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STANDING WAVES WITH A CRITICAL FREQUENCY FOR THE GROSS-PITAEVSKII EQUATION IN TRAPPED DIPOLAR QUANTUM GASES

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Abstract. This article concerns the singularly of perturbed Gross-Pitaevskii equations in trapped dipolar quantum gases,

$$-\varepsilon^2\Delta u+V(x)u+\lambda_1|u|^2u+\lambda_2(K*|u|^2)u=0 \quad \text{in } \mathbb{R}^3,$$
$$u>0, \quad u\in H^1(\mathbb{R}^3),$$

where ε is a small positive parameter, $\lambda_1,\lambda_2\in\mathbb{R}$, $*$ denotes the convolution, $K(x)=\frac{1-3\cos^2\theta}{|x|^3}$ and $\theta=\theta(x)$ is the angle between the dipole axis determined by $(0,0,1)$ and the vector x. Moreover, the potential V satisfies $\liminf_{|x|\to\infty}V(x)>\inf_{\mathbb{R}^3}V(x)=0$. Under certain assumptions on $(\lambda_1,\lambda_2)\in\mathbb{R}^2$, we construct a family of positive solutions $u_\varepsilon\in H^1(\mathbb{R}^3)$ whose L^∞ norm approaches 0 as $\varepsilon\to 0$. Our main results extend the results in Byeon and Wang [6] which dealt with singularly perturbed Schrödinger equations with a local nonlinearity, to the nonlocal Gross-Pitaevskii type equation.

1. Introduction and main result

We study the Gross-Pitaevskii equation arising from Bose-Einstein condensation of trapped dipolar quantum gases,

$$\begin{gathered}-\varepsilon^2\Delta u+V(x)u+\lambda_1|u|^2u+\lambda_2(K*|u|^2)u=0 \quad \text{in } \mathbb{R}^3,\\ u>0, \quad u\in H^1(\mathbb{R}^3),\end{gathered}\tag{1.1}$$

where ε is a small positive parameter, $\lambda_1,\lambda_2\in\mathbb{R}$, $*$ denotes the convolution, $K(x)=\frac{1-3\cos^2\theta}{|x|^3}$ and $\theta=\theta(x)$ is the angle between the dipole axis determined by $(0,0,1)$ and the vector x. Moreover, the potential $V:\mathbb{R}^3\to\mathbb{R}$ is a continuous function satisfying:

- (A1) $\liminf_{|x|\to\infty}V(x)>\inf_{\mathbb{R}^3}V(x)=0$;
- (A2) There is a bounded domain Λ such that $0=\inf_\Lambda V<\min_{\partial\Lambda}V$.

This type of hypothesis on the potential can be regarded as the critical frequency case which was first introduced by Byeon and Wang in [6].

Problem (1.1) arises from the study of the three dimensional Gross-Pitaevskii equation, see e.g. [3, 9, 20, 22, 23],

$$ih\frac{\partial\psi(x,t)}{\partial t}=-\frac{h^2}{2m}\Delta\psi+W(x)\psi+\lambda_0|\psi|^2\psi+(V_{\text{dip}}*|\psi|^2)\psi, \quad x\in\mathbb{R}^3,\ t>0, \tag{1.2}$$

where t is time, h is the Plank constant, m is the mass of a dipolar particle, $W(x)$ is an external trapping potential describing the electromagnetic trap for the condensate, $\lambda_0=4\pi h^2a_s/m$ describes the local interaction between dipoles in the condensate with a_s the s-wave scattering

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length (positive for repulsive interaction and negative for attractive interaction). The long-range dipolar interaction potential between two dipoles is

$$V_{\rm dip}=\frac{\mu_0\mu_{\rm dip}^2}{4\pi}\frac{1-3\cos^2\theta}{|x|^3},\quad x\in\mathbb{R}^3,$$

where μ_0 is the vacuum magnetic permeability, $\mu_{\rm dip}$ is the permanent magnetic dipole moment, $\theta=\theta(x)$ is the angle between the vector $(0,0,1)$ and the vector x.

Carles, Markowich and Sparber [9] studied the existence and uniqueness of solution to the following equation with initial condition $\psi_0\in H^1(\mathbb{R}^3)$,

$$i\partial_t\psi+\frac{1}{2}\Delta\psi=\frac{|x|^2}{2}\psi+\lambda_1|\psi|^2\psi+\lambda_2(K*|\psi|^2)\psi,\quad \psi(0,x)=\psi_0(x). \tag{1.3}$$

They proved that (1.3) has a unique, global solution if $\lambda_1\geq\frac{4}{3}\pi\lambda_2\geq 0$. They called this situation stable regime, referring to the fact that no singularity is formed in finite time. They also showed that in the unstable regime $\big(\lambda_1<\frac{4}{3}\pi\lambda_2\big)$, finite time blow-up may occur.

The physical parameters λ_1 and λ_2 describe the strength of the two nonlinearities in problem (1.1). Inspired by [9], we define the *stable regime* by

$$D_{\rm sr}:=\Big\{(\lambda_1,\lambda_2)\in\mathbb{R}^2:\lambda_1-\frac{4}{3}\pi\lambda_2\geq 0\text{ and }\lambda_1+\frac{8}{3}\pi\lambda_2\geq 0\Big\} \tag{1.4}$$

and the *unstable regime* by

$$D_{\rm ur}:=\Big\{(\lambda_1,\lambda_2)\in\mathbb{R}^2:\lambda_1-\frac{4}{3}\pi\lambda_2< 0\text{ or }\lambda_1+\frac{8}{3}\pi\lambda_2< 0\Big\}. \tag{1.5}$$

In recent years, the following equation related to (1.1)

$$-\frac{1}{2}\Delta u+\frac{a^2}{2}|x|^2u+\lambda_1|u|^2u+\lambda_2(K*|u|^2)u+\mu u=0,\ x\in\mathbb{R}^3, \tag{1.6}$$

with $a\geq 0$, $\mu\in\mathbb{R}$ has been studied by many authors, see [1, 3, 7, 8].

Antonelli and Sparber [1] studied (1.6) with $a=0$ and proved that for any $\mu>0$, $(\lambda_1,\lambda_2)\in D_{\rm ur}$, there exists a positive solution to (1.6) by looking for minimizers of the C^1 functional

$$J(u):=\frac{\Big(\int_{\mathbb{R}^3}|\nabla u|^2\Big)^{3/2}\Big(\int_{\mathbb{R}^3}|u|^2\Big)^{1/2}}{-\lambda_1\int_{\mathbb{R}^3}|u|^4-\lambda_2\int_{\mathbb{R}^3}(K*|u|^2)|u|^2}$$

in $H^1(\mathbb{R}^3)$. Furthermore, some symmetry regularity and decay properties of the solutions to (1.6) were given.

Carles and Hajaiej [8] considered (1.6) and proved that if $(\lambda_1,\lambda_2)\in D_{\rm sr}$, then (1.6) has a non-negative minimal solution with any prescribed L^2 norm which is Steiner symmetric and it is unique provided that either $\lambda_1>\frac{4}{3}\pi\lambda_2>0$ or $\lambda_1>-\frac{8}{3}\pi\lambda_2>0$.

Bellazzini and Jeanjean [7] studied (1.6) in the case where $(\lambda_1,\lambda_2)\in D_{\rm ur}$. They proved that (1.6) has at least one solution with any prescribed L^2 norm with $a=0$ and they also show that there exists $a_0>0$ such that for any $a\in(0,a_0]$, (1.6) has at least two solutions with any prescribed L^2 norm where one is a mountain pass solution and the other one is a topological local minimal solution. Moreover, some stable scattering and asymptotic results were given.

