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BOUNDEDNESS FOR A TWO-SPECIES DOUBLY DEGENERATE DIFFUSION CHEMOTAXIS SYSTEM IN TWO-DIMENSIONS

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Abstract. We study the following two-species doubly degenerate nutrient-taxis system in a smoothly bounded domain $\Omega \subset \mathbb{R}^2$ under homogeneous Neumann boundary conditions:

$$\begin{gathered}
(u_1)_t = \nabla \cdot (u_1 v \nabla u_1) - \chi_1 \nabla \cdot (u_1^2 v \nabla v) + \ell_1 u_1 v, \quad x \in \Omega,\ t > 0,\\
(u_2)_t = \nabla \cdot (u_2 v \nabla u_2) - \chi_2 \nabla \cdot (u_2^2 v \nabla v) + \ell_2 u_2 v, \quad x \in \Omega,\ t > 0,\\
v_t = \Delta v - (u_1 + u_2) v, \quad x \in \Omega,\ t > 0,
\end{gathered}$$

with $\chi_i, \ell_i > 0$ for $i = 1, 2$. This system describes the spatio-temporal dynamics of two interacting populations which consume nutrients. We show that for all reasonably regular initial data, the system possesses a global bounded weak solution. We also identify a criterion regarding the initial smallness of the third component such that the solution stabilizes to a nonconstant steady state.

1. Introduction

In this work, we focus on a two-species doubly degenerate diffusion chemotaxis system in a smooth bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{gathered}
(u_1)_t = \nabla \cdot (u_1 v \nabla u_1) - \chi_1 \nabla \cdot (u_1^2 v \nabla v) + \ell_1 u_1 v, \quad x \in \Omega,\ t > 0,\\
(u_2)_t = \nabla \cdot (u_2 v \nabla u_2) - \chi_2 \nabla \cdot (u_2^2 v \nabla v) + \ell_2 u_2 v, \quad x \in \Omega,\ t > 0,\\
v_t = \Delta v - (u_1 + u_2) v, \quad x \in \Omega,\ t > 0,\\
(u_1 v \nabla u_1 - \chi_1 u_1^2 v \nabla v) \cdot \nu = (u_2 v \nabla u_2 - \chi_2 u_2^2 v \nabla v) \cdot \nu = \nabla v \cdot \nu = 0, \quad x \in \partial\Omega, t > 0,\\
u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x), \quad v(x,0) = v_0(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where $\chi_i > 0$ and $\ell_i > 0$ $(i = 1, 2)$. The initial data u_{10}, u_{20} and v_0 are nonnegative. In this system, u_1 and u_2 stand for the population densities of two bacterial species, and v represents the nutrient concentration. ℓ_1 and ℓ_2 are the conversion rates (growth yield) of consumed nutrient to bacterial growth. The system describes the spatio-temporal dynamics of two interacting populations which consume nutrients. Notably, certain bacterial species –such as *Bacillus subtilis*– are known to exhibit a variety of spatial patterns depending on the softness of the medium, nutrient availability, and temperature. Fujikawa [4], Fujikawa and Matsushita [5], as well as Matsushita and Fujikawa [17] have provided compelling evidence of the intricate collective behavior of *Bacillus subtilis* populations under nutrient-limited conditions. In particular, the emergence of complex structures, such as snowflake-like population distributions, appears to be a generic feature rather than an exception in such settings. Experimental evidence shows that bacterial motility is restricted near low-nutrient regions in relatively soft agar plates.

To account for the emergence of such aggregation patterns, Leyva et al. [12] generalized the degenerate diffusion model of Kawasaki et al. [10] to the doubly degenerate nutrient-taxis system

$$\begin{gathered}
u_t = \nabla \cdot (u v \nabla u) - \chi \nabla \cdot (u^2 v \nabla v) + \ell u v, \quad x \in \Omega,\ t > 0,\\
v_t = \Delta v - u v, \quad x \in \Omega,\ t > 0,
\end{gathered} \tag{1.2}$$

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with $\chi, \ell > 0$. In this system, u and v stand for the density of the species and the concentration of the chemoattractant, respectively. In the one dimensional setting, Winkler [25] revealed the existence of global weak solutions with a particularly remarkable feature being that the first component of the solution asymptotically stabilizes to a nontrivial function coinciding with the spatial profile of a solution to a scalar porous medium-type parabolic equation, which is in sharp contrast to the large-time behavior described in most previous studies. Dropping the requirement $\int_\Omega \ln u_0 > -\infty$, and under the mere assumption that

$$\begin{gathered} u_0 \in W^{1,\infty}(\Omega) \text{ is nonnegative with } u_0 \not\equiv 0, \\ v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega}, \end{gathered}$$

Li and Winkler [13] demonstrated for the one-dimensional version of the system (1.2) that an associated no-flux initial boundary value problem possesses a globally defined and continuous weak solution (u, v), and that moreover there exists $u_\infty \in C^0(\overline{\Omega})$ such that the solution (u, v) approaches the pair $(u_\infty, 0)$ in the large time limit with respect to the topology $(L^\infty(\Omega))^2$. Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Winkler [26] proved that for any $p > 2$ and each fixed nonnegative $u_0 \in W^{1,\infty}(\Omega)$, a smallness condition exclusively involving v_0 can be identified as sufficient to ensure that an associated no-flux type initial-boundary value problem with $(u, v)\big|_{t=0} = (u_0, v_0)$ admits a global weak solution satisfying $\operatorname{ess\,sup}_{t>0} \|u(\cdot, t)\|_{L^p(\Omega)} < \infty$. Zhang and Li [39] showed that for all reasonably regular initial data, the corresponding homogeneous Neumann initial-boundary value problem possesses a global bounded weak solution, and they identified a criterion regarding the initial smallness of the second component such that the solution stabilizes to a nonconstant steady state. For further results on systems related to (1.2) and their generalizations, we refer to [14, 27, 28, 29, 33, 34, 40, 15, 41, 20].

The following two-species chemotaxis model has received considerable attention in the past decades:

$$\begin{gathered} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \quad x \in \Omega,\ t > 0, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \quad x \in \Omega,\ t > 0, \\ w_t = \Delta w + g(u, v, w), \quad x \in \Omega,\ t > 0, \end{gathered} \tag{1.3}$$

with $\chi_i, \mu_i, a_i > 0$, $(i = 1, 2)$, and $\Omega \subset \mathbb{R}^n$ a smoothly bounded domain. Important results on this system can be found in [8, 16, 22, 23]. When $g(u, v, w) = -\beta w + b_1 u + b_2 v$, $\beta, b_i > 0$ $(i = 1, 2)$, Bai and Winkler [1] established global existence of solutions to system (1.3) when $n \leqslant 2$ and asymptotic behavior if μ_1 and μ_2 are sufficiently large. Xiang et al. [42, 35] studied the two-species chemotaxis system with indirect signal production and proved that the solutions are globally bounded and converge to the constant steady state under the locally and globally competitive assumptions. For the case $g(u, v, w) = -w(\alpha u + \beta v)$ with $\alpha, \beta > 0$, it was shown in [24] that the corresponding initial-boundary value problem possesses a unique global bounded classical solution if $\chi_i \|w_0\|_{L^\infty(\Omega)} < \frac{\pi}{\sqrt{n+1}}$ $(i = 1, 2)$, and they proved asymptotic stabilization of solutions. Further studies on various modifications of the system (1.3) can be found in [2, 6, 7, 9, 19, 36, 37]. For the doubly degenerate nutrient taxis model

$$\begin{gathered} (u_1)_t = (u_1 v u_{1x})_x - \left(u_1^2 v v_x\right)_x + \mu_1 u_1 \left(1 - u_1 - a_1 u_2\right), \quad x \in \Omega,\ t > 0, \\ (u_2)_t = (u_2 v u_{2x})_x - \left(u_2^2 v v_x\right)_x + \mu_2 u_2 \left(1 - u_2 - a_2 u_1\right), \quad x \in \Omega,\ t > 0, \\ v_t = v_{xx} - \left(u_1 + u_2\right) v, \quad x \in \Omega,\ t > 0, \end{gathered}$$

under no-flux boundary conditions in an open bounded interval $\Omega \subset \mathbb{R}$ with $a_i > 0$ and $\mu_i > 0$ $(i = 1, 2)$, Xiang [38] demonstrated that for all suitably regular nonnegative initial data (u_{10}, u_{20}, v_0), the system admits at least one global weak solution satisfying the following boundedness property

$$\|u_1(\cdot, t)\|_{L^p(\Omega)} + \|u_2(\cdot, t)\|_{L^p(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leqslant C,$$

and the author showed that the following properties hold: if $a_1, a_2 \in (0, 1)$, there exists a sequence $t_k \to \infty$ such that the global weak solution $(u_1, u_2, v)(\cdot, t_k) \to \left(\frac{1-a_1}{1-a_1 a_2}, \frac{1-a_2}{1-a_1 a_2}, 0\right)$ in $L^2(\Omega)$ as $k \to \infty$, if $a_1 \geqslant 1 > a_2 > 0$, there exists a sequence $t_k \to \infty$ such that the global weak solution $(u_1, u_2, v)(\cdot, t_k) \to (0, 1, 0)$ in $L^2(\Omega)$ as $k \to \infty$.

Inspired by [39], we establish the global existence and large- time behavior of weak solutions for a two-species doubly degenerate nutrient-taxis model.

Main result. Assume that

$$\begin{gathered}
u_{10} \in W^{1,\infty}(\Omega) \text{ is nonnegative, not identically equal to 0 and } \int_\Omega \ln u_{10} > -\infty, \\
u_{20} \in W^{1,\infty}(\Omega) \text{ is nonnegative, not identically equal to 0 and } \int_\Omega \ln u_{20} > -\infty, \\
v_0 \in W^{1,\infty}(\Omega) \text{ is positive in } \overline{\Omega}.
\end{gathered} \tag{1.4}$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\chi_i, \ell_i > 0$ for $i = 1, 2$. Assume that the initial datum (u_{10}, u_{20}, v_0) satisfies* (1.4)*. Then problem* (1.1) *admits a global weak solution (u_1, u_2, v), in the sense of Definition 2.1 below, which is such that*

$$\begin{gathered}
u_1 \in C^0(\overline{\Omega} \times [0,\infty)), \\
u_2 \in C^0(\overline{\Omega} \times [0,\infty)), \\
v \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),
\end{gathered} \tag{1.5}$$

with $u_1, u_2 > 0$ a.e. in $\Omega \times (0,\infty)$ and $v > 0$ in $\overline{\Omega} \times [0,\infty)$, and

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \leqslant C \quad \text{for all } t > 0, \tag{1.6}$$

where $C > 0$ is a constant independent of t.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\chi_i, \ell_i > 0$ for $i = 1, 2$. Assume that the initial datum (u_{10}, u_{20}, v_0) satisfies* (1.4)*. Then there exist nonnegative functions $u_{1\infty}, u_{2\infty} \in L^\infty(\Omega)$ such that*

$$\begin{gathered}
\int_\Omega u_{10} \leqslant \int_\Omega u_{1\infty} \leqslant \int_\Omega u_{10} + \frac{\ell_1}{\ell_2}\int_\Omega u_{20} + \ell_1 \int_\Omega v_0, \\
\int_\Omega u_{20} \leqslant \int_\Omega u_{2\infty} \leqslant \frac{\ell_2}{\ell_1}\int_\Omega u_{10} + \int_\Omega u_{20} + \ell_2 \int_\Omega v_0,
\end{gathered}$$

and that as $t \to \infty$, the solution (u_1, u_2, v) of the system (1.1) *from Theorem 1.1 satisfies*

$$u_1(t) \overset{*}{\rightharpoonup} u_{1\infty}, \quad u_2(t) \overset{*}{\rightharpoonup} u_{2\infty} \quad \text{in } L^\infty(\Omega), \tag{1.7}$$

$$v(t) \to 0 \quad \text{in } L^\infty(\Omega). \tag{1.8}$$

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $\chi_i, \ell_i > 0$ for $i = 1, 2$. Assume that the initial datum (u_{10}, u_{20}, v_0) satisfies* (1.4) *and $u_{10}, u_{20} \neq$ const. Then given $K > 0$, one can find $\delta(K) > 0$ such that if*

$$\|v_0\|_{W^{1,\infty}(\Omega)} \leqslant K, \quad \int_\Omega \frac{|\nabla v_0|^2}{v_0} \leqslant K, \quad \int_\Omega \frac{|\nabla v_0|^4}{v_0^3} \leqslant K, \quad \int_\Omega v_0 \leqslant \delta(K),$$

then the corresponding limit functions obtained in Theorem 1.2, s $u_{1\infty}$ and $u_{2\infty}$, are not constant.

