Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 51, pp. 1–16.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.51

TRAVELING WAVES FOR A GENERALIZED LESLIE-GOWER PREDATOR-PREY MODEL WITH NONLOCAL DISPERSAL

YU-XIA HAO, ZI-MING LIU

Abstract. In this work we study the traveling wave solutions for a generalized nonlocal dispersal Leslie-Gower predator-prey system. Employing the truncation method and Schauder fixed-point theorem, we establish the existence of traveling waves connecting the semi-trivial equilibrium to the coexistence equilibrium for all wave speeds $c \geq c^*$, where c^* denotes the minimal wave speed. Particularly, the right-hand tail limit of the wave profile is obtained by using the idea of a contracting rectangle. Meanwhile, by analyzing the characteristic equation obtained from linearizing the system at the semi-trivial equilibrium, we showed the nonexistence of traveling waves for $0 < c < c_*$. The main technical challenges arise from the combined effects of nonlocal dispersal and the lack of monotonicity in the system.

1. Introduction

In complex ecosystems, predation as an important manifestation of the diverse interactive forms among long-term coexisting species has always been a focal point of ecological research. There are many different kinds of predator-prey systems that have been proposed to model different processes of biological interactions since Lotka [24] and Volterra [27]. For instance, Dunbar [6, 7, 8] demonstrated the existence of various types of traveling waves in a diffusive predator-prey system incorporating Holling type I and II functional responses. By employing Dunbar's approach, Li and Wu [22] established the existence of traveling waves in a diffusive predator-prey model featuring a simplified Holling type III functional response. Lin, Weng, and Wu [23] investigated traveling wave solutions for predator-prey systems with a sigmoidal functional response, while Ai, Du, and Peng [2] explored traveling wave phenomena in Holling-Tanner type predator-prey models. Further important contributions to this field can be found in [5, 9, 15, 17, 28, 33] and the references cited therein.

Research on traveling wave solutions for reaction-diffusion equations dates back to the pioneering works of 1937 [13, 18], and such solutions play a vital role in understanding the long-time asymptotic behavior of population dynamic systems. In recent decades, extensive studies have been devoted to traveling wave solutions in predator-prey systems. Here, we will primarily introduce some works for predator-prey systems with the Leslie-Gower functional response. Notice that the rate of prey consumption by an average predator and the predator growth are identical in most two species predator-prey systems. However, Leslie [19, 20] proposed the following predator-prey system, where the predator's environmental carrying capacity is proportional to the abundance of the prey,

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t) + u(x,t)\Big(1 - u(x,t) - \frac{av(x,t)}{u(x,t)+e_1}\Big),\\ \frac{\partial v(x,t)}{\partial t} &= d\Delta v(x,t) + sv(x,t)\Big(1 - \frac{v(x,t)}{u(x,t)}\Big),\end{aligned}$$

2020 *Mathematics Subject Classification.* 35K57, 35R20, 92D25.
Key words and phrases. Leslie-Gower predator-prey system; nonlocal dispersal; traveling wave solution.

Submitted March 27, 2026. Published July 8, 2026.

where the ratio $\frac{v(x,t)}{u(x,t)}$ is designated as the Leslie-Gower term. Factually, this type of response function cannot adequately reflect the predation phenomena that objectively exist in nature. Based on this, the authors [3, 4] proposed a modified Leslie-Gower predator-prey system, incorporating the positive constant μ into the Leslie-Gower term. Such predators are known as generalist predators [14], as they are capable of exploiting other alternative food sources when their preferred prey is unavailable. Concretely

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t) + u(x,t)\Big(1 - u(x,t) - \frac{av(x,t)}{u(x,t)+e_1}\Big),\\ \frac{\partial v(x,t)}{\partial t} &= d\Delta v(x,t) + sv(x,t)\Big(1 - \frac{v(x,t)}{u(x,t)+\mu}\Big).\end{aligned}$$

Recently, Li et al.[21] considered the generalized Leslie-Gower system

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \Delta u(x,t) + f(u(x,t))\Big(p(u(x,t) - v(x,t)\Big),\\ \frac{\partial v(x,t)}{\partial t} &= d\Delta v(x,t) + sv(x,t)\Big(1 - \frac{v(x,t)}{u(x,t)h(u(x,t))+\mu}\Big),\end{aligned} \tag{1.1}$$

they established the existence of traveling waves connecting the prey-present state and the coexistence state or the prey-present state and the prey-free state by constructing different Lyapunov functions.

