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STABILITY AND DECAY FOR THE 3D ANISOTROPIC MICROPOLAR SYSTEM WITH FRACTIONAL DISSIPATION

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Abstract. This article concerns the stability and long-time behavior of 3D anisotropic micropolar systems with fractional dissipation in $\mathbb{R}^3$. By using delicate energy estimates, we prove the stability of solutions to this system when the H^m $(m \geq 2)$ norm of the initial data is suitably small, and obtain the decay rate for the horizontal derivatives of this solution.

1. Introduction

We study the following Cauchy problem of the 3D anisotropic micropolar system with fractional dissipation:

$$\begin{gathered} \partial_t u + \mu(-\Delta_h)^\alpha u - \chi\Delta u + u\cdot\nabla u + \nabla P - 2\chi\nabla\times\omega = 0, \quad x\in\mathbb{R}^3,\ t>0,\\ \partial_t\omega + \kappa(-\Delta_h)^\beta\omega - \nu\nabla\nabla\cdot\omega + 4\chi\omega + u\cdot\nabla\omega - 2\chi\nabla\times u = 0,\\ \operatorname{div} u = 0,\\ (u,\omega)|_{t=0} = (u_0,\omega_0). \end{gathered} \tag{1.1}$$

Here $u = u(x,t) \in \mathbb{R}^3$ is the velocity, $\omega = \omega(x,t) \in \mathbb{R}^3$ is the micro-rotational velocity and $P = P(x,t) \in \mathbb{R}$ is the scalar pressure function. u_0 is the given initial velocity satisfying $\operatorname{div} u_0 = 0$. The constants μ, χ and ν, κ represent the kinematic viscosity, the vortex viscosity and the angular viscosities, respectively. α and β are the parameters of the fractional dissipations corresponding to the velocity and micro-rotational velocity. The fractional operator $(-\Delta_h)^\alpha$ is defined via the Fourier transform

$$\widehat{(-\Delta_h)^\alpha f}(\xi) = |\xi_h|^{2\alpha}\widehat{f}(\xi),$$

where $\xi_h = (\xi_1,\xi_2)$ and $\widehat{f}$ denotes the Fourier transform $\widehat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,dx$. Micropolar systems were first introduced by Eringen [8], for describing physical phenomena that cannot be treated by the classical Navier-Stokes equations such as fluids consisting of particles suspended in a viscous medium. More physical explanations can be found in [17, 18, 22].

Because of the importance in physics, there has been tremendous interest in developing the mathematical theory of the micropolar system. For the 3D micropolar system with the standard Laplacian dissipation, the authors in [12, 17] obtained the global existence of weak solutions and strong solutions with initial data small. Chen-Price [2] proved the time decay rates for small L^2 strong solutions. Li-Shang [14], Cruz-Novaes [4] and Niche-Perusato [11] established the decay estimates for global weak solutions. We may refer to [1, 3, 6, 15, 20] for more results about the standard micropolar system.

For 3D anisotropic micropolar systems, Wang-Wang [26] considered the 3D micropolar system (1.1) with $\alpha = \beta = 1$, and obtained the global well-posedness with initial data small in $H^1(\mathbb{R}^3)$. Very recently, the smallness was weakened to the L^2 norm of the initial data and its vertical derivative by Shang-Liu in [21]. For the 3D micropolar system with fractional dissipation, the global well-posedness was obtained in [7, 9, 19, 25, 27]. Recently, Liu [16] established the global existence

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and uniqueness of strong solutions to the 3D micropolar system with fractional dissipation in two directions when the exponent is $5/4$.

Motivated by the above works, we shall investigate the stability of system (1.1) with $1/2 \le \alpha = \beta < 1$ in the Sobolev space H^m ($m \ge 2$), and the large-time behavior for system (1.1) in the case $1/2 \le \alpha = \beta \le 2/3$. Now, we state our first result, which establishes global well-posedness for small initial data and stability for system (1.1).

Theorem 1.1. *Suppose that the initial data $(u_0, \omega_0) \in H^m(\mathbb{R}^3)$ with $m \ge 2$ and $1/2 \le \alpha = \beta < 1$. Then there exists a constant $\varepsilon_0 > 0$ such that if*

$$\|u_0\|_{H^m(\mathbb{R}^3)} + \|\omega_0\|_{H^m(\mathbb{R}^3)} \le \varepsilon_0,$$

then system (1.1) has a unique global solution $(u, \omega) \in L^\infty((0,\infty); H^m(\mathbb{R}^3))$ satisfying for any $t > 0$,

$$\|u\|_{H^m}^2 + \|\omega\|_{H^m}^2 + \int_0^t \left(\|\Lambda_h^\alpha u\|_{H^m}^2 + \|\Lambda_h^\alpha \omega\|_{H^m}^2\right) d\tau \le C\varepsilon_0^2,$$

where $\Lambda_h^\alpha = (-\Delta_h)^{\alpha/2}$.

Remark 1.2. For the MHD system of equations, studied in [13],

$$\begin{gathered}
\partial_t u + u\cdot\nabla u + \nu(-\Delta_h)^\alpha u + \nabla P = b\cdot\nabla b, \quad x\in\mathbb{R}^3,\ t>0,\\
\partial_t b + u\cdot\nabla b + \mu(-\Delta_h)^\beta b = b\cdot\nabla u,\\
\operatorname{div} u = \operatorname{div} b = 0,\\
u(x,0) = u_0(x),\ b(x,0) = b_0(x),
\end{gathered}$$

the stability result in Theorem 1.1 is valid provided that for $m \ge 2$, the following smallness condition holds

$$\|u_0\|_{H^m(\mathbb{R}^3)} + \|b_0\|_{H^m(\mathbb{R}^3)} \le \varepsilon_0.$$

Our second result shows that the horizontal gradient of the solution established in Theorem 1.1 decays at the rate of $(1+t)^{-1}$.

Theorem 1.3. *Suppose that the initial data (u_0, ω_0) belongs to $H^m(\mathbb{R}^3)$ with $m \ge 2$ and $\frac{1}{2} \le \alpha = \beta \le \frac{2}{3}$ satisfying the smallness requirement in Theorem 1.1, that is, for a small constant $\varepsilon_0 > 0$ and $m \ge 2$,*

$$\|u_0\|_{H^{m+1}(\mathbb{R}^3)} + \|\omega_0\|_{H^{m+1}(\mathbb{R}^3)} \le \varepsilon_0.$$

Then the corresponding solution (u,ω) satisfies that for any $t \ge 0$,

$$\|\nabla_h u\|_{H^m}^2 + \|\nabla_h \omega\|_{H^m}^2 \le C(\varepsilon_0)(1+t)^{-1}.$$

Remark 1.4. Theorem 1.3 does not provide decay rate in the L^2-norm, which is not surprising. Indeed, we notice that the time-decay property of the L^2-norm of the solution to the heat equation remains unknown when we only set initial data in L^2-norm.

The remainder of this paper is structured as follows. In Section 2, we introduce some notation and inequalities which will be used later. The proof of Theorem 1.1 is given in Section 3. Finally, Section 4 is devoted to the proof of Theorem 1.3.

2. Preliminaries

We introduce some notation to be used in this article. For simplicity, we write Λ^γ (with $\gamma > 0$) for $(-\Delta)^{\frac{\gamma}{2}}$, $\int \cdot dx$ for $\int_{\mathbb{R}^3} \cdot\, dx$, $\|\cdot\|_{L^p}$ for $\|\cdot\|_{L^p(\mathbb{R}^3)}$ and $\|\cdot\|_{H^s}$ for $\|\cdot\|_{H^s(\mathbb{R}^3)}$. The letter C is a positive constant, which may be different on different lines. The following two lemmas provide some anisotropic bounds.

Lemma 2.1 ([13])**.** *Let $0 < \alpha < 1$, then there exist two absolute constants C_1 and C_2 such that*

$$\begin{aligned}
\int_{\mathbb{R}^3} |fgj|\,dx &\le C_1 \|f\|_{L^2}^{1/2} \|\partial_3 f\|_{L^2}^{1/2} \|\Lambda_h^{1-\alpha} g\|_{L^2} \|\Lambda_h^\alpha j\|_{L^2},\\
\int_{\mathbb{R}^3} |fgj|\,dx &\le C_2 \|\Lambda_h^{2-2\alpha} f\|_{L^2}^{1/2} \|\partial_3 f\|_{L^2}^{1/2} \|g\|_{L^2} \|\Lambda_h^\alpha j\|_{L^2}.
\end{aligned}$$

Lemma 2.2 ([21]). *Let f, g and j be smooth functions in $\mathbb{R}^3$, then there holds*

$$\int_{\mathbb{R}^3} |fgj|\,dx \le C_1 \|f\|_{L^2}^{1/2}\|\partial_1 f\|_{L^2}^{1/2}\|g\|_{L^2}^{1/2}\|\partial_2 g\|_{L^2}^{1/2}\|j\|_{L^2}^{1/2}\|\partial_3 j\|_{L^2}^{1/2},$$

$$\int_{\mathbb{R}^3} |fgj|\,dx \le C_2 \|f\|_{L^2}^{1/4}\|\partial_1 f\|_{L^2}^{1/4}\|\partial_2 f\|_{L^2}^{1/4}\|\partial_1\partial_2 f\|_{L^2}^{1/4}\|g\|_{L^2}^{1/2}\|\partial_3 g\|_{L^2}^{1/2}\|j\|_{L^2},$$

where C_1 and C_2 are two absolute constants.

To obtain the decay rate in Theorem 1.3, we need the following lemma.

Lemma 2.3 ([5, 24]). *Assume that a_0, a_1 are positive constants. Let $f(t)$ be a nonnegative function satisfying the following two conditions:*

(i) *Time integrability: $\int_0^\infty f(s)\,ds \le a_0 < \infty$;*
(ii) *Generalized degradation: for $0 \le s \le t$, $f(t) \le a_1 f(s)$.*

Then setting $a_2 = \max\{2a_1 f(0), 2a_0a_1\}$, for any $t > 0$, we have $f(t) \le a_2(1+t)^{-1}$.

3. Proof of Theorem 1.1

As the local well-posedness of (1.1) in $H^m(\mathbb{R}^3)$ with $m \ge 2$ can be established via a standard method (see [10]), it suffices to obtain the a priori estimate of $\|(u,\omega)\|_{H^m}$ with $m \ge 2$. Because of the norm equivalence

$$\|f\|_{H^m}^2 \sim \|f\|_{L^2}^2 + \sum_{i=1}^3 \|\partial_i^m f\|_{L^2}^2,$$

we only need to bound the L^2-norms and H^m-norms.

