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GROWTH OF (α,β,γ)-ORDER SOLUTIONS TO COMPLEX LINEAR DIFFERENTIAL EQUATIONS WITH ANALYTIC COEFFICIENTS IN THE UNIT DISC

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Abstract. In this article, we investigate the growth of solutions of higher-order complex linear differential equations in the unit disc, with analytic coefficients of finite (α,β,γ)-order. By using the concepts of (α,β,γ)-order and (α,β,γ)-type, we establish new results concerning the growth of such solutions. Our results extend and generalize earlier works of Heittokangas et al., Hamouda, Semochko, Tu and Huang, as well as those of the second author with Biswas.

1. Introduction and definitions

Throughout this article, we assume that the reader is familiar with the basic concepts, notation, and fundamental results of Nevanlinna theory in the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$; see for example, [18, 20, 21, 25, 26, 32, 36].

For $k \geq 2$, we consider the complex linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1.1}$$

where the coefficients A_j $(j = 0, 1, \dots, k-1)$ are analytic in the unit disc Δ. It is well known that every solution of (1.1) is analytic in Δ, and that the equation possesses exactly k linearly independent solutions (see [21]). The study of growth and oscillation of solutions of complex linear differential equations in the unit disc has developed rapidly since the 1980s, see [29]. A systematic investigation in this direction was initiated by Heittokangas [21], who introduced suitable function spaces to describe the growth of solutions when the coefficients A_j $(j = 0, 1, \dots, k-1)$ of (1.1) are analytic functions in Δ.

To state the following results, we recall the notion for iterated p-order and iterated p-type of analytic functions in Δ. For $x \in [0, +\infty)$ and $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers, the $k-$fold iterated exponential is recursively defined by $\exp^{[k]} x := \exp(\exp^{[k-1]} x)$ and for sufficiently large $x \in (0, +\infty)$, the k-fold iterated logarithm is defined by $\log^{[k]} x := \log(\log^{[k-1]} x)$. We adopt the conventions $\log^{[0]} x := x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x := x$ and $\exp^{[-1]} x := \log x$.

Let f be a nonconstant analytic function in Δ. The iterated p-order of f is defined by

$$\varrho_{M,p}(f) := \limsup_{r\to 1^-} \frac{\log^{[p+1]} M(r,f)}{-\log(1-r)}, \quad p \in \mathbb{N},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$ denotes the maximum modulus of f.

The iterated p-type of f in Δ, under the assumption $0 < \varrho_{M,p}(f) < +\infty$, is defined by

$$\tau_{M,p}(f) = \limsup_{r\to 1^-} (1-r)^{\varrho_{M,p}(f)} \log^{[p]} M(r,f).$$

Subsequently, Heittokangas et al. [23] investigated the iterated p-order of solutions of equation (1.1), for $k \geq 2$, assuming that the coefficient A_0 is dominant. Their result shows that the growth of every nontrivial solution is completely determined by the iterated p-order of A_0.

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Theorem 1.1 ([23])**.** *Let $p \in \mathbb{N}$. If the coefficients $A_0, A_1, \dots, A_{k-1}$ are analytic functions in Δ such that $\varrho_{M,p}(A_j) < \varrho_{M,p}(A_0)$ for all $j = 1, \dots, k-1$, then every solution $f \not\equiv 0$ of* (1.1) *satisfies $\varrho_{M,p+1}(f) = \varrho_{M,p}(A_0)$.*

Observe that, under the assumptions of Theorem 1.1, $A_0(z)$ is the unique dominant coefficient. Later, Hamouda [19] extended Theorem 1.1 by allowing more than one dominant coefficient, replacing strict dominance by suitable conditions involving iterated p-order and iterated p-type. He proved the following theorem.

Theorem 1.2 ([19])**.** *Let $p \in \mathbb{N}$. If the coefficients $A_0, A_1, \dots, A_{k-1}$ are analytic functions in Δ such that*

$$\varrho_{M,p}(A_j) \le \varrho_{M,p}(A_0) < \infty, \quad j = 1, \dots, k-1,$$
$$\max\{\tau_{M,p}(A_j) : \varrho_{M,p}(A_j) = \varrho_{M,p}(A_0) > 0\} < \tau_{M,p}(A_0),$$

then every solution $f \not\equiv 0$ of (1.1) *satisfies $\varrho_{M,p+1}(f) = \varrho_{M,p}(A_0)$.*

Various further extensions of Theorems 1.1 and 1.2 were obtained using the notion of $[p, q]$-order, see for example [2, 3, 4, 27, 33]. However, these growth indicators are not sufficient to describe arbitrary growth behaviour. In fact, it was shown in [16, Example 1.4], that for every $p \in \mathbb{N}$ there exist functions whose iterated p-order and $[p, q]$-order are both infinite. To overcome this limitation, Chyzhykov and Semochko [16] introduced the concept of φ-order, which provides a more flexible scale for measuring growth, see also [30]. Using this notion, Semochko [30] obtained a result that improves the above-mentioned theorem of Heittokangas et al. by relaxing the dominance condition on the coefficients.

Let Φ be the class of positive, increasing and unbounded functions φ on $(0, +\infty)$ such that $\varphi(e^t)$ is slowly growing, i.e.

$$\lim_{t\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1. \quad \forall c > 0.$$

Let φ be an increasing unbounded function on $(0, +\infty)$. The φ-orders of an analytic function f in Δ are defined by [30]

$$\tilde{\varrho}^0_\varphi(f) = \limsup_{r\to 1^-} \frac{\varphi(M(r, f))}{-\log(1-r)}, \quad \tilde{\varrho}^1_\varphi(f) = \limsup_{r\to 1^-} \frac{\varphi(\log M(r, f))}{-\log(1-r)}.$$

If f is meromorphic in Δ, then the φ-orders are defined by

$$\varrho^0_\varphi(f) = \limsup_{r\to 1^-} \frac{\varphi(e^{T(r,f)})}{-\log(1-r)}, \quad \varrho^1_\varphi(f) = \limsup_{r\to 1^-} \frac{\varphi(T(r, f))}{-\log(1-r)},$$

where $T(r, f)$ denotes the Nevanlinna characteristic function of f.

Note that if $\varphi(r) = \log^{[p+1]} r$, $p \in \mathbb{N}$, and f is an analytic function in Δ, then

$$\tilde{\varrho}^0_\varphi(f) = \varrho_{M,p}(f) \quad \text{and} \quad \tilde{\varrho}^1_\varphi(f) = \varrho_{M,p+1}(f).$$

The following theorem due to Semochko [30] used the concept of φ-order which improves Theorem 1.1.

Theorem 1.3 ([30])**.** *Let $\varphi \in \Phi$ and let $A_0, A_1, \dots, A_{k-1}$ be analytic functions in Δ such that* $\max\{\tilde{\varrho}^0_\varphi(A_j), j = 1, \dots, k-1\} < \tilde{\varrho}^0_\varphi(A_0)$. *Then, every solution $f \not\equiv 0$ of* (1.1) *satisfies* $\tilde{\varrho}^1_\varphi(f) = \tilde{\varrho}^0_\varphi(A_0)$.

The generalized (α, β)-order of an entire function was introduced by Sheremeta [31], and has been studied extensively in recent years; see, for instance, [10, 11]. Applications of this concept to differential equations were initiated by Mulyava et al. [28], who applied the (α, β)-order to the study of solutions of certain second-order heterogeneous differential equations and obtained several remarkable results. For details on the (α, β) -order, we refer to [5, 7, 28, 31].

Motivated by these developments, we study equation (1.1) within a more general growth framework. The main purpose of the present paper is to investigate the growth of nontrivial solutions of equation (1.1) in the unit disc in terms of their (α, β, γ)-order and (α, β, γ)-type. Our results extend and unify several earlier theorems concerning iterated p-order, $[p,q]$-order, and φ-order, and provide a more general description of the growth behaviour of solutions in the unit disc.