In traditional reaction-diffusion frameworks, individuals are assumed to move randomly between adjacent spatial locations. Yet in many practical contexts, such as population ecology, materials science, and epidemic dynamics, individual movement and interaction often occur across extensive ranges making nonlocal dispersal a more suitable description than local diffusion in such scenarios [1, 25]. Within this framework, nonlocal dispersal is typically formulated via integral operators, a representative example is the convolution operator defined by

$$L[u](x) := J * u(x,t) - u(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)dy - u(x,t).$$

Following the interpretation in [1], if $u(x,t)$ represents the population density at location x and time t, and $J(x-y)$ denotes the probability of transitioning from y to x, then the term $\int_{\mathbb{R}} J(x-y)u(y,t)dy$ captures the inflow of individuals to x from elsewhere, while $-\int_{\mathbb{R}} J(x-y)u(x,t)dy$ (where $\int_{\mathbb{R}} J(x)dx = 1$) reflects the outflow of individuals departing from x to other locations. Over the past few years, a substantial body of research has focused on traveling wave phenomena in nonlocal systems, mainly concerning Lotka-Volterra competition, cooperative models, and epidemic models. However, due to the interplay between nonlocal dispersal and the intrinsic nonmonotonicity characteristic of predator-prey interactions, the literature on traveling waves in such systems remains quite limited [10, 16, 26, 29, 30, 35], additional contributions related to forced waves can be found in [12, 32, 11].

This article focuses on the nonlocal version of (1.1), namely, the nonlocal dispersal Leslie-Gower predator-prey system

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= (J * u(x,t) - u(x,t)) + f(u(x,t))\left(p(u(x,t)) - v(x,t)\right),\\ \frac{\partial v(x,t)}{\partial t} &= d\left(J * v(x,t) - v(x,t)\right) + sv(x,t)\Big(1 - \frac{v(x,t)}{u(x,t)h(u(x,t))+\mu}\Big),\end{aligned} \tag{1.2}$$

where the parameters d, s and μ are positive, $u(x,t)$ and $v(x,t)$ are the population densities of the prey and predator, respectively, s is the predator growth rate and d is the diffusion coefficient of predator species. In this paper, we use the following assumptions:

- (H1) $J \in C^1(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(y)dy = 1$ and J is compactly supported;
- (H2) $f(u) \in C^1(\mathbb{R})$, $f(0) = 0$ and $f'(u) > 0$ for $u \in [0, +\infty)$;
- (H3) $p(u) \in C^1(\mathbb{R})$, $p(0) > 1 + \mu$, $p(1) = 0$ and $p'(u) < 0$ for $u \in (0, +\infty)$;
- (H4) $h(u) \in C^1(\mathbb{R})$, $h(0) > 0$ and $h'(u) \geq 0$ for $u \in [0, +\infty)$.

As stated in [21], the following are classical examples of functions satisfying (H2), (H3), and $h(u) = 1$:

(a) The functional response $f(u)$ of Holling-I type
$$f(u) = ru, \quad p(u) = \frac{(1-u)}{r}, \quad r > 0.$$

(b) The functional response $f(u)$ of Holling-II type
$$f(u) = \frac{ru}{u + r_1}, \quad p(u) = \frac{(1-u)(u + r_1)}{r}, \quad r > 0,\ r_1 \geq 1.$$

(c) The functional response $f(u)$ is Ivlev type
$$f(u) = r(1 - e^{-r_2 u}), \quad p(u) = \frac{u(1-u)}{r(1 - e^{-r_2 u})}, \quad r > 0,\ 0 < r_2 < 2.$$

Obviously, system (1.2) has three boundary equilibria $(0,0)$, $(1,0)$ and $(0,\mu)$, under the conditions (H2)–(H4). We can verify that $(1,0)$ is unstable, and that for any $u \in (0,1)$, there is a unique positive stable equilibrium $(u^*, v^*) := (u^*, u^* h(u^*) + \mu)$ with $u^* \in (0,1)$ of system (1.2). We focus on the existence and nonexistence of traveling wave solutions connecting $(1,0)$ to (u^*, v^*) of system (1.2) in the present paper. Note that it is very difficult to determine the final state of traveling waves for system (1.2). The reason is that the system is not monotone. However the nonlocal dispersal leads us not to a suitable Lyapunov function to ensure that traveling waves converge (u^*, v^*) at $+\infty$, which is a common method. Inspired by [5, 31, 12], we derive the right-hand tail limit of the wave profile by using the idea of a contracting rectangle.

The rest of this paper is structured as follows. In Section 2, we establish the existence of traveling waves for $c \geq c_*$. Section 3 is devoted to proving the nonexistence of traveling waves when $0 < c < c_*$. Section 4 summarizes the main conclusions of this work, discusses the existing research limitations, and proposes several promising directions for subsequent research extensions.

2. Existence of traveling waves

In this section we establish the existence of non-negative non-trivial solutions of system (1.2) with the form $u(x,t) = u(x+ct)$ and $v(x,t) = v(x+ct)$. Substituting this solution into (1.2), and using $\xi = x + ct$, we have the wave system
$$\begin{aligned} cu'(\xi) &= \int_{\mathbb{R}} J(y) u(\xi - y) dy - u(\xi) + f(u(\xi)) \left(p(u(\xi)) - v(\xi)\right), \\ cv'(\xi) &= d\Big(\int_{\mathbb{R}} J(y) v(\xi - y) dy - v(\xi)\Big) + sv(\xi)\Big(1 - \frac{v(\xi)}{u(\xi)h(u(\xi)) + \mu}\Big). \end{aligned} \tag{2.1}$$

We aim to determine positive solutions of (2.1) satisfying
$$(u,v)(-\infty) = (1,0), \quad (u,v)(+\infty) = (u^*, v^*). \tag{2.2}$$