Proof. By taking the L^2 inner product of the u and ω in (1.1) with u and ω respectively, we obtain that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) + \mu\|\Lambda_h^\alpha u\|_{L^2}^2 + \kappa\|\Lambda_h^\alpha \omega\|_{L^2}^2 + \chi\|\nabla u\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 + \nu\|\nabla\cdot\omega\|_{L^2}^2 \\
&= 2\chi\int \nabla\times\omega\cdot u\,dx + 2\chi\int \nabla\times u\cdot\omega\,dx.
\end{aligned}$$

From Hölder's and Young's inequalities, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2\right) + \mu\|\Lambda_h^\alpha u\|_{L^2}^2 + \kappa\|\Lambda_h^\alpha \omega\|_{L^2}^2 + \chi\|\nabla u\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2 + \nu\|\nabla\cdot\omega\|_{L^2}^2 \\
&= 4\chi\int \nabla\times u\cdot\omega\,dx \\
&\le \chi\|\nabla u\|_{L^2}^2 + 4\chi\|\omega\|_{L^2}^2.
\end{aligned}$$

By integration in time, we have

$$\begin{aligned}
&\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \mu\int_0^t \|\Lambda_h^\alpha u\|_{L^2}^2\,d\tau + \kappa\int_0^t \|\Lambda_h^\alpha\omega\|_{L^2}^2\,d\tau + \nu\int_0^t \|\nabla\cdot\omega\|_{L^2}^2\,d\tau \\
&\le \|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2.
\end{aligned} \tag{3.1}$$

Acting ∂_i^m $(i = 1,2,3)$ to (1.1) and taking the L^2 inner product of the resulting equations with $\partial_i^m u$ and $\partial_i^m\omega$ respectively, we infer

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\sum_{i=1}^3 \|\partial_i^m u\|_{L^2}^2 + \sum_{i=1}^3 \|\partial_i^m \omega\|_{L^2}^2\Big) + \mu\sum_{i=1}^3 \|\partial_i^m\Lambda_h^\alpha u\|_{L^2}^2 + \kappa\sum_{i=1}^3 \|\partial_i^m\Lambda_h^\alpha\omega\|_{L^2}^2 \\
&+ \chi\sum_{i=1}^3 \|\partial_i^m\nabla u\|_{L^2}^2 + 4\chi\sum_{i=1}^3 \|\partial_i^m\omega\|_{L^2}^2 + \nu\sum_{i=1}^3 \|\partial_i^m\nabla\cdot w\|_{L^2}^2 \\
&= -\sum_{i=1}^3 \int \partial_i^m(u\cdot\nabla u)\cdot\partial_i^m u\,dx + 2\chi\sum_{i=1}^3 \int \partial_i^m(\nabla\times\omega)\cdot\partial_i^m u\,dx
\end{aligned}$$

$$\begin{aligned}&-\sum_{i=1}^{3}\int \partial_i^m(u\cdot\nabla\omega)\cdot\partial_i^m\omega\,dx+2\chi\sum_{i=1}^{3}\int\partial_i^m(\nabla\times u)\cdot\partial_i^m\omega\,dx\\ :=\;& I_1+I_2+I_3+I_4.\end{aligned}$$

By Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}I_2+I_4&=2\chi\sum_{i=1}^{3}\int\partial_i^m\omega\cdot\partial_i^m(\nabla\times u)\,dx+2\chi\sum_{i=1}^{3}\int\partial_i^m(\nabla\times u)\cdot\partial_i^m\omega\,dx\\ &\le 4\chi\sum_{i=1}^{3}\|\partial_i^m\nabla u\|_{L^2}^2\|\partial_i^m\omega\|_{L^2}\\ &\le \chi\sum_{i=1}^{3}\|\partial_i^m\nabla u\|_{L^2}^2+4\chi\sum_{i=1}^{3}\|\partial_i^m\omega\|_{L^2}^2.\end{aligned}$$

Direct computations show that

$$\begin{aligned}I_1&=-\sum_{i=1}^{3}\int\partial_i^m(u\cdot\nabla u)\cdot\partial_i^m u\,dx\\ &=-\sum_{i=1}^{2}\int\partial_i^m(u\cdot\nabla u)\cdot\partial_i^m u\,dx-\int\partial_3^m(u_h\cdot\nabla_h u)\cdot\partial_3^m u\,dx-\int\partial_3^m(u_3\partial_3 u)\cdot\partial_3^m u\,dx\\ &:=I_{11}+I_{12}+I_{13}.\end{aligned}$$

Using Hölder's inequality, Sobolev embedding $\dot H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, and $\dot H^{\frac12}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$, we obtain

$$\begin{aligned}I_{11}&=-\sum_{i=1}^{2}\sum_{l=1}^{m}C_m^l\int\partial_i^l u\cdot\nabla\partial_i^{m-l}u\cdot\partial_i^m u\,dx\\ &\le C\sum_{i=1}^{2}\|\partial_i u\|_{L^6}\|\nabla\partial_i^{m-1}u\|_{L^3}\|\partial_i^m u\|_{L^2}+C\sum_{i=1}^{2}\sum_{l=2}^{m}\|\partial_i^l u\|_{L^3}\|\nabla\partial_i^{m-l}u\|_{L^6}\|\partial_i^m u\|_{L^2}\\ &\le C\sum_{i=1}^{2}\|\Lambda\partial_i u\|_{L^2}\|\Lambda^{3/2}\partial_i^{m-1}u\|_{L^2}\|\partial_i^m u\|_{L^2}+C\sum_{i=1}^{2}\sum_{l=2}^{m}\|\Lambda^{1/2}\partial_i^l u\|_{L^2}\|\Lambda^2\partial_i^{m-l}u\|_{L^2}\|\partial_i^m u\|_{L^2}\\ &\le C\|u\|_{H^m}\|\Lambda_h^\alpha u\|_{H^m}^2,\end{aligned}$$

where $C_m^l=\frac{m!}{l!(m-l)!}$. From Lemma 2.1 we obtain that

$$\begin{aligned}I_{12}&=-\sum_{l=1}^{m}C_m^l\int\partial_3^l u_h\cdot\partial_3^{m-l}\nabla_h u\cdot\partial_3^m u\,dx\\ &=-\sum_{l=1}^{m-1}C_m^l\int\partial_3^l u_h\cdot\partial_3^{m-l}\nabla_h u\cdot\partial_3^m u\,dx-C_m^m\int\partial_3^m u_h\cdot\nabla_h u\cdot\partial_3^m u\,dx\\ &\le C\sum_{l=1}^{m-1}\|\partial_3^l u_h\|_{L^2}^{1/2}\|\partial_3\partial_3^l u_h\|_{L^2}^{1/2}\|\Lambda_h^{2-\alpha}\partial_3^{m-l}u\|_{L^2}\|\Lambda_h^\alpha\partial_3^m u\|_{L^2}\\ &\quad+C\|\Lambda_h^{2-2\alpha}\nabla_h u\|_{L^2}^{1/2}\|\partial_3\nabla_h u\|_{L^2}^{1/2}\|\partial_3^m u\|_{L^2}\|\Lambda_h^\alpha\partial_3^m u\|_{L^2}\\ &\le C\|u\|_{H^m}\|\Lambda_h^\alpha u\|_{H^m}^2,\end{aligned}$$

where we used the fact that, for $\frac12\le\alpha<1$ and $l=1,\cdots,m-1$,

$$\begin{aligned}\|\Lambda_h^{2-2\alpha}\partial_3^{m-l}u\|_{L^2}&=\|\Lambda_h^\alpha\Lambda_h^{2-2\alpha}\partial_3^{m-l}u\|_{L^2}\le C\|\Lambda_h^\alpha u\|_{H^m},\\ \|\Lambda_h^{2-2\alpha}\nabla_h u\|_{L^2}&=\|\Lambda_h^\alpha\Lambda_h^{3-3\alpha}\nabla_h u\|_{L^2}\le C\|\Lambda_h^\alpha\nabla_h u\|_{H^m}.\end{aligned}$$

Thanks to the incompressible condition $\partial_3 u_3 = -\nabla_h \cdot u_h$ and Lemma 2.1, we have

$$
\begin{aligned}
I_{13} &= -\int \partial_3^m (u_3 \partial_3 u) \cdot \partial_3^m u \, dx \\
&= \sum_{l=1}^{m} C_m^l \int \partial_3^l u_3 \partial_3^{m-l} \partial_3 u \partial_3^m u \, dx \\
&= \sum_{l=1}^{m} C_m^l \int \partial_3^{l-1} \nabla_h \cdot u_h \cdot \partial_3^{m-l+1} u \cdot \partial_3^m u \, dx \\
&= C_m^1 \int \nabla_h \cdot u_h \cdot \partial_3^m u \cdot \partial_3^m u \, dx + \sum_{l=2}^{m} C_m^l \int \partial_3^{l-1} \nabla_h \cdot u_h \cdot \partial_3^{m-l+1} u \cdot \partial_3^m u \, dx \\
&\le C \|\Lambda_h^{2-2\alpha} \nabla_h u_h\|_{L^2}^{1/2} \|\partial_3 \nabla_h u_h\|_{L^2}^{1/2} \|\partial_3^m u\|_{L^2} \|\Lambda_h^\alpha \partial_3^m u\|_{L^2} \\
&\quad + C \sum_{l=2}^{m} \|\partial_3^{m-l+1} u\|_{L^2}^{1/2} \|\partial_3 \partial_3^{m-l+1} u\|_{L^2}^{1/2} \|\Lambda_h^{1-\alpha} \partial_3^{l-1} \nabla_h u_h\|_{L^2} \|\Lambda_h^\alpha \partial_3^m u\|_{L^2} \\
&\le C \|u\|_{H^m} \|\Lambda_h^\alpha u\|_{H^m}^2 .
\end{aligned}
$$

From the above, it follows that

$$
I_1 \le C \|u\|_{H^m} \|\Lambda_h^\alpha u\|_{H^m}^2 .
$$

Analogously to the treatments of I_1, a routine computation gives

$$
\begin{aligned}
I_3 &= -\sum_{i=1}^{3} \int \partial_i^m (u \cdot \nabla \omega) \cdot \partial_i^m \omega \, dx \\
&= -\sum_{i=1}^{2} \int \partial_i^m (u \cdot \nabla \omega) \cdot \partial_i^m \omega \, dx - \int \partial_3^m (u_h \cdot \nabla_h \omega) \cdot \partial_3^m \omega \, dx - \int \partial_3^m (u_3 \partial_3 \omega) \cdot \partial_3^m \omega \, dx \\
&:= I_{31} + I_{32} + I_{33}.
\end{aligned}
$$