He and Luo [15] considered the following singularly perturbed Gross-Pitaevskii equation in trapped dipolar quantum gases

$$\begin{gathered}-\varepsilon^2\Delta u+V(x)u+\lambda_1|u|^2u+\lambda_2(K*|u|^2)u=0\quad\text{in }\mathbb{R}^3,\\ u>0,\quad u\in H^1(\mathbb{R}^3),\end{gathered} \tag{1.7}$$

where ε is a small positive parameter, $\lambda_1,\lambda_2\in\mathbb{R}$, $*$ denotes the convolution, $K(x)=\frac{1-3\cos^2\theta}{|x|^3}$ and $\theta=\theta(x)$ is the angle between the dipole axis determined by $(0,0,1)$ and the vector x. Under the condition that the potential V satisfies

(A3) $\inf_{x\in\mathbb{R}^3}V(x)=\alpha>0$;
(A4) There is a bounded domain Λ such that $V_0:=\inf_\Lambda V<\min_{\partial\Lambda}V$.

and $(\lambda_1, \lambda_2) \in D_{\rm ur}$. They constructed a family of bound state solutions which concentrate around the local minimum points of V as $\varepsilon \to 0$ by using the penalization method and a version of quantitative deformation lemma due to [4, 6].

Zhang and Xu [24] studied (1.7) under the assumption that the potential V satisfies

- (A5) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbb{R}^3} V(x) > 0$.

They used mountain pass theorem and Nehari manifold approach to show the existence of ground state solutions to (1.7) for $\varepsilon > 0$ small and describe the concentration phenomenon of ground state solutions as $\varepsilon \to 0$. Moreover, they investigate the relationship between the number of positive solutions and the profile of the potential V under one more assumption

- (A6) There exist $a^1, \dots, a^k$ in $\mathbb{R}^3$ such that $V(a^j) = V_0$, $j = 1, \dots, k$. Moreover, each of $a^1, \dots, a^k$ is a strict global minimum point of V.

Equation (1.1) is related to standing wave solutions for the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2}\Delta\psi - W(x)\psi + |\psi|^{p-1}\psi = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.8}$$

where $\hbar$ denotes the Planck constant, i is the imaginary unit. A solution of the form $\psi(x,t) = \exp(-iEt/\hbar)v(x)$ is called a standing wave. It is easily checked that $\psi(x,t) = \exp(-iEt/\hbar)W(x)$ is a standing wave solution to (1.8) if and only if the function v satisfies

$$\frac{\hbar^2}{2}\Delta\psi - (W(x) - E)v + |v|^{p-1}v = 0, \quad x \in \mathbb{R}^N. \tag{1.9}$$

For simplicity, we let $V(x) = W(x) - E$, $\varepsilon = \hbar/\sqrt{2}$, (1.9) can be rewritten in the form

$$-\varepsilon^2\Delta u + V(x)u = |u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.10}$$

where $2 < q < 2^*$, $N \geq 1$. Under the assumption that

$$\inf_{\mathbb{R}^N} W(x) > E\Big(\inf_{\mathbb{R}^N} V(x) > 0\Big),$$

many mathematicians proved the existence, concentration and multiplicity of solutions to (1.10).

Floer and Weinstein [13] studied (1.9) in the case where $N = 1$, $q = 4$, $V \in L^\infty$ with $\inf V > 0$. They construct a single peak solution which concentrates around any prescribed non-degenerate critical point of the potential V. Y. G. Oh [16, 17] extended this result in higher dimensions when $2 < q < 2N/(N-2)$ and the potential V belongs to Kato class. Furthermore, Oh [18] proved the existence of multi-peak solutions which concentrate around any prescribed finite subsets of the non-degenerate critical points of V. The arguments in [13, 16, 17, 18] are mainly based on a Lyapunov-Schmidt reduction.

Rabinowitz [19] used mountain pass theorem to prove that (1.10) possesses a positive ground state solution for $\varepsilon > 0$ small where the potential V satisfies (A5).

The concentration behavior for the family of positive ground state solutions obtained in [19] was proved by Wang [21]. He proved that the positive ground state solutions to (1.10) must concentrate around the global minimum points of V as $\varepsilon \to 0$.

Under the same condition (A5) on $V(x)$, Cingolani and Lazzo [10] proved the multiplicity of positive ground state solutions to (1.10) by Ljusternik-Schnirelmann theory.

del Pino and Felmer [11] studied (1.10) with some conditions on V replaced by (A3) and (A4). They proved that (1.10) possesses a positive bound state solution for $\varepsilon > 0$ small which concentrates around the local minimum points of V in Λ as $\varepsilon \to 0$.

This article concerns the situation where E is a critical frequency in the sense that

$$\inf_{\mathbb{R}^N} W(x) = E\Big(\inf_{\mathbb{R}^N} V(x) = 0\Big). \tag{1.11}$$

It seems that Byeon and Wang [5] were the first to study the existence, concentration and asymptotic behavior of positive ground state solution to (1.10) under the critical frequency (1.11). They obtained a solution u_ε to (1.10) such that $\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 0$ and $\lim_{\varepsilon\to 0}\varepsilon^{-2/(p-2)}\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} >$

0. And the results on the existence of localized solutions of [5] are extended in [6] to more general nonlinearities via the penalization methods involving the local type

$$g(x,t) := \chi_{D/\varepsilon}(x)\tilde{f}(t) + (1-\chi_{D/\varepsilon}(x))\min\{\tilde{f}(t),\gamma t\} \tag{1.12}$$

and the nonlocal type

$$\Big(\varepsilon^{-6/\mu}\int_{\mathbb{R}^N\setminus\Lambda_\varepsilon} u^2 - 1\Big)_+^{\beta} \tag{1.13}$$

for some D, $\Lambda \subset \mathbb{R}^N$, γ, $\mu > 0$, $1 < \beta < q/2$ and $q \in (2, 2^*)$.

As far as we know, there is no result for (1.1) under the critical frequency situation (1.11). Since the interaction kernel $K(x)$ is highly singular, the singular integral $K * |u|^2$ makes it much more complicated to estimate the uniform boundedness in L^∞ and exponential decay of solutions to (1.1). Moreover, due to the indefinite sign of $K(x)$, λ_1 and λ_2, the local penalization method (1.12) in [6] can not be used in this article.

To study (1.1), we will work with the equivalent equation

$$\begin{gathered} -\Delta v + V(\varepsilon x)v + \lambda_1 v^3 + \lambda_2 (K * v^2)v = 0 \quad \text{in } \mathbb{R}^3,\\ v > 0, \quad v \in H^1(\mathbb{R}^3) \end{gathered} \tag{1.14}$$

with the energy functional

$$I_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)v^2 + \frac{\lambda_1}{4}\int_{\mathbb{R}^3} (v^+)^4 + \frac{\lambda_2}{4}\int_{\mathbb{R}^3} (K * (v^+)^2)(v^+)^2, \quad v \in H_\varepsilon,$$

where $(\lambda_1, \lambda_2) \in D_{\rm ur}$ and $H_\varepsilon := \{v \in H^1(\mathbb{R}^3) | \int_{\mathbb{R}^3} V(\varepsilon x)v^2 < \infty\}$ endowed with the norm

$$\|v\|_\varepsilon := \Big(\int_{\mathbb{R}^3} |\nabla v|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)v^2\Big)^{1/2}.$$

Because of the indefinite sign of $K(x)$, λ_1 and λ_2, we use the nonlocal penalization method (1.13) introduced by Byeon and Wang [6], which helps us to overcome the non-compactness caused by the unboundedness of the domain $\mathbb{R}^3$. For this purpose, we should modify the energy functional.