Linearizing the second equation of system (2.1) at $(u,v) = (1,0)$ gives
$$cv'(\xi) = d\Big(\int_{\mathbb{R}} J(y) v(\xi - y) dy - v(\xi)\Big) + sv(\xi).$$

By setting $v(\xi) = e^{\lambda\xi}$, we obtain the characteristic equation
$$F(\lambda, c) := d\Big(\int_{\mathbb{R}} J(y) e^{-\lambda y} dy - 1\Big) - c\lambda + s = 0. \tag{2.3}$$

Direct computations give
$$\begin{gathered} F(0,c) = s > 0, \\ \frac{\partial F(0,c)}{\partial \lambda} = \Big[d \int_{\mathbb{R}} J(y)(-y) e^{-\lambda y} dy - c\Big]\Big|_{\lambda=0} = -c < 0, \quad \forall c > 0, \\ \frac{\partial F(\lambda,c)}{\partial c} = -\lambda < 0, \quad \forall \lambda > 0, \end{gathered}$$

$$\frac{\partial^2 F(\lambda,c)}{\partial \lambda^2} = d\int_{\mathbb{R}} y^2 J(y)e^{-\lambda y}dy > 0, \quad \forall \lambda > 0.$$

The following lemma can be obtained easily.

Lemma 2.1. *There exists a pair of positive constants (λ^*, c_*) such that*

$$F(\lambda^*, c_*) = 0 \quad \text{and} \quad \frac{\partial f(\lambda^*, c_*)}{\partial \lambda} = 0.$$

Furthermore,

(i) *if $c > c_*$, then $F(\lambda, c) = 0$ has two different real roots $\lambda_1 = \lambda_1(c)$, $\lambda_2 = \lambda_2(c)$ with $0 < \lambda_1 < \lambda^* < \lambda_2$, and*

$$F(\lambda,c)\begin{cases} < 0 & \text{for } \lambda \in (\lambda_1, \lambda_2),\\ > 0 & \text{for } \lambda \in (0,\lambda_1)\cup(\lambda_2,+\infty);\end{cases}$$

(ii) *if $0 < c < c_*$, then $F(\lambda, c) > 0$ for all $\lambda > 0$.*

Let

$$\Delta(\lambda,c) = \int_{\mathbb{R}} J(y)e^{-\lambda y}dy - 1 - c\lambda.$$

Direct calculations yield

$$\frac{\partial \Delta}{\partial \lambda}\Big|_{\lambda=0} = \Big(\int_{\mathbb{R}}(-y)J(y)e^{-\lambda y}dy - c\Big)\Big|_{\lambda=0} = -c < 0 \quad \text{for all } c > 0.$$

Because $\Delta(0, c) = 0$, there is a $\tilde{\lambda} > 0$ such that $\Delta(\tilde{\lambda}, c) < 0$.

2.1. **Lower and upper solutions.** We construct lower and upper solutions for non-critical and critical wave speed scenarios. A pair $((\overline{u}(\xi), \overline{v}(\xi))$ forms an upper solution to (2.1), if $\overline{u}(\xi)$ and $\overline{v}(\xi)$ satisfy (2.1) with the sign = replaced by the sign $\geq$. For lower solutions we replace the sign = with the sign $\leq$. Also for convenience, we denote $q(u) = uh(u) + \mu$.

Non-critical waves ($c > c_*$). We define the continuous functions

$$\overline{u}(\xi) = 1, \quad \overline{v}(\xi) = \begin{cases} q(1)e^{\lambda_1 \xi}, & \xi < 0,\\ q(1), & \xi \geq 0,\end{cases}$$

$$\underline{u}(\xi) = \begin{cases} 1 - \sigma e^{\beta\xi}, & \xi < \xi_1,\\ 0, & \xi \geq \xi_1,\end{cases} \quad \underline{v}(\xi) = \begin{cases} q(1)e^{\lambda_1\xi}(1-\gamma e^{\varepsilon\xi}), & \xi < \xi_2,\\ 0, & \xi \geq \xi_2,\end{cases}$$

where

$$\sigma > \max\Big\{1, \frac{f(1)q(1)}{-\Delta(\beta,c)}\Big\}, \quad 0 < \beta < \min\{\lambda_1, \tilde{\lambda}\},$$

$$0 < \varepsilon < \min\{\lambda_1, \lambda_2 - \lambda_1\}, \quad \gamma > \max\Big\{1, \frac{-sq(1)}{q(0)F(\lambda_1+\varepsilon, c)}\Big\}.$$

Also, we define

$$\xi_1 = \frac{1}{\beta}\ln\frac{1}{\sigma} < 0, \quad \xi_2 = \frac{1}{\varepsilon}\ln\frac{1}{\gamma} < 0,$$

and one can choose $\sigma > 0$ and $\gamma > 0$ large enough such that $\xi_2 < \xi_1$.