In view of Hölder's inequality and the Sobolev embedding inequality, we have

$$
\begin{aligned}
I_{31} &= -\sum_{i=1}^{2} \sum_{l=1}^{m} C_m^l \int \partial_i^l u \cdot \nabla \partial_i^{m-l} \omega \cdot \partial_i^m \omega \, dx \\
&\le C \sum_{i=1}^{2} \|\partial_i u\|_{L^6} \|\nabla \partial_i^{m-1} \omega\|_{L^3} \|\partial_i^m \omega\|_{L^2} + C \sum_{i=1}^{2} \sum_{l=2}^{m} \|\partial_i^l u\|_{L^3} \|\nabla \partial_i^{m-l} \omega\|_{L^6} \|\partial_i^m \omega\|_{L^2} \\
&\le C \sum_{i=1}^{2} \|\Lambda \partial_i u\|_{L^2} \|\Lambda^{3/2} \partial_i^{m-1} \omega\|_{L^2} \|\partial_i^m \omega\|_{L^2} + C \sum_{i=1}^{2} \sum_{l=2}^{m} \|\Lambda^{1/2} \partial_i^l u\|_{L^2} \|\Lambda^2 \partial_i^{m-l} \omega\|_{L^2} \|\partial_i^m \omega\|_{L^2} \\
&\le C \|\omega\|_{H^m} \|\Lambda_h^\alpha u\|_{H^m} \|\Lambda_h^\alpha \omega\|_{H^m} .
\end{aligned}
$$

Using Lemma 2.1 yields

$$
\begin{aligned}
I_{32} &= -\int \partial_3^m (u_h \cdot \nabla_h w) \cdot \partial_3^m \omega \, dx \\
&= -\sum_{l=1}^{m-1} C_m^l \int \partial_3^l u_h \cdot \partial_3^{m-l} \nabla_h \omega \cdot \partial_3^m \omega \, dx - C_m^m \int \partial_3^m u_h \cdot \nabla_h \omega \cdot \partial_3^m \omega \, dx \\
&\le C \sum_{l=1}^{m-1} \|\partial_3^l u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_3^l u_h\|_{L^2}^{1/2} \|\Lambda_h^{2-\alpha} \partial_3^{m-l} \omega\|_{L^2} \|\Lambda_h^\alpha \partial_3^m \omega\|_{L^2} \\
&\quad + C \|\Lambda_h^{2-2\alpha} \nabla_h \omega\|_{L^2}^{\frac{1}{2}} \|\partial_3 \nabla_h \omega\|_{L^2}^{1/2} \|\partial_3^m u_h\|_{L^2} \|\Lambda_h^\alpha \partial_3^m \omega\|_{L^2} \\
&\le C \|u\|_{H^m} \|\Lambda_h^\alpha \omega\|_{H^m}^2 ,
\end{aligned}
$$

and

$$\begin{aligned}
I_{33} &= -\int \partial_3^m (u_3 \partial_3 \omega) \cdot \partial_3^m \omega \, dx \\
&= \sum_{l=1}^m C_m^l \int \partial_3^l u_3 \partial_3^{m-l} \partial_3 \omega \cdot \partial_3^m \omega \, dx = \sum_{l=1}^m C_m^l \int \partial_3^{l-1} \nabla_h \cdot u_h \cdot \partial_3^{m-l+1} \omega \cdot \partial_3^m \omega \, dx \\
&= C_m^1 \int \nabla_h \cdot u_h \cdot \partial_3^m \omega \cdot \partial_3^m \omega \, dx + \sum_{l=2}^m C_m^l \int \partial_3^{l-1} \nabla_h \cdot u_h \cdot \partial_3^{m-l+1} \omega \cdot \partial_3^m \omega \, dx \\
&\le C \|\Lambda_h^{2-2\alpha} \nabla_h u_h\|_{L^2}^{1/2} \|\partial_3 \nabla_h u_h\|_{L^2}^{1/2} \|\partial_3^m \omega\|_{L^2} \|\Lambda_h^\alpha \partial_3^m \omega\|_{L^2} \\
&\quad + C \sum_{l=2}^m \|\partial_3^{m-l+1} \omega\|_{L^2}^{1/2} \|\partial_3 \partial_3^{m-l+1} \omega\|_{L^2}^{1/2} \|\Lambda_h^{1-\alpha} \partial_3^{l-1} \nabla_h u_h\|_{L^2} \|\Lambda_h^\alpha \partial_3^m \omega\|_{L^2} \\
&\le C \|\omega\|_{H^m} \|\Lambda_h^\alpha u\|_{H^m} \|\Lambda_h^\alpha \omega\|_{H^m}.
\end{aligned}$$

Thus, we immediately obtain

$$I_3 \le C\|u\|_{H^m} \|\Lambda_h^\alpha u\|_{H^m}^2 + C\|\omega\|_{H^m} \|\Lambda_h^\alpha u\|_{H^m} \|\Lambda_h^\alpha \omega\|_{H^m}.$$

Gathering these estimates and using Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\sum_{i=1}^3 \|\partial_i^m u\|_{L^2}^2 + \sum_{i=1}^3 \|\partial_i^m u\|_{L^2}^2 \Big) + \mu \sum_{i=1}^3 \|\partial_i^m \Lambda_h^\alpha u\|_{L^2}^2 + \kappa \sum_{i=1}^3 \|\partial_i^m \Lambda_h^\alpha u\|_{L^2}^2 \\
&\le C \left(\|u\|_{H^m} + \|\omega\|_{H^m} \right) \left(\|\Lambda_h^\alpha u\|_{H^m}^2 + \|\Lambda_h^\alpha \omega\|_{H^m}^2 \right).
\end{aligned}$$

Integrating the above inequality in time and adding the resulting inequality to (3.1), we obtain

$$\begin{aligned}
&\|u\|_{H^m}^2 + \|\omega\|_{H^m}^2 + \mu \int_0^t \|\Lambda_h^\alpha u\|_{H^m}^2 \, d\tau + \kappa \int_0^t \|\Lambda_h^\alpha \omega\|_{H^m}^2 \, d\tau \\
&\le \|u_0\|_{H^m}^2 + \|\omega_0\|_{H^m}^2 + C \int_0^t \left(\|u\|_{H^m} + \|\omega\|_{H^m} \right) \left(\|\Lambda_h^\alpha u\|_{H^m}^2 + \|\Lambda_h^\alpha u\|_{H^m}^2 \right) \, d\tau.
\end{aligned} \tag{3.2}$$

If we set

$$E(t) := \sup_{0 \le \tau \le t} \|(u, \omega)\|_{H^m}^2 + \int_0^t \left(\|\Lambda_h^\alpha u\|_{H^m}^2 + \|\Lambda_h^\alpha \omega\|_{H^m}^2 \right) \, d\tau,$$

and assume that

$$E(t) \le M := \frac{1}{(2C_1)^2},$$

then (3.2) leads to

$$E(t) \le C_0 E(0) + C_1 E^{3/2}(t) \le C_0 \varepsilon_0^2 + C_1 M^{1/2} E(t) \le C_0 \varepsilon_0^2 + \frac{1}{2} E(t),$$

which implies $E(t) \le 2C_0 \varepsilon_0^2$. Choosing

$$\varepsilon_0 = \min\big\{1, (\frac{M}{4C_0})^{1/2}\big\},$$

we obtain $E(t) \le \frac{M}{2}$. Using a bootstrapping argument [23], we obtain for any time $t > 0$,

$$E(t) \le C\varepsilon_0^2 \le C\varepsilon_0.$$

This completes the proof. □

4. Proof of Theorem 1.3

By Lemma 2.3, we need to verify that $f(t) := \|\nabla_h u(t)\|_{H^m}^2 + \|\nabla_h \omega(t)\|_{H^m}^2$ satisfying the two conditions in Lemma 2.3. That means, we need to prove that there are two positive constants a_0, a_1 such that

$$\int_0^\infty \left(\|\nabla_h u(s)\|_{H^m}^2 + \|\nabla_h \omega(s)\|_{H^m}^2\right) ds \le a_0, \tag{4.1}$$

and, for any $0 \le s \le t$,

$$\|\nabla_h u(t)\|_{H^m}^2 + \|\nabla_h \omega(t)\|_{H^m}^2 \le a_1\big(\|\nabla_h u(s)\|_{H^m}^2 + \|\nabla_h \omega(s)\|_{H^m}^2\big), \tag{4.2}$$

hold.

Proof. We firstly verify (4.1). Using a calculation method analogous to the proof of Theorem 1.1, we have for any $t > 0$,

$$\begin{aligned} &\|u\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \mu \int_0^t \|\nabla_h u\|_{L^2}^2 \,d\tau + \kappa \int_0^t \|\nabla_h \omega\|_{L^2}^2 \,d\tau + \nu \int_0^t \|\nabla \cdot \omega\|_{L^2}^2 \,d\tau \\ &\le \|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\sum_{i=1}^3 \|\partial_i^m u\|_{L^2}^2 + \sum_{i=1}^3 \|\partial_i^m \omega\|_{L^2}^2\Big) + \mu \sum_{i=1}^3 \|\partial_i^m \nabla_h u\|_{L^2}^2 + \kappa \sum_{i=1}^3 \|\partial_i^m \nabla_h \omega\|_{L^2}^2 \\ &+ \chi \sum_{i=1}^3 \|\partial_i^m \nabla u\|_{L^2}^2 + 4\chi \sum_{i=1}^3 \|\partial_i^m \omega\|_{L^2}^2 + \nu \sum_{i=1}^3 \|\partial_i^m \nabla \cdot w\|_{L^2}^2 \\ &= -\sum_{i=1}^3 \int \partial_i^m (u \cdot \nabla u) \cdot \partial_i^m u \,dx + 2\chi \sum_{i=1}^3 \int \partial_i^m (\nabla \times \omega) \cdot \partial_i^m u \,dx \\ &\quad - \sum_{i=1}^3 \int \partial_i^m (u \cdot \nabla \omega) \cdot \partial_i^m \omega \,dx + 2\chi \sum_{i=1}^3 \int \partial_i^m (\nabla \times u) \cdot \partial_i^m \omega \,dx \\ &:= I_1' + I_2' + I_3' + I_4'. \end{aligned} \tag{4.4}$$