Let L be the class of continuous, non-negative functions $\alpha : (-\infty,+\infty) \to [0,+\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a+b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0,+\infty)$. Further, we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and α is subadditive, that is, $\alpha(a+b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$. Clearly $L_3 \subset L_1$. Particularly, if $\alpha \in L_3$, then for any integer $m \geq 2$, $\alpha(mr) \leq m\alpha(r)$.

Concavity also implies subadditivity: if $\alpha(r)$ is concave on $[0,+\infty)$ with $\alpha(0) \geq 0$, then $\alpha(tx) \geq t\alpha(x)$, for $t \in [0,1]$, which yields

$$\alpha(a+b) \leq \alpha(a) + \alpha(b)$$

for $a, b \geq 0$. Moreover, if α is non-decreasing, subadditive, and unbounded, then for any $R_0 \geq 0$

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

hence $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$.

We assume throughout this paper that α, β and γ satisfy the following conditions:

(i) $\alpha \in L_1$, $\beta \in L_2$, and $\gamma \in L_3$; and

(ii) $\alpha(\log^{[p]} x) = o(\beta(\log \gamma(x)))$, $p \geq 2$, $\alpha(\log x) = o(\alpha(x))$ and $\alpha^{-1}(kx) = o(\alpha^{-1}(x))$ $(0 \leq k < 1)$ as $x \to +\infty$.

Unless otherwise stated, we assume these conditions hold.

Recently, Heittokangas et al. [24] introduced the concept of φ-order of entire and meromorphic functions, where φ is a subadditive function. Building on this idea, the second author and Biswas [6] introduced the (α,β,γ)-order of meromorphic functions in the complex plane. Several works concerning the growth of solutions of higher-order differential equations in terms of the (α,β,γ)-order have since appeared; see [6, 8, 9].

Using these notions, Biswas et al. introduced the (α,β,γ)-order and (α,β,γ)-type of analytic functions f in the unit disc Δ as follows.

Definition 1.4 ([12]). The (α,β,γ)-order denoted by $\varrho_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f in Δ is defined by

$$\varrho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to 1^-} \frac{\alpha(\log T(r,f))}{\beta(\log \gamma(\frac{1}{1-r}))}.$$

If f is an analytic function in Δ, then the (α,β,γ)-order is defined by

$$\varrho^{(M)}_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to 1^-} \frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log \gamma(\frac{1}{1-r}))}.$$

Example 1.5. We consider the analytic function

$$f(z) = \exp^{[4]}\{\frac{1}{1-z}\}, \quad z \in \Delta.$$

This function satisfies

$$M(r,f) = \max_{|z|=r} |f(z)| = \exp^{[4]}\{\frac{1}{1-r}\}.$$

From this, one obtains that the classical iterated order of order 2 (hyper-order) satisfies

$$\begin{aligned}\varrho_{M,2}(f) &= \limsup_{r\to 1^-} \frac{\log^{[3]} M(r,f)}{\log \frac{1}{1-r}}\\ &= \limsup_{r\to 1^-} \frac{\log^{[3]}(\exp^{[4]}\{\frac{1}{1-r}\})}{\log \frac{1}{1-r}}\\ &= \limsup_{r\to 1^-} \frac{\exp\{\frac{1}{1-r}\}}{\log \frac{1}{1-r}} = +\infty,\end{aligned}$$

and similarly $\varrho_{M,[3,2]}(f)$ is also infinite, since

$$\begin{aligned}\varrho_{M,[3,2]}(f) &= \limsup_{r\to 1^-}\frac{\log^{[4]} M(r,f)}{\log^{[2]}\frac{1}{1-r}}\\ &= \limsup_{r\to 1^-}\frac{\log^{[4]}(\exp^{[4]}\{\frac{1}{1-r}\})}{\log^{[2]}\frac{1}{1-r}}\\ &= \limsup_{r\to 1^-}\frac{\frac{1}{1-r}}{\log^{[2]}\frac{1}{1-r}} = +\infty.\end{aligned}$$

On the other hand, if we choose

$$\alpha(r) = \log^{[5]} r, \quad \beta(r) = \log^{[2]} r, \quad \gamma(r) = r,$$

we obtain

$$\begin{aligned}\varrho^{(M)}_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to 1^-}\frac{\alpha(\log^{[2]} M(r,f))}{\beta(\log\gamma(\frac{1}{1-r}))}\\ &= \limsup_{r\to 1^-}\frac{\log^{[5]}(\log^{[2]}\exp^{[4]}\{\frac{1}{1-r}\})}{\log^{[2]}(\log\frac{1}{1-r})}\\ &= \limsup_{r\to 1^-}\frac{\log^{[3]}\frac{1}{1-r}}{\log^{[3]}\frac{1}{1-r}} = 1.\end{aligned}$$

Therefore, the generalized (α,β,γ)-order provides a finer scale for measuring growth in the unit disc. For the function considered above, both the classical hyper-order and the classical $[3,2]$-order are infinite, while the generalized (α,β,γ)-order exists and equals 1.

Similar to Definition 1.4, we can also define the $(\alpha(\log),\beta,\gamma)$-order of a meromorphic function f in Δ in the following way.

Definition 1.6. If f is a meromorphic function in Δ, then

$$\varrho_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to 1^-}\frac{\alpha(\log^{[2]} T(r,f))}{\beta(\log\gamma(\frac{1}{1-r}))}.$$

If f is an analytic function in Δ, then

$$\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to 1^-}\frac{\alpha(\log^{[3]} M(r,f))}{\beta(\log\gamma(\frac{1}{1-r}))}.$$

Proposition 1.7. *If f is an analytic function in Δ, then*

$$\varrho_{(\alpha(\log),\beta,\gamma)}[f] = \varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f].$$

Proof. For nonconstant analytic function f in Δ, it is well known that [20]

$$T(r,f) \le \log M(r,f) \le \frac{R+r}{R-r}T(R,f) \quad (0 < r < R < 1).$$

Choosing $R = \frac{1+r}{2}$, we obtain

$$T(r,f) \le \log M(r,f) \le \frac{1+3r}{1-r}T(\frac{1+r}{2},f). \tag{1.2}$$

Using the double inequality (1.2) and the property

$$\alpha(a+b) \le \alpha(a) + \alpha(b) + c$$

for all $a, b \geq R_0$ and for some fixed $c \in (0, +\infty)$, we derive

$$\begin{aligned}
&\frac{\alpha(\log^{[2]} T(r,f))}{\beta(\log\gamma(\frac{1}{1-r}))}\\
&\leq \frac{\alpha(\log^{[3]} M(r,f))}{\beta(\log\gamma(\frac{1}{1-r}))}\\
&\leq \frac{\alpha(\log^{[2]}\frac{4}{1-r} + \log^{[2]} T(\frac{1+r}{2}, f) + O(1))}{\beta(\log\gamma(\frac{1}{1-r}))}\\
&\leq \frac{\alpha(\log^{[2]}\frac{1}{1-r})}{\beta(\log\gamma(\frac{1}{1-r}))} + \frac{\alpha(\log^{[2]} T(\frac{1+r}{2}, f))}{\beta(\log\gamma(\frac{1}{1-r}))} + \frac{c}{\beta(\log\gamma(\frac{1}{1-r}))}\\
&= \frac{\alpha(\log^{[2]}\frac{1}{1-r})}{\beta(\log\gamma(\frac{1}{1-r}))} + \frac{\alpha(\log^{[2]} T(\frac{1+r}{2}, f))}{\beta(\log\gamma(\frac{1}{1-\frac{1+r}{2}}))} \cdot \frac{\beta(\log\gamma(\frac{2}{1-r}))}{\beta(\log\gamma(\frac{1}{1-r}))} + \frac{c}{\beta(\log\gamma(\frac{1}{1-r}))}.
\end{aligned} \tag{1.3}$$

