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GROWTH OF SOBOLEV NORMS FOR NONLINEAR SCHRÖDINGER EQUATIONS ON EUCLIDEAN SPACES

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Abstract. This article concerns the nonlinear Schrödinger equation (NLS) on $\mathbb{R}^3$ with subcubic and cubic nonlinearities. For the cubic NLS, with initial data in $H^m(\mathbb{R}^3)$, we establish exponential growth of high-order Sobolev norms by using modified energy functionals and lossless Strichartz estimates. For the subcubic NLS, with data in $H^2(\mathbb{R}^3)$, we establish polynomial growth.

1. Introduction

In this article, we focus on the three-dimensional nonlinear Schrödinger equation (NLS) on $\mathbb{R}^3$,

$$\begin{gathered} i\partial_t u + \Delta u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = \varphi \in H^m(\mathbb{R}^3), \end{gathered} \tag{1.1}$$

where Δ is the Laplacian and $H^m(\mathbb{R}^3)$ is the standard Sobolev space of order m on $\mathbb{R}^3$, with $m \geq 2$ an integer. Specifically, we analyze the possible growth of higher-order Sobolev norms for large times, namely the behavior of $\|u(t,x)\|_{H^m(\mathbb{R}^d)}$ for $m \geq 2$ and $t \gg 1$.

The growth problem of high-order Sobolev norms has attracted extensive attention in recent years, primarily because of its close connection with weak wave turbulence. The existing literature has mainly focused on two central research directions: the first concerns a priori bounds for the growth of high-order Sobolev norms in flows governed by Hamiltonian partial differential equations [2, 3, 5, 6, 9]. Bourgain [1] pioneered the investigation into Sobolev norm growth for solutions of the periodic nonlinear Schrödinger equation. By analyzing the Fourier restriction properties of lattice point sets of the form $\{(\xi, |\xi|^2) \mid \xi \in \mathbb{Z}^n, |\xi| \leq N\}$, he obtained constant estimates in different dimensions, and combined with L^2 norm and Hamiltonian mass conservation, proved that the high-order Sobolev norms of solutions grow polynomially when the initial data in $H^s(\mathbb{T}^n)$ is sufficiently small, laying a foundation for research in this direction. Building on this foundation, Thirouin [17] later extended the analysis to the fractional defocusing nonlinear Schrödinger equation and related systems, further enriching the understanding of norm growth behavior in periodic settings. Zhong [18] further generalized the analysis to compact Riemannian manifolds, revealing a quantitative link between norm growth rates, bilinear Strichartz parameters, and manifold dimension. For NLS with external potentials, Zhang and Liu [10] proved polynomial growth for cubic NLS with partially tuned potentials via refined Strichartz estimates and modified energy functionals. Takaoka [16] derived improved growth bounds $\langle t \rangle^{\frac{s-1}{2}+}$ $(s > 1)$ for 2D product-space cubic non-defocusing NLS, using angular Strichartz estimates and the upside-down I-method, while noting open questions on product-space growth. In addition, Duca and Nersesyan [8] showed almost sure unbounded norm growth for torus NLS with external stochastic control, offering a new perspective on intervened dynamics. Finally, NLS arises as the inviscid limit of the energy-critical complex Ginzburg-Landau equation, providing cross-equation theoretical support for NLS dynamics [4]. And the second

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problem concerns the existence of global solutions with unbounded high-order Sobolev norms [7, 11, 12, 13, 14, 15].

The equation admits two fundamental conservation laws: the mass conservation

$$\|u(t)\|_{L^2(\mathbb{R}^3)} = \|\varphi\|_{L^2(\mathbb{R}^3)},$$

and the energy conservation

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^3)} + \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R}^3)} = E(\varphi),$$

where ∇ is the gradient operator in the Euclidean space and $|\cdot|$ denotes the modulus of any complex number. These conservation laws immediately imply that

$$\sup_{\mathbb{R}} \|u(t,x)\|_{H^1(\mathbb{R}^d)} < \infty, \tag{1.2}$$

which together ensure the boundedness of the $H^1(\mathbb{R}^3)$ norm for all time $t \in \mathbb{R}$, leaving only the growth of Hm norms with $m \geq 2$ as the subject of our investigation.

Our analytical approach relies on two key ingredients: modified energy functionals and lossless Strichartz estimates on $\mathbb{R}^3$. Modified energy functionals have proven effective in controlling high-order energy evolutions by eliminating resonant contributions. By constructing a hierarchy of modified energies $\mathcal{E}_{2k}(u)$, we show that its leading term is equivalent to the $H^{2k}(\mathbb{R}^3)$ norm, allowing us to control high-order Sobolev norms without derivative loss.

Specifically, we use integration by parts and the dispersive nature of the NLS to derive sharp estimates for the time derivative of the modified energy. This technique aligns naturally with the structure of the NLS and simplifies the estimation of nonlinear terms, which is crucial for obtaining tight growth bounds. For the cubic case, this leads to exponential growth via a classical iteration argument. For the subcubic case, polynomial growth follows from a similar but adapted iteration scheme that accounts for the weaker nonlinearity.

The main goal of this paper is to establish quantitative growth bounds for high-order Sobolev norms in the three-dimensional Euclidean setting. Our results are summarized in the following theorems.

Theorem 1.1. *For every $m \in \mathbb{N}$ with $m \geq 2$ and for every solution $u(t,x) \in C_t(H^m(\mathbb{R}^3))$ to (1.1), where $p = 3$, we have*

$$\sup_{(0,T)} \|u(t,x)\|_{H^m(\mathbb{R}^3)} \leq C\exp(CT),$$

where $C = C(m, \|\varphi\|_{H^m}) > 0$.

Remark 1.2. The notation $X \lesssim Y$ is used to denote the estimate $X \leq CY$ for some constant C. The proof of Theorem 1.1 can be completed by direct iteration once the following local bound is established: for all $\tau \in (0,1)$,

$$\|u(\tau)\|^2_{H^m(\mathbb{R}^3)} - \|u(0)\|^2_{H^m(\mathbb{R}^3)} \lesssim \tau\|u\|^2_{L^\infty((0,\tau);H^m(\mathbb{R}^3))} + \|u\|^\gamma_{L^\infty((0,\tau);H^m(\mathbb{R}^3))}, \tag{1.3}$$

where $\gamma \in (0,2)$ is a suitable number and the implicit constant depends only on the energy of u. In fact, the key step is to derive the exponential growth via a classical argument, which requires the local well-posedness of the Cauchy problem in the energy space H^1.

Specifically, for sufficiently small τ, the lower-order term can be absorbed to yield

$$\|u\|^2_{L^\infty((0,\tau);H^m(\mathbb{R}^3))} \leq 2\|u(0)\|^2_{H^m(\mathbb{R}^3)} + C.$$

Next, choose $\tau = \tau(\|u(0)\|_{H^1(\mathbb{R}^3)})$ as the existence time provided by the H^1 local Cauchy theory. From the local bound, it follows that

$$\|u(t+\tau)\|^2_{H^m(\mathbb{R}^3)} \leq (1+C)\|u(t)\|^2_{H^m(\mathbb{R}^3)} + C.$$

As a byproduct of the local existence theory in H^1 and conservation of energy, the above inequality holds for all $t \geq 0$. Therefore, the sequence $\alpha_n = \|u(n\tau)\|^2_{H^m(\mathbb{R}^3)}$ satisfies $\alpha_{n+1} \leq 2\alpha_n + C$,which implies the claimed exponential bound.

Remark 1.3. In Theorem 1.1, we establish bounds for the growth of H^m norms corresponding to initial data with H^m regularity, where m is any given integer (either odd or even). A significant portion of this work focuses on the case of even m. In the final section, we briefly outline how our analytical methods can be adjusted to apply to odd m.