Similar to I_2 and I_4 in the proof of Theorem 1.1,

$$\begin{aligned} I_2' + I_4' &= 2\chi \sum_{i=1}^3 \int \partial_i^m \omega \cdot \partial_i^m (\nabla \times u) \,dx + 2\chi \sum_{i=1}^3 \int \partial_i^m (\nabla \times u) \cdot \partial_i^m \omega \,dx \\ &\le \chi \sum_{i=1}^3 \|\partial_i^m \nabla u\|_{L^2}^2 + 4\chi \sum_{i=1}^3 \|\partial_i^m \omega\|_{L^2}^2. \end{aligned}$$

Following the same process as for I_1, we have

$$\begin{aligned} I_1' &= -\sum_{i=1}^3 \int \partial_i^m (u \cdot \nabla u) \cdot \partial_i^m u \,dx \\ &= -\sum_{i=1}^2 \int \partial_i^m (u \cdot \nabla u) \cdot \partial_i^m u \,dx - \int \partial_3^m (u_h \cdot \nabla_h u) \cdot \partial_3^m u \,dx - \int \partial_3^m (u_3 \partial_3 u) \cdot \partial_3^m u \,dx \\ &:= I_{11}' + I_{12}' + I_{13}'. \end{aligned}$$

We use Hölder's inequality, Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, and $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ to bound I_{11}' as follows

$$I_{11}' = -\sum_{i=1}^2 \sum_{l=1}^m C_m^l \int \partial_i^l u \cdot \nabla \partial_i^{m-l} u \cdot \partial_i^m u \,dx$$

$$
\begin{aligned}
&\le C\sum_{i=1}^{2}\|\partial_i u\|_{L^6}\|\nabla\partial_i^{m-1}u\|_{L^3}\|\partial_i^m u\|_{L^2}+C\sum_{i=1}^{2}\sum_{l=2}^{m}\|\partial_i^l u\|_{L^3}\|\nabla\partial_i^{m-l}u\|_{L^6}\|\partial_i^m u\|_{L^2}\\
&\le C\sum_{i=1}^{2}\|\Lambda\partial_i u\|_{L^2}\|\Lambda^{3/2}\partial_i^{m-1}u\|_{L^2}\|\partial_i^m u\|_{L^2}+C\sum_{i=1}^{2}\sum_{l=2}^{m}\|\Lambda^{1/2}\partial_i^l u\|_{L^2}\|\Lambda^2\partial_i^{m-l}u\|_{L^2}\|\partial_i^m u\|_{L^2}\\
&\le C\|u\|_{H^m}\|\nabla_h u\|_{H^m}^2.
\end{aligned}
$$

Applying Lemma 2.2 enables us to conclude that

$$
\begin{aligned}
I_{12}'&=-\sum_{l=1}^{m}C_m^l\int\partial_3^l u_h\cdot\partial_3^{m-l}\nabla_h u\cdot\partial_3^m u\,dx\\
&\le C\sum_{l=1}^{m}\|\partial_3^l u_h\|_{L^2}^{1/2}\|\partial_1\partial_3^l u_h\|_{L^2}^{1/2}\|\partial_3^{m-l}\nabla_h u\|_{L^2}^{1/2}\|\partial_3\partial_3^{m-l}\nabla_h u\|_{L^2}^{1/2}\|\partial_3^m u\|_{L^2}^{1/2}\|\partial_2\partial_3^m u\|_{L^2}^{1/2}\\
&\le C\|u\|_{H^m}\|\nabla_h u\|_{H^m}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{13}'&=-\int\partial_3^m(u_3\partial_3 u)\cdot\partial_3^m u\,dx=\sum_{l=1}^{m}C_m^l\int\partial_3^l u_3\partial_3^{m-l}\partial_3 u\partial_3^m u\,dx\\
&=\sum_{l=1}^{m}C_m^l\int\partial_3^{l-1}\nabla_h\cdot u_h\cdot\partial_3^{m-l+1}u\cdot\partial_3^m u\,dx\\
&\le C\sum_{l=1}^{m}\|\partial_3^{l-1}\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3\partial_3^{l-1}\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3^{m-l+1}u\|_{L^2}^{1/2}\|\partial_1\partial_3^{m-l+1}u\|_{L^2}^{1/2}\|\partial_3^m u\|_{L^2}^{1/2}\|\partial_2\partial_3^m u\|_{L^2}^{1/2}\\
&\le C\|u\|_{H^m}\|\nabla_h u\|_{H^m}^2.
\end{aligned}
$$

Hence, we have

$$
I_1'\le C\|u\|_{H^m}\|\nabla_h u\|_{H^m}^2.
$$

In a same way as for obtaining I_3, we have

$$
\begin{aligned}
I_3'&=-\sum_{i=1}^{3}\int\partial_i^m(u\cdot\nabla\omega)\cdot\partial_i^m\omega\,dx\\
&=-\sum_{i=1}^{2}\int\partial_i^m(u\cdot\nabla\omega)\cdot\partial_i^m\omega\,dx-\int\partial_3^m(u_h\cdot\nabla_h\omega)\cdot\partial_3^m\omega\,dx-\int\partial_3^m(u_3\partial_3\omega)\cdot\partial_3^m\omega\,dx\\
&:=I_{31}'+I_{32}'+I_{33}'.
\end{aligned}
$$

From Hölder's inequality and Sobolev embedding inequality we have

$$
\begin{aligned}
I_{31}'&=-\sum_{i=1}^{2}\sum_{l=1}^{m}C_m^l\int\partial_i^l u\cdot\nabla\partial_i^{m-l}\omega\cdot\partial_i^m\omega\,dx\\
&\le C\sum_{i=1}^{2}\|\partial_i u\|_{L^6}\|\nabla\partial_i^{m-1}\omega\|_{L^3}\|\partial_i^m\omega\|_{L^2}+C\sum_{i=1}^{2}\sum_{l=2}^{m}\|\partial_i^l u\|_{L^3}\|\nabla\partial_i^{m-l}\omega\|_{L^6}\|\partial_i^m\omega\|_{L^2}\\
&\le C\sum_{i=1}^{2}\|\Lambda\partial_i u\|_{L^2}\|\Lambda^{3/2}\partial_i^{m-1}\omega\|_{L^2}\|\partial_i^m\omega\|_{L^2}+C\sum_{i=1}^{2}\sum_{l=2}^{m}\|\Lambda^{1/2}\partial_i^l u\|_{L^2}\|\Lambda^2\partial_i^{m-l}\omega\|_{L^2}\|\partial_i^m\omega\|_{L^2}\\
&\le C\|\omega\|_{H^m}\|\nabla_h u\|_{H^m}\|\nabla_h\omega\|_{H^m}.
\end{aligned}
$$

Applying Lemma 2.2 gives

$$
I_{32}'=-\int\partial_3^m(u_h\cdot\nabla_h w)\cdot\partial_3^m\omega\,dx
$$

$$
\begin{aligned}
&= -\sum_{l=1}^{m} C_m^l \int \partial_3^l u_h \cdot \partial_3^{m-l}\nabla_h \omega \cdot \partial_3^m \omega \, dx \\
&\le C \sum_{l=1}^{m} \|\partial_3^l u_h\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_3^l u_h\|_{L^2}^{1/2} \|\partial_3^{m-l}\nabla_h \omega\|_{L^2}^{1/2} \|\partial_3 \partial_3^{m-l}\nabla_h \omega\|_{L^2}^{1/2} \|\partial_3^m \omega\|_{L^2}^{1/2} \|\partial_2 \partial_3^m \omega\|_{L^2}^{1/2} \\
&\le C \|u\|_{H^m}^{1/2} \|\omega\|_{H^m}^{1/2} \|\nabla_h u\|_{H^m}^{1/2} \|\nabla_h \omega\|_{H^m}^{3/2},
\end{aligned}
$$

and

$$
\begin{aligned}
I'_{33} &= -\int \partial_3^m (u_3 \partial_3 \omega) \cdot \partial_3^m \omega \, dx \\
&= \sum_{l=1}^{m} C_m^l \int \partial_3^l u_3 \partial_3^{m-l} \partial_3 \omega \cdot \partial_3^m \omega \, dx = \sum_{l=1}^{m} C_m^l \int \partial_3^{l-1} \nabla_h \cdot u_h \cdot \partial_3^{m-l+1} \omega \cdot \partial_3^m \omega \, dx \\
&\le C \sum_{l=1}^{m} \|\partial_3^{l-1}\nabla_h u_h\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_3^{l-1}\nabla_h u_h\|_{L^2}^{1/2} \|\partial_3^{m-l+1}\omega\|_{L^2}^{1/2} \|\partial_1 \partial_3^{m-l+1}\omega\|_{L^2}^{1/2} \|\partial_3^m \omega\|_{L^2}^{1/2} \|\partial_2 \partial_3^m \omega\|_{L^2}^{1/2} \\
&\le \|\omega\|_{H^m} \|\nabla_h u\|_{H^m} \|\nabla_h \omega\|_{H^m}.
\end{aligned}
$$

Therefore,

$$
I'_3 \le C\|\omega\|_{H^m}\|\nabla_h u\|_{H^m}\|\nabla_h \omega\|_{H^m} + C\|u\|_{H^m}^{1/2}\|\omega\|_{H^m}^{1/2}\|\nabla_h u\|_{H^m}^{1/2}\|\nabla_h \omega\|_{H^m}^{3/2}.
$$

Gathering these estimates in (4.4) and using Young's inequality, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\sum_{i=1}^{3}\|\partial_i^m u\|_{L^2}^2 + \sum_{i=1}^{3}\|\partial_i^m u\|_{L^2}^2\Big) + \mu \sum_{i=1}^{3}\|\partial_i^m \nabla_h u\|_{L^2}^2 + \kappa \sum_{i=1}^{3}\|\partial_i^m \nabla_h u\|_{L^2}^2 \\
&\le C\left(\|u\|_{H^m} + \|\omega\|_{H^m}\right)\left(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h \omega\|_{H^m}^2\right).
\end{aligned}
$$

Integrating the above inequality in time and adding the resulting inequality with (4.3), we have

$$
\begin{aligned}
&\|u\|_{H^m}^2 + \|\omega\|_{H^m}^2 + \mu\int_0^t \|\nabla_h u\|_{H^m}^2 \, d\tau + \kappa \int_0^t \|\nabla_h \omega\|_{H^m}^2 \, d\tau \\
&\le \|u_0\|_{H^m}^2 + \|\omega_0\|_{H^m}^2 + C\int_0^t \left(\|u\|_{H^m} + \|\omega\|_{H^m}\right)\left(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h u\|_{H^m}^2\right) \, d\tau.
\end{aligned}
\tag{4.5}
$$

By a similar bootstrapping argument in the proof of Theorem 1.1, we set

$$
E'(t) := \sup_{0\le\tau\le t} \|(u,\omega)\|_{H^m}^2 + \int_0^t \left(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h \omega\|_{H^m}^2\right) \, d\tau,
$$

and assume that

$$
E'(t) \le M' := \frac{1}{(2C_1')^2},
$$

then combining the smallness condition of Theorem 1.3, and (4.5) lead to

$$
E'(t) \le C_0' E'(0) + C_1' E'^{3/2}(t) \le C_0' \varepsilon_0^2 + C_1' M^{1/2} E'(t) \le C_0' \varepsilon_0^2 + \frac{1}{2}E'(t),
$$

which implies $E'(t) \le 2C_0\varepsilon_0^2$. Choosing

$$
\varepsilon_0 = \min\big\{1, (\frac{M'}{4C_0'})^{1/2}, (\frac{M}{4C_0})^{1/2}\big\},
$$

we infer that $E'(t) \le \frac{M'}{2}$. Then for each time $t > 0$, we have

$$
\|u\|_{H^m}^2 + \|\omega\|_{H^m}^2 + \int_0^t \left(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h \omega\|_{H^m}^2\right) d\tau \le C'\varepsilon_0^2.
$$

From the entire reasoning process above, it can be seen that all the constants appearing above are independent of time t. Thus, we can deduce that there exists a constant a_0 for which (4.1) holds.