Since $\gamma(\frac{2}{1-r}) \leq 2\gamma(\frac{1}{1-r})$, $\beta(x + O(1)) = (1 + o(1))\beta(x)$ and

$$\alpha(\log^{[2]} x) = o(\beta(\log\gamma(x)) \quad \text{as } x = \frac{1}{1-r} \to +\infty \quad \text{when } r \to 1^-,$$

letting $r \to 1^-$ and taking the limit superior in (1.3) yields the desired equality. □

Proposition 1.8. *Let f_1 and f_2 be nonconstant meromorphic functions in Δ, and let $\varrho_{(\alpha,\beta,\gamma)}[f_1]$ and $\varrho_{(\alpha,\beta,\gamma)}[f_2]$ denote their (α,β,γ)-order. Then*

(i) $\varrho_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] \leq \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1],\ \varrho_{(\alpha,\beta,\gamma)}[f_2]\}$;

(ii) $\varrho_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] \leq \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1],\ \varrho_{(\alpha,\beta,\gamma)}[f_2]\}$;

(iii) *If $\varrho_{(\alpha,\beta,\gamma)}[f_1] \neq \varrho_{(\alpha,\beta,\gamma)}[f_2]$, then*
$$\varrho_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] = \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1],\ \varrho_{(\alpha,\beta,\gamma)}[f_2]\};$$

(iv) *If $\varrho_{(\alpha,\beta,\gamma)}[f_1] \neq \varrho_{(\alpha,\beta,\gamma)}[f_2]$, then*
$$\varrho_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1],\ \varrho_{(\alpha,\beta,\gamma)}[f_2]\}.$$

Proof. Without loss of generality, assume that $\varrho_{(\alpha,\beta,\gamma)}[f_1] \leq \varrho_{(\alpha,\beta,\gamma)}[f_2] < +\infty$. By the definition of the (α,β,γ)-order, for any $\varepsilon > 0$, as $r \to 1^-$

$$T(r, f_1) < \exp\Big\{\alpha^{-1}\Big((\varrho_{(\alpha,\beta,\gamma)}[f_1] + \frac{\varepsilon}{2})\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\} \tag{1.4}$$

and

$$T(r, f_2) < \exp\Big\{\alpha^{-1}\Big((\varrho_{(\alpha,\beta,\gamma)}[f_2] + \frac{\varepsilon}{2})\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\}. \tag{1.5}$$

Since $T(r, f_1 \pm f_2) \leq T(r, f_1) + T(r, f_2) + \log 2$, it follows from (1.4) and (1.5), as $r \to 1^-$

$$T(r, f_1 \pm f_2) < 2\exp\Big\{\alpha^{-1}\Big((\varrho_{(\alpha,\beta,\gamma)}[f_2] + \frac{\varepsilon}{2})\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\} + \log 2.$$

For r sufficiently close to 1, this implies

$$T(r, f_1 \pm f_2) < \exp\Big\{\alpha^{-1}\Big((\varrho_{(\alpha,\beta,\gamma)}[f_2] + \varepsilon)\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\}.$$

Therefore,

$$\alpha(\log T(r, f_1 \pm f_2)) < (\varrho_{(\alpha,\beta,\gamma)}[f_2] + \varepsilon)\beta\big(\log\gamma(\frac{1}{1-r})\big).$$

Taking the limit superior as $r \to 1^-$, we obtain

$$\limsup_{r\to1^-} \frac{\alpha(\log T(r, f_1 \pm f_2))}{\beta(\log\gamma(\frac{1}{1-r}))} \leq\ \varrho_{(\alpha,\beta,\gamma)}[f_2] + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\varrho_{(\alpha,\beta,\gamma)}[f_1 \pm f_2] \leq \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1],\ \varrho_{(\alpha,\beta,\gamma)}[f_2]\}. \tag{1.6}$$

Furthermore, without loss of generality, assume that $\varrho_{(\alpha,\beta,\gamma)}[f_1] < \varrho_{(\alpha,\beta,\gamma)}[f_2] < +\infty$ and set $f(z) = f_1(z) \pm f_2(z)$. By (1.6) we obtain that $\varrho_{(\alpha,\beta,\gamma)}[f] \le \varrho_{(\alpha,\beta,\gamma)}[f_2]$. Since $f_2(z) = \pm(f(z) - f_1(z))$ another application of (1.6) gives

$$\varrho_{(\alpha,\beta,\gamma)}[f_2] \le \max\{\varrho_{(\alpha,\beta,\gamma)}[f], \varrho_{(\alpha,\beta,\gamma)}[f_1]\}.$$

Because $\varrho_{(\alpha,\beta,\gamma)}[f_1] < \varrho_{(\alpha,\beta,\gamma)}[f_2]$, it follows that $\varrho_{(\alpha,\beta,\gamma)}[f_2] \le \varrho_{(\alpha,\beta,\gamma)}[f]$. Thus,

$$\varrho_{(\alpha,\beta,\gamma)}[f] = \varrho_{(\alpha,\beta,\gamma)}[f_2] = \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1], \varrho_{(\alpha,\beta,\gamma)}[f_2]\}.$$

Similarly, since $T(r, f_1 \cdot f_2) \le T(r, f_1) + T(r, f_2)$, we obtain

$$\varrho_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] \le \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1],\ \varrho_{(\alpha,\beta,\gamma)}[f_2]\}.$$

If $\varrho_{(\alpha,\beta,\gamma)}[f_1] \ne \varrho_{(\alpha,\beta,\gamma)}[f_2]$, then the same argument as above yields

$$\varrho_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2] = \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1],\ \varrho_{(\alpha,\beta,\gamma)}[f_2]\}.$$

This completes the proof. □

Definition 1.9 ([12]). The (α,β,γ)-type denoted by $\tau_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f in Δ with $0 < \varrho_{(\alpha,\beta,\gamma)}[f] < +\infty$ is defined by

$$\tau_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to 1^-} \frac{\exp\big(\alpha(\log T(r,f))\big)}{\Big(\exp\big(\beta(\log\gamma(\frac{1}{1-r}))\big)\Big)^{\varrho_{(\alpha,\beta,\gamma)}[f]}}.$$

If f is an analytic function in Δ with $\varrho^{(M)}_{(\alpha,\beta,\gamma)}[f] \in (0,+\infty)$, then the (α,β,γ)-type of f is defined by

$$\tau^{(M)}_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to 1^-} \frac{\exp\big(\alpha(\log^{[2]} M(r,f))\big)}{\Big(\exp\big(\beta(\log\gamma(\frac{1}{1-r}))\big)\Big)^{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[f]}}.$$

Similar to Definition 1.9, we can also define the $(\alpha(\log),\beta,\gamma)$-type of a meromorphic function f in Δ in the following way.