Naturally, if the initial data φ is assumed to be H^{m+1}-regular, the growth of the H^m norm for odd m can be obtained by interpolating between the growth of two adjacent norms, namely H^{m-1} and H^{m+1} (where $m-1$ and $m+1$ are even). Consequently, there is no need to treat the case of odd m separately if the initial data is sufficiently smooth.

However, since only H^m regularity of the initial data is assumed, the case of odd m becomes significantly more complex.

Theorem 1.4. *For every solution $u(t,x) \in C_t(H^2(\mathbb{R}^3))$ to (1) with $p \in (2,3)$ we have*

$$\sup_{(0,T)} \|u(t,x)\|_{H^2(\mathbb{R}^3)} \le C(\max\{1,T\})^{\frac{4}{3-p}},$$

where $C = C(\|\varphi\|_{H^2}) > 0$.

Remark 1.5. The proof of Theorem 1.4 follows once the following local bound is established: for all $\tau \in (0,1)$,

$$\|u(\tau)\|^2_{H^2(\mathbb{R}^3)} - \|u(0)\|^2_{H^2(\mathbb{R}^3)} \lesssim \tau \|u\|^{\frac{p+5}{4}}_{L^\infty((0,\tau);H^2(\mathbb{R}^3))} + \|u\|^{\gamma}_{L^\infty((0,\tau);H^2(\mathbb{R}^3))}, \tag{1.4}$$

for some $\gamma \in \left(0, \frac{p+5}{4}\right)$. To conclude the polynomial growth from 1.4, we can use Remark 1.3. In fact arguing as in Remark 1.2 we obtain

$$\|u\|^2_{L^\infty((0,\tau);H^m(\mathbb{R}^3))} \le 2\|u(0)\|^2_{H^m(\mathbb{R}^3)} + C.$$

The remainder of this article is organized as follows: Section 2 collects preliminary results including Strichartz estimates and the construction of modified energies. Section 3 provides detailed proofs of Theorems 1.1 and 1.4. Section 4 discusses the extension to odd-order Sobolev norms.

2. Preliminaries

In this section, key analytical tools and preliminary results are collected, tailored to the three-dimensional nonlinear Schrödinger equation (NLS) on $\mathbb{R}^3$. All results will be used in the proofs of the main theorems.

2.1. **Strichartz estimates.**

This section begins with the well-established lossless Strichartz estimates for the linear Schrödinger equation on $\mathbb{R}^3$, which capture the dispersive nature of the equation and form the cornerstone of our nonlinear estimates.

Proposition 2.1. *Let $v(t,x)$ be a solution to the nonhomogeneous Schrödinger equation on $\mathbb{R}\times\mathbb{R}^3$,*

$$i\partial_t v + \Delta v = F, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3,$$

where Δ denotes the Euclidean Laplacian. For any $\tau \in (0,1)$, we have

$$\|v\|_{L^2((0,\tau);L^6(\mathbb{R}^3))} \lesssim \|v_0\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^2((0,\tau);L^{6/5}(\mathbb{R}^3))}. \tag{2.1}$$

2.2. **Modified energies.**

The modified energy functionals that lie at the heart of our high-order regularity analysis will be introduced in this section. These functionals are constructed to eliminate resonant terms in the time derivative of the energy, enabling precise control over the growth of high-order Sobolev norms.

$$\begin{gathered} i\partial_t u + \Delta u = |u|^{p-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ u(0,x) = \varphi \in H^{2k}(\mathbb{R}^3). \end{gathered} \tag{2.2}$$

In the subsequent analysis, the following estimate will be used without further explanation

$$\|u\|_{L^\infty(\mathbb{R};H^1(\mathbb{R}^3))} \lesssim_{p,\|\varphi\|_{H^1}} 1. \tag{2.3}$$

We use the energy

$$\mathcal{E}_{2k}(u) = \|\partial_t^k u\|^2_{L^2(\mathbb{R}^3)} - \frac{p-1}{4}\int_{\mathbb{R}^3} \left|\partial_t^{k-1}\nabla(|u|^2)\right|^2 |u|^{p-3}\,dx - \int_{\mathbb{R}^3} \left|\partial_t^{k-1}(|u|^{p-1}u)\right|^2 dx.$$

Proposition 2.2. *Let $u(t,x)$ be a solution to 2.2, where $p = 2n+1 \geq 3$, with initial data $\varphi \in H^{2k}(\mathbb{R}^3)$. Then we have*

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k}(u(t,x)) = &-\frac{p-1}{4}\int_{\mathbb{R}^3}\left|\partial_t^{k-1}\nabla(|u|^2)\right|^2\partial_t(|u|^{p-3})\,dx\\
&+2\int_{\mathbb{R}^3}\partial_t^k(|u|^{p-1})\partial_t^{k-1}(|\nabla u|^2)\,dx\\
&+\sum_{j=0}^{k-1}c_j\int_{\mathbb{R}^d}\partial_t^j\nabla(|u|^2)\partial_t^{k-1}\nabla(|u|^2)\partial_t^{k-j}(|u|^{p-3})\,dx\\
&+\operatorname{Re}\sum_{j=0}^{k-1}c_j\int_{\mathbb{R}^3}\partial_t^j(|u|^{p-1})\partial_t^{k-j}u\partial_t^{k-1}(|u|^{p-1}\bar{u})\,dx\\
&+\operatorname{Re}\sum_{j=0}^{k-2}c_j\int_{\mathbb{R}^3}\partial_t^k(|u|^{p-1})\partial_t^j(\Delta\bar{u})\partial_t^{k-1-j}u\,dx\\
&+\operatorname{Im}\sum_{j=1}^{k-1}c_j\int_{\mathbb{R}^3}\partial_t^j(|u|^{p-1})\partial_t^{k-j}u\partial_t^k\bar{u}\,dx,\end{aligned}\tag{2.4}$$

where c_j denotes explicit constants that may change from line to line.

Proof. Notice that

$$\begin{aligned}\frac{d}{dt}\|\partial_t^k u\|^2_{L^2(\mathbb{R}^3)} &= 2\operatorname{Re}(\partial_t^{k+1}u,\partial_t^k u)\\
&= 2\operatorname{Re}(\partial_t^k(-\Delta u+|u|^{p-1}u), i\partial_t^k u)\\
&= 2\operatorname{Im}\int_{\mathbb{R}^3}(\partial_t^k\nabla u,\partial_t^k\nabla u)\,dx + 2\operatorname{Re}(\partial_t^k(|u|^{p-1}u), i\partial_t^k u)\,,\end{aligned}$$

where (f,g) denotes the usual $L^2(\mathbb{R}^3)$ scalar product $\int_{\mathbb{R}^3} f\cdot\bar{g}\,\mathrm{dx}$. As the first term on the right-hand side vanishes identically, we have

$$\begin{aligned}\frac{d}{dt}\|\partial_t^k u\|^2_{L^2(\mathbb{R}^3)} &= 2\operatorname{Re}(\partial_t^k(|u|^{p-1}u), i\partial_t^k u)\\
&= 2\operatorname{Re}(\partial_t^k(|u|^{p-1}u), i\partial_t^k u) + 2\operatorname{Re}(|u|^{p-1}\partial_t^k u, i\partial_t^k u)\\
&\quad+\operatorname{Re}\sum_{j=1}^{k-1}c_j(\partial_t^j(|u|^{p-1})\partial_t^{k-j}u, i\partial_t^k u),\end{aligned}$$

where c_j are suitable integers. Notice that the second term on the right-hand side vanishes identically and by substitution, we obtain