Lemma 2.2. *The functions $\overline{u}(\xi)$ and $\overline{v}(\xi)$ satisfy*

$$c\overline{u}'(\xi) \geq \int_{\mathbb{R}} J(y)\overline{u}(\xi - y)dy - \overline{u}(\xi) + f(\overline{u}(\xi))\left(p(\overline{u}(\xi)) - \underline{v}(\xi)\right), \quad \xi \not\equiv \xi_2, \tag{2.4}$$

$$c\overline{v}'(\xi) \geq d\Big(\int_{\mathbb{R}} J(y)\overline{v}(\xi-y)dy - \overline{v}(\xi)\Big) + s\overline{v}(\xi)\Big(1 - \frac{\overline{v}(\xi)}{\overline{u}(\xi)h(\overline{u}(\xi)) + \mu}\Big), \quad \xi \not\equiv 0. \tag{2.5}$$

Proof. If $\xi \geq \xi_2$, then $\overline{u}(\xi) = 1, \underline{v}(\xi) = 0$, therefore, (2.4) can be obtained straightforwardly.

In contrast, if $\xi < \xi_2$, then $\overline{u}(\xi) = 1, \underline{v}(\xi) = q(1)e^{\lambda_1 \xi}(1 - \gamma e^{\varepsilon \xi}) > 0$. A direct computation shows that

$$c\overline{u}'(\xi) - \int_{\mathbb{R}} J(y)\overline{u}(\xi - y)dy + \overline{u}(\xi) - f(\overline{u}(\xi))\left(p(\overline{u}(\xi)) - \underline{v}(\xi)\right) > 0.$$

Then (2.4) holds.

Similarly, if $\xi \geq 0$, then $\overline{v}(\xi) = q(1), \overline{u}(\xi) = 1$, thus (2.5) holds naturally.

If $\xi < 0$, then $\overline{u}(\xi) = 1, \overline{v}(\xi) = q(1)e^{\lambda_1 \xi}$. Then

$$\begin{aligned}
&c\overline{v}'(\xi) - d\Big(\int_{\mathbb{R}} J(y)\overline{v}(\xi - y)dy - \overline{v}(\xi)\Big) - s\overline{v}(\xi)\Big(1 - \frac{\overline{v}(\xi)}{\overline{u}(\xi)h(\overline{u}(\xi)) + \mu}\Big) \\
&= c\lambda_1 q(1)e^{\lambda_1 \xi} - d\Big(\int_{\mathbb{R}} J(y)q(1)e^{\lambda_1(\xi - y)}dy - q(1)e^{\lambda_1 \xi}\Big) - sq(1)e^{\lambda_1 \xi}\Big(1 - \frac{q(1)e^{\lambda_1 \xi}}{h(1) + \mu}\Big) \\
&= q(1)e^{\lambda_1 \xi}\Big[c\lambda_1 - d\Big(\int_{\mathbb{R}} J(y)e^{-\lambda_1 y}dy - 1\Big) - s\big(1 - e^{\lambda_1 \xi}\big)\Big] \\
&= q(1)e^{\lambda_1 \xi}\big[-F(\lambda_1, c) + se^{\lambda_1 \xi}\big] > 0.
\end{aligned}$$

Then (2.5) follows. □

Lemma 2.3. *The functions* $\underline{u}(\xi)$ *and* $\underline{v}(\xi)$ *satisfy*

$$c\underline{u}'(\xi) \leq \int_{\mathbb{R}} J(y)\underline{u}(\xi - y)dy - \underline{u}(\xi) + f(\underline{u}(\xi))\left(p(\underline{u}(\xi)) - \overline{v}(\xi)\right), \quad \xi \not\equiv \xi_1, \tag{2.6}$$

$$c\underline{v}'(\xi) \leq d\Big(\int_{\mathbb{R}} J(y)\underline{v}(\xi - y)dy - \underline{v}(\xi)\Big) + s\underline{v}(\xi)\Big(1 - \frac{\underline{v}(\xi)}{\underline{u}(\xi)h(\underline{u}(\xi)) + \mu}\Big), \quad \xi \not\equiv \xi_2. \tag{2.7}$$

Proof. When $\xi \geq \xi_1$, $\underline{u}(\xi) = 0$, hence it is easy to obtain (2.6). When $\xi < \xi_1$, $\underline{u}(\xi) = 1 - \sigma e^{\beta \xi}, \overline{v}(\xi) = q(1)e^{\lambda_1 \xi}$. A direct calculation yields that