Now, to verify (4.2), we first show the L^2 estimate of $(\nabla_h u, \nabla_h \omega)$. Applying ∇_h to (1.1), taking the L^2 inner product of the resulting equations with $\nabla_h u$ and $\nabla_h \omega$ respectively, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_h u\|_{L^2}^2 + \|\nabla_h \omega\|_{L^2}^2\right) + \mu\|\nabla_h \Lambda_h^\alpha u\|_{L^2}^2 + \kappa\|\nabla_h \Lambda_h^\alpha \omega\|_{L^2}^2 + \chi\|\nabla\nabla_h u\|_{L^2}^2\\
&+ 4\chi\|\nabla_h \omega\|_{L^2}^2 + \nu\|\nabla_h \nabla\cdot\omega\|_{L^2}^2\\
&= -\int \nabla_h(u\cdot\nabla u)\cdot\nabla_h u\,dx + 2\chi\int \nabla_h(\nabla\times\omega)\cdot\nabla_h u\,dx\\
&\quad -\int \nabla_h(u\cdot\nabla\omega)\cdot\nabla_h\omega\,dx + 2\chi\int\nabla_h(\nabla\times u)\cdot\nabla_h\omega\,dx\\
&:= J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{4.6}$$

Hölder's and Young's inequalities give rise to

$$\begin{aligned}
J_2 + J_4 &= 2\chi\int\nabla_h\omega\cdot\nabla_h(\nabla\times u)\,dx + 2\chi\int\nabla_h(\nabla\times u)\cdot\nabla_h\omega\,dx\\
&\le 4\chi\|\nabla\nabla_h u\|_{L^2}\|\nabla_h\omega\|_{L^2}\\
&\le \chi\|\nabla\nabla_h u\|_{L^2}^2 + 4\chi\|\nabla_h\omega\|_{L^2}^2.
\end{aligned}$$

Next we write the term J_1 in components,

$$\begin{aligned}
J_1 &= -\int\nabla_h(u\cdot\nabla u)\cdot\nabla_h u\,dx\\
&= -\int\nabla_h u\cdot\nabla u\cdot\nabla_h u\,dx\\
&= -\int\nabla_h u_h\cdot\nabla_h u\cdot\nabla_h u\,dx - \int\nabla_h u_3\partial_3 u\cdot\nabla_h u\,dx\\
&\le C\|\Lambda_h^{2-2\alpha}\nabla_h u_h\|_{L^2}^{\frac12}\|\partial_3\nabla_h u_h\|_{L^2}^{1/2}\|\nabla_h u\|_{L^2}\|\Lambda_h^\alpha\nabla_h u\|_{L^2}\\
&\quad + C\|\Lambda_h^{2-2\alpha}\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3 u\|_{L^2}\|\Lambda_h^\alpha\nabla_h u\|_{L^2}.
\end{aligned}$$

Since $1/2 \le \alpha \le 2/3$, we note that

$$\begin{gathered}
\|\Lambda_h^{2-2\alpha}\nabla_h u_h\|_{L^2} = \|\Lambda_h^\alpha\Lambda_h^{3-3\alpha}u_h\|_{L^2} \le \|\Lambda_h^\alpha u\|_{H^{3-3\alpha}} \le C\|\Lambda_h^\alpha u\|_{H^m},\\
\|\partial_3\nabla_h u_h\|_{L^2} = \|\Lambda_h^\alpha\Lambda_h^{1-\alpha}\partial_3 u_h\|_{L^2} \le \|\Lambda_h^\alpha u\|_{H^{2-\alpha}} \le C\|\Lambda_h^\alpha u\|_{H^m},\\
\|\Lambda_h^{2-2\alpha}\nabla_h u_3\|_{L^2} = \|\Lambda_h^\alpha\Lambda_h^{2-3\alpha}\nabla_h u_3\|_{L^2} \le C\|\Lambda_h^\alpha\nabla_h u\|_{H^m},\\
\|\partial_3\nabla_h u_3\|_{L^2} = \|\Lambda_h^\alpha\nabla_h\Lambda_h^{-\alpha}u_3\|_{L^2} \le C\|\Lambda_h^\alpha\nabla_h u\|_{H^m},\\
\|\partial_3 u\|_{L^2} \le \|u\|_{H^1} \le C\|u\|_{H^m}.
\end{gathered}$$

This and Lemma 2.1, ensure that

$$J_1 \le C\|\Lambda_h^\alpha u\|_{H^m}\|\nabla_h u\|_{L^2}\|\Lambda_h^\alpha\nabla_h u\|_{L^2} + C\|u\|_{H^m}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\nabla_h u\|_{L^2}.$$

Similarly, we deduce that

$$\begin{aligned}
J_3 &= -\int\nabla_h(u\cdot\nabla\omega)\cdot\nabla_h\omega\,dx\\
&= -\int\nabla_h u\cdot\nabla w\cdot\nabla_h\omega\,dx\\
&= -\int\nabla_h u_h\cdot\nabla_h\omega\cdot\nabla_h\omega\,dx - \int\nabla_h u_3\partial_3\omega\cdot\nabla_h\omega\,dx\\
&\le C\|\Lambda_h^{2-2\alpha}\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3\nabla_h u_h\|_{L^2}^{1/2}\|\nabla_h\omega\|_{L^2}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2}\\
&\quad + C\|\Lambda_h^{2-2\alpha}\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3\omega\|_{L^2}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2}\\
&\le C\|\Lambda_h^\alpha u\|_{H^m}\|\nabla_h\omega\|_{L^2}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2} + C\|\omega\|_{H^m}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2}.
\end{aligned}$$

Putting all the estimates above into (4.6), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_h u\|_{L^2}^2+\|\nabla_h\omega\|_{L^2}^2\right)+\mu\|\nabla_h\Lambda_h^\alpha u\|_{L^2}^2+\kappa\|\nabla_h\Lambda_h^\alpha\omega\|_{L^2}^2+\nu\|\nabla_h\nabla\cdot\omega\|_{L^2}^2\\
&\le C\|\Lambda_h^\alpha u\|_{H^m}\|\nabla_h u\|_{L^2}\|\Lambda_h^\alpha\nabla_h u\|_{L^2}+C\|u\|_{H^m}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\nabla_h u\|_{L^2}\\
&\quad+C\|\Lambda_h^\alpha u\|_{H^m}\|\nabla_h\omega\|_{L^2}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2}+C\|\omega\|_{H^m}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2}\\
&\le C\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\nabla_h u\|_{L^2}\|\Lambda_h^\alpha\nabla_h u\|_{L^2}+C\|u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\nabla_h u\|_{L^2}\\
&\quad+C\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\nabla_h\omega\|_{L^2}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2}+C\|\omega\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\nabla_h\omega\|_{L^2}.
\end{aligned}$$

Next, we focus on the $\dot H^m$ estimate of $(\nabla_h u,\nabla_h\omega)$. Acting $\partial_m^i\nabla_h$ $(i=i,2,3)$ to (1.1), a standard L^2 energy estimate process gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\sum_{i=1}^3\|\partial_i^m\nabla_h u\|_{L^2}^2+\sum_{i=1}^3\|\partial_i^m\nabla_h\omega\|_{L^2}^2\Big)+\mu\sum_{i=1}^3\|\partial_i^m\nabla_h\Lambda_h^\alpha u\|_{L^2}^2+\kappa\sum_{i=1}^3\|\partial_i^m\nabla_h\Lambda_h^\alpha\omega\|_{L^2}^2\\
&+\chi\sum_{i=1}^3\|\partial_i^m\nabla\nabla_h u\|_{L^2}^2+4\chi\sum_{i=1}^3\|\partial_i^m\nabla_h\omega\|_{L^2}^2+\nu\sum_{i=1}^3\|\partial_i^m\nabla_h\nabla\cdot\omega\|_{L^2}^2\\
&=-\sum_{i=1}^3\int\partial_i^m\nabla_h(u\cdot\nabla u)\cdot\partial_i^m\nabla_h u\,dx+2\chi\sum_{i=1}^3\int\partial_i^m\nabla_h(\nabla\times\omega)\cdot\partial_i^m\nabla_h u\,dx\\
&\quad-\sum_{i=1}^3\int\partial_i^m\nabla_h(u\cdot\nabla\omega)\cdot\partial_i^m\nabla_h\omega\,dx+2\chi\sum_{i=1}^3\int\partial_i^m\nabla_h(\nabla\times u)\cdot\partial_i^m\nabla_h\omega\,dx\\
&:=K_1+K_2+K_3+K_4.
\end{aligned}$$