Following [6], we define $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ by

$$J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v),$$

where I_ε is energy functional defined above,

$$Q_\varepsilon(v) = \Big(\int_{\mathbb{R}^3} \chi_\varepsilon v^2 - 1\Big)_+^{\beta}, \quad \chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Lambda/\varepsilon,\\ \varepsilon^{-18} & \text{if } x \notin \Lambda/\varepsilon, \end{cases}$$

and $1 < \beta < 2$. It will be shown that the functional Q_ε acts as a penalization to recover the (PS) condition for the modified functional J_ε. Moreover, because of the mass concentration quantity derived from Q_ε, the solutions to the modified problem can be estimated suitably by elliptic estimates. After suitable estimates of solutions to the modified problem, we can verify that for $\varepsilon > 0$ small the solutions we constructed for the modified problem are indeed solutions to the original problem (1.1).

Before stating our results, we give some notation. We define $\mathcal{Z} = \{x \in \mathbb{R}^3 : V(x) = 0\}$ and $\mathcal{M} := \mathcal{Z} \cap \Lambda = \{x \in \Lambda : V(x) = 0\}$. For any set $B \subset \mathbb{R}^3$ and $\delta > 0$, we denote $B^\delta = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, B) \le \delta\}$.

Theorem 1.1. *Assume that the potential V satisfies (A1), (A2) and $(\lambda_1, \lambda_2) \in D_{\rm ur}$, then there exists an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$,* (1.1) *possesses a positive bound state solution $u_\varepsilon \in H^1(\mathbb{R}^3)$. Moreover,*

(i) *for each $\delta > 0$, there exists $C_1, C_2 > 0$, such that*

$$u_\varepsilon(x) \le C_1 \exp\Big(-\frac{C_2}{\varepsilon}\operatorname{dist}(x, \mathcal{M}^\delta)\Big),$$

where C_1, C_2 are independent of ε;

(ii) $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} = 0$ *and* $\lim_{\varepsilon\to 0} \varepsilon^{-1}\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N)} > 0$.

This article is organized as follows. In Section 2, we give some preliminary results. In Section 3, we prove the main result Theorem 1.1.

2. Preliminaries

In this section, we give some preliminary results. Let

$$C(u) := \lambda_1 \int_{\mathbb{R}^3} |u|^4 + \lambda_2 \int_{\mathbb{R}^3} (K * |u|^2)|u|^2$$

and define the Fourier transform of u by

$$\mathcal{F}(u)(\xi) := \widehat{u}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x)dx,$$

we have the following results:

Lemma 2.1 ([9, Lemma 2.3])**.** *The Fourier transform of K is*

$$\widehat{K}(\xi) = \frac{4\pi}{3}(3\cos^2\theta - 1) = \frac{4\pi}{3}\Big(\frac{3\xi_3^2}{|\xi|^2} - 1\Big) = \frac{4\pi}{3}\Big(\frac{2\xi_3^2 - \xi_1^2 - \xi_2^2}{|\xi|^2}\Big) \in \big[-\frac{4}{3}\pi, \frac{8}{3}\pi\big],$$

where θ is the angle between ξ and the vector $(0,0,1)$.

Lemma 2.2. *Let $u \in H^1(\mathbb{R}^3)$, then*

$$C(u) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} [\lambda_1 + \lambda_2 \widehat{K}(\xi)] |\widehat{u^2}|^2 d\xi$$

and $|C(u)| \le C\|u\|^4_{L^4(\mathbb{R}^3)}$. Moreover, if $(\lambda_1, \lambda_2) \in D_{\rm sr}$ given in (1.4), then $C(u) \ge 0$.

Proof. The first part is obtained by Plancherel identity (see for example [2, Theorem 1.25]) and the detailed proof can be found in [7]. The second part is a direct conclusion from Lemma 2.1. □

Lemma 2.3 ([9, Lemma 2.1])**.** *The operator $\mathcal{K} : u \to K * u$ can be extended as a continuous operator on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$.*

The equation for $m > 0$,

$$\begin{gathered} -\Delta u + mu + \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u = 0 \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3) \end{gathered} \tag{2.1}$$

is the limiting problem to (1.1) with the energy functional

$$I_m(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{m}{2}\int_{\mathbb{R}^3} u^2 + \frac{\lambda_1}{4}\int_{\mathbb{R}^3} |u^+|^4 + \frac{\lambda_2}{4}\int_{\mathbb{R}^3} (K * |u^+|^2)|u^+|^2, \quad u \in H^1(\mathbb{R}^3).$$

Denoting c_m the ground state level of (2.1), that is

$$c_m = \inf\{I_m(u) : u \in H^1(\mathbb{R}^N)\backslash\{0\} \text{ is a solution to (2.1)}\}.$$

We say $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ is a ground state solution to (2.1) if u is a solution to (2.1) and satisfies $I_m(u) = c_m$. He and Luo [15], Zhang and Xu [24] proved that (2.1) possesses a ground state solution for $(\lambda_1, \lambda_2) \in D_{\rm ur}$. Furthermore, He and Luo [15] proved that the ground state level c_m and the mountain pass value coincide, i.e.

$$c_m = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I_m(\gamma(t)),$$

where the set of paths is defined as

$$\Gamma := \{\gamma \in C([0,1], H^1(\mathbb{R}^N)) : \gamma(0) = 0 \text{ and } I_m(\gamma(1)) < 0\}.$$

In view of the fact that $C(u)$ is homogeneous, Zhang and Xu [24] defined the restricted set by

$$\mathcal{O} = \{u \in H^1(\mathbb{R}^3) : C(u) < 0\}.$$

Note that [15, Lemma 2.3] implies $\mathcal{O} \neq \emptyset$, they used the set $\mathcal{O}$ and the Nehari manifold method to characterize the ground state level c_m, i.e. if $(\lambda_1, \lambda_2) \in D_{\rm ur}$,

$$c_m = \inf_{u\in\mathcal{N}_m} I_m(u) = \inf_{w\in\mathcal{O}} \max_{t>0} I_m(tw) > 0,$$

where $\mathcal{N}_m$ is the Nehari manifold corresponding to I_m defined by

$$\mathcal{N}_m := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'_m(u), u\rangle = 0\}.$$

Lemma 2.4. *Suppose that there is a nonnegative bounded sequence $\{u_n\}_{n=1}^\infty \subset D^{1,2}(\mathbb{R}^3)$ and satisfies*

$$-\Delta u \le C(u^3 + |K * u^2|u) \tag{2.2}$$

in the weak sense, then

$$\|u_n\|_{L^\infty(\mathbb{R}^3)} \le C\|u_n\|_{L^6(\mathbb{R}^3)}, \tag{2.3}$$

where C is independent of n.

Proof. We let $u_L = \min(u, L)$ where $L > 0$. Taking $u_n(u_n)_L^{2(\beta-1)}$ for $\beta \ge 1$ as a test function in (2.2) and by Young's inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u_n|^2 (u_n)_L^{2(\beta-1)} + C(\beta-1)\int_{\mathbb{R}^3} |\nabla (u_n)_L|^2 (u_n)_L^{2(\beta-1)} \\ &\le C\int_{\mathbb{R}^3} u_n^4 (u_n)_L^{2(\beta-1)} + C\int_{\mathbb{R}^3} |K * u_n^2| u_n^2 (u_n)_L^{2(\beta-1)}. \end{aligned} \tag{2.4}$$

Letting $W_L = u_n(u_n)_L^{(\beta-1)}$, by Sobolev's inequality and (2.4), we see that

$$\begin{aligned} \|W_L\|_{L^6}^2 &\le C\int_{\mathbb{R}^3} |\nabla W_L|^2 \\ &\le C(\beta-1)^2 \int_{\mathbb{R}^3} |\nabla (u_n)_L|^2 (u_n)_L^{2(\beta-1)} + C\int_{\mathbb{R}^3} |\nabla u_n|^2 (u_n)_L^{2(\beta-1)} \\ &\le C\beta^2\Big(\int_{\mathbb{R}^3} u_n^4 (u_n)_L^{2(\beta-1)} + \int_{\mathbb{R}^3} |K * u_n^2| u_n^2 (u_n)_L^{2(\beta-1)}\Big). \end{aligned}$$