$$\begin{aligned}
&c\underline{u}'(\xi) - \int_{\mathbb{R}} J(y)\underline{u}(\xi - y)dy + \underline{u}(\xi) - f(\underline{u}(\xi))\left(p(\underline{u}(\xi)) - \overline{v}(\xi)\right) \\
&= -c\sigma\beta e^{\beta \xi} + \int_{\mathbb{R}} J(y)\sigma e^{\beta(\xi - y)}dy - \sigma e^{\beta \xi} - f(1 - \sigma e^{\beta \xi})\left(p(1 - \sigma e^{\beta \xi}) - q(1)e^{\lambda_1 \xi}\right) \\
&= \sigma e^{\beta \xi}\Big[-c\beta + \int_{\mathbb{R}} J(y)e^{-\beta y}dy - 1\Big] - f(1 - \sigma e^{\beta \xi})\left(p(1 - \sigma e^{\beta \xi}) - q(1)e^{\lambda_1 \xi}\right) \\
&\leq \sigma e^{\beta \xi}\Big[-c\beta + \int_{\mathbb{R}} J(y)e^{-\beta y}dy - 1\Big] + q(1)e^{\lambda_1 \xi}f(1 - \sigma e^{\beta \xi}) \\
&= e^{\beta \xi}\Big[\sigma\left(-c\beta + \int_{\mathbb{R}} J(y)e^{-\beta y}dy - 1\right) + q(1)e^{(\lambda_1 - \beta)\xi}f(1 - \sigma e^{\beta \xi})\Big] \\
&\leq e^{\beta \xi}\Big[\sigma\Big(-c\beta + \int_{\mathbb{R}} J(y)e^{-\beta y}dy - 1\Big) + q(1)f(1)\Big] \\
&= e^{\beta \xi}[\sigma\Delta(\beta, c) + q(1)f(1)] \leq 0.
\end{aligned}$$

Consequently, (2.6) holds.

In the same way, when $\xi \geq \xi_2$, $\underline{v}(\xi) = 0$. It is obvious that (2.7) is true. When $\xi < \xi_2$, $\underline{u}(\xi) = 1 - \sigma e^{\beta \xi}, \underline{v}(\xi) = q(1)e^{\lambda_1 \xi}(1 - \gamma e^{\varepsilon \xi})$, then

$$\begin{aligned}
&c\underline{v}'(\xi) - d\Big(\int_{\mathbb{R}} J(y)\underline{v}(\xi - y)dy - \underline{v}(\xi)\Big) - s\underline{v}(\xi)\Big(1 - \frac{\underline{v}(\xi)}{\underline{u}(\xi)h(\underline{u}(\xi)) + \mu}\Big) \\
&= cq(1)\lambda_1 e^{\lambda_1 \xi} - c\gamma q(1)(\lambda_1 + \varepsilon)e^{(\lambda_1 + \varepsilon)\xi} - q(1)d\int_{\mathbb{R}} J(y)\left(e^{\lambda_1(\xi - y)} - \gamma e^{(\lambda_1 + \varepsilon)(\xi - y)}\right)dy \\
&\quad + q(1)d\Big(e^{\lambda_1 \xi} - \gamma e^{(\lambda_1 + \varepsilon)\xi}\Big) - sq(1)(e^{\lambda_1 \xi} - e^{(\lambda_1 + \varepsilon)\xi})\Big[1 - \frac{q(1)e^{\lambda_1 \xi}(1 - \gamma e^{\varepsilon \xi})}{(1 - \sigma e^{\beta \xi})h(1 - \sigma e^{\beta \xi}) + \mu}\Big] \\
&= q(1)e^{\lambda_1 \xi}\Big[c\lambda_1 - d\int_{\mathbb{R}} J(y)e^{-\lambda_1 y} + d - s\Big(1 - \frac{q(1)e^{\lambda_1 \xi}(1 - \gamma e^{\varepsilon \xi})}{(1 - \sigma e^{\beta \xi})h(1 - \sigma e^{\beta \xi}) + \mu}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \gamma q(1)e^{(\lambda_1+\varepsilon)\xi}\Big[c(\lambda_1+\varepsilon) - d\int_{\mathbb{R}} J(y)e^{-(\lambda_1+\varepsilon)y} + d - s\Big(1 - \frac{q(1)e^{\lambda_1\xi}(1-\gamma e^{\varepsilon\xi})}{(1-\sigma e^{\beta\xi})h(1-\sigma e^{\beta\xi})+\mu}\Big)\Big]\\
&= q(1)e^{\lambda_1\xi}\Big[-F(\lambda_1,c) + s\frac{q(1)e^{\lambda_1\xi}(1-\gamma e^{\varepsilon\xi})}{(1-\sigma e^{\beta\xi})h(1-\sigma e^{\beta\xi})+\mu}\Big]\\
&\quad + \gamma q(1)e^{(\lambda_1+\varepsilon)\xi}\Big[F(\lambda_1+\varepsilon,c) - s\frac{q(1)e^{\lambda_1\xi}(1-\gamma e^{\varepsilon\xi})}{(1-\sigma e^{\beta\xi})h(1-\sigma e^{\beta\xi})+\mu}\Big]\\
&= q(1)e^{(\lambda_1+\varepsilon)\xi}\Big[s\frac{q(1)e^{(\lambda_1-\varepsilon)\xi}(1-\gamma e^{\varepsilon\xi})}{(1-\sigma e^{\beta\xi})h(1-\sigma e^{\beta\xi})+\mu} + \gamma F(\lambda_1+\varepsilon,c) - \gamma s\frac{q(1)e^{\lambda_1\xi}(1-\gamma e^{\varepsilon\xi})}{(1-\sigma e^{\beta\xi})h(1-\sigma e^{\beta\xi})+\mu}\Big]\\
&\le q(1)e^{(\lambda_1+\varepsilon)\xi}\Big[s\frac{q(1)e^{(\lambda_1-\varepsilon)\xi}(1-\gamma e^{\varepsilon\xi})}{(1-\sigma e^{\beta\xi})h(1-\sigma e^{\beta\xi})+\mu} + \gamma F(\lambda_1+\varepsilon,c)\Big]\\
&\le q(1)e^{(\lambda_1+\varepsilon)\xi}\Big[s\frac{q(1)e^{(\lambda_1-\varepsilon)\xi}}{(1-\sigma e^{\beta\xi})h(1-\sigma e^{\beta\xi})+\mu} + \gamma F(\lambda_1+\varepsilon,c)\Big]\\
&\le q(1)e^{(\lambda_1+\varepsilon)\xi}\Big[s\frac{q(1)}{\mu} + \gamma F(\lambda_1+\varepsilon,c)\Big] \le 0.
\end{aligned}
$$