Definition 1.10. The $(\alpha(\log),\beta,\gamma)$-type denoted by $\tau_{(\alpha(\log),\beta,\gamma)}[f]$ of a meromorphic function f in Δ with $0 < \varrho_{(\alpha(\log),\beta,\gamma)}[f] < +\infty$ is defined by

$$\tau_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to 1^-} \frac{\exp\big(\alpha(\log^{[2]} T(r,f))\big)}{\Big(\exp\big(\beta(\log\gamma(\frac{1}{1-r}))\big)\Big)^{\varrho_{(\alpha(\log),\beta,\gamma)}[f]}}.$$

If f is an analytic function in Δ with $\varrho^{M}_{(\alpha(\log),\beta,\gamma)}[f] = \varrho_{(\alpha(\log),\beta,\gamma)}[f] \in (0,+\infty)$, then the $(\alpha(\log),\beta,\gamma)$-type of f is defined by

$$\tau^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to 1^-} \frac{\exp\big(\alpha(\log^{[3]} M(r,f))\big)}{\Big(\exp\big(\beta(\log\gamma(\frac{1}{1-r}))\big)\Big)^{\varrho^{M}_{(\alpha(\log),\beta,\gamma)}[f]}}.$$

Remark 1.11. The (α,β,γ)-order provides a generalized logarithmic-iteration scale for describing the growth of analytic and meromorphic functions in the unit disc, extending the classical notions of order, iterated p-order, and $[p,q]$-order. In contrast, the integrated growth theory developed by Chyzhykov, Gröhn, Heittokangas and Rättyä [17] is based on integral characteristics involving logarithmic derivatives rather than on an explicit iteration-based scale. Consequently, the two approaches capture different aspects of growth behavior, although potential connections between them remain largely unexplored. Establishing such connections may provide further insight into the interplay between growth and oscillation phenomena in complex analysis.

More recently, growth problems have also been investigated in the broader setting of nonlinear partial differential equations in several complex variables. Xu and his collaborators obtained explicit descriptions and existence criteria for transcendental entire solutions of various classes of nonlinear PDEs in $\mathbb{C}^2$ and $\mathbb{C}^3$, together with detailed analyses of their growth properties [34, 35].

These developments further illustrate the importance of growth-theoretic methods in the study of complex differential equations.

The remainder of the paper is organized as follows. Section 2 contains the main results, Section 3 presents several auxiliary lemmas, and Section 4 is devoted to the proofs of the main theorems.

2. Main Results

In this section, we present the main results of the paper. The following theorems generalize Theorems 1.1-1.3 and [33, Theorem 2.3]. They may also be regarded as analogues in the unit disc of the corresponding results obtained in [9] for entire functions.

Theorem 2.1. *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions in Δ such that*

$$\max\big\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : j = 1, \dots, k-1\big\} < \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0].$$

Then every solution $f(z) \not\equiv 0$ of (1.1) satisfies

$$\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0].$$

Remark 2.2. By choosing $\alpha(r) = \log^{[p-1]} r$ ($p \geq 1$ is an integer) and $\beta(r) = \gamma(r) = r$ in Theorem 2.1, we recover Theorem 1.1 . Furthermore, by taking $\alpha(r) = \varphi(\exp^{[2]} r)$, where $\varphi \in \Phi$, and assuming that α satisfies conditions (i) and (ii), while setting $\beta(r) = \gamma(r) = r$, Theorem 2.1 reduces to Theorem 1.3.

Theorem 2.3. *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions. Assume that*

$$\max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : j = 1, \dots, k-1\} \leq \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0 \quad (0 < \varrho_0 < +\infty)$$

and

$$\max\{\tau^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0]\} < \tau^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \tau_0 \quad (0 < \tau_0 < +\infty).$$

Then every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0]$.

Remark 2.4. By choosing $\alpha(r) = \log^{[p-1]} r$ ($p \geq 1$ is an integer) and $\beta(r) = \gamma(r) = r$ in Theorem 2.3, we recover Theorem 1.2. Furthermore, by taking $\alpha(r) = \log^{[p-1]} r$ and $\beta(r) = \log^{[q-1]} r$ ($p \geq q \geq 1$, where p and q are integers) and $\gamma(r) = r$, [33, Theorem 2.3] follows as a special case of Theorem 2.3.

By combining Theorems 2.1 and 2.3, we obtain the following result.

Corollary 2.5. *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions. Assume that either*

$$\max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : j = 1, \dots, k-1\} < \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0 < +\infty,$$

or

$$\max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : j = 1, \dots, k-1\} \leq \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0 < +\infty \quad (0 < \varrho_0 < +\infty)$$

and

$$\begin{aligned}\max\{\tau^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0]\}&\\ < \tau^{(M)}_{(\alpha,\beta,\gamma)}[A_0]&\\ = \tau_0 \quad (0 < \tau_0 < +\infty).&\end{aligned}$$

Then every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0]$.

3. Auxiliary lemmas

In this section, we present several lemmas that will be needed later. The following lemma is a direct consequence of [15, Theorem 3.1].

Lemma 3.1 ([15])**.** *Let k and j be integers satisfying $k > j \geq 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If f is a meromorphic in Δ such that $f^{(j)} \not\equiv 0$, then*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq \Big[\Big(\frac{1}{1-|z|}\Big)^{2+\varepsilon} \max\Big\{\log\frac{1}{1-|z|}; T(s(|z|), f)\Big\}\Big]^{k-j}$$

for $|z| \notin F$, where $F \subset [0,1)$ is a set of finite logarithmic measure $m_l(F) = \int_F \frac{dr}{1-r} < \infty$, and where $s(|z|) = 1 - d(1-r)$. Moreover, if $\varrho(f) < \infty$, then

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq \Big(\frac{1}{1-|z|}\Big)^{(k-j)(\varrho(f)+2+\varepsilon)}.$$

Lemma 3.2 ([20, 21])**.** *Let f be a nonconstant meromorphic function in the unit disc Δ and let $k \in \mathbb{N}$. Then*

$$m(r, \frac{f^{(k)}}{f}) = S(r, f),$$

where $S(r,f) = O(\log T(r,f) + \log(\frac{1}{1-r}))$, possibly outside a set $F_1 \subset [0,1)$ with finite logarithmic measure.

Here, we present a generalized lemma on the logarithmic derivative in terms of the (α, β, γ)-order in the unit disc Δ.

Lemma 3.3. *Let f be a meromorphic function in Δ of order $\varrho_{(\alpha(\log),\beta,\gamma)}[f] = \varrho^*$ and $k \in \mathbb{N}$. Then, for any given $\varepsilon > 0$,*

$$m\big(r, \frac{f^{(k)}}{f}\big) = O\Big(\exp\Big\{\alpha^{-1}\Big((\varrho^* + \varepsilon)\beta\big(\log\gamma(\frac{1}{1-r})\big)\Big)\Big\}\Big),$$

outside, possibly, an exceptional set $F_2 \subset [0,1)$ of finite logarithmic measure.

Proof. Let $k = 1$. By the definition of the generalized $(\alpha(\log), \beta, \gamma)$-order

$$\varrho_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to 1^-} \frac{\alpha\big(\log^{[2]} T(r,f)\big)}{\beta\big(\log\gamma(\frac{1}{1-r})\big)} = \varrho^*,$$

it follows that for any $\varepsilon > 0$ and all r sufficiently close to 1,

$$\alpha\Big(\log^{[2]} T(r,f)\Big) \leq (\varrho^* + \varepsilon)\beta\Big(\log\gamma(\frac{1}{1-r})\Big).$$

Since α is increasing and invertible, we obtain

$$\log^{[2]} T(r,f) \leq \alpha^{-1}\Big[(\varrho^* + \varepsilon)\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\Big],$$

and consequently,

$$T(r,f) \leq \exp^{[2]}\Big\{\alpha^{-1}\Big[(\varrho^* + \varepsilon)\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\Big]\Big\}. \tag{3.1}$$

From (3.1) and Lemma 3.2 of the logarithmic derivative and the assumption

$$\alpha\big(\log^{[2]} x\big) = o(\beta(\log\gamma(x)) \quad \text{as } x = \frac{1}{1-r} \to +\infty \quad \text{when } r \to 1^-,$$

we deduce that

$$\begin{aligned} m\big(r, \frac{f'}{f}\big) &= O\big(\log T(r,f) + \log(\frac{1}{1-r})\big) \\ &= O\Big(\exp\{\alpha^{-1}((\varrho^* + \varepsilon)\beta(\log\gamma(\frac{1}{1-r})))\}\Big), \quad r \notin F_2, \end{aligned} \tag{3.2}$$

where $F_2 \subset [0,1)$ is of finite linear logarithmic measure.