$$\begin{aligned}\frac{d}{dt}\|\partial_t^k u\|^2_{L^2(\mathbb{R}^3)} &= 2\ \operatorname{Re}(\partial_t^k(|u|^{p-1}u), -\Delta(\partial_t^{k-1}u)) + 2\ \operatorname{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^{k-1}(|u|^{p-1}u))\\
&\quad+\operatorname{Re}\sum_{j=1}^{k-1}c_j(\partial_t^j(|u|^{p-1})\partial_t^{k-j}u, i\partial_t^k u)\\
&= 2\operatorname{Re}(\partial_t^k(|u|^{p-1}u), -\Delta(\partial_t^{k-1}u)) + 2\ \operatorname{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^{k-1}(|u|^{p-1}u))\\
&\quad+\operatorname{Re}\sum_{j=0}^{k-1}c_j(\partial_t^j(|u|^{p-1})\partial_t^{k-j}u, \partial_t^{k-1}(|u|^{p-1}u))\\
&\quad+\operatorname{Re}\sum_{j=1}^{k-1}c_j(\partial_t^j(|u|^{p-1})\partial_t^{k-j}u, i\partial_t^k u)\\
&= 2\operatorname{Re}(\partial_t^k(|u|^{p-1}u), -\Delta(\partial_t^{k-1}u)) + \int_{\mathbb{R}^3}\partial_t|\partial_t^{k-1}(|u|^{p-1}u)|^2\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \operatorname{Re} \sum_{j=0}^{k-1} c_j (\partial_t^j (|u|^{p-1}) \partial_t^{k-j} u, \partial_t^{k-1}(|u|^{p-1}u)) \\
&+ \operatorname{Re} \sum_{j=1}^{k-1} c_j (\partial_t^j (|u|^{p-1}) \partial_t^{k-j} u, i\partial_t^k u).
\end{aligned}
$$

Firstly, we pay attention to the first term on the right-hand side

$$
2 \operatorname{Re}(\partial_t^k (|u|^{p-1})u, -\Delta(\partial_t^{k-1}u)) = \int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})(-\bar{u}\partial_t^{k-1}(\Delta u) - u\partial_t^{k-1}(\Delta \bar{u}))\, dx,
$$

and notice that

$$
-\bar{u}\Delta(\partial_t^{k-1}u) - u\Delta(\partial_t^{k-1}\bar{u}) = \partial_t^{k-1}(-\bar{u}\Delta u - u\Delta\bar{u}) + \operatorname{Re}\sum_{j=0}^{k-2} c_j \partial_t^j(\Delta u)\partial_t^{k-1-j}\bar{u}.
$$

Using the

$$
\Delta(|u|^2) = u\Delta\bar{u} + \bar{u}\Delta u + 2|\nabla u|^2.
$$

we obtain

$$
\begin{aligned}
&2 \operatorname{Re}(\partial_t^k (|u|^{p-1})u, -\Delta\partial_t^{k-1}u) \\
&= -\int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^{k-1}\Delta(|u|^2)\, \mathrm{dx} + 2\int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^{k-1}(|\nabla u|^2)\, dx \\
&\quad + \operatorname{Re}\sum_{j=0}^{k-2} c_j \int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^j(\Delta u)\partial_t^{k-1-j}\bar{u}\, dx \\
&= \int_{\mathbb{R}^3} \big(\partial_t^k\nabla(|u|^{p-1}), \partial_t^{k-1}\nabla(|u|^2)\big)\, dx + 2\int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^{k-1}(|\nabla u|^2)\, dx \\
&\quad + \operatorname{Re}\sum_{j=0}^{k-2} c_j \int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^j(\Delta u)\partial_t^{k-1-j}\bar{u}\, dx, \\
&= \frac{p-1}{2}\int_{\mathbb{R}^3} \big(\partial_t^k\big(\nabla(|u|^2)|u|^{p-3}\big), \partial_t^{k-1}\nabla(|u|^2)\big)\, \mathrm{d} + 2\int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^{k-1}(|\nabla u|^2)\, dx \\
&\quad + \operatorname{Re}\sum_{j=0}^{k-2} c_j \int_{\mathbb{R}^d} \partial_t^k(|u|^{p-1})\partial_t^j(\Delta u)\partial_t^{k-1-j}\bar{u}\, dx.
\end{aligned}
$$

Then using the Leibniz rule to develop ∂_t^k, we obtain

$$
\begin{aligned}
&2 \operatorname{Re}(\partial_t^k (|u|^{p-1})u, -\Delta\partial_t^{k-1}u) \\
&= \frac{p-1}{2}\int_{\mathbb{R}^3} \big(\partial_t^k\nabla(|u|^2)|u|^{p-3}, \partial_t^{k-1}\nabla(|u|^2)\big)\, dx \\
&\quad + \sum_{j=0}^{k-1} c_j \int_{\mathbb{R}^3} \Big(\partial_t^j\nabla(|u|^2), \partial_t^{k-1}\nabla(|u|^2)\Big)\, \partial_t^{k-j}(|u|^{p-3})\, dx \\
&\quad + 2\int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^{k-1}(|\nabla u|^2)\, \mathrm{dx} + \operatorname{Re}\sum_{j=0}^{k-2} c_j \int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^j(\Delta u)\partial_t^{k-1-j}\bar{u}\, dx \\
&= \frac{p-1}{4}\int_{\mathbb{R}^3} \partial_t\big|\partial_t^{k-1}\nabla(|u|^2)\big|^2|u|^{p-3}\, dx \\
&\quad + \sum_{j=0}^{k-1} c_j \int_{\mathbb{R}^3} \Big(\partial_t^j\nabla(|u|^2), \partial_t^{k-1}\nabla(|u|^2)\Big)\, \partial_t^{k-j}(|u|^{p-3})\, dx \\
&\quad + 2\int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^{k-1}(|\nabla u|^2)\, dx + \operatorname{Re}\sum_{j=0}^{k-2} c_j \int_{\mathbb{R}^3} \partial_t^k(|u|^{p-1})\partial_t^j(\Delta u)\partial_t^{k-1-j}\bar{u}\, dx,
\end{aligned}
$$

and we can finally conclude the proof by combining this identity with (2.5). $\square$

Relying on the lossless Strichartz estimate (Proposition 2.1) in Euclidean space, the proof achieves direct control of the nonlinear term with no derivative loss.

Remark 2.3. In the cubic NLS, we may make simplifications, and more precisely, we have

$$\mathcal{E}_{2k}(u)=\|\partial_t^k u\|_{L^2(\mathbb{R}^3)}^2-\frac{1}{2}\int_{\mathbb{R}^3}\left|\partial_t^{k-1}\nabla(|u|^2)\right|^2\,\mathrm{dx}-\int_{\mathbb{R}^3}\left|\partial_t^{k-1}(|u|^2u)\right|^2\,dx,$$

and

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{2k}(u(t,x))&=2\int_{\mathbb{R}^3}\partial_t^k(|u|^2)\partial_t^{k-1}(|\nabla u|^2)\,dx\\&\quad+\operatorname{Re}\sum_{j=0}^{k-2}c_j\int_{\mathbb{R}^3}\partial_t^k(|u|^2)\partial_t^j(\Delta u)\partial_t^{k-1-j}\bar{u}\,dx\\&\quad+\operatorname{Re}\sum_{j=0}^{k-1}c_j\int_{\mathbb{R}^3}\partial_t^j(|u|^2)\partial_t^{k-j}u\partial_t^{k-1}(|u|^2\bar{u})\,dx\\&\quad+\operatorname{Im}\sum_{j=1}^{k-1}c_j\int_{\mathbb{R}^3}\partial_t^j(|u|^2)\partial_t^{k-j}u\partial_t^k\bar{u}\,dx.\end{aligned}\tag{2.5}$$

The norms $\|\partial_t^k u\|_{L^2}$ and $\|u\|_{H^{2k}}$ are comparable. Our purpose here is to demonstrate that for the solutions $u(t,x)$ of 2.2 with $p=3$, the leading term in our modified energy $\mathcal{E}_{2k}(u)$ is essentially equivalent to the Sobolev norm $\|u\|_{H^{2k}}$.

