathcal{L}^\rho\Big([0,\infty),\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\Big)}+\|u\otimes u\|_{\mathcal{L}^\rho\Big([0,\infty),\hat{\dot{F}}^{5+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho}}_{p,q,\lambda}\Big)}\\
&\quad+\|b\otimes b\|_{\mathcal{L}^\rho\Big([0,\infty),\hat{\dot{F}}^{5+\frac{\lambda-3}{p}-4\alpha+\frac{2\alpha}{\rho}}_{p,q,\lambda}\Big)}\Big)\\
&\lesssim |t_2-t_1|^{1/\rho'}\Big(\|u\|_{\mathcal{L}^\rho\Big([0,\infty),\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\Big)}+\|u\|_{\mathcal{L}^\rho\Big(I,\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\Big)}\|u\|_{\mathcal{L}^\infty\Big(I,\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\\
&\quad+\|b\|_{\mathcal{L}^\rho\Big(I,\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-\frac{2\alpha}{\rho'}}_{p,q,\lambda}\Big)}\|b\|_{\mathcal{L}^\infty\Big(I,\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big)}\Big).
\end{aligned}
$$

For all $\varepsilon>0$, choosing an appropriate δ such that when $|t_2-t_1|<\delta$,

$$
\Big\|\Big(\sum_{j\le N}2^{jq(4+\frac{\lambda-3}{p}-2\alpha)}\big|\widehat{\dot{\Delta}_j u}(t_2)-\widehat{\dot{\Delta}_j u}(t_1)\big|^q\Big)^{1/q}\Big\|_{M^{p,\lambda}}<\frac{\varepsilon}{2}. \tag{4.15}
$$

Combining (4.14) and (4.15), we obtain $\|u(t_2)-u(t_1)\|_{\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}}<\varepsilon$. It may be concluded that

$$
(u,b)\in C\Big([0,\infty),\hat{\dot{F}}^{4+\frac{\lambda-3}{p}-2\alpha}_{p,q,\lambda}\Big).
$$

The proof is complete. □

5. proof of Theorem 1.4

Proof. If $v(x,t)=e^{\sqrt{t}\Lambda^\alpha}u(x,t)$ and $w(x,t)=e^{\sqrt{t}\Lambda^\alpha}b(x,t)$, then

$$
\begin{aligned}
v(x,t)=e^{\sqrt{t}\Lambda^\alpha}e^{-t(-\Delta)^\alpha}u_0-e^{\sqrt{t}\Lambda^\alpha}\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\mathbb{P}\operatorname{div}(e^{-\sqrt{\tau}\Lambda^\alpha}v\\
\otimes e^{-\sqrt{\tau}\Lambda^\alpha}v-e^{-\sqrt{\tau}\Lambda^\alpha}w\otimes e^{-\sqrt{\tau}\Lambda^\alpha}w)(\tau)d\tau,
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
w(x,t)=e^{\sqrt{t}\Lambda^\alpha}e^{-t(-\Delta)^\alpha}b_0-e^{\sqrt{t}\Lambda^\alpha}\int_0^t e^{-(t-\tau)(-\Delta)^\alpha}\operatorname{div}(e^{-\sqrt{\tau}\Lambda^\alpha}v\\
\otimes e^{-\sqrt{\tau}\Lambda^\alpha}w-e^{-\sqrt{\tau}\Lambda^\alpha}w\otimes e^{-\sqrt{\tau}\Lambda^\alpha}v)(\tau)d\tau.
\end{aligned} \tag{5.2}
$$