The proof is complete. □

Critical waves ($c = c_*$). In this case, $\lambda_1 = \lambda_2 = \lambda^*$. We define functions

$$
\overline{u}(\xi) = 1, \quad \overline{v}(\xi) = \begin{cases} -bq(1)\xi e^{\lambda^*\xi}, & \xi < -\frac{2}{\lambda^*},\\ q(1), & \xi \ge -\frac{2}{\lambda^*},\end{cases}
$$

$$
\underline{u}(\xi) = \begin{cases} 1-\sigma e^{\beta\xi}, & \xi < \xi_1,\\ 0, & \xi \ge \xi_1,\end{cases} \quad \underline{v}(\xi) = \begin{cases} q(1)e^{\lambda^*\xi}\left(-b\xi - \gamma(-\xi)^{1/2}\right), & \xi < \xi_3,\\ 0, & \xi \ge \xi_3,\end{cases}
$$

where

$$
b = \frac{\lambda^* e^2}{2}, \quad \sigma > \max\Big\{e^{\frac{2\beta}{\lambda^*}}, -\frac{f(1)q(1)b}{(\lambda^*-\beta)e\Delta(\beta,c_*)}\Big\},
$$

$$
0 < \beta < \min\{\lambda^*, \tilde{\lambda}\}, \quad \gamma > \max\Big\{b\sqrt{\frac{2}{\lambda^*}}, \frac{8sq(1)b^2}{q(0)d\int_{\mathbb{R}} J(y)y^2e^{-\lambda^* y}dy}\Big(\frac{7}{2\lambda^* e}\Big)^{7/2}\Big\}.
$$

Meanwhile, we define $\xi_1 = \frac{1}{\beta}\ln\frac{1}{\sigma}, \xi_3 = -\left(\frac{\gamma}{b}\right)^2$ with $\xi_1, \xi_3 < -\frac{2}{\lambda^*}$ by the appropriate choice of σ and γ.

Lemma 2.4. *The functions $\overline{u}(\xi)$ and $\overline{v}(\xi)$ satisfy*

$$
c_*\overline{u}'(\xi) \ge \int_{\mathbb{R}} J(y)\overline{u}(\xi-y)dy - \overline{u}(\xi) + f(\overline{u}(\xi))\left(p(\overline{u}(\xi)) - \underline{v}(\xi)\right), \tag{2.8}
$$

$$
c_*\overline{v}'(\xi) \ge d\Big(\int_{\mathbb{R}} J(y)\overline{v}(\xi-y)dy - \overline{v}(\xi)\Big) + s\overline{v}(\xi)\Big(1 - \frac{\overline{v}(\xi)}{\overline{u}(\xi)h(\overline{u}(\xi))+\mu}\Big), \quad \xi \not\equiv -\frac{2}{\lambda^*}. \tag{2.9}
$$

Proof. For any $\xi \in \mathbb{R}$, $\overline{u}(\xi) = 1, \underline{v}(\xi) \ge 0$, then (2.8) holds. Furthermore, for $\xi < -\frac{2}{\lambda^*}$, $\overline{u}(\xi) = 1, \overline{v}(\xi) = -bq(1)\xi e^{\lambda^*\xi}$, one has

$$
\begin{aligned}
&c_*\overline{v}'(\xi) - d\Big(\int_{\mathbb{R}} J(y)\overline{v}(\xi-y)dy - \overline{v}(\xi)\Big) - s\overline{v}(\xi)\Big(1 - \frac{\overline{v}(\xi)}{\overline{u}(\xi)h(\overline{u}(\xi))+\mu}\Big)\\
&= -c_*bq(1)e^{\lambda^*\xi} - c_*\lambda_1 b\xi q(1)e^{\lambda^*\xi} + d\int_{\mathbb{R}} J(y)bq(1)(\xi-y)e^{\lambda^*(\xi-y)}dy\\
&\quad - dbq(1)\xi e^{\lambda^*\xi} + sbq(1)\xi e^{\lambda^*\xi} + sb^2q(1)\xi^2e^{2\lambda^*\xi}\\
&= bq(1)e^{\lambda^*\xi}\Big[-c_* - c_*\lambda^*\xi + d\int_{\mathbb{R}} J(y)(\xi-y)e^{-\lambda^* y}dy - d\xi + s\xi\Big] + sb^2q(1)\xi^2e^{2\lambda^*\xi}\\
&= bq(1)\xi e^{\lambda^*\xi}F(\lambda^*,c_*) + bq(1)e^{\lambda^*\xi}\Big[-c_* - d\int_{\mathbb{R}} J(y)ye^{-\lambda^* y}dy\Big] + sb^2q(1)\xi^2e^{2\lambda^*\xi}\\
&= sb^2q(1)\xi^2e^{2\lambda^*\xi} \ge 0.
\end{aligned}
$$