Proposition 2.4. *Let $u(t,x)$ be solution to 2.2 with $p=3$. Then for every $k,s\in\mathbb{N}$, we have*

$$\|\partial_t^k u-i^k\Delta^k u\|_{H^s(\mathbb{R}^3)}\lesssim_{\|\varphi\|_{H^1}}\|u\|_{H^{s+2k-1}(\mathbb{R}^3)}.\tag{2.6}$$

Proof. We use the identity

$$\partial_t^h u=i^h\Delta^h u+\sum_{j=0}^{h-1}c_j\partial_t^j\Delta^{h-j-1}(u|u|^{p-1}),\tag{2.7}$$

where $c_j\in\mathbb{C}$ are suitable coefficients. The elementary proof follows by induction on h and by using the equation solved by $u(t,x)$.

Assume the result is true for k, then we are reduced to proving

$$\|\partial_t^j(u|u|^2)\|_{H^{2k-2j+s}(\mathbb{R}^3)}\lesssim\|u\|_{H^{s+2k+1}(\mathbb{R}^3)},\quad j=0,\dots,k.\tag{2.8}$$

Expanding the time and space derivatives on the left-hand side, we use the Sobolev embedding $H^1(\mathbb{R}^3)\subset L^6(\mathbb{R}^3)$ and the induction hypothesis at the last step, we obtain the estimate

$$\begin{aligned}&\|\partial_t^{j_1}u\|_{W^{k_1,6}(\mathbb{R}^3)}\times\|\partial_t^{j_2}u\|_{W^{k_2,6}(\mathbb{R}^3)}\times\|\partial_t^{j_3}u\|_{W^{k_3,6}(\mathbb{R}^3)}\\&\lesssim\|\partial_t^{j_1}u\|_{H^{k_1+1}(\mathbb{R}^3)}\times\|\partial_t^{j_2}u\|_{H^{k_2+1}(\mathbb{R}^3)}\times\|\partial_t^{j_3}u\|_{H^{k_3+1}(\mathbb{R}^3)}\\&\lesssim\|u\|_{H^{2j_1+k_1+1}(\mathbb{R}^3)}\|u\|_{H^{2j_2+k_2+1}(\mathbb{R}^3)}\|u\|_{H^{2j_3+k_3+1}(\mathbb{R}^3)},\end{aligned}$$

where $j_1+j_2+j_3=j$ and $k_1+k_2+k_3=2k-2j+s$.

By interpolation, it is easy to see that

$$\|u\|_{H^{2j_l+k_l+1}(\mathbb{R}^3)}\lesssim\|u\|_{H^{s+2k+1}(\mathbb{R}^3)}^{\theta_l}\|u\|_{H^1(\mathbb{R}^3)}^{1-\theta_l},\quad l=1,2,3,$$

where

$$\theta_l(s+2k+1)+(1-\theta_l)=2j_l+k_l+1,$$

and we conclude as above since $\sum_{l=1}^3\theta_l=1$ for $j=0,\dots,k$. $\square$

2.3. **Strichartz estimates for nonlinear solutions.** In this subsection a priori bounds for the Strichartz norms of solutions to (2.2) with cubic nonlinearity will be established. The notation $L^p_\tau X$ stands for the space $L^p((0,\tau);X)$, where X is a Banach space and $p\in[1,\infty]$.

Proposition 2.5. *We have the following estimate for every solution $u(t,x)$ to 2.2 for $(p,l)=(3,3)$ and for every $\epsilon>0$, $\tau\in(0,1)$*

$$\begin{aligned}\|\partial_t^j u\|_{L^2_\tau L^6(\mathbb{R}^3)} \lesssim_{\epsilon,\|\varphi\|_{H^1}} & \|\partial_t^j u_0\|_{L^2(\mathbb{R}^3)} \\ &+\sqrt{\tau}\sum_{\substack{j_1+j_2+j_3=j\\ j_1=\max\{j_1,j_2,j_3\}}} \|u\|_{L^\infty_\tau H^{2j_1}(\mathbb{R}^3)}\|u\|_{L^\infty_\tau H^{2j_2+1}(\mathbb{R}^3)}\|u\|_{L^\infty_\tau H^{2j_3+1}(\mathbb{R}^3)},\end{aligned}\tag{2.9}$$

and

$$\begin{aligned}\|\partial_t^j u\|_{L^2_\tau W^{1,6}(\mathbb{R}^3)} \lesssim_{\epsilon,\|\varphi\|_{H^1}} & \|\partial_t^j u_0\|_{H^1(\mathbb{R}^3)} \\ &+\sqrt{\tau}\sum_{j_1+j_2+j_3=j} \|u\|_{L^\infty_\tau H^{2j_1+1}(\mathbb{R}^3)}\|u\|_{L^\infty_\tau H^{2j_2+1}(\mathbb{R}^3)}\|u\|_{L^\infty_\tau H^{2j_3+1}(\mathbb{R}^3)}.\end{aligned}\tag{2.10}$$

Proof. Firstly, we prove (2.10). Using Strichartz estimates and the equation solved by $\partial_t^j u$ yields

$$\|\partial_t^j u\|_{L^2_\tau W^{1,6}(\mathbb{R}^3)} \lesssim \|\partial_t^j u_0\|_{H^1(\mathbb{R}^3)} + \|\partial_t^j (u|u|^2)\|_{L^2_\tau W^{1,6/5}(\mathbb{R}^3)}.$$

Expanding the time derivative and making use of Hölder's inequality yields

$$\begin{aligned}\|\partial_t^j(u|u|^2)\|_{W^{1,6/5}(\mathbb{R}^3)} &\lesssim \sum_{\substack{j_1+j_2+j_3=j\\ j_1=\max\{j_1,j_2,j_3\}}} \|\partial_t^{j_1}u\|_{H^1(\mathbb{R}^3)}\|\partial_t^{j_2}u\|_{L^6(\mathbb{R}^3)}\|\partial_t^{j_3}u\|_{L^6(\mathbb{R}^3)} \\ &\lesssim \sum_{\substack{j_1+j_2+j_3=j\\ j_1=\max\{j_1,j_2,j_3\}}} \|\partial_t^{j_1}u\|_{H^1(\mathbb{R}^3)}\|\partial_t^{j_2}u\|_{H^1(\mathbb{R}^3)}\|\partial_t^{j_3}u\|_{H^1(\mathbb{R}^3)}.\end{aligned}$$

We then conclude the proof by using Proposition 2.4 in the special case of the cubic NLS on $\mathbb{R}^3$. The proof of (2.9) is similar. □

3. Proof of Theorems 1.1 and 1.4

In this section, prove our main theorems for the three-dimensional NLS on $\mathbb{R}^3$. Firstly, we establish the exponential growth bound for the cubic NLS ($p=3$), then prove the polynomial growth bound for the subcubic NLS ($2<p<3$), and finally extend the results to odd-order Sobolev norms. For the three-dimensional Euclidean space, for brevity: L^q, $W^{s,q}$, H^s denote the spaces $L^q(\mathbb{R}^3)$, $W^{s,q}(\mathbb{R}^3)$, $H^s(\mathbb{R}^3)$ respectively.

3.1. **Exponential growth for H^{2k} norms of solutions to the cubic NLS in $\mathbb{R}^3$.** Start with the proof of Theorem 1.1 in the case $m=2k$, which establishes exponential growth of high-order Sobolev norms for the cubic NLS on $\mathbb{R}^3$. The key idea is to use modified energy functionals and iterative energy estimates, combined with the Strichartz estimates.