By Hölder's inequality, Lemma 2.3 and the boundedness of $\{u_n\}_{n=1}^\infty$ in $D^{1,2}(\mathbb{R}^3)$, we have

$$\Big[\int_{\mathbb{R}^3} u_n^6 (u_n)_L^{6(\beta-1)}\Big]^{1/3} \le C\beta^2\Big[\int_{\mathbb{R}^3} u_n^3 (u_n)_L^{3(\beta-1)}\Big]^{2/3}.$$

Furthermore, if $u_n \in L^{3\beta}(\mathbb{R}^3)$, letting $L \to +\infty$, we see that

$$\|u_n\|_{L^{6\beta}(\mathbb{R}^3)} \le C^{1/\beta}\beta^{1/\beta}\|u_n\|_{L^{3\beta}(\mathbb{R}^3)}.$$

Choosing $\beta = 2^m$, we have

$$\|u_n\|_{L^{3\cdot 2^{m+1}}(\mathbb{R}^3)} \le C^{2^{-m}} 2^{m2^{-m}} \|u_n\|_{L^{3\cdot 2^m}(\mathbb{R}^3)}. \tag{2.5}$$

Iterating by (2.5), we obtain

$$\|u_n\|_{L^{3\cdot 2^{m+1}}(\mathbb{R}^3)} \le C^{\sum_{i=1}^m 2^{-i}} 2^{\sum_{i=1}^m i2^{-i}} \|u_n\|_{L^6(\mathbb{R}^3)}.$$

Letting $m \to \infty$, we obtain (2.3). □

Lemma 2.5 ([[14, Lemma 8.17]])**.** *] Let Ω be an open subset of $\mathbb{R}^N (N \ge 2)$. Suppose that $t > N$, $h \in L^{t/2}(\Omega)$ and $u \in H^1(\Omega)$ satisfies $-\Delta u(x) \le h(x), \quad x \in \Omega$ in the weak sense. Then for each ball $B_{2r}(y) \subset \Omega$,*

$$\sup_{B_r(y)} u \le C\big(\|u^+\|_{L^2(B_{2r}(y))} + \|h\|_{L^{t/2}(B_{2r}(y))}\big),$$

where $C = C(N, t, r)$ is independent of y.

3. Singularly perturbed problem

In this section, we assume that $(\lambda_1, \lambda_2) \in D_{\rm ur}$. (1.1) can be rewritten as

$$\begin{gathered} -\Delta v + V(\varepsilon x)v + \lambda_1 v^3 + \lambda_2 (K * v^2)v = 0 \quad \text{in } \mathbb{R}^3, \\ v > 0, \quad v \in H^1(\mathbb{R}^3) \end{gathered} \tag{3.1}$$

with the corresponding energy functional

$$I_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)v^2 + \frac{\lambda_1}{4}\int_{\mathbb{R}^3} (v^+)^4 + \frac{\lambda_2}{4}\int_{\mathbb{R}^3} (K * (v^+)^2)(v^+)^2, \quad v \in H_\varepsilon,$$

where

$$H_\varepsilon := \{v \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)v^2 < \infty\}$$

endowed with the norm

$$\|v\|_\varepsilon := \Big(\int_{\mathbb{R}^3} |\nabla v|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)v^2\Big)^{1/2}.$$

We define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Lambda/\varepsilon, \\ \varepsilon^{-18} & \text{if } x \notin \Lambda/\varepsilon \end{cases} \quad Q_\varepsilon(v) = \Big(\int_{\mathbb{R}^3} \chi_\varepsilon v^2 - 1\Big)_+^\beta,$$

where $1 < \beta < 2$. Finally, we set $J_\varepsilon : H_\varepsilon \to \mathbb{R}$ by

$$J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v).$$

Moreover, for each $R > 0$, we regard $H_0^1(B_R(0))$ as a subspace of H_ε. Namely, for any $u \in H_0^1(B_R(0))$, we extend u by defining $u(x) = 0$ for $|x| > R$, then $\|\cdot\|_\varepsilon$ is equivalent to the standard norm of $H_0^1(B_R(0))$ for each $R > 0$, $\varepsilon > 0$. Next we will show that $J_\varepsilon\big|_{H_0^1(B_R(0))}$ possesses the mountain pass geometry for $\varepsilon > 0$ small and $R > 0$ large.

Lemma 3.1. *For each fixed $\varepsilon > 0$ and $R > 0$, there exists a constat $r(\varepsilon) > 0$ such that*

$$\inf\{J_\varepsilon(u) : u \in H_0^1(B_R(0)), \|u\|_\varepsilon = r\} > 0.$$

Proof. From (A1), we choose $b > 0$ such that $\liminf_{|x|\to+\infty} V(x) \geq 2b$, we can take $R_0 > 0$ such that $V(\varepsilon x) \geq b$ for $|x| \geq \varepsilon^{-1}R_0$ and $B_{R_0}(0) \supset \Lambda$. By Lemma 2.3, Sobolev's imbedding theorem and Young's inequality, we see that for any $\eta > 0$, there exists $C_\eta > 0$ such that

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)v^2 + \frac{1}{4}C(v) + Q_\varepsilon(v) \\ &\geq \frac{1}{2}\|v\|_\varepsilon^2 - C\int_{\mathbb{R}^3} v^4 \\ &\geq \frac{1}{2}\|v\|_\varepsilon^2 - \eta\int_{\mathbb{R}^3} v^2 - C_\eta\int_{\mathbb{R}^3} v^6 \\ &\geq \frac{1}{4}\|v\|_\varepsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3\setminus B_{R_0/\varepsilon}(0)} V(\varepsilon x)v^2 - \eta\int_{\mathbb{R}^3\setminus B_{R_0/\varepsilon}(0)} v^2 - \eta\int_{B_{R_0/\varepsilon}(0)} v^2 - C_\eta\int_{\mathbb{R}^3} v^6 \\ &\geq \frac{1}{4}\|v\|_\varepsilon^2 + \frac{1}{4}\int_{\mathbb{R}^3\setminus B_{R_0/\varepsilon}(0)} V(\varepsilon x)v^2 - \eta\int_{\mathbb{R}^3\setminus B_{R_0/\varepsilon}(0)} v^2 - \frac{C_0\delta}{\varepsilon^2}\|v\|_\varepsilon^2 - C_\eta\|v\|_\varepsilon^6 \end{aligned}$$

Choosing $\eta > 0$ such that $\eta < \{\frac{b}{4}, \frac{\varepsilon^2}{8C_0}\}$, we see that

$$J_\varepsilon(u) \geq \frac{1}{8}\|v\|_\varepsilon^2 - C_\varepsilon\|v\|_\varepsilon^6,$$

the lemma holds. □

Lemma 3.2. *For each fixed $\varepsilon > 0$ small and $R > 0$ large, there exists a $\phi_\varepsilon \in H_0^1(B_R(0))$ such that $J_\varepsilon(\phi_\varepsilon) < 0$.*