Our two-species analysis yields an extension of the one-species results in [39].

The remainder of this paper is organized as follows. In Section 2, we define the weak solutions of the system (1.1). Then we show the local existence for the regularized problem. Furthermore, we recall some basic estimates and inequalities. In Section 3, we demonstrate the uniform $L^\infty \times L^\infty \times W^{1,\infty}$ bound for (u_1, u_2, v). In Section 4, we obtain the large-time behavior of global weak solutions to system (1.1).

2. Preliminaries

This section is devoted to the local well-posedness of the regularized problem and presents some auxiliary lemmas. First, we define the weak solutions, which arise in a natural manner.

Definition 2.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $u_{10}, u_{20}, v_0 \in L^1(\Omega)$ with $u_{10}, u_{20}, v_0 \geqslant 0$. We call a triple of nonnegative functions

$$\begin{gathered} u_1 \in L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)), \\ u_2 \in L^1_{\rm loc}(\overline{\Omega} \times [0,\infty)), \\ v \in L^\infty_{\rm loc}(\overline{\Omega} \times [0,\infty)) \cap L^1_{\rm loc}([0,\infty); W^{1,1}(\Omega)) \end{gathered} \tag{2.1}$$

a global weak solution (u_1, u_2, v) of system (1.1) if

$$u_1^2,\, u_2^2 \in L^1_{\rm loc}([0,\infty); W^{1,1}(\Omega)) \quad \text{and} \quad u_1^2\nabla v,\, u_2^2 \nabla v \in L^1_{\rm loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^2), \tag{2.2}$$

and if for all test functions $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ we have

$$\begin{aligned} &-\int_0^\infty \int_\Omega u_1 \varphi_t - \int_\Omega u_{10}\varphi(\cdot,0) \\ &= -\frac{1}{2}\int_0^\infty \int_\Omega v \nabla u_1^2 \cdot \nabla\varphi + \chi_1 \int_0^\infty \int_\Omega u_1^2 v \nabla v \cdot \nabla \varphi + \ell_1 \int_0^\infty \int_\Omega u_1 v \varphi, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &-\int_0^\infty \int_\Omega u_2 \varphi_t - \int_\Omega u_{20}\varphi(\cdot,0) \\ &= -\frac{1}{2}\int_0^\infty \int_\Omega v \nabla u_2^2 \cdot \nabla\varphi + \chi_2 \int_0^\infty \int_\Omega u_2^2 v \nabla v \cdot \nabla \varphi + \ell_2 \int_0^\infty \int_\Omega u_2 v \varphi, \end{aligned} \tag{2.4}$$

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi + \int_0^\infty \int_\Omega (u_1+u_2) v \varphi. \tag{2.5}$$

For $\varepsilon \in (0,1)$ we introduce a regularized version of (1.1):

$$\begin{gathered} u_{1\varepsilon t} = \nabla \cdot (u_{1\varepsilon} v_\varepsilon \nabla u_{1\varepsilon}) - \chi_1 \nabla \cdot (u_{1\varepsilon}^2 v_\varepsilon \nabla v_\varepsilon) + \ell_1 u_1 v, \quad x \in \Omega,\ t>0, \\ u_{2\varepsilon t} = \nabla \cdot (u_{2\varepsilon} v_\varepsilon \nabla u_{2\varepsilon}) - \chi_2 \nabla \cdot (u_{2\varepsilon}^2 v_\varepsilon \nabla v_\varepsilon) + \ell_2 u_2 v, \quad x \in \Omega,\ t>0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - (u_{1\varepsilon} + u_{2\varepsilon}) v_\varepsilon, \quad x \in \Omega,\ t>0, \\ \frac{\partial u_{1\varepsilon}}{\partial \nu} = \frac{\partial u_{2\varepsilon}}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t>0, \\ u_{1\varepsilon}(x,0) = u_{10}(x) + \varepsilon, \quad u_{2\varepsilon}(x,0) = u_{20}(x)+\varepsilon, \quad v_\varepsilon(x,0) = v_0(x), \quad x \in \Omega, \end{gathered} \tag{2.6}$$

which, based on established methods, allows us to state the following result regarding existence, extensibility, and fundamental properties, see [26, Lemma 2.1].

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and assume that* (1.4) *holds. Then, for each $\varepsilon \in (0,1)$, there exists $T_{\max,\varepsilon} \in (0,\infty]$ and at least one triple $(u_{1\varepsilon}, u_{2\varepsilon}, v_\varepsilon)$ of functions*

$$\begin{gathered} u_{1\varepsilon} \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ u_{2\varepsilon} \in C^0(\overline{\Omega} \times [0, T_{\max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})), \\ v_\varepsilon \in \cap_{q>2} C^0\big([0, T_{\max,\varepsilon}); W^{1,q}(\Omega)\big) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max,\varepsilon})) \end{gathered} \tag{2.7}$$

such that $u_{1\varepsilon}, u_{2\varepsilon}, v_\varepsilon > 0$ in $\overline{\Omega} \times (0, T_{\max,\varepsilon})$, that $(u_{1\varepsilon}, u_{2\varepsilon}, v_\varepsilon)$ solves (2.6) *in the classical sense, and that if $T_{\max,\varepsilon} < \infty$, then*

$$\limsup_{t \nearrow T_{\max,\varepsilon}} \Big(\|u_{1\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} + \|u_{2\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{2.8}$$

Moreover, this solution satisfies

$$\int_\Omega u_{10} \leqslant \int_\Omega u_{1\varepsilon}(t) \leqslant m_1 = \int_\Omega (u_{10}+1) + \frac{\ell_1}{\ell_2} \int_\Omega (u_{20}+1) + \ell_1 \int_\Omega v_0, \quad \forall t \in (0, T_{\max,\varepsilon}) \tag{2.9}$$

$$\int_\Omega u_{20} \leqslant \int_\Omega u_{2\varepsilon}(t) \leqslant m_2 = \frac{\ell_2}{\ell_1} \int_\Omega (u_{10}+1) + \int_\Omega (u_{20}+1) + \ell_2 \int_\Omega v_0 \quad \forall t \in (0, T_{\max,\varepsilon}), \tag{2.10}$$

$$\|v_\varepsilon(t)\|_{L^\infty(\Omega)} \leqslant \|v_0\|_{L^\infty(\Omega)} \quad llt \in (0, T_{\max,\varepsilon}), \tag{2.11}$$

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega (u_{1\varepsilon} + u_{2\varepsilon}) v_\varepsilon \leqslant \int_\Omega v_0 \quad \forall \varepsilon \in (0,1). \tag{2.12}$$

Lemma 2.3 ([39, Lemma A.1]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $p \geqslant 1$. For any $\phi, \psi \in C^1(\overline{\Omega})$ satisfying $\phi, \psi > 0$ in $\overline{\Omega}$, it holds*

$$\int_\Omega \phi^{p+1}\psi \leqslant c\Big\{ \int_\Omega \frac{\phi}{\psi}|\nabla\psi|^2 + \int_\Omega \frac{\psi}{\phi}|\nabla\phi|^2 + \int_\Omega \phi\psi \Big\} \int_\Omega \phi^p$$

for a constant $c = c(p,\Omega) > 0$.

Lemma 2.4 ([39, Lemma A.2]). *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $p \geqslant 1$. For each $\eta > 0$ and any $\phi, \psi \in C^1(\overline{\Omega})$ satisfying $\phi, \psi > 0$ in $\overline{\Omega}$, it holds*

$$\begin{aligned} \int_\Omega \phi^{p+1}\psi|\nabla\psi|^2 &\leqslant \eta \int_\Omega \phi^{p-1}\psi|\nabla\phi|^2 + c\big\{\|\psi\|_{L^\infty(\Omega)} + \frac{\|\psi\|_{L^\infty(\Omega)}^3}{\eta}\big\} \int_\Omega \phi^{p+1}\psi \int_\Omega \frac{|\nabla\psi|^4}{\psi^3} \\ &\quad + c\|\psi\|_{L^\infty(\Omega)}^2 \big\{ \int_\Omega \phi \big\}^{2p+1} \int_\Omega \frac{|\nabla\psi|^4}{\psi^3} + c\|\psi\|_{L^\infty(\Omega)}^2 \int_\Omega \phi\psi \end{aligned}$$

for some constant $c = c(p,\Omega) > 0$.

Lemma 2.5 ([18, Lemma 4.2]). *Assume that Ω is bounded and let $w \in C^2(\overline{\Omega})$ satisfy $\frac{\partial w}{\partial v} = 0$ on $\partial\Omega$. Then we have*

$$\frac{\partial|\nabla w|^2}{\partial\nu} \leqslant 2\kappa|\nabla w|^2 \quad \text{on } \partial\Omega,$$

where $\kappa = \kappa(\Omega) > 0$ is an upper bound for the curvatures of $\partial\Omega$.

Lemma 2.6 ([3, Theorem 1, p.272]). *Assume Ω is bounded and $\partial\Omega$ is C^1. Let $p \in [1,\infty)$. Then there exists a bounded linear operator $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that*

(i) *$Tw = w|_{\partial\Omega}$ if $w \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$,*
(i) *$\|Tw\|_{L^p(\partial\Omega)} \leqslant C\|w\|_{W^{1,p}(\Omega)}$ for each $w \in W^{1,p}(\Omega)$, with the constant C depending only on p and Ω.*

3. Existence of bounded weak solutions

This section we establish a uniform $L^\infty \times L^\infty$ bound for $(u_{1\varepsilon}, u_{2\varepsilon})$, which in turn yields $W^{1,\infty}$ estimates for v_ε. Winkler [25] found a gradient-like structure in system (2.6). We use this structure to obtain uniform estimates for the temporal-spatial integral of $\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2$ and $\frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2$.

Lemma 3.1. *Assume that (1.4) holds. Then we have*

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 \leqslant C, \quad \int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2 \leqslant C \quad \forall \varepsilon \in (0,1), \tag{3.1}$$

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega v_\varepsilon \leqslant C \quad \forall \varepsilon \in (0,1), \tag{3.2}$$

$$\int_\Omega \Big(\ln\frac{1}{u_{1\varepsilon}(t)}\Big)_+ \leqslant C, \quad \int_\Omega \Big(\ln\frac{1}{u_{2\varepsilon}(t)}\Big)_+ \leqslant C \quad \forall t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1), \tag{3.3}$$

where C is a positive constant depending on $-\int_\Omega \ln u_{10}$, $\int_\Omega u_{10}$, $-\int_\Omega \ln u_{20}$, $\int_\Omega u_{20}$, $\int_\Omega v_0$ and $\int_\Omega |\nabla v_0|^2$ but independent of ε.