We apply Hölder's and Young's inequalities to obtain

$$\begin{aligned}
K_2+K_4&=2\chi\sum_{i=1}^3\int\partial_i^m\nabla_h(\nabla\times\omega)\cdot\partial_i^m\nabla_h u\,dx+2\chi\sum_{i=1}^3\int\partial_i^m\nabla_h(\nabla\times u)\cdot\partial_i^m\nabla_h\omega\,dx\\
&=2\chi\sum_{i=1}^3\int\partial_i^m\nabla_h\omega\cdot\partial_i^m\nabla_h(\nabla\times u)\,dx+2\chi\sum_{i=1}^3\int\partial_i^m\nabla_h(\nabla\times u)\cdot\partial_i^m\nabla_h\omega\,dx\\
&\le 4\chi\sum_{i=1}^3\|\partial_i^m\nabla_h\nabla u\|_{L^2}\|\partial_i^m\nabla_h\omega\|_{L^2}\\
&\le\chi\sum_{i=1}^3\|\partial_i^m\nabla\nabla_h u\|_{L^2}^2+4\chi\sum_{i=1}^3\|\partial_i^m\nabla_h\omega\|_{L^2}^2.
\end{aligned}$$

For the term K_1, we can deduce from a routine computation that

$$\begin{aligned}
K_1&=-\sum_{i=1}^3\int\partial_i^m\nabla_h(u\cdot\nabla u)\cdot\partial_i^m\nabla_h u\,dx\\
&=-\sum_{i=1}^2\int\partial_i^m\nabla_h(u\cdot\nabla u)\cdot\partial_i^m\nabla_h u\,dx-\int\partial_3^m\nabla_h(u_h\cdot\nabla_h u)\cdot\partial_3^m\nabla_h u\,dx\\
&\quad-\int\partial_3^m\nabla_h(u_3\partial_3 u)\cdot\partial_3^m\nabla_h u\,dx\\
&:=K_{11}+K_{12}+K_{13}.
\end{aligned}$$

It follows from Hölder's inequality, Sobolev embedding $\dot H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ and $\dot H^{\frac12}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$ that

$$K_{11}=-\sum_{i=1}^2\int\left(\partial_i^m(\nabla_h u\cdot\nabla u)\cdot\partial_i^m\nabla_h u+\partial_i^m(u\cdot\nabla\nabla_h u)\cdot\partial_i^m\nabla_h u\right)dx$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2}\sum_{l=0}^{m} C_m^l \int \partial_i^l \nabla_h u \cdot \partial_i^{m-l}\nabla u \cdot \partial_i^m \nabla_h u \, dx - \sum_{i=1}^{2}\sum_{l=1}^{m} C_m^l \int \partial_i^l u \cdot \partial_i^{m-l}\nabla\nabla_h u \cdot \partial_i^m \nabla_h u \, dx \\
&\le C\sum_{i=1}^{2} \|\nabla_h u\|_{L^6} \|\partial_i^m \nabla u\|_{L^3} \|\partial_i^m \nabla_h u\|_{L^2} + C\sum_{i=1}^{2}\sum_{i=1}^{m} \|\partial_i^l \nabla_h u\|_{L^3} \|\partial_i^{m-l}\nabla u\|_{L^6} \|\partial_i^m \nabla_h u\|_{L^2} \\
&\quad + C\sum_{i=1}^{2} \|\partial_i u\|_{L^6} \|\partial_i^{m-1}\nabla\nabla_h u\|_{L^3} \|\partial_i^m \nabla_h u\|_{L^2} \\
&\quad + C\sum_{i=1}^{2}\sum_{l=2}^{m} \|\partial_i^l u\|_{L^3} \|\partial_i^{m-l}\nabla\nabla_h u\|_{L^6} \|\partial_i^m \nabla_h u\|_{L^2} \\
&\le C\sum_{i=1}^{2} \|\Lambda\nabla_h u\|_{L^2} \|\Lambda^{3/2}\partial_i^m u\|_{L^2} \|\partial_i^m \nabla_h u\|_{L^2} \\
&\quad + C\sum_{i=1}^{2}\sum_{l=1}^{m} \|\Lambda^{1/2}\partial_i^l \nabla_h u\|_{L^2} \|\Lambda^2 \partial_i^{m-l} u\|_{L^2} \|\partial_i^m \nabla_h u\|_{L^2} \\
&\quad + C\sum_{i=1}^{2} \|\Lambda\partial_i u\|_{L^2} \|\Lambda^{3/2}\partial_i^{m-1}\nabla_h u\|_{L^2} \|\partial_i^m \nabla_h u\|_{L^2} \\
&\quad + C\sum_{i=1}^{2}\sum_{l=2}^{m} \|\Lambda^{1/2}\partial_i^l u\|_{L^2} \|\Lambda^2 \nabla_h \partial_i^{m-l} u\|_{L^2} \|\partial_i^m \nabla_h u\|_{L^2} \\
&\le C\|u\|_{H^{m+1}} \|\Lambda_h^\alpha \nabla_h u\|_{H^m}^2 .
\end{aligned}
$$

We write the term K_{12} as two terms,

$$
\begin{aligned}
K_{12} &= -\int \partial_3^m \nabla_h (u_h \cdot \nabla_h u) \cdot \partial_3^m \nabla_h u \, dx \\
&= -\int \partial_3^m (\nabla_h u_h \cdot \nabla_h u) \cdot \partial_3^m \nabla_h u \, dx - \int \partial_3^m (u_h \cdot \nabla_h \nabla_h u) \cdot \partial_3^m \nabla_h u \, dx \\
&= -\sum_{l=0}^{m} C_m^l \int \partial_3^l \nabla_h u_h \cdot \nabla_h \partial_3^{m-l} u \cdot \partial_3^m \nabla_h u \, dx - \sum_{l=0}^{m} C_m^l \int \partial_3^l u_h \cdot \nabla_h \nabla_h \partial_3^{m-l} u \cdot \partial_3^m \nabla_h u \, dx \\
&:= K_{121} + K_{122}.
\end{aligned}
$$

By Lemma 2.1, we see that

$$
\begin{aligned}
K_{121} &\le C\|\nabla_h u_h\|_{L^2}^{1/2} \|\partial_3 \nabla_h u_h\|_{L^2}^{1/2} \|\Lambda_h^{1-\alpha}\nabla_h \partial_3^m u\|_{L^2} \|\Lambda_h^\alpha \nabla_h \partial_3^m u\|_{L^2} \\
&\quad + C\sum_{l=1}^{m} \|\nabla_h \partial_3^{m-l} u\|_{L^2}^{1/2} \|\partial_3 \nabla_h \partial_3^{m-l} u\|_{L^2}^{1/2} \|\Lambda_h^{1-\alpha}\partial_3^l \nabla_h u_h\|_{L^2} \|\Lambda_h^\alpha \nabla_h \partial_3^m u\|_{L^2} \\
&\le C\|\nabla_h u\|_{H^m} \|\Lambda_h^\alpha u\|_{H^{m+1}} \|\Lambda_h^\alpha \nabla_h u\|_{H^m} + C\|u\|_{H^{m+1}} \|\Lambda_h^\alpha \nabla_h u\|_{H^m}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
K_{122} &= -\sum_{l=0}^{m} C_m^l \int \partial_3^l u_h \cdot \nabla_h \nabla_h \partial_3^{m-l} u \cdot \partial_3^m \nabla_h u \, dx \\
&= -\int u_h \cdot \nabla_h \nabla_h \partial_3^m u \cdot \partial_3^m \nabla_h u \, dx - \sum_{l=1}^{m} C_m^l \int \partial_3^l u_h \cdot \nabla_h \nabla_h \partial_3^{m-l} u \cdot \partial_3^m \nabla_h u \, dx \\
&= \frac{1}{2}\int (\nabla_h \cdot u_h) \partial_3^m \nabla_h u \cdot \partial_3^m \nabla_h u \, dx - \sum_{l=1}^{m} C_m^l \int \partial_3^l u_h \cdot \nabla_h \nabla_h \partial_3^{m-l} u \cdot \partial_3^m \nabla_h u \, dx \\
&\le C\|\nabla_h u_h\|_{L^2}^{1/2} \|\partial_3 \nabla_h u_h\|_{L^2}^{1/2} \|\Lambda_h^{1-\alpha}\nabla_h \partial_3^m u\|_{L^2} \|\Lambda_h^\alpha \nabla_h \partial_3^m u\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{l=1}^{m}\|\nabla_h\partial_3^{m-l}\nabla_h u\|_{L^2}^{1/2}\|\partial_3\nabla_h\partial_3^{m-l}\nabla_h u\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\partial_3^l u_h\|_{L^2}\|\Lambda_h^\alpha\nabla_h\partial_3^m u\|_{L^2}\\
&\le C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m} + C\|u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}^2.
\end{aligned}
$$

Direct computations imply

$$
\begin{aligned}
K_{13} &= -\int \partial_3^m\nabla_h\left(u_3\partial_3 u\right)\cdot\partial_3^m\nabla_h u\,dx\\
&= -\int \partial_3^m\left(\nabla_h u_3\partial_3 u\right)\cdot\partial_3^m\nabla_h u - \int \partial_3^m\left(u_3\partial_3\nabla_h u\right)\cdot\partial_3^m\nabla_h u\,dx\\
&= -\sum_{l=0}^{m} C_m^l\int \partial_3^l\nabla_h u_3\partial_3^{m-l+1}u\cdot\partial_3^m\nabla_h u\,dx - \sum_{l=0}^{m} C_m^l\int \partial_3^l u_3\partial_3^{m-l+1}\nabla_h u\cdot\partial_3^m\nabla_h u\,dx\\
&:= K_{131} + K_{132}.
\end{aligned}
$$