Proof. Arguing as in the proof of [15, Lemma 2.3], we can select a nonnegative $u_0 \in C_c^\infty(\mathbb{R}^3)\backslash\{0\}$ to ensure that $C(u_0) < 0$. Then we choose $R > 0$ large such that $\mathrm{supp} u_0 \subset B_R(0)$, at this time, u_0 can be regarded as a function in $H_0^1(B_R(0))$, then

$$\begin{aligned} J_\varepsilon(tu_0) &= \frac{t^2}{2}\|u_0\|_\varepsilon^2 + \frac{t^4}{4}C(u_0) + \Big(\int_{\mathbb{R}^3} (\chi_\varepsilon t u_0)^2 - 1\Big)_+^\beta \\ &\le \frac{t^2}{2}\|u_0\|_\varepsilon^2 + \frac{t^4}{4}C(u_0) + Ct^{2\beta}\Big(\int_{\mathbb{R}^3} (\chi_\varepsilon u_0)^2\Big)^\beta + C. \end{aligned}$$

Note that $C(u_0) < 0$ and $1 < \beta < 2$, we can choose a large T_ε such that $J_\varepsilon(T_\varepsilon u_0) < 0$, $T_\varepsilon u_0$ is the desired ϕ_ε. □

Similar to the argument in Lemmas 3.1 and 3.2, we see that J_ε possesses the mountain pass geometry in H_ε. Hence, we can define the mountain pass value of J_ε and $J_\varepsilon|_{H_0^1(B_R(0))}$ as follows:

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon} \max_{s\in[0,1]} J_\varepsilon(\gamma(s)),$$

where

$$\begin{gathered} \Gamma_\varepsilon := \{\gamma \in C([0,1], H_\varepsilon) : \gamma(0) = 0, \quad J_\varepsilon(\gamma(1)) < 0\}, \\ c_{\varepsilon,R} := \inf_{\gamma\in\Gamma_{\varepsilon,R}} \max_{0\le t\le 1} J_\varepsilon(\gamma(t)), \\ \Gamma_{\varepsilon,R} := \big\{\gamma \in C([0,1],\ H_0^1(B_R(0))) : \gamma(0) = 0,\ J_\varepsilon(\gamma(1)) < 0\big\}. \end{gathered}$$

It is easy to see that for each $\varepsilon > 0$ small and $R > 0$ large, $c_{\varepsilon,R} \ge c_\varepsilon$, then $\lim_{R\to+\infty} c_{\varepsilon,R} := d_\varepsilon \ge c_\varepsilon$.

Lemma 3.3. *For each $\varepsilon > 0$ small and $R > 0$ large, $c_{\varepsilon,R}$ is a critical value of J_ε on $H_0^1(B_R(0))$.*

Proof. By Lemmas 3.1 and 3.2, and the mountain pass theorem, for each fixed $\varepsilon > 0$ small and $R > 0$ large, there exists a sequence $\{v_{n,\varepsilon}^R\}_{n=1}^\infty \subset H_0^1(B_R(0))$ such that $J_\varepsilon(v_{n,\varepsilon}^R) \to c_{\varepsilon,R}$ and $J'_\varepsilon(v_{n,\varepsilon}^R) \to 0$ in $(H_0^1(B_R(0)))^{-1}$ as $n \to \infty$. Then

$$\begin{aligned} &c_{\varepsilon,R} + o(1) + o(1)\|v_{n,\varepsilon}^R\|_\varepsilon \\ &= J_\varepsilon(v_{n,\varepsilon}^R) - \frac{1}{4}\langle J'_\varepsilon(v_{n,\varepsilon}^R), v_{n,\varepsilon}^R\rangle \\ &= \frac{1}{4}\|v_{n,\varepsilon}^R\|_\varepsilon^2 + \Big(\int_{\mathbb{R}^3} \chi_\varepsilon (v_{n,\varepsilon}^R)^2 - 1\Big)_+^\beta - \frac{\beta}{2}\Big(\int_{\mathbb{R}^3} \chi_\varepsilon (v_{n,\varepsilon}^R)^2 - 1\Big)_+^{\beta-1} \int_{\mathbb{R}^3} \chi_\varepsilon (v_{n,\varepsilon}^R)^2 \\ &\ge \frac{1}{4}\|v_{n,\varepsilon}^R\|_\varepsilon^2 + \Big(1 - \frac{\beta}{2}\Big)\Big(\int_{\mathbb{R}^3} \chi_\varepsilon (v_{n,\varepsilon}^R)^2 - 1\Big)_+^\beta - \frac{\beta}{2}\Big(\int_{\mathbb{R}^3} \chi_\varepsilon (v_{n,\varepsilon}^R)^2 - 1\Big)_+^{\beta-1}, \end{aligned} \tag{3.2}$$

where $o(1) \to 0$ as $n \to \infty$. From (3.2), we see that

$$\begin{aligned} &\Big(1 - \frac{\beta}{2}\Big)\Big(\int_{\mathbb{R}^3} \chi_\varepsilon (v_{n,\varepsilon}^R)^2 - 1\Big)_+^\beta - \frac{\beta}{2}\Big(\int_{\mathbb{R}^3} \chi_\varepsilon (v_{n,\varepsilon}^R)^2 - 1\Big)_+^{\beta-1} \\ &\le c_{\varepsilon,R} + o(1) + o(1)\|v_{n,\varepsilon}^R\|_\varepsilon - \frac{1}{4}\|v_{n,\varepsilon}^R\|_\varepsilon^2. \end{aligned} \tag{3.3}$$

Noting that since $1 < \beta < 2$, the left side of (3.3) is bounded from below, then we see that $\{v_{n,\varepsilon}^R\}_{n=1}^\infty$ is bounded in $H_0^1(B_R(0))$. Up to a subsequence, as $n \to \infty$, we see that $v_{n,\varepsilon}^R \rightharpoonup v_\varepsilon^R$ in $H_0^1(B_R(0))$, $v_{n,\varepsilon}^R \to v_\varepsilon^R$ in $L^s(B_R(0))(1 \le s < 6)$ and $v_{n,\varepsilon}^R \to v_\varepsilon^R$ a.e. $B_R(0)$. Using standard argument, we verify that

$$v_{n,\varepsilon}^R \to v_\varepsilon^R \quad \text{in } H_0^1(B_R(0)) \text{ as } n \to \infty \tag{3.4}$$

and $v_\varepsilon^R \ge 0$ satisfies

$$\begin{gathered} -\Delta v_\varepsilon^R + V(\varepsilon x)v_\varepsilon^R + 2\beta\Big(\int_{\mathbb{R}^3} \chi_\varepsilon (v_\varepsilon^R)^2 dx - 1\Big)_+^{\beta-1} \chi_\varepsilon v_\varepsilon^R \\ = -\lambda_1 (v_\varepsilon^R)^3 - \lambda_2 (K * |v_\varepsilon^R|^2)v_\varepsilon^R \quad \text{in } B_R(0), \\ v_\varepsilon^R = 0 \quad \text{on } \partial B_R(0) \end{gathered} \tag{3.5}$$

and $J_\varepsilon(v_\varepsilon^R) = c_{\varepsilon,R}$. □

Lemma 3.4. *For each $\varepsilon > 0$ small, d_ε is a critical value of J_ε on H_ε.*

Proof. Noting that $\lim_{\varepsilon\to 0} d_\varepsilon = 0$ (Lemma 3.5 below) and $\lim_{R\to+\infty} c_{\varepsilon,R} := d_\varepsilon$, by (3.3), we choose $\bar\varepsilon > 0$ such that for each $\varepsilon \in (0,\bar\varepsilon]$, there exists $R_\varepsilon > 0$ such that

$$\|v_\varepsilon^R\|_{H_\varepsilon} \le C \text{ uniformly for } \varepsilon \in (0,\bar\varepsilon] \text{ and } R > R_\varepsilon. \tag{3.6}$$

By Lemma 2.4, we see that

$$\|v_\varepsilon^R\|_{L^\infty(\mathbb{R}^3)} \le C \text{ uniformly for } \varepsilon \in (0,\bar\varepsilon] \text{ and } R > R_\varepsilon. \tag{3.7}$$