Proof. Using the first three equations in (2.6) and integrating by parts, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big\{ -\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega \ln u_{1\varepsilon} - \frac{\chi_1+\chi_2}{\chi_2}\int_\Omega \ln u_{2\varepsilon} + \frac{\chi_1+\chi_2}{2}\int_\Omega |\nabla v_\varepsilon|^2\Big\} \\ &- \frac{\chi_1+\chi_2}{\chi_1}\int_\Omega \frac{1}{u_{1\varepsilon}} \cdot \{\nabla\cdot(u_{1\varepsilon}v_\varepsilon\nabla u_{1\varepsilon} - \chi_1 u_{1\varepsilon}^2 v_\varepsilon \nabla v_\varepsilon) + \ell_1 u_{1\varepsilon}v_\varepsilon\} \\ &\quad - \frac{\chi_1+\chi_2}{\chi_2}\int_\Omega \frac{1}{u_{2\varepsilon}} \cdot \{\nabla\cdot(u_{2\varepsilon}v_\varepsilon\nabla u_{2\varepsilon} - \chi_2 u_{2\varepsilon}^2 v_\varepsilon \nabla v_\varepsilon) + \ell_2 u_{2\varepsilon}v_\varepsilon\} \end{aligned}$$

$$
\begin{aligned}
&+ (\chi_1+\chi_2)\int_\Omega \nabla v_\varepsilon\cdot\nabla\{\Delta v_\varepsilon-(u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon\}\\
&=-\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\Big\{\frac{\nabla u_{1\varepsilon}}{u_{1\varepsilon}^2}\cdot(u_{1\varepsilon}v_\varepsilon\nabla u_{1\varepsilon}-\chi_1u_{1\varepsilon}^2v_\varepsilon\nabla v_\varepsilon)+\ell_1v_\varepsilon\Big\}\\
&\quad-\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\Big\{\frac{\nabla u_{2\varepsilon}}{u_{2\varepsilon}^2}\cdot(u_{2\varepsilon}v_\varepsilon\nabla u_{2\varepsilon}-\chi_2u_{2\varepsilon}^2v_\varepsilon\nabla v_\varepsilon)+\ell_2v_\varepsilon\Big\}\\
&\quad+(\chi_1+\chi_2)\int_\Omega\nabla v_\varepsilon\cdot\{\nabla\Delta v_\varepsilon-(\nabla u_{1\varepsilon}+\nabla u_{2\varepsilon})v_\varepsilon-(u_{1\varepsilon}+u_{2\varepsilon})\nabla v_\varepsilon\}\\
&=-\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\Big(\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2+\ell_1v_\varepsilon\Big)+(\chi_1+\chi_2)\int_\Omega v_\varepsilon\nabla u_{1\varepsilon}\cdot\nabla v_\varepsilon\\
&\quad-\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\Big(\frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2+\ell_2v_\varepsilon\Big)+(\chi_1+\chi_2)\int_\Omega v_\varepsilon\nabla u_{2\varepsilon}\cdot\nabla v_\varepsilon\\
&\quad-(\chi_1+\chi_2)\int_\Omega\{|\Delta v_\varepsilon|^2+v_\varepsilon(\nabla u_{1\varepsilon}+\nabla u_{2\varepsilon})\cdot\nabla v_\varepsilon+(u_{1\varepsilon}+u_{2\varepsilon})|\nabla v_\varepsilon|^2\}\\
&=-\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\Big(\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2+\ell_1v_\varepsilon\Big)-\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\Big(\frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2+\ell_2v_\varepsilon\Big)\\
&\quad-(\chi_1+\chi_2)\int_\Omega\{|\Delta v_\varepsilon|^2+(u_{1\varepsilon}+u_{2\varepsilon})|\nabla v_\varepsilon|^2\}
\end{aligned}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Integration of the above differential equality on $(0,t)$ leads to

$$
\begin{aligned}
&-\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\ln u_{1\varepsilon}(t)-\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\ln u_{2\varepsilon}(t)+\frac{\chi_1+\chi_2}{2}\int_\Omega|\nabla v_\varepsilon(t)|^2\\
&+\frac{\chi_1+\chi_2}{\chi_1}\int_0^t\int_\Omega\Big(\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2+\ell_1v_\varepsilon\Big)+\frac{\chi_1+\chi_2}{\chi_2}\int_0^t\int_\Omega\Big(\frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2+\ell_2v_\varepsilon\Big)\\
&+(\chi_1+\chi_2)\int_0^t\int_\Omega\{|\Delta v_\varepsilon|^2+(u_{1\varepsilon}+u_{2\varepsilon})|\nabla v_\varepsilon|^2\}\\
&=-\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\ln(u_{10}+\varepsilon)-\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\ln(u_{20}+\varepsilon)+\frac{\chi_1+\chi_2}{2}\int_\Omega|\nabla v_0|^2
\end{aligned}
\tag{3.4}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Hence, according to (2.9) and $\ln s\leqslant s$ for all $s>0$, we have

$$
\begin{aligned}
&\frac{\chi_1+\chi_2}{2}\int_\Omega|\nabla v_\varepsilon(t)|^2+\frac{\chi_1+\chi_2}{\chi_1}\int_0^t\int_\Omega\Big(\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2+\ell_1v_\varepsilon\Big)\\
&+\frac{\chi_1+\chi_2}{\chi_2}\int_0^t\int_\Omega\Big(\frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2+\ell_2v_\varepsilon\Big)+(\chi_1+\chi_2)\int_0^t\int_\Omega\{|\Delta v_\varepsilon|^2+(u_{1\varepsilon}+u_{2\varepsilon})|\nabla v_\varepsilon|^2\}\\
&\leqslant-\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\ln u_{10}-\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\ln u_{20}\\
&\quad+\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\ln u_{1\varepsilon}(t)+\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\ln u_{2\varepsilon}(t)+\frac{\chi_1+\chi_2}{2}\int_\Omega|\nabla v_0|^2\\
&\leqslant-\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\ln u_{10}-\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\ln u_{20}+\frac{\chi_1+\chi_2}{2}\int_\Omega|\nabla v_0|^2+\frac{\chi_1+\chi_2}{\chi_1}m_1+\frac{\chi_1+\chi_2}{\chi_2}m_2
\end{aligned}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Therefore, (3.1) and (3.2) hold. Using (3.4), we obtain

$$
\begin{aligned}
&\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega\Big(\ln\frac{1}{u_{1\varepsilon}(t)}\Big)_++\frac{\chi_1+\chi_2}{\chi_2}\int_\Omega\Big(\ln\frac{1}{u_{2\varepsilon}(t)}\Big)_+\\
&=\frac{\chi_1+\chi_2}{\chi_1}\Big(-\int_\Omega\ln u_{1\varepsilon}(t)+\int_{\{u_{1\varepsilon}(t)>1\}}\ln u_{1\varepsilon}(t)\Big)\\
&\quad+\frac{\chi_1+\chi_2}{\chi_2}\Big(-\int_\Omega\ln u_{2\varepsilon}(t)+\int_{\{u_{2\varepsilon}(t)>1\}}\ln u_{2\varepsilon}(t)\Big)\\
&\leqslant\frac{\chi_1+\chi_2}{\chi_1}\Big(-\int_\Omega\ln u_{1\varepsilon}(t)+\int_\Omega u_{1\varepsilon}(t)\Big)+\frac{\chi_1+\chi_2}{\chi_2}\Big(-\int_\Omega\ln u_{2\varepsilon}(t)+\int_\Omega u_{2\varepsilon}(t)\Big)
\end{aligned}
$$

$$\leqslant -\frac{\chi_1+\chi_2}{\chi_1}\int_\Omega \ln u_{10} - \frac{\chi_1+\chi_2}{\chi_2}\int_\Omega \ln u_{20} + \frac{\chi_1+\chi_2}{2}\int_\Omega |\nabla v_0|^2 + \frac{\chi_1+\chi_2}{\chi_1}m_1 + \frac{\chi_1+\chi_2}{\chi_2}m_2$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. The proof is complete. □

Winkler [29] obtained a uniform estimate for the temporal-spatial integral of $\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2$ in a doubly degenerate reaction-diffusion system. According to Lemma 3.1, and applying a similar method, we also get such an estimate for the system (2.6).

Lemma 3.2. *Assume that* (1.4) *holds. Then*

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 \leqslant C \quad \text{for all } \varepsilon \in (0,1), \tag{3.5}$$

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leqslant C \quad \text{for all } \varepsilon \in (0,1), \tag{3.6}$$

where C is a positive constant depending on $-\int_\Omega \ln u_{10}$, $\int_\Omega u_{10}$, $-\int_\Omega \ln u_{20}$, $\int_\Omega u_{20}$, $\int_\Omega v_0$, $\int_\Omega |\nabla v_0|^2$ and $\int_\Omega \frac{|\nabla v_0|^2}{v_0}$, but independent of ε.

Proof. Owing to $v_\varepsilon > 0$ in $\overline{\Omega} \times (0, T_{\max,\varepsilon})$ and, by standard parabolic regularity, actually belongs to $C^3(\overline{\Omega} \times (0, T_{\max,\varepsilon}))$, we use the third equation in the system (2.6) to compute

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\
&= \int_\Omega \frac{1}{v_\varepsilon}\nabla v_\varepsilon \cdot \nabla\{\Delta v_\varepsilon - (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon\} - \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon^2}|\nabla v_\varepsilon|^2 \cdot \{\Delta v_\varepsilon - (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon\} \\
&= \int_\Omega \frac{1}{v_\varepsilon}\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - \int_\Omega \frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 - \int_\Omega \nabla(u_{1\varepsilon}+u_{2\varepsilon}) \cdot \nabla v_\varepsilon \\
&\quad - \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon^2}|\nabla v_\varepsilon|^2 \cdot \Delta v_\varepsilon + \frac{1}{2}\int_\Omega \frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\
&= \int_\Omega \frac{1}{v_\varepsilon}\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon^2}|\nabla v_\varepsilon|^2 \cdot \Delta v_\varepsilon \\
&\quad - \frac{1}{2}\int_\Omega \frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 - \int_\Omega \nabla(u_{1\varepsilon}+u_{2\varepsilon}) \cdot \nabla v_\varepsilon
\end{aligned} \tag{3.7}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. According to [30, Lemma 3.2], we obtain the integral identity

$$\int_\Omega \frac{1}{v_\varepsilon}\nabla v_\varepsilon \cdot \nabla \Delta v_\varepsilon - \frac{1}{2}\int_\Omega \frac{1}{v_\varepsilon^2}|\nabla v_\varepsilon|^2 \Delta v_\varepsilon = -\int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2}\int_{\partial\Omega} \frac{1}{v_\varepsilon}\frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu}.$$

Hence, (3.7) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2}\int_\Omega \frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\
&= \frac{1}{2}\int_{\partial\Omega} \frac{1}{v_\varepsilon}\frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu} - \int_\Omega \nabla(u_{1\varepsilon}+u_{2\varepsilon}) \cdot \nabla v_\varepsilon
\end{aligned} \tag{3.8}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Based on [30, Lemma 3.3] and [26, Lemma 3.4], we have

$$\int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \geqslant c_1 \int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} + c_1 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}, \tag{3.9}$$

where $c_1 = \frac{1}{28+16\sqrt{2}}$. By Lemmas 2.5 and 2.6, we obtain

$$\frac{\partial |\nabla v_\varepsilon|^2}{\partial \nu} \leqslant c_2 |\nabla v_\varepsilon|^2 \quad \text{on } \partial\Omega,$$

$$\int_{\partial\Omega} \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \leqslant c_3 \int_\Omega \Big|\nabla\Big(\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\Big)\Big| + c_3 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon},$$

where $c_2 = c_2(\Omega) > 0$ and $c_3 = c_3(\Omega) > 0$. Hence, by Young inequality, we have

$$
\begin{aligned}
\frac{1}{2}\int_{\partial\Omega}\frac{1}{v_\varepsilon}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} &\leqslant \frac{c_2}{2}\int_{\partial\Omega}\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\\
&\leqslant \frac{c_2c_3}{2}\int_\Omega\Big|\nabla\Big(\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\Big)\Big| + \frac{c_2c_3}{2}\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\\
&\leqslant c_2c_3\int_\Omega\frac{|D^2v_\varepsilon\cdot\nabla v_\varepsilon|}{v_\varepsilon} + \frac{c_2c_3}{2}\int_\Omega\frac{|\nabla v_\varepsilon|^3}{v_\varepsilon^2} + \frac{c_2c_3}{2}\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\\
&\leqslant c_1\int_\Omega\frac{|D^2v_\varepsilon|^2}{v_\varepsilon} + \frac{c_1}{4}\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_4\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\\
&\leqslant c_1\int_\Omega\frac{|D^2v_\varepsilon|^2}{v_\varepsilon} + \frac{c_1}{2}\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_5\int_\Omega v_\varepsilon,
\end{aligned}
\tag{3.10}
$$

where $c_4 = \frac{c_2^2c_3^2}{2c_1} + \frac{c_2c_3}{2}$ and $c_5 = \frac{c_4^2}{c_1}$. Notice that

$$
-\int_\Omega\nabla(u_{1\varepsilon}+u_{2\varepsilon})\cdot\nabla v_\varepsilon \leqslant \frac{1}{4}\int_\Omega\frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}+u_{2\varepsilon}}|\nabla(u_{1\varepsilon}+u_{2\varepsilon})|^2. \tag{3.11}
$$