For $\xi \geq -\frac{2}{\lambda^*}$, $\overline{u}(\xi) = 1, \overline{v}(\xi) = q(1)$, then (2.9) is true. This completes the proof. $\square$

Lemma 2.5. *The functions $\underline{u}(\xi)$ and $\underline{v}(\xi)$ satisfy*

$$c_*\underline{u}'(\xi) \geq \int_{\mathbb{R}} J(y)\underline{u}(\xi-y)dy - \underline{u}(\xi) + f(\underline{u}(\xi))\left(p(\underline{u}(\xi)) - \overline{v}(\xi)\right), \quad \xi \not\equiv \xi_1, \tag{2.10}$$

$$c_*\underline{v}'(\xi) \geq d\Big(\int_{\mathbb{R}} J(y)\underline{v}(\xi-y)dy - \underline{v}(\xi)\Big) + s\underline{v}(\xi)\Big(1 - \frac{\underline{v}(\xi)}{\underline{u}(\xi)h(\underline{u}(\xi)) + \mu}\Big), \quad \xi \not\equiv \xi_3. \tag{2.11}$$

Proof. When $\xi \geq \xi_1$, $\underline{u}(\xi) = 0$, (2.10) is established. When $\xi < \xi_1$, $\underline{u}(\xi) = 1 - \sigma e^{\beta\xi}, \overline{v}(\xi) = -bq(1)\xi e^{\lambda^*\xi}$. Then

$$\begin{aligned}
&c_*\underline{u}'(\xi) - \int_{\mathbb{R}} J(y)\underline{u}(\xi-y)dy + \underline{u}(\xi) - f(\underline{u}(\xi))\left(p(\underline{u}(\xi)) - \overline{v}(\xi)\right)\\
&= -c_*\sigma\beta e^{\beta\xi} - \int_{\mathbb{R}} J(y)(1 - \sigma e^{\beta(\xi-y))}dy + (1 - \sigma e^{\beta\xi})\\
&\quad - f(1 - \sigma e^{\beta\xi})\Big(p(1 - \sigma e^{\beta\xi}) + bq(1)\xi e^{\lambda^*\xi}\Big)\\
&= \sigma e^{\beta\xi}\Big[-c_*\beta + \int_{\mathbb{R}} J(y)e^{-\beta y}dy - 1\Big] - f(1 - \sigma e^{\beta\xi})p(1 - \sigma e^{\beta\xi}) - f(1 - \sigma e^{\beta\xi})bq(1)\xi e^{\lambda^*\xi}\\
&\leq \sigma e^{\beta\xi}\Delta(\beta, c_*) - b\xi q(1)f(1)e^{\lambda^*\xi}\\
&= e^{\beta\xi}\Big[\sigma\Delta(\beta, c_*) + q(1)f(1)b\frac{1}{(\lambda^* - \beta)e}\Big] \leq 0.
\end{aligned}$$

In addition, when $\xi \geq \xi_3$, $\underline{v}(\xi) = 0$. It is obvious that (2.11) holds. If $\xi < \xi_3$ and $\underline{u}(\xi) \geq 1 - \sigma e^{\beta\xi}, \underline{v}(\xi) = q(1)e^{\lambda^*\xi}(-b\xi - \gamma(-\xi)^{1/2})$, then