Since $Q_\varepsilon(v_\varepsilon^R)$ is uniformly bounded for all $\varepsilon > 0$ small and $R > R_\varepsilon$, we have

$$\int_{\mathbb{R}^3\setminus(\Lambda/\varepsilon)} (v_\varepsilon^R)^2 \le C\varepsilon^{18}. \tag{3.8}$$

We see from (3.5) and (3.7) that

$$-\Delta v_\varepsilon^R \le C(v_\varepsilon^R)^3 + C(|K * |v_\varepsilon^R|^2|v_\varepsilon^R) \le C(v_\varepsilon^R)^{2/3} + |K * v_\varepsilon^R|(v_\varepsilon^R)^{1/3},$$

in the weak sense. Letting $t = 6$ and $r = 1$ in Lemma 2.5, we have

$$\begin{aligned}
\sup_{B_1(y)} v_\varepsilon^R &\le C\Big(\|v_\varepsilon^R\|_{L^2(B_2(y))} + \|v_\varepsilon^R\|_{L^2(B_2(y))}^{2/3} + \Big(\int_{B_2(y)} |K * v_\varepsilon^R|^3(v_\varepsilon^R)\Big)^{1/3}\Big) \\
&\le C(\|v_\varepsilon^R\|_{L^2(B_2(y))} + \|v_\varepsilon^R\|_{L^2(B_2(y))}^{2/3} + \|K * v_\varepsilon^R\|_{L^6(\mathbb{R}^3)}\|v_\varepsilon^R\|_{L^2(B_2(y))}^{1/3}), y \in \mathbb{R}^3.
\end{aligned}$$

By (3.8) and Lemma 2.3, we check that for $\varepsilon > 0$ small,

$$v_\varepsilon^R(x) \le C\varepsilon^3 \quad \text{for all } |x| \ge R_0/\varepsilon + 2 \text{ and } R > R_\varepsilon. \tag{3.9}$$

By (3.6), (3.7) and Lemma 2.3, we verify that $-\lambda_1(v_\varepsilon^R)^3 - \lambda_2(K * |v_\varepsilon^R|^2)v_\varepsilon^R$ is bounded in $L^q_{\rm loc}(\mathbb{R}^3)$ for every $q \ge 1$ and the L^p-estimate implies the boundedness of $\{v_\varepsilon^R\}$ in $W^{2,q}_{\rm loc}(\mathbb{R}^3)(q \ge 1)$, then by the Sobolev's imbedding theorem, there exists $\alpha \in (0,1)$ such that

$$\|v_\varepsilon^R\|_{C^{1,\alpha}_{\rm loc}(\mathbb{R}^3)} \le C \quad \text{uniformly for } \varepsilon \in (0,\bar\varepsilon] \text{ and } R > R_\varepsilon. \tag{3.10}$$

Let $\bar\theta$ the angle between $x - y$ and the dipole axis $(0,0,1)$ and note that the average of $1 - 3\cos^2\bar\theta$ on spheres $\partial B_r(x)(r > 0)$ vanishes, that is

$$\int_{\partial B_r(x)} (1 - 3\cos^2\bar\theta)ds = 0. \tag{3.11}$$

Using differential mean value theorem, (3.9), (3.10) and (3.11) yield that for $|x| \ge 2R_0/\varepsilon$ and $R > R_\varepsilon$,

$$\begin{aligned}
&\lim_{\delta\to 0}\int_{B_1(x)\setminus B_\delta(x)} \frac{1 - 3\cos^2\bar\theta}{|x-y|^3}|v_\varepsilon^R(y)|^2dy \\
&= \lim_{\delta\to 0}\int_{B_1(x)\setminus B_\delta(x)} \frac{1 - 3\cos^2\bar\theta}{|x-y|^3}(|v_\varepsilon^R(y)|^2 - |v_\varepsilon^R(x)|^2)dy \\
&\le C\|v_\varepsilon^R\|_{L^\infty(B_1(x))}\|\nabla v_\varepsilon^R\|_{L^\infty(B_1(x))}\int_{B_1(x)} \frac{1}{|x-y|^2}dy \le C\varepsilon^3.
\end{aligned} \tag{3.12}$$

Recalling $\Lambda \subset B_{R_0}(0)$, we see from (3.6) that for $|x| \ge 2R_0/\varepsilon$,

$$\int_{\Lambda/\varepsilon} \frac{1 - 3\cos^2\bar\theta}{|x-y|^3}|v_\varepsilon^R(y)|^2dy \le C\frac{\varepsilon^3}{R_0^3}\int_{\Lambda/\varepsilon}|v_\varepsilon^R(y)|^2dy \le C\varepsilon^3. \tag{3.13}$$

Moreover, (3.8) implies that for $|x| \ge 2R_0/\varepsilon$,

$$\int_{\mathbb{R}^3\setminus((\Lambda/\varepsilon)\cup B_1(x))} \frac{1 - 3\cos^2\bar\theta}{|x-y|^3}|v_\varepsilon^R(y)|^2dy \le C\int_{\mathbb{R}^3\setminus(\Lambda/\varepsilon)}|v_\varepsilon^R(y)|^2dy \le C\varepsilon^{18}. \tag{3.14}$$

Finally, from (3.9), (3.12), (3.13) and (3.14), for $\varepsilon > 0$ small,

$$-\lambda_1(v_\varepsilon^R)^3 - \lambda_2(K * |v_\varepsilon^R|^2)v_\varepsilon^R \le \frac{1}{2}V(\varepsilon x)v_\varepsilon^R \quad \text{for } |x| \ge 2R_0/\varepsilon \text{ and } R > R_\varepsilon.$$

The Maximum Principle shows that

$$0 \le v_\varepsilon^R(x) \le C_1(\varepsilon) e^{-C_2(\varepsilon)|x|} \text{ for } |x| \ge 2R_0/\varepsilon \text{ and } R > R_\varepsilon, \tag{3.15}$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are independent of R.

Choosing a cut-off function $\varphi_A \in C^\infty(\mathbb{R}^3)$ such that $0 \le \varphi_A \le 1$, $\varphi_A = 0$ for $|x| \le A$, $\varphi_A = 1$ for $|x| \ge 2A$ and $|\nabla \varphi_A| \le C/A$. It follows from (3.15), (3.6), Lemma 2.3 and the fact $\langle J_\varepsilon'(v_\varepsilon^R), \varphi_A v_\varepsilon^R \rangle = 0$ that as $A \to \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}^3 \setminus B_{2A}(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 \\
&\le \frac{C}{A} \int_{\mathbb{R}^3 \setminus B_A(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 + C \int_{\mathbb{R}^3 \setminus B_A(0)} |v_\varepsilon^R|^4 + C \int_{\mathbb{R}^3 \setminus B_A(0)} (K * |v_\varepsilon^R|^2)|v_\varepsilon^R|^2 \\
&\le \frac{C}{A} \int_{\mathbb{R}^3 \setminus B_A(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 + C \int_{\mathbb{R}^3 \setminus B_A(0)} |v_\varepsilon^R|^4 \\
&\quad + C \Big(\int_{\mathbb{R}^3} (K * |v_\varepsilon^R|^2)^2 \Big)^{1/2} \Big(\int_{\mathbb{R}^3 \setminus B_A(0)} |v_\varepsilon^R|^4 \Big)^{1/2} \\
&\le \frac{C}{A} \int_{\mathbb{R}^3 \setminus B_A(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 + C(\varepsilon) \int_{\mathbb{R}^3 \setminus B_A(0)} e^{-C(\varepsilon)|x|} \\
&\quad + C(\varepsilon) \Big(\int_{\mathbb{R}^3 \setminus B_A(0)} e^{-C(\varepsilon)|x|} \Big)^{1/2} \to 0,
\end{aligned}$$

that is for $\varepsilon > 0$ small,

$$\lim_{A \to \infty} \int_{\mathbb{R}^3 \setminus B_{2A}(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 = 0. \tag{3.16}$$