This and Lemma 2.1 yield

$$
\begin{aligned}
k_{131} &= -\sum_{l=0}^{m} C_m^l\int \partial_3^l\nabla_h u_3\partial_3^{m-l+1}u\cdot\partial_3^m\nabla_h u\,dx\\
&= -\int \nabla_h u_3\partial_3^{m+1}u\cdot\partial_3^m\nabla_h u\,dx - \sum_{l=1}^{m} C_m^l\int \partial_3^l\nabla_h u_3\partial_3^{m-l+1}u\cdot\partial_3^m\nabla_h u\,dx\\
&= -\int \nabla_h u_3\partial_3^{m+1}u\cdot\partial_3^m\nabla_h u\,dx + \sum_{l=1}^{m} C_m^l\int \partial_3^{l-1}\nabla_h\nabla_h\cdot u_h\partial_3^{m-l+1}u\cdot\partial_3^m\nabla_h u\,dx\\
&\le C\|\Lambda_h^{2-2\alpha}\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3^{m+1}u\|_{L^2}\|\Lambda_h^\alpha\partial_3^m\nabla_h u\|_{L^2}^2\\
&\quad + C\sum_{l=1}^{m}\|\partial_3^{m-l+1}u\|_{L^2}^{1/2}\|\partial_3\partial_3^{m-l+1}u\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\partial_3^{l-1}\nabla_h\nabla_h\cdot u_h\|_{L^2}\|\Lambda_h^\alpha\partial_3^m\nabla_h u\|_{L^2}\\
&\le C\|u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}^2,
\end{aligned}
$$

where we used the fact that, for $\frac{1}{2}\le\alpha<\frac{2}{3}$ and $l=1,\cdots,m$,

$$
\begin{gathered}
\|\Lambda_h^{2-2\alpha}\nabla_h u_3\|_{L^2} = \|\Lambda_h^\alpha\Lambda_h^{2-3\alpha}\nabla_h u_3\|_{L^2} \le \|\Lambda_h^\alpha\nabla_h u\|_{H^{2-3\alpha}} \le \|\Lambda_h^\alpha\nabla_h u\|_{H^m},\\
\|\partial_3\nabla_h u_3\|_{L^2} \le \|\nabla_h\nabla_h u_h\|_{L^2} = \|\Lambda_h^\alpha\Lambda_h^{1-\alpha}\nabla_h u_h\|_{L^2} \le \|\Lambda_h^\alpha\nabla_h u\|_{H^{1-\alpha}} \le \|\Lambda_h^\alpha\nabla_h u\|_{H^m},\\
\|\Lambda_h^{1-\alpha}\partial_3^{l-1}\nabla_h\nabla_h\cdot u_h\|_{L^2} \le \|\Lambda_h^\alpha\Lambda_h^{2-2\alpha}\partial_3^{l-1}\nabla_h u_h\|_{L^2} \le \|\Lambda_h^\alpha\nabla_h u\|_{H^{1-2\alpha+l}} \le \|\Lambda_h^\alpha\nabla_h u\|_{H^m},\\
\|\partial_3\partial_3^{m-l+1}u\|_{L^2} \le \|u\|_{H^{m+1+1-l}} \le \|u\|_{H^{m+1}}.
\end{gathered}
$$

Similarly,

$$
\begin{aligned}
K_{132} &= -\sum_{l=0}^{m} C_m^l\int \partial_3^l u_3\partial_3^{m-l+1}\nabla_h u\cdot\partial_3^m\nabla_h u\,dx\\
&= -\int u_3\partial_3\partial_3^m\nabla_h u\cdot\partial_3^m\nabla_h u\,dx - \sum_{l=1}^{m} C_m^l\int \partial_3^l u_3\partial_3^{m-l+1}\nabla_h u\cdot\partial_3^m\nabla_h u\,dx\\
&= -\frac{1}{2}\int \nabla_h\cdot u_h\partial_3^m\nabla_h u\cdot\partial_3^m\nabla_h u\,dx - \sum_{l=1}^{m} C_m^l\int \partial_3^l u_3\partial_3^{m-l+1}\nabla_h u\cdot\partial_3^m\nabla_h u\,dx\\
&\le C\|\Lambda_h^{2-2\alpha}\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3^m\nabla_h u\|_{L^2}\|\Lambda_h^\alpha\partial_3^m\nabla_h u\|_{L^2}\\
&\quad + C\sum_{l=1}^{m}\|\partial_3^l u_3\|_{L^2}^{1/2}\|\partial_3^{l+1}u_3\|_{L^2}^{1/2}\|\Lambda_h^{l-\alpha}\partial_3^{m-l+1}\nabla_h u\|_{L^2}\|\Lambda_h^\alpha\partial_3^m\nabla_h u\|_{L^2}\\
&\le C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}.
\end{aligned}
$$

Thus, we immediately obtain

$$
K_1 \le C\|u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}^2 + C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}.
$$

A routine computation gives

$$
\begin{aligned}
K_3 &= -\sum_{i=1}^{3} \int \partial_i^m \nabla_h \left(u \cdot \nabla \omega\right) \cdot \partial_i^m \nabla_h \omega \, dx \\
&= -\sum_{i=1}^{2} \int \partial_i^m \nabla_h \left(u \cdot \nabla \omega\right) \cdot \partial_i^m \nabla_h \omega \, dx - \int \partial_3^m \nabla_h \left(u_h \cdot \nabla_h \omega\right) \cdot \partial_3^m \nabla_h \omega \, dx \\
&\quad - \int \partial_3^m \nabla_h \left(u_3 \partial_3 \omega\right) \cdot \partial_3^m \nabla_h \omega \, dx \\
&:= K_{31} + K_{32} + K_{33}.
\end{aligned}
$$

The term K_{31} can be similarly estimated as K_{11}:

$$
\begin{aligned}
K_{31} &= -\sum_{i=1}^{2} \int \left(\partial_i^m \left(\nabla_h u \cdot \nabla \omega\right) \cdot \partial_i^m \nabla_h \omega + \partial_i^m \left(u \cdot \nabla \nabla_h \omega\right) \cdot \partial_i^m \nabla_h \omega\right) dx \\
&= -\sum_{i=1}^{2} \sum_{l=0}^{m} C_m^l \int \partial_i^l \nabla_h u \cdot \partial_i^{m-l} \nabla \omega \cdot \partial_i^m \nabla_h \omega \, dx - \sum_{i=1}^{2} \sum_{l=1}^{m} C_m^l \int \partial_i^l u \cdot \partial_i^{m-l} \nabla \nabla_h \omega \cdot \partial_i^m \nabla_h \omega \, dx \\
&\le C \sum_{i=1}^{2} \|\nabla_h u\|_{L^6} \|\partial_i^m \nabla \omega\|_{L^3} \|\partial_i^m \nabla_h \omega\|_{L^2} + C \sum_{i=1}^{2} \sum_{l=1}^{m} \|\partial_i^l \nabla_h u\|_{L^3} \|\partial_i^{m-l} \nabla \omega\|_{L^6} \|\partial_i^m \nabla_h \omega\|_{L^2} \\
&\quad + C \sum_{i=1}^{2} \|\partial_i u\|_{L^6} \|\partial_i^{m-1} \nabla \nabla_h \omega\|_{L^3} \|\partial_i^m \nabla_h \omega\|_{L^2} \\
&\quad + C \sum_{i=1}^{2} \sum_{l=2}^{m} \|\partial_i^l u\|_{L^3} \|\partial_i^{m-l} \nabla \nabla_h \omega\|_{L^6} \|\partial_i^m \nabla_h \omega\|_{L^2} \\
&\le C \sum_{i=1}^{2} \|\Lambda \nabla_h u\|_{L^6} \|\Lambda^{3/2} \partial_i^m \omega\|_{L^3} \|\partial_i^m \nabla_h \omega\|_{L^2} \\
&\quad + C \sum_{i=1}^{2} \sum_{l=1}^{m} \|\Lambda^{1/2} \partial_i^l \nabla_h u\|_{L^2} \|\Lambda^2 \partial_i^{m-l} \omega\|_{L^2} \|\partial_i^m \nabla_h \omega\|_{L^2} \\
&\quad + C \sum_{i=1}^{2} \|\Lambda \partial_i u\|_{L^2} \|\Lambda^{3/2} \partial_i^{m-1} \nabla_h \omega\|_{L^2} \|\partial_i^m \nabla_h \omega\|_{L^2} \\
&\quad + C \sum_{i=1}^{2} \sum_{l=2}^{m} \|\Lambda^{1/2} \partial_i^l u\|_{L^2} \|\Lambda^2 \partial_i^{m-l} \nabla_h \omega\|_{L^2} \|\partial_i^m \nabla_h \omega\|_{L^2} \\
&\le C \|u\|_{H^{m+1}} \|\Lambda_h^\alpha \nabla_h \omega\|_{H^m}^2 + C \|u\|_{H^{m+1}} \|\Lambda_h^\alpha \nabla_h u\|_{H^m} \|\Lambda_h^\alpha \nabla_h \omega\|_{H^m} \\
&\quad + C \|\omega\|_{H^{m+1}} \|\Lambda_h^\alpha \nabla_h u\|_{H^m} \|\partial_i^m \nabla_h \omega\|_{H^m}.
\end{aligned}
$$

For the term K_{32}, we find

$$
\begin{aligned}
K_{32} &= -\int \partial_3^m \nabla_h (u_h \cdot \nabla_h \omega) \cdot \partial_3^m \nabla_h \omega \, dx \\
&= -\int \partial_3^m (\nabla_h u_h \cdot \nabla_h \omega) \cdot \partial_3^m \nabla_h \omega \, dx - \int \partial_3^m (u_h \cdot \nabla_h \nabla_h \omega) \cdot \partial_3^m \nabla_h \omega \, dx \\
&= -\sum_{l=0}^{m} C_m^l \int \partial_3^l \nabla_h u_h \cdot \nabla_h \partial_3^{m-l} \omega \cdot \partial_3^m \nabla_h \omega \, dx - \sum_{l=0}^{m} C_m^l \int \partial_3^l u_h \cdot \nabla_h \nabla_h \partial_3^{m-l} \omega \cdot \partial_3^m \nabla_h \omega \, dx \\
&:= K_{321} + K_{322}.
\end{aligned}
$$

By applying Lemma 2.1, we obtain

$$\begin{aligned}K_{321} &\le C\|\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3\nabla_h u_h\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\nabla_h\partial_3^m\omega\|_{L^2}\|\Lambda_h^{\alpha}\nabla_h\partial_3^m\omega\|_{L^2}\\
&\quad + C\sum_{l=1}^{m}\|\nabla_h\partial_i^{m-l}\omega\|_{L^2}^{1/2}\|\partial_3\nabla_h\partial_3^{m-l}\omega\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\partial_3^l\nabla_h u_h\|_{L^2}\|\Lambda_h^{\alpha}\nabla_h\partial_3^m\omega\|_{L^2}\\
&\le C\|\nabla_h u\|_{H^m}\|\Lambda_h^{\alpha}\omega\|_{H^m}\|\Lambda_h^{\alpha}\nabla_h\omega\|_{H^m} + C\|\omega\|_{H^{m+1}}\|\Lambda_h^{\alpha}\nabla_h u\|_{H^m}\|\Lambda_h^{\alpha}\nabla_h\omega\|_{H^m},\end{aligned}$$

and

$$\begin{aligned}K_{322} &= -\sum_{l=0}^{m}C_m^l\int\partial_3^l u_h\cdot\nabla_h\nabla_h\partial_3^{m-l}\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&= -\int u_h\cdot\nabla_h\nabla_h\partial_3^m\omega\cdot\partial_3^m\nabla_h\omega\,dx - \sum_{l=1}^{m}C_m^l\int\partial_3^l u_h\cdot\nabla_h\nabla_h\partial_3^{m-l}\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&= \frac{1}{2}\int(\nabla_h\cdot u_h)\,\partial_3^m\nabla_h\omega\cdot\partial_3^m\nabla_h\omega\,dx - \sum_{l=1}^{m}C_m^l\int\partial_3^l u_h\cdot\nabla_h\nabla_h\partial_3^{m-l}\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&\le C\|\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3\nabla_h u_h\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\partial_3^m\nabla_h\omega\|_{L^2}\|\Lambda_h^{\alpha}\partial_3^m\nabla_h\omega\|_{L^2}\\
&\quad + C\sum_{l=1}^{m}\|\nabla_h\partial_3^{m-l}\nabla_h\omega\|_{L^2}^{1/2}\|\partial_3\nabla_h\partial_3^{m-l}\nabla_h\omega\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\partial_3^l u_h\|_{L^2}\|\Lambda_h^{\alpha}\nabla_h\partial_3^m\omega\|_{L^2}\\
&\le C\|\nabla_h u\|_{H^m}\|\Lambda_h^{\alpha}\omega\|_{H^{m+1}}\|\Lambda_h^{\alpha}\nabla_h\omega\|_{H^m} + C\|\omega\|_{H^{m+1}}\|\Lambda_h^{\alpha}\nabla_h u\|_{H^m}\|\Lambda_h^{\alpha}\nabla_h\omega\|_{H^m}.\end{aligned}$$