Proposition 3.1. *Let us assume that $u(t,x)$ solves 2.2 with $p=3$. Then we have the following bound for every $\tau\in(0,1)$,*

$$\int_0^\tau |\text{right-hand side of (2.4)}|\,ds \lesssim \tau\|u\|^2_{L^\infty_\tau H^{2k}} + \|u\|^\gamma_{L^\infty_\tau H^{2k}},$$

for some $\gamma\in(0,2)$.

Proof. Throughout the proof, we use the following inequality, which is derived by combining a basic interpolation inequality with the estimate (2.3),

$$\|u\|_{L^\infty_\tau H^s} \lesssim_{\|\varphi\|_{H^1}} \|u\|^{\frac{s-1}{2k-1}}_{L^\infty_\tau H^{2k}}, \quad s\in[1,2k],\tag{3.1}$$

By combining Proposition 2.4, Proposition 2.5, and the inequality (3.1), the following estimates hold for $j \geq 1$,

$$\begin{aligned} \|\partial_t^j u\|_{L^2_\tau L^6} &\lesssim_\epsilon \|u_0\|_{H^{2j}} + \sqrt{\tau} \sum_{\substack{j_1+j_2+j_3=j \\ j_1=\max\{j_1,j_2,j_3\}}} \|u\|_{L^\infty_\tau H^{2j_1}} \|u\|_{L^\infty_\tau H^{2j_2+1}} \|u\|_{L^\infty_\tau H^{2j_3+1}} \\ &\lesssim \|u_0\|_{H^{2k}}^{\frac{2j-1}{2k-1}} + \sqrt{\tau} \|u\|_{L^\infty_\tau H^{2k}}^{\frac{2j-1}{2k-1}}. \end{aligned} \tag{3.2}$$

Additionally, for the $W^{1,6}$ norm, we have

$$\begin{aligned} \|\partial_t^j u\|_{L^2_\tau W^{1,6}} &\lesssim_\epsilon \|u_0\|_{H^{2j+1}} + \sqrt{\tau} \sum_{j_1+j_2+j_3=j} \|u\|_{L^\infty_\tau H^{2j_1+1}} \|u\|_{L^\infty_\tau H^{2j_2+1}} \|u\|_{L^\infty_\tau H^{2j_3+1}} \\ &\lesssim \|u_0\|_{L^\infty_\tau H^{2k}}^{\frac{2j}{2k-1}} + \sqrt{\tau} \|u\|_{L^\infty_\tau H^{2k}}^{\frac{2j}{2k-1}}. \end{aligned} \tag{3.3}$$

We denote the four terms on the right-hand side of equation (2.5) as I, II, III, and IV, and proceed to estimate each term individually.

We estimate term I. By expanding the time derivatives ∂_t^k and ∂_t^{k-1}, and applying Hölder's inequality, the estimation of Term I reduces to bounding the integral

$$\int_0^T \|\partial_t^{k_1} u\|_{L^2} \|\partial_t^{k_2} u\|_{L^6} \|\partial_t^{j_1} u\|_{W^{1,6}} \|\partial_t^{j_2} u\|_{W^{1,6}} ds, \tag{3.4}$$

where $k_1 \geq k_2$, $j_1 + j_2 = k - 1$, and $k_1 + k_2 = k$.

By combining the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$, Proposition 2.4 and the inequality (3.1), we find that

$$\begin{aligned} (3.4) &\lesssim \|u\|_{L^\infty_\tau H^{2k_1}} \|u\|_{L^\infty_\tau H^{2k_2+1}} \|\partial_t^{j_1} u\|_{L^2_\tau W^{1,6}} \|\partial_t^{j_2} u\|_{L^2_\tau W^{1,6}} \\ &\lesssim \|u\|_{L^\infty_\tau H^{2k}} \|\partial_t^{j_1} u\|_{L^2_\tau W^{1,6}} \|\partial_t^{j_2} u\|_{L^2_\tau W^{1,6}}. \end{aligned}$$

Using (3.3), we obtain

$$(3.4) \lesssim \tau \|u\|_{L^\infty_\tau H^{2k}}^2 + \|u\|_{L^\infty_\tau H^{2k}}^\gamma,$$

for a suitable $\gamma \in (0, 2)$.

Next, we estimate term II, which can be reduced to bounding integrals of the form

$$\int_0^T \|\partial_t^{k_1} u\|_{L^2} \|\partial_t^{k_2} u\|_{L^6} \|\partial_t^j \Delta u\|_{L^6} \|\partial_t^{k-1-j} u\|_{L^6} ds, \tag{3.5}$$

where $k_1 \leq k_2$,$j = 0, \dots, k-2$ and $k_1 + k_2 = k$.

By using the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ in conjunction with Proposition 2.4, we obtain

$$(3.5) \lesssim \|u\|_{L^\infty_\tau H^{2k_1}} \|\partial_t^{k_2} u\|_{L^2_\tau L^6} \|u\|_{L^\infty_\tau H^{2j+3}} \|\partial_t^{k-1-j} u\|_{L^2_\tau L^6}.$$

Utilizing (3.2) and (3.1), we deduce that

$$(3.5) \lesssim \|u\|_{L^\infty_\tau H^{2k_1}} \|u\|_{L^\infty_\tau H^{2k}}^{\frac{2k_2-1}{2k-1}} \|u\|_{L^\infty_\tau H^{2j+3}} \|u\|_{L^\infty_\tau H^{2k}}^{\frac{2(k-1-j)-1}{2k-1}} \lesssim \tau \|u\|_{L^\infty_\tau H^{2k}}^2 + \|u\|_{L^\infty_\tau H^{2k}}^\gamma,$$

where $\gamma \in (0, 2)$. estimating term III is reduced to bounding integral

$$\int_0^T \|\partial_t^{j_1} u\|_{L^\infty} \|\partial_t^{j_2} u\|_{L^\infty} \|\partial_t^{k-j} u\|_{L^2} \|\partial_t^{k_1} u\|_{L^6} \|\partial_t^{k_2} u\|_{L^6} \|\partial_t^{k_3} u\|_{L^6} ds, \tag{3.6}$$

where $j_1 + j_2 = j, 0 \leq j \leq k - 1$, and $k_1 + k_2 + k_3 = k - 1$.

By applying the Sobolev embeddings $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ and $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, along with Proposition 2.4, we obtain

$$(3.6) \lesssim \|u\|_{L^\infty_\tau H^{2j_1+2}} \|u\|_{L^\infty_\tau H^{2j_2+2}} \|u\|_{L^\infty_\tau H^{2k-2j}} \|\partial_t^{k_1} u\|_{L^2_\tau L^6} \|\partial_t^{k_2} u\|_{L^2_\tau L^6} \|u\|_{L^\infty_\tau H^{2k_3+1}}.$$

Combining inequality (3.1) and (3.2), we find that

$$(3.6) \lesssim \tau \|u\|_{L^\infty_\tau H^{2k}}^2 + \|u\|_{L^\infty_\tau H^{2k}}^\gamma,$$

for $\gamma \in (0,2)$. For term IV, it suffices to estimate integral

$$\int_0^T \|\partial_t^k u\|_{L^2} \|\partial_t^{k-j} u\|_{L^6} \|\partial_t^{j_1} u\|_{L^6} \|\partial_t^{j_2} u\|_{L^6} ds, \tag{3.7}$$

where $j_1 + j_2 = j$, and $1 \le j \le k-1$.