Since $\{v_\varepsilon^R\}$ is bounded in H_ε, we can assume that as $R \to \infty$, $v_\varepsilon^R \rightharpoonup v_\varepsilon$ in H_ε, $v_\varepsilon^R \to v_\varepsilon$ in $L^s_{\rm loc}(\mathbb{R}^3)$ for all $s \in [1, 6)$ and $v_\varepsilon^R \to v_\varepsilon$ a.e. $\mathbb{R}^3$. By (3.16) and Sobolev's imbedding theorem, we obtain $v_\varepsilon^R \to v_\varepsilon$ in $L^s(\mathbb{R}^3)$ for all $s \in [2, 6)$ as $R \to \infty$. Using standard argument, we see that $v_\varepsilon^R \to v_\varepsilon$ in H_ε as $R \to \infty$. Hence, $v_\varepsilon \in H_\varepsilon$ satisfies

$$-\Delta v_\varepsilon + V(\varepsilon x) v_\varepsilon + 2\beta \Big(\int_{\mathbb{R}^3} \chi_\varepsilon v_\varepsilon^2 - 1 \Big)_+^{\beta - 1} \chi_\varepsilon v_\varepsilon + \lambda_1 v_\varepsilon^3 + \lambda_2 (K * v_\varepsilon^2) v_\varepsilon = 0 \quad \text{in } \mathbb{R}^3 \tag{3.17}$$

and $J_\varepsilon(v_\varepsilon) = d_\varepsilon$. □

Lemma 3.5. $\lim_{\varepsilon \to 0} d_\varepsilon = 0$.

Proof. Without loss of generality, we may assume that $0 \in \mathcal{M}$, i.e. $V(0) = 0$. We let $w \in H_0^1(B_1(0))$ a positive ground state solution to

$$-\Delta u + \lambda_1 u^3 + \lambda_2 (K * u^2) u = 0 \quad \text{in } B_1(0), \quad u = 0 \text{ on } \partial B_1(0).$$

Then for each $R > 0$, $w_R := \frac{1}{R} w\left(\frac{x}{R}\right)$ satisfies

$$-\Delta w_R + \lambda_1 w_R^3 + \lambda_2 (K * w_R^2) w_R = 0 \quad \text{in } B_R(0), \quad w_R = 0 \text{ on } \partial B_R(0).$$

It is obvious that

$$\int_{B_R(0)} |\nabla w_R|^2 = \frac{1}{R} \int_{B_1(0)} |\nabla w|^2, \tag{3.18}$$

$$\int_{B_R(0)} w_R^4 = \frac{1}{R} \int_{B_1(0)} w^4, \tag{3.19}$$

$$\int_{B_R(0)} (K * w_R^2) w_R^2 = \frac{1}{R} \int_{B_1(0)} (K * w^2) w^2. \tag{3.20}$$

We regard w_R as a function in $H^1(\mathbb{R}^N)$ by defining $w_R = 0$ for $|x| \ge R$. Then for $\varepsilon > 0$ and $t > 0$,

$$J_\varepsilon(t w_R) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla w_R|^2 + \frac{t^2}{2} \int_{\mathbb{R}^3} V(\varepsilon x) w_R^2 + \frac{t^4}{4} C(w_R).$$

It is easy to see that there exists $t_0 > 0$ such that $J_\varepsilon(tw_R) < 0$ for $t > t_0$ and $\varepsilon > 0$ small. Then

$$\begin{aligned}
\limsup_{\varepsilon\to 0} d_\varepsilon &\le \limsup_{\varepsilon\to 0} c_{\varepsilon,R} \\
&\le \limsup_{\varepsilon\to 0} \max_{t\in(0,+\infty)} J_\varepsilon(tw_R) \\
&\le \max_{t\in(0,+\infty)} \Big(\frac{t^2}{2}\int_{\mathbb{R}^3}|\nabla w_R|^2 + \frac{t^4}{4}C(w_R)\Big) + \limsup_{\varepsilon\to 0}\frac{t_0^2}{2}\max_{x\in B_R(0)} V(\varepsilon x)\int_{\mathbb{R}^3} w_R^2 \\
&= 0 + \frac{1}{2}\int_{\mathbb{R}^3}|\nabla w_R|^2 + \frac{1}{4}C(w_R).
\end{aligned}$$

Recalling the fact that $d_\varepsilon \ge c_\varepsilon > 0$ and letting $R \to +\infty$ in the above inequality, we see from (3.18), (3.19) and (3.20) that $\lim_{\varepsilon\to 0} d_\varepsilon = 0$. □

Lemma 3.6. *$\{\|v_\varepsilon\|_\varepsilon\}_{\varepsilon>0}$ and $\{Q_\varepsilon(v_\varepsilon)\}_{\varepsilon>0}$ are bounded for $\varepsilon > 0$ small.*

Proof. Noting that $d_\varepsilon = J_\varepsilon(v_\varepsilon) - \frac{1}{4}\langle J'_\varepsilon(v_\varepsilon), v_\varepsilon\rangle$ and arguing as in (3.2) and (3.3), the conclusion follows. □

Lemma 3.7. *$\{\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\}_{\varepsilon>0}$ is bounded for $\varepsilon > 0$ small.*

The above lemma is a direct conclusion from Lemmas 2.4 and 3.6.

Lemma 3.8. *For each $\delta > 0$, $\lim_{\varepsilon\to 0}\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3\setminus(\mathcal{M}^{\delta/2}/\varepsilon))} = 0$.*

Proof. If follows from Lemma 3.7 that

$$\int_{\mathbb{R}^3\setminus(\Lambda/\varepsilon)} v_\varepsilon^2 \le C\varepsilon^{18}.$$

Then by the boundedness of $\{\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\}_{\varepsilon>0}$, we see that for any $q \ge 2$,

$$\int_{\mathbb{R}^3\setminus(\Lambda/\varepsilon)} v_\varepsilon^q \le C\int_{\mathbb{R}^3\setminus(\Lambda/\varepsilon)} v_\varepsilon^2 \le C\varepsilon^{18},$$

which implies

$$\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3\setminus(\Lambda/\varepsilon))} \to 0 \quad \text{as } \varepsilon \to 0.$$

Suppose that there is $\{y_\varepsilon\}_{\varepsilon>0} \subset (\Lambda/\varepsilon)\setminus(\mathcal{M}^{\delta/2}/\varepsilon)$ such that $\liminf_{\varepsilon\to 0} v_\varepsilon(y_\varepsilon) > 0$ with $V(\varepsilon y_\varepsilon) \to m > 0$, by elliptic estimate, $\{v_\varepsilon(x+y_\varepsilon)\}_{\varepsilon>0}$ is bounded in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^3)$ for some $\alpha \in (0,1)$, then $v_\varepsilon(x+y_\varepsilon) \to v \ne 0$ in $C^1_{\rm loc}(\mathbb{R}^3)$ sense. It is easy to see that v is a solution to (2.1). From [11, Lemma 2.2], we see that $\liminf_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) \ge c_m > 0$ which contradicts Lemma 3.5. □

Lemma 3.9. *For each $\delta > 0$, there exist constants $C_1, C_2 > 0$ such that*

$$v_\varepsilon(x) \le C_1\exp(-C_2 \mathrm{dist}(x, \mathcal{Z}^{3\delta/4}/\varepsilon)).$$

Proof. Noting that for each $\delta > 0$, $\inf\{V(x) : x \notin \mathcal{Z}^{3\delta/4}\} > 0$. Arguing as for (3.8) to (3.14), we see that

$$\lim_{\varepsilon\to 0}\big\|\lambda_1 v_\varepsilon^2 + \lambda_2(K * v_\varepsilon^2)\big\|_{L^\infty(\mathbb{R}^3\setminus(\mathcal{Z}^{3\delta/4}/\varepsilon))} = 0.$$

Thus, we obtain decay estimate by applying comparison principle. We refer to [5, Lemma 2.7] for details. □

If $\mathcal{Z}\setminus\mathcal{M} \ne \emptyset$, we have the following estimate of v_ε on $(\mathcal{Z}^\delta/\varepsilon)\setminus(\mathcal{M}^\delta/\varepsilon)$. This estimate mainly comes from [6, 12], but for completeness, we give a proof.