Summing (3.8)-(3.11), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + c_1\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_\Omega\frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2\\
&\leqslant 2\int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}+u_{2\varepsilon}}|\nabla(u_{1\varepsilon}+u_{2\varepsilon})|^2 + 2c_5\int_\Omega v_\varepsilon
\end{aligned}
\tag{3.12}
$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0, 1)$. Integrating (3.12) on $(0, t)$ yields

$$
\begin{aligned}
&\int_\Omega\frac{|\nabla v_\varepsilon(t)|^2}{v_\varepsilon(t)} + c_1\int_0^t\int_\Omega\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{1}{2}\int_0^t\int_\Omega\frac{u_{1\varepsilon}+u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2\\
&\leqslant \int_\Omega\frac{|\nabla v_0|^2}{v_0} + 2\int_0^t\int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}+u_{2\varepsilon}}|\nabla(u_{1\varepsilon}+u_{2\varepsilon})|^2 + 2c_5\int_0^t\int_\Omega v_\varepsilon
\end{aligned}
\tag{3.13}
$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0, 1)$. Notice that

$$
\begin{aligned}
\int_0^t\int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}+u_{2\varepsilon}}|\nabla(u_{1\varepsilon}+u_{2\varepsilon})|^2 &\leqslant 2\int_0^t\int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}+u_{2\varepsilon}}|\nabla u_{1\varepsilon}|^2 + 2\int_0^t\int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}+u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2\\
&\leqslant 2\int_0^t\int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 + 2\int_0^t\int_\Omega\frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2.
\end{aligned}
$$

Therefore, (3.5) and (3.6) hold because of Lemma 3.1 and (3.13). □

Applying Lemma 2.3 with $p = 1$ and the estimates derived in Lemmas 3.1 and 3.2, we obtain the following temporal-spatial L^2 bounds for $u_{1\varepsilon}$ and $u_{2\varepsilon}$, weighted by the factor v_ε.

Lemma 3.3. *Assume that* (1.4) *holds. Then*

$$
\int_0^{T_{\max,\varepsilon}}\int_\Omega u_{1\varepsilon}^2v_\varepsilon \leqslant C \quad \text{and} \quad \int_0^{T_{\max,\varepsilon}}\int_\Omega u_{2\varepsilon}^2v_\varepsilon \leqslant C \quad \text{for all } \varepsilon \in (0,1),
$$

where C is a positive constant depending on $-\int_\Omega\ln u_{10}$, $\int_\Omega u_{10}$, $-\int_\Omega\ln u_{20}$, $\int_\Omega u_{20}$, $\int_\Omega v_0$, $\int_\Omega|\nabla v_0|^2$ and $\int_\Omega\frac{|\nabla v_0|^2}{v_0}$, but independent of ε.

Proof. Letting $p = 1$ in Lemma 2.3, we obtain

$$
\int_\Omega u_{1\varepsilon}^2v_\varepsilon \leqslant c_1(\Omega)\Big\{\int_\Omega\frac{u_{1\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 + \int_\Omega u_{1\varepsilon}v_\varepsilon\Big\}\int_\Omega u_{1\varepsilon}
$$

for a constant $c_1(\Omega) > 0$. By (2.9) and (2.12), (3.1) and (3.5), we have

$$
\int_0^{T_{\max,\varepsilon}}\Big\{\Big(\int_\Omega\frac{u_{1\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \int_\Omega\frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 + \int_\Omega u_{1\varepsilon}v_\varepsilon\Big)\int_\Omega u_{1\varepsilon}\Big\} \leqslant c_2
$$

for all $\varepsilon \in (0,1)$ with a constant $c_2 > 0$ independent of ε. A completely analogous estimate holds for $u_{2\varepsilon}$. This completes the proof. $\square$

Though (2.12), (3.1), and Lemma 3.2 we establish that

$$\int_\Omega \Big(\frac{u_{1\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 + u_{1\varepsilon} v_\varepsilon\Big) \in L^1(0,T_{\max,\varepsilon}),$$
$$\int_\Omega \Big(\frac{u_{2\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \frac{v_\varepsilon}{u_{2\varepsilon}}|\nabla u_{2\varepsilon}|^2 + u_{2\varepsilon} v_\varepsilon\Big) \in L^1(0,T_{\max,\varepsilon})$$
$$\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \in L^1(0,T_{\max,\varepsilon})$$

for all $\varepsilon \in (0,1)$, these are insufficient to get the estimates for $\int_\Omega u_{1\varepsilon}^p$ and $\int_\Omega u_{2\varepsilon}^p$, respectively. What is needed next is

$$\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \in L^\infty(0,T_{\max,\varepsilon}).$$

Hence, we consider an energy-like functional

$$8b\Big(\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon} + \int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon}\Big) + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}.$$

Lemma 3.4. *It holds*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big\{8b\Big(\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon} + \int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon}\Big) + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}\Big\}\\
&\leqslant -2b\int_\Omega v_\varepsilon(|\nabla u_{1\varepsilon}|^2 + |\nabla u_{2\varepsilon}|^2) + 8b(\chi_1^2+\chi_2^2)\int_\Omega (u_{1\varepsilon}^2 + u_{2\varepsilon}^2) v_\varepsilon |\nabla v_\varepsilon|^2\\
&\quad + 8b\int_\Omega (\ell_1 u_{1\varepsilon}\ln u_{1\varepsilon} + \ell_2 u_{2\varepsilon}\ln u_{2\varepsilon})v_\varepsilon + 8b(\ell_1+\ell_2)\int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v\\
&\quad - \int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2\left|D^2\ln v_\varepsilon\right|^2 - \int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon^{-3}|\nabla v_\varepsilon|^4 + c\int_\Omega v_\varepsilon
\end{aligned}$$

for all $t \in (0,T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, where b and $c = c(b,\Omega)$ are positive constants independent of ε.

Proof. According to [27, Lemma 3.4], there exists a constant $b > 0$ independent of ε such that

$$\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \leqslant b\int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2 |D^2\ln v_\varepsilon|^2, \tag{3.14}$$
$$\int_\Omega v_\varepsilon^{-3}|\nabla v_\varepsilon|^2|D^2 v_\varepsilon|^2 \leqslant b\int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2|D^2\ln v_\varepsilon|^2. \tag{3.15}$$

By [14, Lemma 2.3], we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + 4\int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2\left|D^2\ln v_\varepsilon\right|^2 + \int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon^{-3}|\nabla v_\varepsilon|^4\\
&\leqslant -4\int_\Omega v_\varepsilon^{-2}|\nabla v_\varepsilon|^2\left[(\nabla u_{1\varepsilon}+\nabla u_{2\varepsilon})\cdot\nabla v_\varepsilon\right] + 2\int_{\partial\Omega} v_\varepsilon^{-3}|\nabla v_\varepsilon|^2\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}
\end{aligned} \tag{3.16}$$

for all $t \in (0,T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. From Young's inequality and (3.14), we have

$$\begin{aligned}
&-4\int_\Omega v_\varepsilon^{-2}|\nabla v_\varepsilon|^2\left[(\nabla u_{1\varepsilon}+\nabla u_{2\varepsilon})\cdot\nabla v_\varepsilon\right]\\
&\leqslant \frac{2}{b}\int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} + 2b\int_\Omega v_\varepsilon|\nabla u_{1\varepsilon}+\nabla u_{2\varepsilon}|^2\\
&\leqslant 2\int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2\left|D^2\ln v_\varepsilon\right|^2 + 4b\int_\Omega v_\varepsilon(|\nabla u_{1\varepsilon}|^2 + |\nabla u_{2\varepsilon}|^2).
\end{aligned} \tag{3.17}$$

In view of (3.14)-(3.15) and [27, Lemma 3.5], we pick $c = c(b,\Omega) > 0$ such that

$$\begin{aligned} 2\int_{\partial\Omega} v_\varepsilon^{-3}|\nabla v_\varepsilon|^2 \frac{\partial |\nabla v_\varepsilon|^2}{\partial\nu} &\leqslant \frac{1}{2b}\int_\Omega v_\varepsilon^{-3}|\nabla v_\varepsilon|^2|D^2 v_\varepsilon|^2 + \frac{1}{2b}\int_\Omega v_\varepsilon^{-5}|\nabla v_\varepsilon|^6 + c\int_\Omega v_\varepsilon \\ &\leqslant \int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2 \left|D^2 \ln v_\varepsilon\right|^2 + c\int_\Omega v_\varepsilon. \end{aligned} \tag{3.18}$$