Applying Fourier transform to (5.1), (5.2), and using Lemma 2.15, it follows that

$$
\begin{aligned}
|\widehat{v}(\xi,t)|&\le e^{\sqrt{t}|\xi|^\alpha-t|\xi|^{2\alpha}}|\widehat{u}_0|\\
&\quad+\int_0^t e^{|\xi|^\alpha\sqrt{t}-(t-\tau)|\xi|^{2\alpha}}|\xi|\Big(\int_{\mathbb{R}^3}e^{-\sqrt{\tau}|\xi-y|^\alpha}|\widehat{v}(\xi-y,\tau)|\otimes e^{-\sqrt{\tau}|y|^\alpha}|\widehat{v}(y,\tau)|dy\\
&\quad+\int_{\mathbb{R}^3}e^{-\sqrt{\tau}|\xi-y|^\alpha}|\widehat{w}(\xi-y,\tau)|\otimes e^{-\sqrt{\tau}|y|^\alpha}|\widehat{w}(y,\tau)|dy\Big)d\tau\\
&=e^{\sqrt{t}|\xi|^\alpha-\frac12 t|\xi|^{2\alpha}-\frac12 t|\xi|^{2\alpha}}|\widehat{u_0}|\\
&\quad+\int_0^t e^{-\frac12(t-\tau)|\xi|^{2\alpha}}|\xi|\Big(\int_{\mathbb{R}^3}e^{|\xi|^\alpha\sqrt{t}-\frac12(t-\tau)|\xi|^{2\alpha}-\sqrt{\tau}(|\xi-y|^\alpha+|y|^\alpha)}||\widehat{v}(\xi-y)|\otimes|\widehat{v}(y)|dy
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^3} e^{|\xi|^\alpha\sqrt{t}-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}-\sqrt{\tau}(|\xi-y|^\alpha+|y|^\alpha)}|\widehat{w}(\xi-y)|\otimes|\widehat{w}(y)|dy\Big)d\tau\\
&\lesssim e^{-\frac{1}{2}t|\xi|^{2\alpha}}|\widehat{u_0}|+\int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}|\xi||\widehat{v\otimes v}|+|\widehat{w\otimes w}|d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
|\widehat{w}(\xi,t)| &\le e^{\sqrt{t}|\xi|^\alpha-t|\xi|^{2\alpha}}|\widehat{b}_0|\\
&\quad+\int_0^t e^{|\xi|^\alpha\sqrt{t}-(t-\tau)|\xi|^{2\alpha}}|\xi|\Big(\int_{\mathbb{R}^3} e^{-\sqrt{\tau}|\xi-y|^\alpha}|\widehat{v}(\xi-y,\tau)|\otimes e^{-\sqrt{\tau}|y|^\alpha}|\widehat{w}(y,\tau)|dy\\
&\quad+\int_{\mathbb{R}^3} e^{-\sqrt{\tau}|\xi-y|^\alpha}|\widehat{w}(\xi-y,\tau)|\otimes e^{-\sqrt{\tau}|y|^\alpha}|\widehat{v}(y,\tau)|dy\Big)d\tau\\
&= e^{\sqrt{t}|\xi|^\alpha-\frac{1}{2}t|\xi|^{2\alpha}-\frac{1}{2}t|\xi|^{2\alpha}}|\widehat{b_0}|\\
&\quad+\int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}|\xi|\Big(\int_{\mathbb{R}^3} e^{|\xi|^\alpha\sqrt{t}-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}-\sqrt{\tau}(|\xi-y|^\alpha+|y|^\alpha)}|\widehat{v}(\xi-y)|\otimes|\widehat{w}(y)|dy\\
&\quad+\int_{\mathbb{R}^3} e^{|\xi|^\alpha\sqrt{t}-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}-\sqrt{\tau}(|\xi-y|^\alpha+|y|^\alpha)}|\widehat{w}(\xi-y)|\otimes|\widehat{v}(y)|dy\Big)d\tau\\
&\lesssim e^{-\frac{1}{2}t|\xi|^{2\alpha}}|\widehat{b_0}|+\int_0^t e^{-\frac{1}{2}(t-\tau)|\xi|^{2\alpha}}|\xi||\widehat{v\otimes w}|+|\widehat{w\otimes v}|d\tau.
\end{aligned}
$$

By applying the proof method of Theorem 1.1, the proof of Theorem 1.4 is straightforward. ☐

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