Lemma 3.10. *For $\delta > 0$ small, there exist constants $C_1, C_2 > 0$ such that*

$$\|v_\varepsilon\|_{L^\infty((\mathcal{Z}^\delta/\varepsilon)\setminus(\mathcal{M}^\delta/\varepsilon))} \le C_1\exp\Big(-\frac{C_2}{\varepsilon}\Big).$$

Proof. Taking $2\delta > 0$ to ensure that $(\mathcal{Z}^{2\delta}\backslash\mathcal{M}^{2\delta}) \cap \Lambda^{2\delta} = \emptyset$. Let $(\varphi, \bar{\lambda}_1)$ be the pair of the first eigenfunction and eigenvalue of $-\Delta$ on $\mathcal{Z}^{2\delta}\backslash\mathcal{M}^{2\delta}$ with Dirichlet boundary condition, we may assume that

$$\inf\{\varphi(x) : x \in \partial(\mathcal{Z}^{\delta}\backslash\mathcal{M}^{\delta})\} = 1. \tag{3.21}$$

It follows from Lemma 3.6, Lemma 3.7 and arguing as for (3.8) to (3.14), we have

$$\|\lambda_1 v_\varepsilon^2 + \lambda_2(K * v_\varepsilon^2)\|_{L^\infty((\mathcal{Z}^{2\delta}/\varepsilon)\backslash(\Lambda^{\delta}/\varepsilon))} \le C\varepsilon^3.$$

From Lemma 3.9, we see that $v_\varepsilon(x) \le C_1 \exp\left(-\frac{C_2\delta}{4\varepsilon}\right)$ on $\partial(\mathcal{Z}^{\delta}/\varepsilon)$. Then we define $\varphi_\varepsilon(x) = C_1 \exp\left(-\frac{C_2\delta}{4\varepsilon}\right)\varphi(\varepsilon x)$. Then $-\Delta\varphi_\varepsilon = \bar{\lambda}_1\varepsilon^2\varphi_\varepsilon$. Noting that $-\Delta v_\varepsilon \le C\varepsilon^3 v_\varepsilon$, we have

$$-\Delta(v_\varepsilon - \varphi_\varepsilon) \le C\varepsilon^3 v_\varepsilon - \bar{\lambda}_1\varepsilon^2\varphi_\varepsilon \le 0$$

according to Lemma 3.7 and (**??**), then by a comparison principle, we obtain the estimate. □

Proof of Theorem 1.1. From Lemmas 3.9 and 3.10, we see that as $\varepsilon \to 0$,

$$\begin{aligned}
\varepsilon^{-18}\int_{\mathbb{R}^3\backslash(\Lambda/\varepsilon)} v_\varepsilon^2 &\le \varepsilon^{-18}\int_{\mathbb{R}^3\backslash(\Lambda/\varepsilon)\backslash(\mathcal{Z}^{\delta}/\varepsilon)} v_\varepsilon^2 + \varepsilon^{-18}\int_{(\mathcal{Z}^{\delta}/\varepsilon)\backslash(\Lambda/\varepsilon)} v_\varepsilon^2 \\
&\le \varepsilon^{-18}\int_{\mathbb{R}^3\backslash(\mathcal{Z}^{\delta}/\varepsilon)} v_\varepsilon^2 + \varepsilon^{-18}\int_{(\mathcal{Z}^{\delta}/\varepsilon)\backslash(\mathcal{M}^{\delta}/\varepsilon)} v_\varepsilon^2 \\
&\le C\varepsilon^{-18}\int_{\mathbb{R}^3\backslash(\mathcal{Z}^{\delta}/\varepsilon)} e^{-C\frac{\delta}{\varepsilon}} + C\varepsilon^{-18}\int_{(\mathcal{Z}^{\delta}/\varepsilon)\backslash(\mathcal{M}^{\delta}/\varepsilon)} e^{-\frac{C}{\varepsilon}} \to 0.
\end{aligned}$$

This implies that $Q_\varepsilon(v_\varepsilon) = 0$ for $\varepsilon > 0$ small. Then

$$\frac{1}{4}\|v_\varepsilon\|_\varepsilon = J_\varepsilon(v_\varepsilon) - \frac{1}{4}\langle J'_\varepsilon(v_\varepsilon), v_\varepsilon\rangle = d_\varepsilon \to 0$$

as $\varepsilon \to 0$, by Lemma 2.4, we see that $\lim_{\varepsilon\to 0}\|v_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = 0$. Noting that v_ε satisfies

$$-\Delta v_\varepsilon + V(\varepsilon x)v_\varepsilon + \lambda_1 v_\varepsilon^3 + \lambda_2(K * v_\varepsilon^2)v_\varepsilon = 0 \quad \text{in } \mathbb{R}^3.$$

Letting $w_\varepsilon(x) = \varepsilon^{-1}v_\varepsilon(x/\varepsilon)$, we see that

$$-\Delta w_\varepsilon + \frac{V(x)}{\varepsilon^2}w_\varepsilon + \lambda_1 w_\varepsilon^3 + \lambda_2(K * w_\varepsilon^2)w_\varepsilon = 0 \quad \text{in } \mathbb{R}^3 \tag{3.22}$$

Multiplying w_ε on both sides of (3.22) and integrating by parts, we see from Lemma 2.3 that

$$\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}V(x)w_\varepsilon^2 \le C\|w_\varepsilon\|^2_{L^\infty(\mathbb{R}^3)}\int_{\mathbb{R}^3}w_\varepsilon^2. \tag{3.23}$$

We denote $\phi \in C_c^\infty(\mathbb{R}^N, [0,1])$ such that $\phi(x) = 1$ on $\mathcal{Z}^{4\delta}$, then it follows that

$$\begin{aligned}
\int_{\mathbb{R}^3} w_\varepsilon^2 &\le C\int_{\mathbb{R}^3}\phi^2 w_\varepsilon^2 + C\int_{\mathbb{R}^3}(1-\phi)^2 w_\varepsilon^2 \\
&\le C\int_{\mathbb{R}^3}|\nabla(\phi w_\varepsilon)|^2 + C\frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}V(x)w_\varepsilon^2 \\
&\le C\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2 + C\int_{\mathbb{R}^3}|\nabla\phi|^2 w_\varepsilon^2 + C\frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}V(x)w_\varepsilon^2 \\
&\le C\int_{\mathbb{R}^3}|\nabla w_\varepsilon|^2 + C\frac{1}{\varepsilon^2}\int_{\mathbb{R}^3}V(x)w_\varepsilon^2.
\end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24), we see that $\liminf_{\varepsilon\to 0}\|w_\varepsilon\|_{L^\infty(\mathbb{R}^3)} > 0$. Recalling that $u_\varepsilon(x) = v_\varepsilon(x/\varepsilon)$, we obtain $\liminf_{\varepsilon\to 0}\varepsilon^{-1}\|u_\varepsilon\|_{L^\infty(\mathbb{R}^3)} > 0$. □

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