Using the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ and Proposition 2.4,we can control this integral as follows

$$(3.7) \lesssim \|u\|_{L_\tau^\infty H^{2k}} \|u\|_{L_\tau^\infty H^{2k-2j+1}} \|\partial_t^{j_1} u\|_{L_\tau^2 L^6} \|\partial_t^{j_1} u\|_{L_\tau^2 L^6}.$$

Again by (3.1) and (3.2) we arrive at

$$(3.7) \lesssim \tau \|u\|_{L_\tau^\infty H^{2k}}^2 + \|u\|_{L_\tau^\infty H^{2k}}^\gamma,$$

for some $\gamma \in (0,2)$. □

To complete the proof of Theorem 1.1, the modified energy $\mathcal{E}_{2k}(u)$ should be decomposed as $\mathcal{E}_{2k}(u) = \|\partial_t^k u\|_{L^2}^2 + \mathcal{R}_{2k}(u)$, where

$$\mathcal{R}_{2k}(u) = -\frac{1}{2} \int \left|\partial_t^{k-1} \nabla(|u|^2)\right|^2 \mathrm{dx} - \int \left|\partial_t^{k-1}(|u|^2 u)\right|^2 \mathrm{dx}.$$

We need to establish an estimate for $\mathcal{R}_{2k}(u)$. Through detailed analysis, it follows that

$$|\mathcal{R}_{2k}(u)| \lesssim \|u\|_{H^{2k}}^{\frac{4k-3}{2k-1}+\epsilon} + \|u\|_{H^{2k}}^{\frac{4k-5}{2k-1}+\epsilon}.$$

To verify this, we first estimate the first integral in $\mathcal{R}_{2k}(u)$,

$$\begin{aligned}\int \left|\partial_t^{k-1} \nabla(|u|^2)\right|^2 \mathrm{dx} &\lesssim \sum_{k_1+k_2=k-1} \|\partial_t^{k_1} u\|_{W^{1,2}}^2 \|\partial_t^{k_2} u\|_{L^\infty}^2 \\ &\lesssim \sum_{k_1+k_2=k-1} \|u\|_{H^{2k_1+1}}^2 \|u\|_{H^{2k_2+1}}^{1-\epsilon} \|u\|_{H^{2k_2+2}}^{1+\epsilon} \lesssim \|u\|_{H^{2k}}^{\frac{4k-3}{2k-1}+\epsilon}.\end{aligned}$$

Here, the interpolation estimate has been used

$$\|v\|_{L^\infty} \lesssim \|v\|_{H^1}^{\frac{1-\epsilon}{2}} \|v\|_{H^2}^{\frac{1+\epsilon}{2}}, \tag{3.8}$$

which is derived by combining interpolation techniques with Sobolev embedding, Proposition 2.4, and inequality (3.1). Next, estimate the second integral in $\mathcal{R}_{2k}(u)$,

$$\begin{aligned}\int \left|\partial_t^{k-1}(|u|^2 u)\right|^2 \mathrm{dx} &\lesssim \sum_{j_1+j_2+j_3=k-1} \|\partial_t^{j_1} u\|_{L^2}^2 \|\partial_t^{j_2} u\|_{L^\infty}^2 \|\partial_t^{j_3} u\|_{L^\infty}^2 \\ &\lesssim \sum_{j_1+j_2+j_3=k-1} \|u\|_{H^{2j_1}}^2 \|u\|_{H^{2j_2+1}}^{1-\epsilon} \|u\|_{H^{2j_3+1}}^{1-\epsilon} \|u\|_{H^{2j_2+2}}^{1+\epsilon} \|u\|_{H^{2j_3+2}}^{1+\epsilon} \\ &\lesssim \|u\|_{H^{2k}}^{\frac{4k-5}{2k-1}+\epsilon}.\end{aligned}$$

This estimation also relies on the interpolation inequality (3.8), Proposition 2.4, and inequality (3.1).

3.2. **Polynomial growth of H^2 for the subcubic NLS on $\mathbb{R}^3$.** We now proceed to establish the proof of Theorem 1.4, which concerns the polynomial growth of the H2 norm for subcubic NLS. Firstly, a specialized energy functional tailored to this case will be introduced

$$\mathcal{F}_2(v(t,x)) = \int_{\mathbb{R}^3} |\partial_t v|^2 \, dx - (p-1) \int_{\mathbb{R}^3} |v|^{p-1} \left|\nabla |v|\right|^2 \, dx - \frac{p-1}{p} \int_{\mathbb{R}^3} |v|^{2p} \, dx.$$

Proposition 3.2. *Let $u(t,x)$ be solution to 2.2 for $2 < p < 3$. Then*

$$\begin{aligned}\frac{d}{dt} \mathcal{F}_2 u(t,x) = (p-1)(p-3) &\int_{\mathbb{R}^3} |u|^{p-2} \partial_t |u| \left|\nabla |u|\right|^2 \, dx \\ &+ 2(p-1) \int_{\mathbb{R}^3} |u|^{p-2} \partial_t |u| \left|\nabla u\right|^2 \, \mathrm{dx}.\end{aligned} \tag{3.9}$$

Proof. We start by computing the time derivative of $\|\partial_t u\|_{L^2}^2$,

$$\begin{aligned}\frac{d}{dt}\|\partial_t u\|_{L^2}^2 &= 2\operatorname{Re}(\partial_t^2 u, \partial_t u)\\ &= 2\operatorname{Re}(\partial_t(-\Delta u + |u|^{p-1}u), i\partial_t u)\\ &= 2\operatorname{Im}\int_{\mathbb{R}^3}(\partial_t\nabla u, \partial_t\nabla u)\,dx + 2\operatorname{Re}(\partial_t(|u|^{p-1}u), i\partial_t u),\end{aligned}$$

where $(f,g)=\int_{\mathbb{R}^3} f\bar{g}\,\mathrm{dx}$. Since the first term on the right-hand side vanishes identically, it can be simplified to

$$\begin{aligned}\frac{d}{dt}\|\partial_t u\|_{L^2}^2 &= 2\operatorname{Re}(\partial_t(|u|^{p-1}u), i\partial_t u) + 2\operatorname{Re}(|u|^{p-1}\partial_t u, i\partial_t u)\\ &= 2\operatorname{Re}(\partial_t(|u|^{p-1})u, -\Delta u) + 2\operatorname{Re}(\partial_t(|u|^{p-1})u, |u|^{p-1}u)\\ &= 2\operatorname{Re}(\partial_t(|u|^{p-1})u, -\Delta u) + \frac{p-1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|u|^{2p}\,dx.\end{aligned}$$

Notice that

$$\Delta(|u|^2) = u\Delta\bar{u} + \bar{u}\Delta u + 2|\nabla u|^2.$$

Next, we evaluate the integral of $2\operatorname{Re}(\partial_t(|u|^{p-1})u, -\Delta u)$. Using integration by parts and the identity for the Laplacian of a product, this integral can be rewritten as

$$\begin{aligned}&2\operatorname{Re}(\partial_t(|u|^{p-1})u, -\Delta u)\\ &= -(\partial_t|u|^{p-1}, \Delta|u|^2) + 2(\partial_t|u|^{p-1}, |\nabla u|^2)\\ &= (\partial_t\nabla|u|^{p-1}, \nabla|u|^2) + 2(\partial_t|u|^{p-1}, |\nabla u|^2)\\ &= 2(p-1)(\partial_t(|u|^{p-2}\nabla|u|), |u|\nabla|u|) + 2(\partial_t|u|^{p-1}, |\nabla u|^2)\\ &= 2(p-1)\frac{d}{dt}(|u|^{p-2}\nabla|u|, |u|\nabla|u|) - 2(p-1)(|u|^{p-2}\nabla|u|, \partial_t|u|\nabla|u|)\\ &\quad - 2(p-1)(|u|^{p-2}\nabla|u|, |u|\nabla\partial_t|u|) + 2(\partial_t|u|^{p-1}, |\nabla u|^2)\\ &= 2(p-1)\frac{d}{dt}(|u|^{p-2}\nabla|u|, |u|\nabla|u|) - 2(p-1)(|u|^{p-2}\nabla|u|, \partial_t|u|\nabla|u|)\\ &\quad - (p-1)\frac{d}{dt}(|u|^{p-1}, |\nabla|u||^2) + (p-1)(\partial_t|u|^{p-1}, |\nabla|u||^2) + 2(\partial_t|u|^{p-1}, |\nabla u|^2).\end{aligned}$$

Substituting this result back into the expression for $\frac{d}{dt}\|\partial_t u\|_{L^2}^2$ and rearranging terms, leads to (3.9). $\square$

The following proposition serves as a counterpart to Proposition 3.1 in the context of the subcubic NLS.