In light of (2.6) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon} + \int_\Omega v_\varepsilon |\nabla u_{1\varepsilon}|^2 \\ &= \chi_1 \int_\Omega u_{1\varepsilon} v_\varepsilon \nabla u_{1\varepsilon}\cdot\nabla v_\varepsilon + \ell_1\int_\Omega u_{1\varepsilon}v_\varepsilon \ln u_{1\varepsilon} + \ell_1\int_\Omega u_{1\varepsilon}v_\varepsilon \\ &\leqslant \frac14\int_\Omega v_\varepsilon|\nabla u_{1\varepsilon}|^2 + \chi_1^2\int_\Omega u_{1\varepsilon}^2 v_\varepsilon|\nabla v_\varepsilon|^2 + \ell_1\int_\Omega u_{1\varepsilon}v_\varepsilon\ln u_{1\varepsilon} + \ell_1\int_\Omega u_{1\varepsilon}v_\varepsilon \\ &\leqslant \frac14\int_\Omega v_\varepsilon|\nabla u_{1\varepsilon}|^2 + (\chi_1^2+\chi_2^2)\int_\Omega u_{1\varepsilon}^2 v_\varepsilon|\nabla v_\varepsilon|^2 + \ell_1\int_\Omega u_{1\varepsilon}v_\varepsilon\ln u_{1\varepsilon} + (\ell_1+\ell_2)\int_\Omega u_{1\varepsilon}v_\varepsilon \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon} + \int_\Omega v_\varepsilon |\nabla u_{2\varepsilon}|^2 \\ &= \chi_2 \int_\Omega u_{2\varepsilon} v_\varepsilon \nabla u_{2\varepsilon}\cdot\nabla v_\varepsilon + \ell_2\int_\Omega u_{2\varepsilon}v_\varepsilon \ln u_{2\varepsilon} + \ell_2\int_\Omega u_{2\varepsilon}v_\varepsilon \\ &\leqslant \frac14\int_\Omega v_\varepsilon|\nabla u_{2\varepsilon}|^2 + \chi_2^2\int_\Omega u_{2\varepsilon}^2 v_\varepsilon|\nabla v_\varepsilon|^2 + \ell_2\int_\Omega u_{2\varepsilon}v_\varepsilon\ln u_{2\varepsilon} + \ell_2\int_\Omega u_{2\varepsilon}v_\varepsilon \\ &\leqslant \frac14\int_\Omega v_\varepsilon|\nabla u_{2\varepsilon}|^2 + (\chi_1^2+\chi_2^2)\int_\Omega u_{2\varepsilon}^2 v_\varepsilon|\nabla v_\varepsilon|^2 + \ell_2\int_\Omega u_{2\varepsilon}v_\varepsilon\ln u_{2\varepsilon} + (\ell_1+\ell_2)\int_\Omega u_{2\varepsilon}v_\varepsilon \end{aligned} \tag{3.20}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. From (3.16)-(3.20), we deduce that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Big\{8b\Big(\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon} + \int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon}\Big) + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}\Big\} \\ &\leqslant -6b\int_\Omega v_\varepsilon(|\nabla u_{1\varepsilon}|^2 + |\nabla u_{2\varepsilon}|^2) + 8b(\chi_1^2+\chi_2^2)\int_\Omega (u_{1\varepsilon}^2 + u_{2\varepsilon}^2)v_\varepsilon|\nabla v_\varepsilon|^2 \\ &\quad + 8b\int_\Omega (\ell_1 u_{1\varepsilon}\ln u_{1\varepsilon} + \ell_2 u_{2\varepsilon}\ln u_{2\varepsilon})v_\varepsilon + 8b(\ell_1+\ell_2)\int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon \\ &\quad - \int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2\left|D^2\ln v_\varepsilon\right|^2 - \int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon^{-3}|\nabla v_\varepsilon|^4 \\ &\quad + 4b\int_\Omega v_\varepsilon(|\nabla u_{1\varepsilon}|^2 + |\nabla u_{2\varepsilon}|^2) + c\int_\Omega v_\varepsilon \\ &= -2b\int_\Omega v_\varepsilon(|\nabla u_{1\varepsilon}|^2 + |\nabla u_{2\varepsilon}|^2) + 8b(\chi_1^2+\chi_2^2)\int_\Omega (u_{1\varepsilon}^2 + u_{2\varepsilon}^2)v_\varepsilon|\nabla v_\varepsilon|^2 \\ &\quad + 8b\int_\Omega (\ell_1 u_{1\varepsilon}\ln u_{1\varepsilon} + \ell_2 u_{2\varepsilon}\ln u_{2\varepsilon})v_\varepsilon + 8b(\ell_1+\ell_2)\int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon \\ &\quad - \int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2\left|D^2\ln v_\varepsilon\right|^2 - \int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon^{-3}|\nabla v_\varepsilon|^4 + c\int_\Omega v_\varepsilon \end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. This completes the proof. $\square$

We now aim to derive that

$$\frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \in L^\infty((0,T_{\max,\varepsilon}); L^1(\Omega)).$$

To achieve this, we use the differential inequality in Lemma 3.4.

Lemma 3.5. *It holds*

$$\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \leqslant C \quad \text{for all } t \in (0, T_{\max}) \text{ and } \varepsilon \in (0,1) \tag{3.21}$$

$$\int_0^{T_{\max,\varepsilon}} \int_\Omega \frac{|\nabla v_\varepsilon|^6}{v_\varepsilon^5} \leqslant C \quad \text{for all } \varepsilon \in (0,1), \tag{3.22}$$

where C is a positive constant depending on

$$-\int_\Omega \ln u_{10}, \quad \int_\Omega u_{10} \ln u_{10}, \quad \int_\Omega u_{10}, \quad -\int_\Omega \ln u_{20}, \quad \int_\Omega u_{20} \ln u_{20},$$
$$\int_\Omega u_{20}, \quad \int_\Omega v_0, \quad \int_\Omega |\nabla v_0|^2, \quad \int_\Omega \frac{|\nabla v_0|^2}{v_0}, \quad \int_\Omega \frac{|\nabla v_0|^4}{v_0^3},$$

but independent of ε.

Proof. The estimates in this lemma follow from Lemma 3.4. Applying Lemma 2.4 with $p = 1$ and $\eta = \frac{1}{8(\chi_1^2+\chi_2^2)}$, we obtain

$$\begin{aligned}
&8b(\chi_1^2+\chi_2^2) \int_\Omega u_{1\varepsilon}^2 v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\leqslant b \int_\Omega v_\varepsilon |\nabla u_{1\varepsilon}|^2 + \Big\{ c_1 b(\chi_1^2+\chi_2^2)\|v_\varepsilon\|_{L^\infty(\Omega)} + 8c_1 b(\chi_1^2+\chi_2^2)^2 \|v_\varepsilon\|_{L^\infty(\Omega)}^3 \Big\} \int_\Omega u_{1\varepsilon}^2 v_\varepsilon \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad + c_1 b(\chi_1^2+\chi_2^2)\|v_\varepsilon\|_{L^\infty(\Omega)}^2 \Big\{ \int_\Omega u_{1\varepsilon} \Big\}^3 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_1 b(\chi_1^2+\chi_2^2)\|v_\varepsilon\|_{L^\infty(\Omega)}^2 \int_\Omega u_{1\varepsilon} v_\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&8b(\chi_1^2+\chi_2^2) \int_\Omega u_{2\varepsilon}^2 v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\leqslant b \int_\Omega v_\varepsilon |\nabla u_{2\varepsilon}|^2 + \Big\{ c_1 b(\chi_1^2+\chi_2^2)\|v_\varepsilon\|_{L^\infty(\Omega)} + 8c_1 b(\chi_1^2+\chi_2^2)^2 \|v_\varepsilon\|_{L^\infty(\Omega)}^3 \Big\} \int_\Omega u_{2\varepsilon}^2 v_\varepsilon \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad + c_1 b(\chi_1^2+\chi_2^2)\|v_\varepsilon\|_{L^\infty(\Omega)}^2 \Big\{ \int_\Omega u_{2\varepsilon} \Big\}^3 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_1 b(\chi_1^2+\chi_2^2)\|v_\varepsilon\|_{L^\infty(\Omega)}^2 \int_\Omega u_{2\varepsilon} v_\varepsilon
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, where $c_1 = c_1(\Omega) > 0$. Collecting these two inequalities, Lemma 3.4, the elementary estimate $s \ln s + \frac{1}{e} \geqslant 0$ ($s > 0$), and the basic bounds (2.9)-(2.11), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \Big\{ 8b\Big(\int_\Omega u_{1\varepsilon} \ln u_{1\varepsilon} + \int_\Omega u_{2\varepsilon} \ln u_{2\varepsilon} \Big) + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \Big\} + b \int_\Omega v_\varepsilon (|\nabla u_{1\varepsilon}|^2 + |\nabla u_{2\varepsilon}|^2) \\
&+ \int_\Omega v_\varepsilon^{-1} |\nabla v_\varepsilon|^2 \left| D^2 \ln v_\varepsilon \right|^2 + \int_\Omega (u_{1\varepsilon} + u_{2\varepsilon}) v_\varepsilon^{-3} |\nabla v_\varepsilon|^4 \\
&\leqslant B \int_\Omega (u_{1\varepsilon}^2 + u_{2\varepsilon}^2) v_\varepsilon \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_1 b(\chi_1^2+\chi_2^2)(m_1^3+m_2^3)\|v_0\|_{L^\infty(\Omega)}^2 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad + b\{c_1(\chi_1^2+\chi_2^2)\|v_0\|_{L^\infty(\Omega)}^2 + 8(\ell_1+\ell_2)\} \int_\Omega (u_{1\varepsilon} + u_{2\varepsilon}) v_\varepsilon \\
&\quad + 8b(\ell_1+\ell_2) \int_\Omega (u_{1\varepsilon}^2 + u_{2\varepsilon}^2) v_\varepsilon + c \int_\Omega v_\varepsilon \\
&\leqslant B \int_\Omega (u_{1\varepsilon}^2 + u_{2\varepsilon}^2) v_\varepsilon \Big[8b\Big(\int_\Omega u_{1\varepsilon} \ln u_{1\varepsilon} + \int_\Omega u_{2\varepsilon} \ln u_{2\varepsilon} \Big) + \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + \frac{16b|\Omega|}{e} \Big] \\
&\quad + c_1 b(\chi_1^2+\chi_2^2)(m_1^3+m_2^3)\|v_0\|_{L^\infty(\Omega)}^2 \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad + b\{c_1(\chi_1^2+\chi_2^2)\|v_0\|_{L^\infty(\Omega)}^2 + 8(\ell_1+\ell_2)\} \int_\Omega (u_{1\varepsilon} + u_{2\varepsilon}) v_\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&+8b(\ell_1+\ell_2)\int_\Omega (u_{1\varepsilon}^2+u_{2\varepsilon}^2)v_\varepsilon + c\int_\Omega v_\varepsilon \\
={}& B\int_\Omega (u_{1\varepsilon}^2+u_{2\varepsilon}^2)v_\varepsilon\Big[8b\left(\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon}+\int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon}\right)+\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}\Big] \\
&+c_1 b(\chi_1^2+\chi_2^2)(m_1^3+m_2^3)\|v_0\|_{L^\infty(\Omega)}^2\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&+b\{c_1(\chi_1^2+\chi_2^2)\|v_0\|_{L^\infty(\Omega)}^2+8(\ell_1+\ell_2)\}\int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon \\
&+8b\left(\ell_1+\ell_2+\frac{2B|\Omega|}{e}\right)\int_\Omega (u_{1\varepsilon}^2+u_{2\varepsilon}^2)v_\varepsilon + c\int_\Omega v_\varepsilon
\end{aligned} \tag{3.23}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$, where $B=c_1b(\chi_1^2+\chi_2^2)\|v_0\|_{L^\infty(\Omega)}^2+8c_1b(\chi_1^2+\chi_2^2)^2\|v_0\|_{L^\infty(\Omega)}^4$. We set

$$
\begin{gathered}
F_\varepsilon(t):=8b\Big(\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon}+\int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon}\Big)+\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3},\\
L_\varepsilon(t):=B\int_\Omega (u_{1\varepsilon}^2+u_{2\varepsilon}^2)v_\varepsilon,\\
\begin{aligned}
M_\varepsilon(t):={}&c_1 b(\chi_1^2+\chi_2^2)(m_1^3+m_2^3)\|v_0\|_{L^\infty(\Omega)}^2\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}\\
&+b\{c_1(\chi_1^2+\chi_2^2)\|v_0\|_{L^\infty(\Omega)}^2+8(\ell_1+\ell_2)\}\int_\Omega (u_{1\varepsilon}+u_{2\varepsilon})v_\varepsilon\\
&+8b\Big(\ell_1+\ell_2+\frac{2B|\Omega|}{e}\Big)\int_\Omega (u_{1\varepsilon}^2+u_{2\varepsilon}^2)v_\varepsilon + c\int_\Omega v_\varepsilon
\end{aligned}
\end{gathered}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Inequality (3.23) can be rewritten as

$$
F_\varepsilon'(t)\leqslant L_\varepsilon(t)F_\varepsilon(t)+M_\varepsilon(t)\quad \text{for all } t\in(0,T_{\max,\varepsilon}) \text{ and } \varepsilon\in(0,1).
$$

Integrating the above differential inequality yields

$$
F_\varepsilon(t)\leqslant F_\varepsilon(0)e^{\int_0^t L_\varepsilon(s)\,\mathrm{d}s}+\int_0^t e^{\int_s^t L_\varepsilon(\sigma)\,\mathrm{d}\sigma}M_\varepsilon(s)\,\mathrm{d}s \tag{3.24}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Notice that

$$
\int_s^t L_\varepsilon(\sigma)\,\mathrm{d}\sigma\leqslant c_2\quad \text{for all } t\in(0,T_{\max,\varepsilon}),\ s\in[0,t) \text{ and } \varepsilon\in(0,1) \tag{3.25}
$$

by Lemma 3.3, and

$$
\int_0^t M_\varepsilon(s)\,\mathrm{d}s\leqslant c_3\quad \text{for all } t\in(0,T_{\max,\varepsilon}) \text{ and } \varepsilon\in(0,1), \tag{3.26}
$$

thanks to (2.12), (3.2), (3.6), and Lemma 3.3. Therefore, (3.24) implies that

$$
\begin{aligned}
&F_\varepsilon(t)\\
&:=8b\Big(\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon}+\int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon}\Big)+\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}\\
&\leqslant \Big\{8b\Big[\int_\Omega (u_{10}+1)\ln(u_{10}+1)+\int_\Omega (u_{20}+1)\ln(u_{20}+1)\Big]+\int_\Omega \frac{|\nabla v_0|^4}{v_0^3}\Big\}\cdot e^{c_2}+c_3e^{c_2}
\end{aligned} \tag{3.27}
$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Notice that

$$
8b\Big(\int_\Omega u_{1\varepsilon}\ln u_{1\varepsilon}+\int_\Omega u_{2\varepsilon}\ln u_{2\varepsilon}\Big)\geqslant -\frac{16b|\Omega|}{e}\quad \text{for all } t\in(0,T_{\max,\varepsilon}) \text{ and } \varepsilon\in(0,1) \tag{3.28}
$$

Hence, we obtain (3.21). From (3.25)-(3.28), an integration of (3.23) implies

$$
\int_0^{T_{\max,\varepsilon}}\int_\Omega v_\varepsilon^{-1}|\nabla v_\varepsilon|^2|D^2\ln v_\varepsilon|^2\leqslant c_4\quad \text{for all } \varepsilon\in(0,1).
$$

Therefore, (3.14) yields (3.22). □

By (3.21) we obtain the following lemma.