Now assume that for some $k\in\mathbb{N}$,

$$m\big(r,\frac{f^{(k)}}{f}\big)=O\Big(\exp\Big\{\alpha^{-1}\Big((\varrho^*+\varepsilon)\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\}\Big),\quad r\notin F_2. \tag{3.3}$$

Since $N(r,f^{(k)})\le (k+1)N(r,f)$, we have

$$\begin{aligned}T(r,f^{(k)})&=m(r,f^{(k)})+N(r,f^{(k)})\\&\le m(r,\frac{f^{(k)}}{f})+m(r,f)+(k+1)N(r,f)\\&\le (k+1)T(r,f)+O\Big(\exp\Big\{\alpha^{-1}\Big((\varrho^*+\varepsilon)\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\}\Big)\end{aligned}$$

and hence, by (3.1),

$$T(r,f^{(k)})=O\Big(\exp^{[2]}\Big\{\alpha^{-1}\Big((\varrho^*+\varepsilon)\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\}\Big). \tag{3.4}$$

Applying the logarithmic derivative lemma to $f^{(k)}$ and using (3.4), we obtain

$$\begin{aligned}m(r,\frac{f^{(k+1)}}{f^{(k)}})&=m(r,\frac{(f^{(k)})'}{f^{(k)}})\\&=O(\log T(r,f^{(k)})+\log(\frac{1}{1-r}))\\&=O\Big(\exp\{\alpha^{-1}((\varrho^*+\varepsilon)\beta(\log\gamma(\frac{1}{1-r})))\}\Big),\quad r\notin F_2.\end{aligned} \tag{3.5}$$

Finally, combining (3.3) and (3.5), we obtain

$$\begin{aligned}m(r,\frac{f^{(k+1)}}{f})&\le m(r,\frac{f^{(k+1)}}{f^{(k)}})+m(r,\frac{f^{(k)}}{f})\\&=O\Big(\exp\Big\{\alpha^{-1}\Big((\varrho^*+\varepsilon)\beta(\log\gamma(\frac{1}{1-r}))\Big)\Big\}\Big),\quad r\notin F_2.\end{aligned}$$

This completes the induction and hence the proof. □

To avoid some problems of the exceptional sets, we need the following lemma.

Lemma 3.4 ([1, 21])**.** *Let $g:[0,1)\mapsto\mathbb{R}$ and $h:[0,1)\mapsto\mathbb{R}$ be monotone non-decreasing functions such that $g(r)\le h(r)$ holds outside of an exceptional set $F_3\subset[0,1)$ of finite logarithmic measure. Then there exists a $d\in(0,1)$ such that if $s(r)=1-d(1-r)$, then $g(r)\le h(s(r))$ for all $r\in[0,1)$.*

Lemma 3.5. *Let f be a nonconstant meromorphic function in Δ. Then $\varrho_{(\alpha,\beta,\gamma)}[f']=\varrho_{(\alpha,\beta,\gamma)}[f]$.*

Proof. Let $\varrho_{(\alpha,\beta,\gamma)}[f]=\varrho$. By the definition of the (α,β,γ)-order, for any given $\varepsilon>0$ and all r sufficiently close to 1,

$$\alpha(\log T(r,f))\le(\varrho+\varepsilon)\beta\Big(\log\gamma(\frac{1}{1-r})\Big),$$

which implies

$$\log T(r,f)\le\alpha^{-1}\Big[(\varrho+\varepsilon)\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\Big]. \tag{3.6}$$

Clearly,

$$T(r,f')\le 2T(r,f)+m(r,\frac{f'}{f}).$$

By Lemma 3.2 on the logarithmic derivative, we obtain

$$\begin{aligned}\log T(r,f')&\le\log T(r,f)+\log\Big\{O\Big(\log T(r,f)+\log(\frac{1}{1-r})\Big)\Big\}+O(1)\\&\le\log T(r,f)+\log^{[2]}T(r,f)+\log^{[2]}(\frac{1}{1-r})+O(1),\quad r\notin F_4.\end{aligned}$$

Using (3.6), together with the assumption

$$\alpha(\log^{[2]} x) = o(\beta(\log \gamma(x)) \quad \text{as } x = \frac{1}{1-r} \to +\infty \text{ when } r \to 1^-,$$

it follows that

$$\log T(r, f') \le \alpha^{-1}\Big[(\varrho + 4\varepsilon)\beta\Big(\log \gamma(\frac{1}{1-r})\Big)\Big], \quad r \notin F_4, \tag{3.7}$$

where $F_4 \subset [0, 1)$ is a set of finite logarithmic measure. By Lemma 3.4, choosing $d = 1/2$ and for all $r \in [0, 1)$, inequality (3.7) yields

$$\begin{aligned} \log T(r, f') &\le \alpha^{-1}\Big[(\varrho + 4\varepsilon)\beta\Big(\log \gamma(\frac{1}{1-s(r)})\Big)\Big] \\ &\le \alpha^{-1}\Big[(\varrho + 4\varepsilon)\beta\Big(\log \gamma\Big(\frac{1}{1-(1-\frac{1}{2}(1-r))}\Big)\Big)\Big] \\ &\le \alpha^{-1}\Big[(\varrho + 4\varepsilon)\beta\Big(\log \gamma(\frac{2}{1-r})\Big)\Big]. \end{aligned} \tag{3.8}$$

Since $\gamma(\frac{2}{1-r}) \le 2\gamma(\frac{1}{1-r})$ and

$$\beta(x + O(1)) = (1 + o(1))\beta(x) \quad \text{as } x = \frac{1}{1-r} \to +\infty \quad \text{when } r \to 1^-,$$

we obtain

$$\begin{aligned} \alpha(\log T(r, f') &\le (\varrho + 4\varepsilon)\beta\Big(\log \gamma(\frac{2}{1-r})\Big) \\ &\le (\varrho + 4\varepsilon)\beta\Big(\log\Big(2\gamma(\frac{1}{1-r})\Big)\Big) \\ &= (\varrho + 4\varepsilon)\beta\Big(\log 2 + \log \gamma(\frac{1}{1-r})\Big) \\ &= (\varrho + 4\varepsilon)\beta\Big(O(1) + \log \gamma(\frac{1}{1-r})\Big) \\ &= (\varrho + 4\varepsilon)(1 + o(1))\beta\Big(\log \gamma(\frac{1}{1-r})\Big). \end{aligned}$$

Consequently,

$$\frac{\alpha(\log T(r, f')}{\beta(\log \gamma(\frac{1}{1-r}))} \le (\varrho + 4\varepsilon)(1 + o(1)).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\varrho_{(\alpha,\beta,\gamma)}[f'] \le \varrho_{(\alpha,\beta,\gamma)}[f]$.