Proposition 3.3. *We have for every $\tau \in (0,1)$*

$$\int_0^\tau |\textit{right-hand side of } (3.9)|\,ds \lesssim \tau\|u\|_{L_\tau^\infty H^2}^{\frac{p+5}{4}} + \|u\|_{L_\tau^\infty H^2}^{\gamma},$$

for some $\gamma \in \big(0, \frac{p+5}{4}\big)$.

Proof. Denote the two terms on the right-hand side of (3.9) as I and II. A detailed estimation of term I will be shown as below, as the estimation of term II follows a similar line of reasoning. To estimate term I, the diamagnetic inequality will be used to eliminate the modulus inside the derivatives ∇ and ∂_t. Applying Hölder's inequality, we obtain

$$|I| \lesssim \|\partial_t u\|_{L_\tau^\infty L^2}\|u\|^2_{L_\tau^2 W^{1,\frac{12}{5-p}}}\|u\|_{L^6}^{p-2} \lesssim \tau^{\frac{6-2p}{8}}\|\partial_t u\|_{L_\tau^\infty L^2}\|u\|^2_{L_\tau^{\frac{p+1}{4}} W^{1,\frac{12}{5-p}}},$$

where the pair $\big(\frac{8}{p+1}, \frac{12}{5-p}\big)$ is Strichartz admissible.

Using the equation solved by $u(t,x)$, we can replace $\|\partial_t u\|_{L_\tau^\infty L^2}$ with $\|u\|_{L_\tau^\infty H^2}$, leading to

$$|I| \lesssim \tau^{\frac{6-2p}{8}}\|u\|_{L_\tau^\infty H^2}\|u\|^2_{L_\tau^{\frac{8}{p+1}} W^{1,\frac{12}{5-p}}}.$$

The bound is further estimated

$$\|u\|_{L_\tau^{\frac{8}{p+1}} W^{1,\frac{12}{5-p}}} \lesssim \|u\|_{L_\tau^\infty H^1}^{\frac{3-p}{4}} \|u\|_{L_\tau^2 W^{1,6}}^{\frac{p+1}{4}}.$$

By the energy conservation, the estimate can be simplified to

$$|I| \lesssim \tau^{\frac{6-2p}{8}} \|u\|_{L_\tau^\infty H^2} \|u\|_{L_\tau^2 W^{1,6}}^{\frac{p+1}{2}}.$$

Finally, applying the Strichartz estimate (3.3) with $j=0$ (which remains valid for subcubic NLS solutions), it follows that

$$|I| \lesssim \tau \|u\|_{L_\tau^\infty H^2} \|u\|_{L_\tau^\infty H^2}^{\frac{p+1}{4}} + \|u\|_{L_\tau^\infty H^2}^\gamma,$$

where $\gamma \in \left(0, \frac{p+5}{4}\right)$.

Here, $\|u\|_{L_\tau^2 W^{1,6}}$ has been estimated through the middle term on the right-hand side in (3.3), as it involves the highest power of $\|u\|_{L_\tau^\infty H^2}$, while lower powers are absorbed into the term $\|u\|_{L_\tau^\infty H^2}^\gamma$. □

Integrating (3.9) on $[0,\tau]$ and arguing exactly as in the proof of Theorems 1.1 yields the bound

$$\|u(\tau)\|_{H^2(\mathbb{R}^3)}^2 - \|u(0)\|_{H^2(\mathbb{R}^3)}^2 \lesssim \tau \|u\|_{L^\infty((0,\tau);H^2(\mathbb{R}^3))}^{\frac{p+5}{4}} + \|u\|_{L^\infty((0,\tau);H^2(\mathbb{R}^3))}^\gamma,$$

for some $\gamma \in \left(0, \frac{p+5}{4}\right)$. This bound is sufficient from completing the proof of Theorem 1.4, as outlined in Remark 1.5.

3.3. Growth of odd Sobolev norms H^{2k+1}.

We first note that if the initial datum φ has H^{2k+2} regularity, then the bound for H^{2k+1} norm

$$\sup_{(0,T)} \|u(t,x)\|_{H^{2k+1}(\mathbb{R}^2)} \le C(\max\{1,T\})^{2k+\epsilon},$$

can be derived by interpolating between the already established growth bounds for the adjacent even-order norms

$$\begin{aligned} \sup_{(0,T)} \|u(t,x)\|_{H^{2k+2}(\mathbb{R}^2)} &\le C(\max\{1,T\})^{2k+1+\epsilon}, \\ \sup_{(0,T)} \|u(t,x)\|_{H^{2k}(\mathbb{R}^2)} &\le C(\max\{1,T\})^{2k-1+\epsilon}. \end{aligned}$$

A similar argument can be employed to extend Theorem 1.1 to the case of odd $m=2k+1$, provided the initial data possesses sufficient regularity. However, the key distinction in this subsection is that the initial data φ is only assumed to be in H^{2k+1}, which renders the aforementioned interpolation argument inapplicable.

To address the case of odd-order regularity $m=2k+1$, the proof of Theorem 1.1 (previously established for even $m=2k$) is adapted by introducing a modified energy functional tailored to odd-order norms

$$\begin{aligned} \mathcal{E}_{2k+1}(u) = \frac{1}{2}\|\partial_t^k \nabla u\|_{L^2}^2 &+ \frac{1}{2}\int |u|^{p-1}|\partial_t^k u|^2 \,\mathrm{dx} + \frac{p-1}{8}\int |u|^{p-3}\left|\partial_t^k(|u|^2)\right|^2 \,dx \\ &- \operatorname{Re}\sum_{j=1}^{k-1} c_j \int \partial_t^j u \partial_t^{k-j}(|u|^{p-1})\partial_t^k \bar{u}\,dx \\ &- \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\partial_t^j(|u|^2)\partial_t^k(|u|^2)\,dx. \end{aligned} \tag{3.10}$$

The following proposition provides the time evolution of this modified energy functional, from which the proof of Theorem 1.1 for odd $m=2k+1$ can be completed using the same techniques as in the even-order case. We omit the detailed proof and leave it to the reader.

Proposition 3.4. *Let $u(t,x)$ be a solution to* (1.1) *with initial datum φ in H^{2k+1}. Then*

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{E}_{2k+1}(u(t,x))\\
&= \frac{1}{2}\int \partial_t(|u|^{p-1})|\partial_t^k u|^2\,\mathrm{dx} - \mathrm{Re}\sum_{j=1}^{k-1} c_j \int \partial_t^{j+1}u\partial_t^{k-j}(|u|^{p-1})\partial_t^k\bar{u}\,dx\\
&\quad - \mathrm{Re}\sum_{j=1}^{k-1} c_j \int \partial_t^{j}u\partial_t^{k-j+1}(|u|^{p-1})\partial_t^k\bar{u}\,\mathrm{dx} + \frac{p-1}{8}\int_{\mathbb{R}^2}\partial_t(|u|^{p-3})\left|\partial_t^k(|u|^2)\right|^2\,dx\\
&\quad + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j+1}(|u|^{p-3})\partial_t^{j}(|u|^2)\partial_t^k(|u|^2)\,dx\\
&\quad + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j}(|u|^{p-3})\partial_t^{j+1}(|u|^2)\partial_t^k(|u|^2)\,\mathrm{dx} + \sum_{j=1}^{k} c_j \int \partial_t^{k}(|u|^{p-1})\partial_t^{j}u\partial_t^{k+1-j}\bar{u}\,dx,
\end{aligned}
$$

where $c_j \in \mathbb{R}$ are explicit real numbers that can change in different lines.