Lemma 3.6. *For all $p > 1$, we have*

$$\int_\Omega u_{1\varepsilon}^p(t) \leqslant C(p) \quad \text{and} \quad \int_\Omega u_{2\varepsilon}^p(t) \leqslant C(p) \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1),$$

where $C(p)$ is a positive constant depending on $-\int_\Omega \ln u_{10}$, $\int_\Omega u_{10} \ln u_{10}$, $\int_\Omega u_{10}$, $-\int_\Omega \ln u_{20}$, $\int_\Omega u_{20} \ln u_{20}$, $\int_\Omega u_{20}$, $\int_\Omega v_0$, $\int_\Omega |\nabla v_0|^2$, $\int_\Omega \frac{|\nabla v_0|^2}{v_0}$ and $\int_\Omega \frac{|\nabla v_0|^4}{v_0^3}$, but independent of ε.

Proof. We begin by testing the first equation of the system (2.6) with $u_{1\varepsilon}^{p-1}$. Then, after integrating by parts and applying Young's inequality, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_{1\varepsilon}^p &= p \int_\Omega u_{1\varepsilon}^{p-1} \left\{ \nabla \cdot (u_{1\varepsilon} v_\varepsilon \nabla u_{1\varepsilon}) - \chi_1 \nabla \cdot \left(u_{1\varepsilon}^2 v_\varepsilon \nabla v_\varepsilon \right) + \ell_1 u_{1\varepsilon} v_\varepsilon \right\} \\
&\leqslant -p(p-1) \int_\Omega u_{1\varepsilon}^{p-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2 + p(p-1)\chi_1 \int_\Omega u_{1\varepsilon}^p v_\varepsilon \nabla u_{1\varepsilon} \cdot \nabla v_\varepsilon + p\ell_1 \int_\Omega u_{1\varepsilon}^p v_\varepsilon \\
&\leqslant -\frac{p(p-1)}{2} \int_\Omega u_{1\varepsilon}^{p-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2 + \frac{p(p-1)\chi_1^2}{2} \int_\Omega u_{1\varepsilon}^{p+1} v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\quad + p\ell_1 \Big(\int_\Omega u_{1\varepsilon}^{p+1} v_\varepsilon + \int_\Omega v_\varepsilon \Big)
\end{aligned} \tag{3.29}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Using Lemma 2.4 with $\eta = \frac{1}{2\chi_1^2}$ and invoking (2.11), we obtain

$$\begin{aligned}
&\frac{p(p-1)\chi_1^2}{2} \int_\Omega u_{1\varepsilon}^{p+1} v_\varepsilon |\nabla v_\varepsilon|^2 \\
&\leqslant \frac{p(p-1)}{4} \int_\Omega u_{1\varepsilon}^{p-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2 \\
&\quad + \Big\{ c_1 \chi_1^2 \|v_0\|_{L^\infty(\Omega)} + 2c_1 \chi_1^4 \|v_0\|_{L^\infty(\Omega)}^3 \Big\} \int_\Omega u_{1\varepsilon}^{p+1} v_\varepsilon \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad + c_1 \chi_1^2 \|v_0\|_{L^\infty(\Omega)}^2 \Big\{ \int_\Omega u_{1\varepsilon} \Big\}^{2p+1} \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + c_1 \chi_1^2 \|v_0\|_{L^\infty(\Omega)}^2 \int_\Omega u_{1\varepsilon} v_\varepsilon,
\end{aligned}$$

where $c_1 = c_1(p, \Omega) > 0$. The above inequality and (3.29) show that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_{1\varepsilon}^p + \frac{p(p-1)}{4} \int_\Omega u_{1\varepsilon}^{p-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2 \\
&\leqslant \Big\{ c_1 \chi_1^2 \|v_0\|_{L^\infty(\Omega)} + 2c_1 \chi_1^4 \|v_0\|_{L^\infty(\Omega)}^3 \Big\} \int_\Omega u_{1\varepsilon}^{p+1} v_\varepsilon \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad + c_1 \chi_1^2 \|v_0\|_{L^\infty(\Omega)}^2 \Big\{ \int_\Omega u_{1\varepsilon} \Big\}^{2p+1} \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} \\
&\quad + c_1 \chi_1^2 \|v_0\|_{L^\infty(\Omega)}^2 \int_\Omega u_{1\varepsilon} v_\varepsilon + p\ell_1 \int_\Omega u_{1\varepsilon}^{p+1} v_\varepsilon + p\ell_1 \int_\Omega v_\varepsilon
\end{aligned} \tag{3.30}$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Based on Lemma 2.3, we conclude that

$$\int_\Omega u_{1\varepsilon}^{p+1} v_\varepsilon \leqslant c_2 \Big\{ \int_\Omega \frac{u_{1\varepsilon}}{v_\varepsilon} |\nabla v_\varepsilon|^2 + \int_\Omega \frac{v_\varepsilon}{u_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2 + \int_\Omega u_{1\varepsilon} v_\varepsilon \Big\} \int_\Omega u_{1\varepsilon}^p,$$

where $c_2 = c_2(p, \Omega) > 0$. The above inequality and (3.30) yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u_{1\varepsilon}^p + \frac{p(p-1)}{4} \int_\Omega u_{1\varepsilon}^{p-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2 \\
&\leqslant A \Big\{ \int_\Omega \frac{u_{1\varepsilon}}{v_\varepsilon} |\nabla v_\varepsilon|^2 + \int_\Omega \frac{v_\varepsilon}{u_{1\varepsilon}} |\nabla u_{1\varepsilon}|^2 + \int_\Omega u_{1\varepsilon} v_\varepsilon \Big\} \int_\Omega u_{1\varepsilon}^p \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}
\end{aligned}$$

$$
\begin{aligned}
&+ A\Big\{ \int_\Omega \frac{u_{1\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \int_\Omega \frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 + \int_\Omega u_{1\varepsilon}v_\varepsilon\Big\} \int_\Omega u_{1\varepsilon}^p \\
&+ A\Big\{ \int_\Omega u_{1\varepsilon}\Big\}^{2p+1} \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + A\int_\Omega u_{1\varepsilon}v_\varepsilon + p\ell_1 \int_\Omega v_\varepsilon
\end{aligned}
$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$, where

$$
A = \max\big\{ c_1c_2\chi_1^2\|v_0\|_{L^\infty(\Omega)} + 2c_1c_2\chi_1^4\|v_0\|_{L^\infty(\Omega)}^3, c_1\chi_1^2\|v_0\|^2_{L^\infty(\Omega)}, c_2p\ell_1\big\}.
$$

Using the above inequality, (2.9) and (3.21), we can find $c_3 > 0$ such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{1\varepsilon}^p + \frac{p(p-1)}{4}\int_\Omega u_{1\varepsilon}^{p-1}v_\varepsilon|\nabla u_{1\varepsilon}|^2 \\
&\leqslant (c_3+1)A\left\{ \int_\Omega \frac{u_{1\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \int_\Omega \frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 + \int_\Omega u_{1\varepsilon}v_\varepsilon\right\} \int_\Omega u_{1\varepsilon}^p \\
&\quad + Am_1^{2p+1}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + A\int_\Omega u_{1\varepsilon}v_\varepsilon + p\ell_1\int_\Omega v_\varepsilon
\end{aligned}
\tag{3.31}
$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. For $\varepsilon \in (0,1)$, we set

$$
\begin{aligned}
Y_\varepsilon(t) &:= \int_\Omega u_{1\varepsilon}^p, \\
H_\varepsilon(t) &:= (c_3+1)A\Big\{ \int_\Omega \frac{u_{1\varepsilon}}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \int_\Omega \frac{v_\varepsilon}{u_{1\varepsilon}}|\nabla u_{1\varepsilon}|^2 + \int_\Omega u_{1\varepsilon}v_\varepsilon\Big\}, \\
G_\varepsilon(t) &:= Am_1^{2p+1}\int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3} + A\int_\Omega u_{1\varepsilon}v_\varepsilon + p\ell_1\int_\Omega v_\varepsilon.
\end{aligned}
$$

Hence, (3.31) becomes

$$
Y_\varepsilon'(t) \leqslant H_\varepsilon(t)Y_\varepsilon(t) + G_\varepsilon(t) \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1).
$$

Integrating the above differential inequality yields

$$
Y_\varepsilon(t) \leqslant Y_\varepsilon(0)e^{\int_0^t H_\varepsilon(s)\,\mathrm{d}s} + \int_0^t e^{\int_s^t H_\varepsilon(\sigma)\,\mathrm{d}\sigma}G_\varepsilon(s)\,\mathrm{d}s \tag{3.32}
$$

for all $t \in (0, T_{\max,\varepsilon})$ and $\varepsilon \in (0,1)$. Notice that

$$
\int_s^t H_\varepsilon(\sigma)\,\mathrm{d}\sigma \leqslant c_4(p) \quad \text{for all } t \in (0, T_{\max,\varepsilon}),\ s \in [0,t) \text{ and } \varepsilon \in (0,1)
$$

by (2.12), (3.1), (3.5), and

$$
\int_0^t G_\varepsilon(s)\,\mathrm{d}s \leqslant c_5(p) \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1)
$$

thanks to (2.12), (3.2) and (3.6). Hence, by (3.32), we obtain

$$
\int_\Omega u_{1\varepsilon}^p(t) \leqslant e^{c_4(p)}\int_\Omega (u_{10}+1)^p + c_5(p)e^{c_4(p)} \quad \text{for all } t \in (0, T_{\max,\varepsilon}) \text{ and } \varepsilon \in (0,1).
$$

An analogous bound for $u_{2\varepsilon}$ follows similarly, completing the proof. □

Applying standard heat semigroup estimates, we deduce L^∞ bounds for ∇v_ε.