We now prove the reverse inequality $\varrho_{(\alpha,\beta,\gamma)}[f'] \ge \varrho_{(\alpha,\beta,\gamma)}[f]$. Let $\varrho_{(\alpha,\beta,\gamma)}[f'] = \varrho'$. By definition, for any given $\varepsilon > 0$ and as $r \to 1^-$, we have

$$T(r, f') \le \exp\Big\{\alpha^{-1}\Big[(\varrho' + \varepsilon)\beta\Big(\log \gamma(\frac{1}{1-r})\Big)\Big]\Big\}. \tag{3.9}$$

By a result of Chuang in [14, Theorem 4.1] (see also [13, p. 281]),

$$T(r, f) \le O\Big(T(\frac{r+3}{4}, f') + \log(\frac{1}{1-r})\Big), \quad r \to 1^-. \tag{3.10}$$

Using $\gamma(\frac{4}{1-r}) \le 4\gamma(\frac{1}{1-r})$ together with $\beta(x + O(1)) = (1 + o(1))\beta(x)$ and the assumption

$$\alpha(\log^{[2]} x) = o(\beta(\log \gamma(x)) \quad \text{as } x = \frac{1}{1-r} \to +\infty \quad \text{when } r \to 1^-,$$

we deduce from (3.9) and (3.10) that

$$\begin{aligned} T(r, f) &\le O\Big(\exp\Big\{\alpha^{-1}\Big[(\varrho' + \varepsilon)\beta\Big(\log \gamma(\frac{4}{1-r})\Big)\Big]\Big\} + \log(\frac{1}{1-r})\Big) \\ &\le \exp\Big\{\alpha^{-1}\Big[(\varrho' + 2\varepsilon)\beta\Big(\log\Big(\gamma(\frac{4}{1-r})\Big)\Big)\Big]\Big\} \\ &\le \exp\Big\{\alpha^{-1}\Big[(\varrho' + 2\varepsilon)(1 + o(1))\beta\Big(\log \gamma(\frac{1}{1-r})\Big)\Big]\Big\}. \end{aligned}$$

Consequently,

$$\log T(r,f) \le \alpha^{-1}\Big[(\varrho' + 2\varepsilon)(1+o(1))\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\Big],$$

and hence

$$\alpha(\log T(r,f)) \le (\varrho' + 2\varepsilon)(1+o(1))\beta\Big(\log\gamma(\frac{1}{1-r})\Big), \quad r \to 1^-.$$

Since, $\varepsilon > 0$ is arbitrary, this yields $\varrho_{(\alpha,\beta,\gamma)}[f'] \ge \varrho_{(\alpha,\beta,\gamma)}[f]$. Combining both inequalities, we conclude that $\varrho_{(\alpha,\beta,\gamma)}[f'] = \varrho_{(\alpha,\beta,\gamma)}[f]$ which completes the proof. □

Remark 3.6. From Lemma 3.5, one readily deduces that $\varrho_{(\alpha(\log),\beta,\gamma)}[f'] = \varrho_{(\alpha(\log),\beta,\gamma)}[f]$, where f is a meromorphic function in Δ.

Lemma 3.7. *Let f be an analytic function in the unit disc Δ such that $0 < \varrho^{(M)}_{(\alpha,\beta,\gamma)}[f] = \varrho < +\infty$. Then, for any $0 < \mu < \varrho$, there exists a set $I \subset [0,1)$ of infinite logarithmic measure $m_l(I) = \int_I \frac{dr}{1-r} = +\infty$, such that for all $r \in I$ one has*

$$\alpha\Big(\log^{[2]} M(r,f)\Big) > \mu\beta\Big(\log\gamma(\frac{1}{1-r})\Big).$$

Proof. By the definition of the limit superior, there exists an increasing sequence $\{r_m\}$ with $r_m \to 1^-$ as $m \to +\infty$ such that

$$1 - (1 - \frac{1}{m})(1 - r_m) < r_{m+1}$$

and

$$\lim_{m\to+\infty} \frac{\alpha(\log^{[2]} M(r_m,f))}{\beta\big(\log\gamma(\frac{1}{1-r_m})\big)} = \varrho.$$

Hence, there exists an integer m_0 such that for all $m \ge m_0$ and for any given ε satisfying $0 < \varepsilon < \varrho - \mu$,

$$\alpha\Big(\log^{[2]} M(r_m,f)\Big) > (\varrho - \varepsilon)\beta\Big(\log\gamma(\frac{1}{1-r_m})\Big). \tag{3.11}$$

For $r \in [r_m, 1 - (1 - \frac{1}{m})(1 - r_m)]$, we observe that

$$\lim_{m\to+\infty} \frac{\beta\big(\log\gamma((1-\frac{1}{m})\frac{1}{1-r})\big)}{\beta\big(\log\gamma(\frac{1}{1-r})\big)} = 1.$$

Therefore, for a given μ with $0 < \mu < \varrho - \varepsilon$, there exists an integer m_1 such that for $m \ge m_1$ we have

$$\frac{\beta\big(\log\gamma((1-\frac{1}{m})\frac{1}{1-r})\big)}{\beta\big(\log\gamma(\frac{1}{1-r})\big)} > \frac{\mu}{\varrho - \varepsilon}. \tag{3.12}$$

Combining (3.11) and (3.12) for all $m \ge m_2 := \max\{m_0, m_1\}$ and for any

$$r \in [r_m, 1 - (1 - \frac{1}{m})(1 - r_m)],$$

we obtain

$$\begin{aligned}
\alpha\Big(\log^{[2]} M(r,f)\Big) &\ge \alpha\Big(\log^{[2]} M(r_m,f)\Big)\\
&> (\varrho - \varepsilon)\beta\Big(\log\gamma(\frac{1}{1-r_m})\Big)\\
&\ge (\varrho - \varepsilon)\beta\Big(\log\gamma((1-\frac{1}{m})\frac{1}{1-r})\Big)\\
&> (\varrho - \varepsilon)\frac{\mu}{\varrho - \varepsilon}\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\\
&= \mu\beta\Big(\log\gamma(\frac{1}{1-r})\Big).
\end{aligned}$$

Finally, let $I = \cup_{m=m_2}^{+\infty} I_m$ where $I_m = [r_m, 1-(1-\frac{1}{m})(1-r_m)]$. Then,

$$m_l(I) = \sum_{m=m_2}^{+\infty} \int_{I_m} \frac{dr}{1-r} = \sum_{m=m_2}^{+\infty} \log\big(\frac{m}{m-1}\big) = +\infty.$$

This completes the proof. ☐

Lemma 3.8. *Let f be an analytic function in Δ with $\varrho^{(M)}_{(\alpha,\beta,\gamma)}[f] = \varrho \in (0,+\infty)$ and $\tau^{(M)}_{(\alpha,\beta,\gamma)}[f] \in (0,+\infty)$. Then for any given $\omega < \tau^{(M)}_{(\alpha,\beta,\gamma)}[f]$, there exists a set $I_1 \subset [0,1)$ of infinite logarithmic measure such that for all $r \in I_1$*

$$\exp\Big\{\alpha\Big(\log^{[2]}(M(r,f))\Big)\Big\} > \omega\Big(\exp\Big\{\beta\Big(\log\Big(\gamma(\frac{1}{1-r})\Big)\Big)\Big\}\Big)^{\varrho}.$$

Proof. By the definition of the generalized (α,β,γ)-type,

$$\tau^{(M)}_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to1^-} \frac{\exp\big(\alpha\big(\log^{[2]} M(r,f)\big)\big)}{\Big(\exp\big(\beta\big(\log\gamma(\frac{1}{1-r})\big)\big)\Big)^{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[f]}},$$

where $\varrho = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[f]$.