Proof. Begin by noting the following key identity

$$
\begin{aligned}
\mathrm{Re}(i\partial_t^{k+1}u, \partial_t^k u) &= \mathrm{Re}(\partial_t^k(-\Delta u), \partial_t^k u) + \mathrm{Re}(\partial_t^k(|u|^{p-1}), \partial_t^k u)\\
&= \|\partial_t^k \nabla u\|_{L^2}^2 + \mathrm{Re}(\partial_t^k(|u|^{p-1}), \partial_t^k u).
\end{aligned}
$$

Taking the time derivative of both sides of this identity yields

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\partial_t^k\nabla u\|_{L^2}^2 + \mathrm{Re}(\partial_t^k(|u|^{p-1}), \partial_t^k u)\right)\\
&= \frac{d}{dt}\mathrm{Re}(i\partial_t^{k+1}u, \partial_t^k u) = \mathrm{Re}(i\partial_t^{k+2}u, \partial_t^k u)\\
&= \mathrm{Re}(\partial_t^{k+1}(-\Delta u), \partial_t^k u) + \mathrm{Re}(\partial_t^{k+1}(|u|^{p-1}u), \partial_t^k u)\\
&= \frac{1}{2}\frac{d}{dt}\|\partial_t^{k+1}\nabla u\|_{L^2}^2 + \mathrm{Re}(\partial_t^{k+1}(|u|^{p-1}u), \partial_t^k u).
\end{aligned}
$$

Next, we expand the term $\mathrm{Re}(\partial_t^{k+1}(|u|^{p-1}u), \partial_t^k u)$ using the Leibniz rule for differentiation and properties of the complex inner product.

$$
\begin{aligned}
&\mathrm{Re}(\partial_t^{k+1}(|u|^{p-1}u), \partial_t^k u)\\
&= \frac{d}{dt}\mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^{k+1}u)\\
&= \frac{d}{dt}\mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \mathrm{Re}(\partial_t^k(|u|^{p-1})u, \partial_t^{k+1}u) - \mathrm{Re}(|u|^{p-1}\partial_t^k u, \partial_t^{k+1}u)\\
&\quad + \mathrm{Re}\sum_{j=1}^{k-1} c_j(\partial_t^j u\partial_t^{k-j}(|u|^{p-1}), \partial_t^{k+1}u)\\
&= \frac{d}{dt}\mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \mathrm{Re}(\partial_t^k(|u|^{p-1})u, \partial_t^{k+1}u) - \frac{1}{2}\frac{d}{dt}\int |u|^{p-1}|\partial_t^k u|^2\,\mathrm{dx}\\
&\quad + \frac{1}{2}\int \partial_t(|u|^{p-1})|\partial_t^k u|^2\,\mathrm{dx} + \mathrm{Re}\sum_{j=1}^{k-1} c_j(\partial_t^j u\partial_t^{k-j}(|u|^{p-1}), \partial_t^{k+1}u)\\
&= \frac{d}{dt}\mathrm{Re}(\partial_t^k(|u|^{p-1}u), \partial_t^k u) - \mathrm{Re}(\partial_t^k(|u|^{p-1})u, \partial_t^{k+1}u) - \frac{1}{2}\frac{d}{dt}\int |u|^{p-1}|\partial_t^k u|^2\,dx\\
&\quad + \frac{1}{2}\int \partial_t(|u|^{p-1})|\partial_t^k u|^2\,dx + \frac{d}{dt}\mathrm{Re}\sum_{j=1}^{k-1} c_j(\partial_t^j u\partial_t^{k-j}(|u|^{p-1}), \partial_t^k u)
\end{aligned}
$$

$$- \operatorname{Re} \sum_{j=1}^{k-1} c_j (\partial_t^{j+1} u \partial_t^{k-j} (|u|^{p-1}), \partial_t^k u) - \operatorname{Re} \sum_{j=1}^{k-1} c_j (\partial_t^j u \partial_t^{k-j+1} (|u|^{p-1}), \partial_t^k u).$$

Then we handle the third term on the right-hand side as

$$\begin{aligned} & - \operatorname{Re}(\partial_t^k (|u|^{p-1}) u, \partial_t^{k+1} u) \\ & = -\frac{1}{2} \int \partial_t^k (|u|^{p-1}) \partial_t^{k+1} (|u|^2) \, \mathrm{dx} + \sum_{j=1}^{k} c_j \int \partial_t^k (|u|^{p-1}) \partial_t^j u \partial_t^{k+1-j} \bar{u} \, \mathrm{dx}. \end{aligned}$$

By using the identity

$$\partial_t^k (|u|^{p-1}) = \frac{1}{2}(p-1) \partial_t^{k-1} (\partial_t (|u|^2) |u|^{p-3}),$$

we obtain

$$\begin{aligned} & - \operatorname{Re}(\partial_t^k (|u|^{p-1}) u, \partial_t^{k+1} u) \\ & = -\frac{p-1}{4} \int |u|^{p-3} \partial_t^k (|u|^2) \partial_t^{k+1} (|u|^2) \, dx + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j} (|u|^{p-3}) \partial_t^j (|u|^2) \partial_t^{k+1} (|u|^2) \, dx \\ & \quad + \sum_{j=1}^{k} c_j \int \partial_t^k (|u|^{p-1}) \partial_t^j u \partial_t^{k+1-j} \bar{u} \, dx \\ & = -\frac{p-1}{8} \frac{d}{dt} \int |u|^{p-3} \left| \partial_t^k (|u|^2) \right|^2 \mathrm{dx} + \frac{p-1}{8} \int \partial_t (|u|^{p-3}) \left| \partial_t^k (|u|^2) \right|^2 \, dx \\ & \quad + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j} (|u|^{p-3}) \partial_t^j (|u|^2) \partial_t^{k+1} (|u|^2) \, \mathrm{dx} + \sum_{j=1}^{k} c_j \int \partial_t^k (|u|^{p-1}) \partial_t^j u \partial_t^{k+1-j} \bar{u} \, dx. \end{aligned}$$

Then by elementary computations,

$$\begin{aligned} & - \operatorname{Re}(\partial_t^k (|u|^{p-1}) u, \partial_t^{k+1} u) \\ & = -\frac{p-1}{8} \frac{d}{dt} \int |u|^{p-3} \left| \partial_t^k (|u|^2) \right|^2 \, dx + \frac{p-1}{8} \int \partial_t (|u|^{p-3}) \left| \partial_t^k (|u|^2) \right|^2 \, dx \\ & \quad + \frac{d}{dt} \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j} (|u|^{p-3}) \partial_t^j (|u|^2) \partial_t^k (|u|^2) \, dx \\ & \quad + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j+1} (|u|^{p-3}) \partial_t^j (|u|^2) \partial_t^k (|u|^2) \, dx \\ & \quad + \sum_{j=1}^{k-1} c_j \int \partial_t^{k-j} (|u|^{p-3}) \partial_t^{j+1} (|u|^2) \partial_t^k (|u|^2) \, dx \\ & \quad + \sum_{j=1}^{k} c_j \int \partial_t^k (|u|^{p-1}) \partial_t^j u \partial_t^{k+1-j} \bar{u} \, dx. \end{aligned}$$

□

4. Conclusion

In summary, this paper studies the growth of high-order Sobolev norms for solutions to the three-dimensional subcubic and cubic NLS. Using modified energy functionals and lossless Strichartz estimates, explicit growth bounds are established: exponential growth of H^m norms for cubic NLS ($p = 3$) and polynomial growth of H^2 norms for subcubic NLS ($2 < p < 3$). Results are extended to both even and odd-order regularity via tailored energy functionals, complementing existing NLS regularity theory.

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