Lemma 3.7. *Assume that* (1.4) *holds. Then there exists constant $C > 0$, independent of ε, such that*

$$
\|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} \leqslant C \quad \textit{for all } t \in (0, T_{\max,\varepsilon}) \textit{ and } \varepsilon \in (0,1).
$$

Proof. By well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\geqslant 0}$ on Ω (cf. [31]), for a given $p>1$ one can find $c_1=c_1(\Omega)>0$ satisfying

$$\begin{aligned}
&\|\nabla v_\varepsilon(t)\|_{L^\infty(\Omega)}\\
&=\Big\|\nabla e^{t(\Delta-1)}v_0-\int_0^t\nabla e^{(t-s)(\Delta-1)}\left\{(u_{1\varepsilon}(s)+u_{2\varepsilon}(s))v_\varepsilon(s)-v_\varepsilon(s)\right\}\,\mathrm{d}s\Big\|_{L^\infty(\Omega)}\\
&\leqslant c_1\|v_0\|_{W^{1,\infty}(\Omega)}\\
&\quad+c_1\int_0^t\Big(1+(t-s)^{-\frac12-\frac1p}\Big)e^{-(t-s)}\|(u_{1\varepsilon}(s)+u_{2\varepsilon}(s))v_\varepsilon(s)-v_\varepsilon(s)\|_{L^p(\Omega)}\,\mathrm{d}s
\end{aligned}\tag{3.33}$$

for all $t\in(0,T_{\max,\varepsilon})$ and $\varepsilon\in(0,1)$. Thanks to (2.11) and Lemma 3.6, we obtain

$$\begin{aligned}
&\|(u_{1\varepsilon}(s)+u_{2\varepsilon}(s))v_\varepsilon(s)-v_\varepsilon(s)\|_{L^p(\Omega)}\\
&\leqslant\|u_{1\varepsilon}(s)+u_{2\varepsilon}(s)\|_{L^p(\Omega)}\|v_\varepsilon(s)\|_{L^\infty(\Omega)}+|\Omega|^{1/p}\|v_\varepsilon(s)\|_{L^\infty(\Omega)}\\
&\leqslant c_2\|v_0\|_{L^\infty(\Omega)}+|\Omega|^{1/p}\|v_0\|_{L^\infty(\Omega)}
\end{aligned}$$

where $c_2=\sup_{\varepsilon\in(0,1)}\sup_{t\in(0,T_{\max,\varepsilon})}(\|u_{1\varepsilon}(s)\|_{L^p(\Omega)}+\|u_{2\varepsilon}(s)\|_{L^p(\Omega)})<+\infty$. Therefore, (3.33) implies Lemma 3.7. □

A Moser-type iterative argument [39, Lemma 4.4] yields the boundedness of $u_{1\varepsilon}$ and $u_{2\varepsilon}$.

Lemma 3.8. *Assume that* (1.4) *holds. Then we have*

$$\|u_{1\varepsilon}(t)\|_{L^\infty(\Omega)}\leqslant C\quad\textit{and}\quad\|u_{2\varepsilon}(t)\|_{L^\infty(\Omega)}\leqslant C\quad\textit{for all } t\in(0,T_{\max,\varepsilon})\textit{ and }\varepsilon\in(0,1),$$

where C is a positive constant depending on $-\int_\Omega\ln u_{10}$, $\int_\Omega u_{10}\ln u_{10}$, $\int_\Omega u_{10}$, $-\int_\Omega\ln u_{20}$, $\int_\Omega u_{20}\ln u_{20}$, $\int_\Omega u_{20}$, $\int_\Omega v_0$, $\int_\Omega|\nabla v_0|^2$, $\int_\Omega\frac{|\nabla v_0|^2}{v_0}$ and $\int_\Omega\frac{|\nabla v_0|^4}{v_0^3}$, but independent of ε.

By (2.8) and Lemma 3.8, we deduce the global existence of $(u_{1\varepsilon},u_{2\varepsilon},v_\varepsilon)$ for every $\varepsilon\in(0,1)$.

Lemma 3.9. *Assume that* (1.4) *holds. Then $T_{\max,\varepsilon}=+\infty$ for all $\varepsilon\in(0,1)$.*

Based on a comparison argument and the L^∞ bound from Lemma 3.8, we obtain a lower bound for v_ε.

Lemma 3.10. *Let $T>0$ and assume that* (1.4) *holds. Then we have*

$$v_\varepsilon(x,t)\geqslant C(T)\quad\textit{for all } x\in\Omega,\ t\in(0,T)\textit{ and }\varepsilon\in(0,1),$$

where $C(T)$ is a positive constant depending on v_0, but independent of ε.

Proof. By Lemma 3.8, we have

$$v_{\varepsilon t}\geqslant\Delta v_\varepsilon-c_1v_\varepsilon\quad\text{in }\Omega\times(0,\infty)\quad\text{for all }\varepsilon\in(0,1),$$

where $c_1=\sup_{\varepsilon\in(0,1)}\sup_{t>0}(\|u_{1\varepsilon}(t)\|_{L^\infty(\Omega)}+\|u_{2\varepsilon}(t)\|_{L^\infty(\Omega)})<+\infty$. Applying a comparison principle we obtain

$$v_\varepsilon(x,t)\geqslant\Big(\inf_\Omega v_0\Big)\cdot e^{-c_1t}\quad\text{for all }x\in\Omega,\ t>0\text{ and }\varepsilon\in(0,1).$$

The proof is complete. □

Using the boundedness of $u_{1\varepsilon}$, $u_{2\varepsilon}$ and v_ε obtained in Lemmas 3.7, 3.8 and 3.10, we can apply standard parabolic regularity theory to get Hölder continuity for $u_{1\varepsilon}$, $u_{2\varepsilon}$, and v_ε.

Lemma 3.11. *Let $T>0$ and assume that* (1.4) *holds. Then one can find $\theta_1=\theta(T)\in(0,1)$ such that*

$$\|u_{1\varepsilon}\|_{C^{\theta_1,\theta_1/2}(\overline\Omega\times[0,T])}\leqslant C_1(T)\quad\textit{for all }\varepsilon\in(0,1),\tag{3.34}$$

$$\|u_{2\varepsilon}\|_{C^{\theta_1,\theta_1/2}(\overline\Omega\times[0,T])}\leqslant C_1(T)\quad\textit{for all }\varepsilon\in(0,1),\tag{3.35}$$

$$\|v_\varepsilon\|_{C^{\theta_1,\theta_1/2}(\overline\Omega\times[0,T])}\leqslant C_1(T)\quad\textit{for all }\varepsilon\in(0,1),\tag{3.36}$$

where $C_1(T)$ is a positive constant independent of ε. In addition, for each $\tau > 0$ and all $T > \tau$ one can also fix $\theta_2 = \theta_2(\tau, T) \in (0, 1)$ such that

$$\|v_\varepsilon\|_{C^{2+\theta_2,1+\frac{\theta_2}{2}}(\overline{\Omega}\times[\tau,T])} \leqslant C_2(\tau, T) \quad \text{for all } \varepsilon \in (0, 1), \tag{3.37}$$

where $C_2(\tau, T)$ is a positive constant independent of ε.

Proof. Lemmas 3.7, 3.8, 3.10, (1.4) and parabolic Hölder regularity theory (cf. [21]) imply (3.34)-(3.36). The standard Schauder estimates [11], together with (3.34), (3.35) and a cut-off argument, imply (3.37). □

By applying a standard extraction argument to the family $\{(u_{1\varepsilon}, u_{2\varepsilon}, v_\varepsilon)\}_{\varepsilon\in(0,1)}$ and using the uniform bounds established above, we obtain a global bounded weak solution (u_1, u_2, v) of system (1.1).

Lemma 3.12. *Assume that the initial value (u_{10}, u_{20}, v_0) satisfies (1.4). Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0, 1)$ as well as functions u_1, u_2 and v which satisfy (1.5) with $u_1, u_2 > 0$ a.e. in $\Omega \times (0,\infty)$ and $v > 0$ in $\overline{\Omega} \times [0,\infty)$ such that*

$$u_{1\varepsilon} \to u_1 \quad \text{and} \quad u_{2\varepsilon} \to u_2 \quad \text{in } C^0_{\rm loc}(\overline{\Omega} \times [0,\infty)), \tag{3.38}$$

$$v_\varepsilon \to v \quad \text{in } C^0_{\rm loc}(\overline{\Omega} \times [0,\infty)) \text{ and in } C^{2,1}_{\rm loc}(\overline{\Omega} \times (0,\infty)), \tag{3.39}$$

$$\nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla v \quad \text{in } L^\infty(\Omega \times (0,\infty)), \tag{3.40}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and that (u_1, u_2, v) is a global weak solution of the system (1.1) as defined in Definition 2.1. Additionally, we have

$$\int_\Omega u_{10} \leqslant \int_\Omega u_1(t) \leqslant \int_\Omega u_{10} + \frac{\ell_1}{\ell_2}\int_\Omega u_{20} + \ell_1 \int_\Omega v_0, \tag{3.41}$$

$$\int_\Omega u_{20} \leqslant \int_\Omega u_2(t) \leqslant \frac{\ell_2}{\ell_1}\int_\Omega u_{10} + \int_\Omega u_{20} + \ell_2 \int_\Omega v_0 \quad \text{for all } t > 0, \tag{3.42}$$

$$\|v(t)\|_{L^\infty(\Omega)} \leqslant \|v_0\|_{L^\infty(\Omega)} \quad \text{for all } t > 0, \tag{3.43}$$

$$\int_0^\infty \int_\Omega (u_1 + u_2)v \leqslant \int_\Omega v_0. \tag{3.44}$$

Proof. According to Lemmas 3.7 and 3.11 and a diagonal extraction argument, we obtain a subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ and nonnegative limit functions u_1,u_2 and v that satisfy (1.5) and (3.38)-(3.40). (3.3), (3.38) and Fatou's lemma show that $u_1, u_2 > 0$ a.e. in $\Omega \times (0,\infty)$. From Lemma 3.10 and (3.39), we obtain $v > 0$ in $\overline{\Omega} \times [0,\infty)$.

By integrating the system (2.6), we obtain

$$\begin{gathered}
\int_\Omega u_{10} \leqslant \int_\Omega u_{1\varepsilon}(t) \leqslant \int_\Omega (u_{10} + \varepsilon) + \frac{\ell_1}{\ell_2}\int_\Omega (u_{20} + \varepsilon) + \ell_1 \int_\Omega v_0, \\
\int_\Omega u_{20} \leqslant \int_\Omega u_{2\varepsilon}(t) \leqslant \frac{\ell_2}{\ell_1}\int_\Omega (u_{20} + \varepsilon) + \int_\Omega (u_{20} + \varepsilon) + \ell_2 \int_\Omega v_0
\end{gathered}$$

for all $t \in (0, T_{\max,\varepsilon})$. Hence, (3.38) yields (3.41). Applying (2.11) and (3.39), we have (3.43). Finally, (2.12), (3.38)-(3.39) imply (3.44).

By integrating (3.31) and combining (2.12), (3.1)-(3.2), Lemmas 3.2, 3.6 and 3.9, we find a constant $c_1 > 0$ such that

$$\int_0^\infty \int_\Omega u_{1\varepsilon}^{p-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2 \leqslant c_1 \quad \text{for all } \varepsilon \in (0, 1). \tag{3.45}$$

With $\beta \in (1, 2)$ and $\gamma \in (1, 3)$ fixed, Young's inequality gives

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla u_{1\varepsilon}^2|^\beta &= 2^\beta \int_0^T \int_\Omega \Big(u_{1\varepsilon}^{\gamma-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2\Big)^{\beta/2} u_{1\varepsilon}^{\frac{(3-\gamma)\beta}{2}} v_\varepsilon^{-\beta/2} \\
&\leqslant 2^\beta \int_0^T \int_\Omega u_{1\varepsilon}^{\gamma-1} v_\varepsilon |\nabla u_{1\varepsilon}|^2 + 2^\beta \int_0^T \int_\Omega u_{1\varepsilon}^{\frac{(3-\gamma)\beta}{2-\beta}} v_\varepsilon^{-\frac{\beta}{2-\beta}}
\end{aligned}$$

for all $T > 0$ and $\varepsilon \in (0,1)$. Using Lemmas 3.8, 3.10 and (3.45), we obtain

$$\big(u_{1\varepsilon}^2\big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^\beta\big((0,T);W^{1,\beta}(\Omega)\big) \quad \text{for all } T>0. \tag{3.46}$$

Likewise, we have

$$\big(u_{2\varepsilon}^2\big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^\beta\big((0,T);W^{1,\beta}(\Omega)\big) \quad \text{for all } T>0. \tag{3.47}$$

Conditions (2.1) and (2.2) for Definition 2.1 are immediately from (3.38), (3.39), (3.46) and (3.47); whereas (2.3)-(2.5) follow from (3.38)-(3.40) together with (3.46) and (3.47). □

Now the proof of Theorem 1.1 follows directly from Lemmas 3.7-3.9, 3.12.