The argument used in the proof of Lemma 3.7 can then be applied verbatim to the above lim sup. Therefore, for every $\omega < \tau^{(M)}_{(\alpha,\beta,\gamma)}[f]$, there exists a set $I_1 \subset [0,1)$ of infinite logarithmic measure such that

$$\exp\Big\{\alpha\Big(\log^{[2]}(M(r,f))\Big)\Big\} > \omega\Big(\exp\Big\{\beta\Big(\log\Big(\gamma(\frac{1}{1-r})\Big)\Big)\Big\}\Big)^{\varrho}$$

for all $r \in I_1$. This completes the proof. ☐

Lemma 3.9 ([22]). *Let f be a solution of equation (1.1), where the coefficients $A_j(z)$ $(j = 0,\dots,k-1)$ are analytic functions on the disc $\Delta_R = \{z \in \mathbb{C} : |z| < R\}, 0 < R \le \infty$. Let $n_c \in \{1,\dots,k\}$ be the number of nonzero coefficients $A_j(z)$ $(j = 0,\dots,k-1)$, and let $\theta \in [0,2\pi)$ and $\varepsilon > 0$. If $z_\theta = \nu e^{i\theta} \in \Delta_R$ is such that $A_j(z_\theta) \ne 0$ for some $j = 0,\dots,k-1$, then for all $\nu < r < R$,*

$$|f(re^{i\theta})| \le C\exp\Big(n_c\int_\nu^r \max_{j=0,\dots,k-1} |A_j(te^{i\theta})|^{\frac{1}{k-j}}\,dt\Big),$$

where $C > 0$ is a constant satisfying

$$C \le (1+\varepsilon)\max_{j=0,\dots,k-1}\Big(\frac{|f^{(j)}(z_\theta)|}{(n_c)^j \max\limits_{n=0,\dots,k-1} |A_n(z_\theta)|^{\frac{j}{k-n}}}\Big).$$

Lemma 3.10. *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ be analytic functions in the disc Δ. Then every nontrivial solution f of (1.1) satisfies*

$$\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] \le \max\big\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j], j = 0,\dots,k-1\big\}.$$

Proof. Set $\varrho = \max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j], j = 0,\dots,k-1\}$. Let $f \not\equiv 0$ be a solution of (1.1). Let $\theta_0 \in [0,2\pi]$ be such that $|f(re^{i\theta_0})| = M(r,f)$. By Lemma 3.9, we have

$$\begin{aligned} M(r,f) &\le C\exp\Big(n_c\int_\nu^r \max_{j=0,\dots,k-1} |A_j(te^{i\theta})|^{\frac{1}{k-j}}\,dt\Big) \\ &\le C\exp\Big(n_c\int_\nu^r \max_{j=0,\dots,k-1} (M(r,A_j))^{\frac{1}{k-j}}\,dt\Big) \\ &\le C\exp\Big(n_c(r-\nu)\max_{j=0,\dots,k-1} M(r,A_j)\Big). \end{aligned} \tag{3.13}$$

By the definition of $\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j]$,

$$M(r,A_j) \le \exp^{[2]}\Big\{\alpha^{-1}\Big((\varrho+\frac{\varepsilon}{2})\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\Big)\Big\}, \quad j = 0,\dots,k-1 \tag{3.14}$$

holds for all $\varepsilon > 0$ and all r sufficiently close to 1. Hence from (3.13) and (3.14) we obtain

$$\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] \le \varrho + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the desired result

$$\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] \le \max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j], j = 0, \dots, k-1\}. \qquad \square$$

4. Proof of Main Results

Proof of Theorem 2.1. Set $\varrho_j = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j]$ $(j = 0, \dots, k-1)$ and $\varrho^*_f = \varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f]$.

First, we prove that $\varrho^*_f \ge \varrho_0$. Suppose the contrary $\varrho^*_f < \varrho_0$. Let $f \not\equiv 0$ be a solution of (1.1). In accordance with (1.1), we have

$$|A_0(z)| \le |\frac{f^{(k)}(z)}{f(z)}| + |A_{k-1}(z)||\frac{f^{(k-1)}(z)}{f(z)}| + \dots + |A_1(z)||\frac{f'(z)}{f(z)}|. \tag{4.1}$$

Since the functions A_j are analytic in Δ and satisfy $\varrho_j < \varrho_0$ $(j = 1, \dots, k-1)$, there exists a constant $\lambda_1 > 0$ such that $\varrho_j < \lambda_1 < \varrho_0$ $(j = 1, \dots, k-1)$. Hence, as $r \to 1^-$,

$$M(r, A_j) < \exp^{[2]}\Big\{\alpha^{-1}\Big(\lambda_1\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\Big)\Big\}. \tag{4.2}$$

Without loss of generality, we assume that

$$\varrho^*_f < \lambda_1 < \varrho_0. \tag{4.3}$$

Applying Lemma 3.7 to the coefficient $A_0(z)$ with a constant λ_2 satisfying $\lambda_1 < \lambda_2 < \varrho_0$, we obtain

$$M(r, A_0) > \exp^{[2]}\Big\{\alpha^{-1}\Big(\lambda_2\beta\big(\log\gamma(\frac{1}{1-r})\big)\Big)\Big\}, \quad r \in I,\ r \to 1^-, \tag{4.4}$$

where $I \subset [0, 1)$ is a set of infinite logarithmic measure. The Lemma 3.1 implies the following estimate for $j = 1, \dots, k$

$$|\frac{f^{(j)}(z)}{f(z)}| \le \Big((\frac{1}{1-|z|})^{2+\varepsilon} T(s(|z|), f)\Big)^j, \quad |z| \notin F, \tag{4.5}$$

where $F \subset [0, 1)$ is a set of finite logarithmic measure.

Since $I \setminus F$ has infinite logarithmic measure, there exists a sequence $\{z_n\}$ with $|z_n| = r_n \in I \setminus F$ such that $r_n \to 1^-$. Let $s(|z_n|) = R_n$. Then $1 - |z_n| = \frac{1}{d}(1 - R_n)$, $d \in (0, 1)$.

Using (4.2), (4.4), (4.5) together with assumption (4.3), we derive from (4.1) that

$$\begin{aligned}
&\exp^{[2]}\Big\{\alpha^{-1}\Big(\lambda_2\beta\Big(\log\gamma(\frac{d}{1-R_n})\Big)\Big)\Big\} \\
&\le \Big((\frac{d}{1-R_n})^{2+\varepsilon} T(R_n, f)\Big)^k + \Big(\Big((\frac{d}{1-R_n})^{2+\varepsilon} T(R_n, f)\Big)^{k-1} + \cdots \\
&\quad + (\frac{d}{1-R_n})^{2+\varepsilon} T(R_n, f)\Big) \exp^{[2]}\Big\{\alpha^{-1}\Big(\lambda_1\beta\Big(\log\gamma(\frac{d}{1-R_n})\Big)\Big)\Big\} \\
&\le k\Big((\frac{d}{1-R_n})^{2+\varepsilon} T(R_n, f)\Big)^k \exp^{[2]}\Big\{\alpha^{-1}\Big(\lambda_1\beta(\log\gamma(\frac{d}{1-R_n}))\Big)\Big\} \\
&\le k\Big((\frac{d}{1-R_n})^{2+\varepsilon} \exp^{[2]}\Big\{\alpha^{-1}\Big((\varrho^*_f + \varepsilon)\beta(\log\gamma(\frac{1}{1-R_n}))\Big)\Big\}\Big)^k \\
&\quad \times \exp^{[2]}\Big\{\alpha^{-1}\Big(\lambda_1\beta\Big(\log\gamma(\frac{d}{1-R_n})\Big)\Big)\Big\} \\
&\le \Big(\exp^{[2]}\Big\{\alpha^{-1}\Big((\lambda_1 + \varepsilon)\beta(\log\gamma(\frac{d}{1-R_n}))\Big)\Big\}\Big)^{k+2} \\
&\le \exp^{[2]}\Big\{\alpha^{-1}\Big((\lambda_1 + 2\varepsilon)\beta\Big(\log\gamma(\frac{d}{1-R_n})\Big)\Big)\Big\}, \quad R_n \in I \setminus F,\ R_n \to 1^-.
\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$ and the monotonicity of the function α^{-1}, we obtain that $\lambda_1 \ge \lambda_2$. This contradiction proves the inequality $\varrho_0 \le \varrho^*_f$.