Similarly,

$$\begin{aligned}K_{33} &= -\int\partial_3^m\nabla_h(u_3\partial_3\omega)\cdot\partial_3^m\nabla_h\omega\,dx\\
&= -\int\partial_3^m(\nabla_h u_3\partial_3\omega)\cdot\partial_3^m\nabla_h\omega\,dx - \int\partial_3^m(u_3\partial_3\nabla_h\omega)\cdot\partial_3^m\nabla_h\omega\,dx\\
&= -\sum_{l=0}^{m}C_m^l\int\partial_3^l\nabla_h u_3\partial_3^{m-l+1}\omega\cdot\partial_3^m\nabla_h\omega\,dx - \sum_{l=0}^{m}C_m^l\int\partial_3^l u_3\partial_3^{m-l+1}\nabla_h\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&:= K_{331} + K_{332}.\end{aligned}$$

By Lemma 2.1 we conclude that

$$\begin{aligned}K_{331} &= -\int\nabla_h u_3\partial_3^{m+1}\omega\cdot\partial_3^m\nabla_h\omega\,dx - \sum_{l=1}^{m}C_m^l\int\partial_3^l\nabla_h u_3\partial_3^{m-l+1}\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&= -\int\nabla_h u_3\partial_3^{m+1}\omega\cdot\partial_3^m\nabla_h\omega\,dx - \sum_{l=1}^{m}C_m^l\int\partial_3^{l-1}\nabla_h\nabla_h\cdot u_h\partial_3^{m-l+1}\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&\le C\|\Lambda_h^{2-2\alpha}\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3\nabla_h u_3\|_{L^2}^{1/2}\|\partial_3^{m+1}\omega\|_{L^2}\|\Lambda_h^{\alpha}\partial_3^m\nabla_h\omega\|_{L^2}\\
&\quad + C\sum_{l=1}^{m}\|\partial_3^{m-l+1}\omega\|_{L^2}^{1/2}\|\partial_3\partial_3^{m-l+1}\omega\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\partial_3^{l-1}\nabla_h\nabla_h\cdot u_h\|_{L^2}\|\Lambda_h^{\alpha}\partial_3^m\nabla_h\omega\|_{L^2}\\
&\le C\|\omega\|_{H^{m+1}}\|\Lambda_h^{\alpha}\nabla_h u\|_{H^m}\|\Lambda_h^{\alpha}\nabla_h\omega\|_{H^m},\end{aligned}$$

and

$$\begin{aligned}K_{332} &= -\sum_{l=0}^{m}C_m^l\int\partial_3^l u_3\partial_3^{m-l+1}\nabla_h\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&= -\int u_3\partial_3^m\partial_3\nabla_h\omega\cdot\partial_3^m\nabla_h\omega\,dx - \sum_{l=1}^{m}C_m^l\int\partial_3^l u_3\partial_3^{m-l+1}\nabla_h\omega\cdot\partial_3^m\nabla_h\omega\,dx\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int \nabla_h\cdot u_h \partial_3^m\nabla_h\omega\cdot\partial_3^m\nabla_h\omega\,dx - \sum_{l=1}^m C_m^l\int \partial_3^l u_3\partial_3^{m-l+1}\nabla_h\omega\cdot\partial_3^m\nabla_h\omega\,dx\\
&\le C\|\Lambda_h^{2-2\alpha}\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3\nabla_h u_h\|_{L^2}^{1/2}\|\partial_3^m\nabla_h\omega\|_{L^2}\|\Lambda_h^\alpha\partial_3^m\nabla_h\omega\|_{L^2}\\
&\quad + C\sum_{l=1}^m\|\partial_3^l u_3\|_{L^2}^{1/2}\|\partial_3^{l+1}u_3\|_{L^2}^{1/2}\|\Lambda_h^{1-\alpha}\partial_3^{m-l+1}\nabla_h\omega\|_{L^2}\|\Lambda_h^\alpha\partial_3^m\nabla_h\omega\|_{L^2}\\
&\le C\|\nabla_h\omega\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m} + C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\omega\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
K_3 &\le C\|u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}^2 + C\left(\|u\|_{H^{m+1}} + \|u\|_{H^{m+1}}\right)\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}\\
&\quad + C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\omega\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m} + C\|\nabla_h\omega\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}.
\end{aligned}
$$

Substituting the above estimates into (4.6), we infer that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\sum_{i=1}^3\|\partial_i^m\nabla_h u\|_{L^2}^2 + \sum_{i=1}^3\|\partial_i^m\nabla_h\omega\|_{L^2}^2\Big) + \mu\sum_{i=1}^3\|\partial_i^m\nabla_h\Lambda_h^\alpha u\|_{L^2}^2\\
&+ \kappa\sum_{i=1}^3\|\partial_i^m\nabla_h\Lambda_h^\alpha\omega\|_{L^2}^2 + \nu\sum_{i=1}^3\|\partial_i^m\nabla_h\nabla\cdot\omega\|_{L^2}^2\\
&\le C\|u\|_{H^{m+1}}\left(\|\Lambda_h^\alpha\nabla_h u\|_{H^m}^2 + \|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}^2\right) + C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\\
&\quad + C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\omega\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m} + C\|\nabla_h\omega\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}\\
&\quad + C\left(\|u\|_{H^{m+1}} + \|\omega\|_{H^{m+1}}\right)\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\|\Lambda^\alpha\nabla_h\omega\|_{H^m}.
\end{aligned}
$$

Adding the above inequality to (3.2) and using Young's inequality, we have

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h\omega\|_{H^m}^2\Big) + \mu\|\nabla_h\Lambda_h^\alpha u\|_{H^m}^2 + \kappa\|\nabla_h\Lambda_h^\alpha\omega\|_{H^m}^2\\
&\le C\|u\|_{H^{m+1}}\left(\|\Lambda_h^\alpha\nabla_h u\|_{H^m}^2 + \|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}^2\right) + C\left(\|u\|_{H^{m+1}} + \|\omega\|_{H^{m+1}}\right)\|\Lambda_h^\alpha\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}\\
&\quad + C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h u\|_{H^m} + C\|\nabla_h u\|_{H^m}\|\Lambda_h^\alpha\omega\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}\\
&\quad + C\|\nabla_h\omega\|_{H^m}\|\Lambda_h^\alpha u\|_{H^{m+1}}\|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}\\
&\le C(\frac{1}{\delta})\left(\|\Lambda_h^\alpha u\|_{H^{m+1}}^2 + \|\Lambda_h^\alpha\omega\|_{H^{m+1}}^2\right)\left(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h\omega\|_{H^m}^2\right)\\
&\quad + \delta\left(\|\nabla_h\Lambda_h^\alpha u\|_{H^m}^2 + \|\nabla_h\Lambda_h^\alpha\omega\|_{H^m}^2\right) + C\left(\|u\|_{H^{m+1}} + \|\omega\|_{H^{m+1}}\right)\left(\|\Lambda_h^\alpha\nabla_h u\|_{H^m}^2 + \|\Lambda_h^\alpha\nabla_h\omega\|_{H^m}^2\right).
\end{aligned} \tag{4.7}
$$

With Theorem 1.1 in hand, we know that the solution (u,ω) of (1.1) satisfies

$$
\|u\|_{H^m}^2 + \|\omega\|_{H^m}^2 + \int_0^t\left(\|\Lambda_h^\alpha u\|_{H^m}^2 + \|\Lambda_h^\alpha\omega\|_{H^m}^2\right)d\tau \le C(\|u_0\|_{H^m}^2 + \|\omega_0\|_{H^m}^2),
$$

for small enough initial data (u_0,ω_0) and $m\ge 2$. Therefore, there exist two sufficiently small constants $\varepsilon_0>0$ and $\delta>0$ satisfying

$$
\begin{gathered}
\|u_0\|_{H^{m+1}} + \|\omega_0\|_{H^{m+1}} < \varepsilon_0,\\
\min\{\mu,\kappa\} - \delta - C\varepsilon_0 \ge 0.
\end{gathered}
$$

Then (4.7) leads to

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h w\|_{H^m}^2\Big) + \big(\min\{\mu,\kappa\} - \delta - C\varepsilon_0\big)\left(\|\nabla_h\Lambda_h^\alpha u\|_{H^m}^2 + \|\nabla_h\Lambda_h^\alpha w\|_{H^m}^2\right)\\
&\le C(\frac{1}{\delta})\left(\|\Lambda_h^\alpha u\|_{H^{m+1}}^2 + \|\Lambda_h^\alpha\omega\|_{H^{m+1}}^2\right)\left(\|\nabla_h u\|_{H^m}^2 + \|\nabla_h\omega\|_{H^m}^2\right).
\end{aligned}
$$

Because

$$
f(t) = \|\nabla_h u\|_{H^m}^2 + \|\nabla_h\omega\|_{H^m}^2,
$$

the Gronwall inequality ensures that

$$f(t) \leq f(s) \exp\Big(\int_s^t C(\frac{1}{\delta})(\|\Lambda_h^\alpha u\|_{H^{m+1}}^2 + \|\Lambda_h^\alpha \omega\|_{H^{m+1}}^2)\, d\tau \Big) < C(\varepsilon_0, \delta) f(s),$$

which implies the desired result (4.2). □

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