4. Long-time behavior

In this section, we study the long-time behavior of global bounded weak solutions of system (1.1). We begin with a lemma on the integrability of $u_{1\varepsilon t}$ and $u_{2\varepsilon t}$.

Lemma 4.1. *Assume that the initial value (u_{10}, u_{20}, v_0) satisfies (1.4). Then one can find $\lambda \in (0,1)$ such that*

$$\begin{gathered}\int_0^\infty \|u_{1\varepsilon t}(t)\|_{(W^{1,\infty}(\Omega))^*}\,\mathrm{d}t \leqslant L\Big\{\int_\Omega v_0\Big\}^\lambda,\\ \int_0^\infty \|u_{2\varepsilon t}(t)\|_{(W^{1,\infty}(\Omega))^*}\,\mathrm{d}t \leqslant L\Big\{\int_\Omega v_0\Big\}^\lambda\end{gathered}$$

for all $\varepsilon \in (0,1)$, where L is a positive constant independent of ε.

The proof of the above lemma uses an argument analogous to that of [29, Lemma 5.1], we omit it here. Applying Lemma 4.1, we can obtain that $u_1(t_k)$ and $u_2(t_k)$ form two Cauchy sequences in $(W^{1,\infty}(\Omega))^*$ for any nondecreasing sequence $\{t_k\}_{k\in\mathbb{N}}$, which is essential for the long-term behavior of u_1 and u_2.

Lemma 4.2. *Let λ and L be as in Lemma 4.1, and assume that the initial value (u_{10}, u_{20}, v_0) satisfies (1.4). Then for any choice of $\{t_k\}_{k\in\mathbb{N}} \subset [0,\infty)$ satisfying $t_{k+1} \geqslant t_k$ for all $k \in \mathbb{N}$, we have*

$$\begin{gathered}\sum_{k\in\mathbb{N}} \|u_1(t_{k+1}) - u_1(t_k)\|_{(W^{1,\infty}(\Omega))^*} \leqslant L\Big\{\int_\Omega v_0\Big\}^\lambda,\\ \sum_{k\in\mathbb{N}} \|u_2(t_{k+1}) - u_2(t_k)\|_{(W^{1,\infty}(\Omega))^*} \leqslant L\Big\{\int_\Omega v_0\Big\}^\lambda.\end{gathered}$$

Proof. Having fixed such a nondecreasing sequence $\{t_k\}_{k\in\mathbb{N}}$, it follows from Lemma 4.1 that for each $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=1}^N \|u_{1\varepsilon}(t_{k+1}) - u_{1\varepsilon}(t_k)\|_{(W^{1,\infty}(\Omega))^*} &= \sum_{k=1}^N \Big\|\int_{t_k}^{t_{k+1}} u_{1\varepsilon t}(t)\,\mathrm{d}t\Big\|_{(W^{1,\infty}(\Omega))^*}\\ &\leqslant \sum_{k=1}^N \int_{t_k}^{t_{k+1}} \|u_{1\varepsilon t}(t)\|_{(W^{1,\infty}(\Omega))^*}\,\mathrm{d}t\\ &\leqslant L\Big(\int_\Omega v_0\Big)^\lambda \quad \text{for all } \varepsilon\in(0,1).\end{aligned} \tag{4.1}$$

Using (3.38), we obtain $u_{1\varepsilon}(t_k) \to u_1(t_k)$ in $L^1(\Omega) \hookrightarrow (W^{1,\infty}(\Omega))^*$ as $\varepsilon_j \searrow 0$ for each $k \in \mathbb{N}$. Therefore, the first estimate in this lemma is obtained from (4.1) by passing to the limits $\varepsilon = \varepsilon_j \searrow 0$ and $N \to \infty$. The same estimate also holds for u_2. □

Because of the the quantitative dependence on v_0, Lemma 4.2 simultaneously provides large-time stabilization of the first two components and estimates for the distance between the resulting limits and the initial value.

Lemma 4.3. *Let λ and L be as in Lemma 4.1, and assume that the initial value (u_{10}, u_{20}, v_0) satisfies* (1.4)*. Then one can find two nonnegative functions $u_{1\infty}, u_{2\infty} \in (W^{1,\infty}(\Omega))^*$ such that*

$$u_1(t) \to u_{1\infty}, \quad u_2(t) \to u_{2\infty} \quad \text{in } (W^{1,\infty}(\Omega))^* \quad \text{as } t \to \infty, \tag{4.2}$$

$$\|u_{1\infty} - u_{10}\|_{(W^{1,\infty}(\Omega))^*} \leqslant L\Big\{ \int_\Omega v_0 \Big\}^\lambda, \tag{4.3}$$

$$\|u_{2\infty} - u_{20}\|_{(W^{1,\infty}(\Omega))^*} \leqslant L\Big\{ \int_\Omega v_0 \Big\}^\lambda. \tag{4.4}$$

Proof. For any nondecreasing sequence $\{t_k\}_{k\in\mathbb{N}}$ with $t_k \to \infty$, Lemma 4.2 implies that $\{u_1(t_k)\}_{k\in\mathbb{N}}$ forms a Cauchy sequence in $(W^{1,\infty}(\Omega))^*$. Consequently,

$$u_1(t_k) \to u_{1\infty} \quad \text{in } (W^{1,\infty}(\Omega))^* \quad \text{as } t_k \to \infty. \tag{4.5}$$

Lemma 4.2 also yields the uniqueness of the limit function, hence we have

$$u_1(t) \to u_{1\infty} \quad \text{in } (W^{1,\infty}(\Omega))^* \quad \text{as } t \to \infty.$$

Another application of Lemma 4.2 yields

$$\|u_{1\infty} - u_{10}\|_{(W^{1,\infty}(\Omega))^*} \leqslant \sum_{k\in\mathbb{N}} \|u_1(k+1) - u_1(k)\|_{(W^{1,\infty}(\Omega))^*} \leqslant L\Big\{ \int_\Omega v_0 \Big\}^\lambda$$

i.e. (4.3) holds. Similar results hold for u_2. □

The following lemma establishes the long-time behavior of v.

Lemma 4.4. *Assume that the initial value (u_{10}, u_{20}, v_0) satisfies* (1.4)*. Then we have*

$$v(t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty.$$

Proof. From (3.6), (3.39), (3.43), Lemma 3.9 and Fatou's lemma, yield that $\int_0^\infty \int_\Omega |\nabla v|^4 < +\infty$. Using the Poincaré inequality, we obtain

$$\int_t^{t+1} \|v(s) - \bar{v}(s)\|_{L^4(\Omega)} \,\mathrm{d}s \leqslant c_1 \Big(\int_t^{t+1} \int_\Omega |\nabla v(x,s)|^4 \,\mathrm{d}x \,\mathrm{d}s \Big)^{1/4} \to 0 \quad \text{as } t \to \infty,$$

where $\bar{f} = \frac{1}{|\Omega|} \int_\Omega f$. From (1.6), (3.41), (3.44) and Hölder's inequality, we obtain

$$\begin{aligned}
&\bar{u}_{10} \int_t^{t+1} \|v(s)\|_{L^1(\Omega)} \,\mathrm{d}s \\
&\leqslant \int_t^{t+1} \int_\Omega u_1(x,s)\bar{v}(s) \,\mathrm{d}x \,\mathrm{d}s \\
&= \int_t^{t+1} \int_\Omega u_1 v - \int_t^{t+1} \int_\Omega u_1(x,s)(v(x,s) - \bar{v}(s)) \,\mathrm{d}x \,\mathrm{d}s \\
&\leqslant \int_t^{t+1} \int_\Omega u_1 v + \sup_{s>0} \|u_1(s)\|_{L^\infty(\Omega)} |\Omega|^{\frac{3}{4}} \int_t^{t+1} \|v(s) - \bar{v}(s)\|_{L^4(\Omega)} \,\mathrm{d}s \to 0 \quad \text{as } t \to \infty.
\end{aligned} \tag{4.6}$$

By the Gagliardo-Nirenberg inequality, we obtain

$$\|\phi\|_{L^\infty(\Omega)} \leqslant c\|\phi\|_{W^{1,4}(\Omega)}^{4/5} \|\phi\|_{L^1(\Omega)}^{1/5} \quad \text{for all } \phi \in W^{1,4}(\Omega).$$

Hence, based on (1.6) and (4.6), we can find $c_2 > 0$ such that

$$\|v(t)\|_{L^\infty(\Omega)} \leqslant c \cdot c_2^{4/5} \|v(t)\|_{L^1(\Omega)}^{1/5} \to 0 \quad \text{as } t \to \infty.$$

The proof is complete. □

Proof of Theorem 1.2. Having established (1.8) via Lemma 4.4, we now focus on (1.7). From (1.6) we infer that $\{u_1(t)\}_{t>0}$ is relatively compact in the weak $*$ topology of $L^\infty(\Omega)$. Therefore, for any nondecreasing sequence $t_k \to \infty$, we can find a subsequence $\{t_{k_j}\}_{j\in\mathbb{N}} \subset (0,\infty)$ and an element $\tilde{u}_{1\infty} \in L^\infty(\Omega)$ such that

$$u_1(t_{k_j}) \stackrel{*}{\rightharpoonup} \widetilde{u}_{1\infty} \quad \text{in } L^\infty(\Omega) \quad \text{as } j \to \infty.$$

Therefore, for any $\phi \in W^{1,\infty}(\Omega)$ thanks to the embedding $W^{1,\infty}(\Omega) \hookrightarrow L^1(\Omega)$, we have

$$\int_\Omega u_1(t_{k_j})\phi \to \int_\Omega \widetilde{u}_{1\infty}\phi \quad \text{as } j \to \infty.$$

Applying (4.2), for any $\phi \in W^{1,\infty}(\Omega)$, we also obtain

$$\int_\Omega u_1(t_{k_j})\phi \to u_{1\infty}(\phi) \quad \text{as } j \to \infty.$$

Hence, for any $\phi \in W^{1,\infty}(\Omega)$, it holds that

$$u_{1\infty}(\phi) = \int_\Omega \widetilde{u}_{1\infty}\phi,$$

which yields that $u_{1\infty} = \widetilde{u}_{1\infty} \in L^\infty(\Omega)$, and

$$\int_\Omega u_1(t)\phi \to \int_\Omega u_{1\infty}\phi \quad \text{as } t \to \infty \tag{4.7}$$

i.e.

$$u_1(t) \overset{*}{\rightharpoonup} u_{1\infty} \quad \text{in } L^\infty(\Omega).$$

Using (3.41) and (4.7) with $\phi \equiv 1$, we have

$$\int_\Omega u_{10} \leqslant \int_\Omega u_{1\infty} \leqslant \int_\Omega u_{10} + \frac{\ell_1}{\ell_2}\int_\Omega u_{20} + \ell_1 \int_\Omega v_0.$$

Similar results hold for u_2. □

Based on (4.3)-(4.4), we can obtain that under an appropriate smallness assumption on the initial datum v_0, the limits obtained as $t \to \infty$ cannot be constants.

Theorem 1.3 is proved by doing a slight adaptation of [32, Lemma 5.4]; we omit it.

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