Second, we prove that $\varrho_0 \geq \varrho_f^*$. By using Lemma 3.10, we obtain

$$\varrho_f^* = \varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] \leq \max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j], j=0,\dots,k-1\} = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0.$$

From the last inequality and $\varrho_0 \leq \varrho_f^*$, we obtain $\varrho_f^* = \varrho_0$, that is $\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0]$. □

Proof of Theorem 2.3. Suppose that f is a nontrivial solution of (1.1). From (1.1), we can write

$$|A_0(z)| \leq |\frac{f^{(k)}(z)}{f(z)}| + |A_{k-1}(z)||\frac{f^{(k-1)}(z)}{f(z)}| + \dots + |A_1(z)||\frac{f'(z)}{f(z)}|. \tag{4.6}$$

If

$$\max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j], j=1,\dots,k-1\} < \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0 < +\infty,$$

then by Theorem 2.1, we obtain that

$$\varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0].$$

Suppose that

$$\max\{\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j], j=1,\dots,k-1\} = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0 < +\infty$$

and

$$\max\big\{\tau^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] > 0\big\} < \tau^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \tau_0 < +\infty.$$

First, we prove that

$$\varrho_f^* = \varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] \geq \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0.$$

Suppose on the contrary $\varrho_f^* = \varrho^{(M)}_{(\alpha(\log),\beta,\gamma)}[f] < \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0$. By assumption, there exists a set $K \subseteq \{1,2,\dots,k-1\}$ such that

$$\begin{gathered}\varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] = \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0, \quad \tau^{(M)}_{(\alpha,\beta,\gamma)}[A_j] < \tau^{(M)}_{(\alpha,\beta,\gamma)}[A_0], \quad j \in K,\\ \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_j] < \varrho^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \varrho_0, \quad j \in \{1,2,\dots,k-1\} \setminus K.\end{gathered}$$

Thus, we choose λ_3 and λ_4 satisfying

$$\max\{\tau^{(M)}_{(\alpha,\beta,\gamma)}[A_j] : j \in K\} < \lambda_3 < \lambda_4 < \tau^{(M)}_{(\alpha,\beta,\gamma)}[A_0] = \tau_0.$$

For $r \to 1^-$, we have

$$|A_j(z)| \leq \exp^{[2]}\Big\{\alpha^{-1}\Big(\log\Big(\lambda_3\Big(\exp\Big(\beta\big(\log\gamma(\frac{1}{1-r})\big)\Big)\Big)^{\varrho_0}\Big)\Big)\Big\}, \quad j \in K \tag{4.7}$$

and

$$\begin{aligned}|A_j(z)| &\leq \exp^{[2]}\Big\{\alpha^{-1}\Big(\xi\beta\Big(\log\gamma(\frac{1}{1-r})\Big)\Big)\Big\}\\ &= \exp^{[2]}\Big\{\alpha^{-1}\Big(\log\big(\exp\big(\beta(\log\gamma(\frac{1}{1-r}))\big)\big)^{\xi}\Big)\Big\}\\ &\leq \exp^{[2]}\Big\{\alpha^{-1}\Big(\log\big(\lambda_3(\exp(\beta(\log\gamma(\frac{1}{1-r}))))^{\varrho_0}\big)\Big)\Big\},\end{aligned} \tag{4.8}$$

for $j \in \{1,2,\dots,k-1\} \setminus K$, where $0 < \xi < \varrho_0$. By Lemma 3.8, there exists a set $I_1 \subset [0,1)$ with infinite logarithmic measure, such that for all $r \in I_1$, it holds

$$|A_0(z)| > \exp^{[2]}\Big\{\alpha^{-1}\Big(\log\Big(\lambda_4\big(\exp(\beta(\log\gamma(\frac{1}{1-r})))\big)^{\varrho_0}\Big)\Big)\Big\}. \tag{4.9}$$

By (4.5), the following estimate holds for $j=1,\dots,k$,

$$|\frac{f^{(j)}(z)}{f(z)}| \leq \Big(\big(\frac{1}{1-|z|}\big)^{2+\varepsilon}T(s(|z|),f)\Big)^{j}, \quad |z| \notin F, \tag{4.10}$$

where $F \subset [0, 1)$ is a set of finite logarithmic measure. By (4.10) and the Definition 1.6 of $(\alpha(\log), \beta, \gamma)$-order of f, for any given $\varepsilon \in (0, \max\{\frac{\lambda_4 - \lambda_3}{2}, \varrho_0 - \varrho_f^*\})$, as $s(|z|) \to 1^-$ we have

$$\begin{aligned} |\frac{f^{(j)}(z)}{f(z)}| &\le \Big((\frac{1}{1-|z|})^{2+\varepsilon} T(s(|z|), f) \Big)^j \\ &\le \Big(\exp^{[2]} \Big\{ \alpha^{-1} \Big((\varrho_f^* + \varepsilon) \beta \Big(\log \gamma (\frac{1}{1 - s(|z|)}) \Big) \Big) \Big\} \Big)^{k+1}, \ j = 1, \dots, k. \end{aligned} \tag{4.11}$$

Since $I_1 \setminus F$ is of infinite logarithmic measure, there exists a sequence $\{z_n\}$ with $|z_n| = r_n \in I_1 \setminus F$ such that $r_n \to 1^-$. Setting $s(|z_n|) = R_n$, we obtain $1 - |z_n| = \frac{1}{d}(1 - R_n)$, $d \in (0, 1)$. Therefore, by substituting (4.7), (4.8), (4.9) and (4.11) into (4.6), we obtain, for $R_n \in I_1 \setminus F$ with $R_n \to 1^-$ that

$$\begin{aligned} &\exp^{[2]} \Big\{ \alpha^{-1} \Big(\log \Big(\lambda_4 (\exp(\beta(\log \gamma(\frac{d}{1-R_n}))))^{\varrho_0} \Big) \Big) \Big\} \\ &\le k \exp^{[2]} \Big\{ \alpha^{-1} \Big(\log \Big(\lambda_3 \Big(\exp \Big(\beta(\log \gamma(\frac{d}{1-R_n})) \Big) \Big)^{\varrho_0} \Big) \Big) \Big\} \\ &\quad \times \Big[\exp^{[2]} \Big\{ \alpha^{-1} \Big((\varrho_f^* + \varepsilon) \beta \Big(\log \gamma (\frac{1}{1-R_n}) \Big) \Big) \Big\} \Big]^{k+1} \\ &\le \exp^{[2]} \Big\{ \alpha^{-1} \Big(\log \Big((\lambda_3 + 2\varepsilon)(\exp(\beta(\log \gamma(\frac{d}{1-R_n}))))^{\varrho_0} \Big) \Big) \Big\}. \end{aligned} \tag{4.12}$$

From (4.12), we obtain that $\lambda_3 \ge \lambda_4$. This contradiction implies

$$\varrho^{(M)}_{(\alpha(\log), \beta, \gamma)}[f] \ge \varrho^{(M)}_{(\alpha, \beta, \gamma)}[A_0].$$

On the other hand, by Lemma 3.10, we obtain that

$$\varrho^{(M)}_{(\alpha(\log), \beta, \gamma)}[f] \le \max\big\{ \varrho^{(M)}_{(\alpha, \beta, \gamma)}[A_j], j = 0, \dots, k-1 \big\} = \varrho^{(M)}_{(\alpha, \beta, \gamma)}[A_0].$$

Hence every nontrivial solution f of (1.1) satisfies $\varrho^{(M)}_{(\alpha(\log), \beta, \gamma)}[f] = \varrho^{(M)}_{(\alpha, \beta, \gamma)}[A_0]$. ☐

5. Conclusion

The results obtained in this paper describe the growth of solutions in terms of the generalized (α, β, γ)-order under suitable dominance conditions on the coefficients. Several questions remain open. In particular, it would be interesting to investigate whether these results can be extended to other generalized growth scales and growth–oscillation frameworks, including those considered by Chyzhykov et al. [17]. Their theory is based on integrated estimates involving logarithmic derivatives, whereas the present work relies on a logarithmic-iteration scale. Such investigations may lead to a deeper understanding of the relationships between different growth theories and the oscillatory behavior of solutions of complex differential equations.

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