$$\begin{aligned}
&c_*\underline{v}'(\xi) - d\Big(\int_{\mathbb{R}} J(y)\underline{v}(\xi-y)dy - \underline{v}(\xi)\Big) - s\underline{v}(\xi)\Big(1 - \frac{\underline{v}(\xi)}{\underline{u}(\xi)h(\underline{u}(\xi)) + \mu}\Big)\\
&\leq c_*\Big[q(1)\lambda^* e^{\lambda^*\xi}(-b\xi - \gamma(-\xi)^{1/2}) + q(1)e^{\lambda^*\xi}(-b + \frac{1}{2}\gamma(-\xi)^{-1/2})\Big]\\
&\quad - d\int_{\mathbb{R}} J(y)q(1)e^{\lambda^*(\xi-y)}\Big(-b(\xi-y) - \gamma(y-\xi)^{1/2}\Big)dy + q(1)de^{\lambda^*\xi}\Big(-b\xi - \gamma(-\xi)^{1/2}\Big)\\
&\quad - sq(1)e^{\lambda^*\xi}(-b\xi - \gamma(-\xi)^{1/2}) + s\underline{v}^2(\xi)\frac{1}{q(0)}\\
&= q(1)e^{\lambda^*\xi}b\xi\Big(-c_*\lambda^* + d\int_{\mathbb{R}} J(y)e^{-\lambda^* y}dy - d + s\Big)\\
&\quad + e^{\lambda^*\xi}\Big[-c_*\lambda^*\gamma q(1)(-\xi)^{1/2} + dq(1)(-\gamma(-\xi)^{1/2}) + sq(1)\gamma(-\xi)^{1/2}\Big]\\
&\quad + e^{\lambda^*\xi}\Big[c_*q(1)(-b) - d\int_{\mathbb{R}} J(y)bq(1)ye^{-\lambda^* y}dy\Big]\\
&\quad + c_*\frac{\gamma}{2}(-\xi)^{1/2}q(1)e^{\lambda^*\xi} + d\gamma\int_{\mathbb{R}} J(y)q(1)e^{\lambda^*(\xi-y)}(y-\xi)^{1/2}dy + s\underline{v}^2(\xi)\frac{1}{q(0)}\\
&= q(1)e^{\lambda^*\xi}b\xi F(\lambda^*, c_*) - d\int_{\mathbb{R}} J(y)\gamma q(1)e^{-\lambda^*(\xi-y)}(-\xi)^{1/2}dy\\
&\quad + c_*\frac{\gamma}{2}(-\xi)^{-1/2}q(1)e^{\lambda^*\xi} + d\gamma\int_{\mathbb{R}} J(y)q(1)e^{\lambda^*(\xi-y)}(y-\xi)^{1/2}dy + s\underline{v}^2(\xi)\frac{1}{q(0)}\\
&= c_*q(1)\frac{\gamma}{2}(-\xi)^{-1/2}e^{\lambda^*\xi} + d\gamma\int_{\mathbb{R}} J(y)q(1)e^{\lambda^*(\xi-y)}[(y-\xi)^{1/2} - (-\xi)^{1/2}]dy + s\frac{\underline{v}^2(\xi)}{q(0)}\\
&= q(1)\frac{\gamma}{2}e^{\lambda^*\xi}d\int_{\mathbb{R}} J(y)(-y)e^{-\lambda^* y}dy(-\xi)^{-1/2}\\
&\quad + d\gamma q(1)e^{\lambda^*\xi}\int_{\mathbb{R}} J(y)\big[\frac{1}{2}(-\xi)^{-1/2}y - \frac{1}{8}(-\xi)^{-\frac{3}{2}}y^2\big]e^{-\lambda^* y}dy + s\underline{v}^2(\xi)\frac{1}{q(0)}
\end{aligned}$$

$$
\begin{aligned}
&= -q(1)\frac{\gamma}{8}e^{\lambda^*\xi}d\int_{\mathbb{R}} J(y)(-\xi)^{-\frac{3}{2}}y^2e^{-\lambda^* y}dy + s\underline{v}^2(\xi)\frac{1}{q(0)} \\
&\le (-\xi)^{-\frac{3}{2}}e^{\lambda^*\xi}\Big[-q(1)\frac{\gamma}{8}d\int_{\mathbb{R}} J(y)y^2e^{-\lambda^* y}dy + \frac{s}{q(0)}(q(1))^2b^2e^{\lambda^*\xi}(\xi)^{2+\frac{3}{2}}\Big] \\
&\le (-\xi)^{-\frac{3}{2}}q(1)e^{\lambda^*\xi}\Big[-\frac{\gamma}{8}d\int_{\mathbb{R}} J(y)y^2e^{-\lambda^* y}dy + \frac{s}{q(0)}q(1)b^2(\frac{7}{2e\lambda^*})^{\frac{7}{2}}\Big] \le 0.
\end{aligned}
$$

Notice that (2.10) and (2.11) hold when $\sup_{\xi\le 0}\{(-\xi)^n e^{\sigma\xi}\} = (\frac{n}{\sigma e})^n$ and the proof is complete. □

2.2. **Existence of traveling waves.** By using the lower and upper solutions constructed in subsection 2.1 and Schauder fixed-point theorem, we can get the existence of traveling wave solutions of (2.1) with speed $c \ge c_*$. Choose D satisfying

$$
D > \max\Big\{\sup_{u\in[0,1]}\{f(1)|p'(u)| + q(1)|f'(u)|\}, s\Big(\frac{2q(1)}{q(0)} - 1\Big)\Big\}
$$

large enough such that

$$
\begin{gathered}
H_1(\phi,\psi) = D\phi + f(\phi)(p(\phi) - \psi) \text{is increasing on } \phi, \\
H_2(\phi,\psi) = D\psi + s\psi\Big(1 - \frac{\psi}{q(\phi)}\Big) \text{ is increasing on } \psi.
\end{gathered}
$$

Then, system (2.1) can be rewritten as

$$
\begin{gathered}
\int_{\mathbb{R}} J(y)\phi(\xi - y)dy - \phi(\xi) - c\phi'(\xi) - D\phi + H_1(\phi,\psi) = 0, \\
d\Big(\int_{\mathbb{R}} J(y)\psi(\xi - y)dy - \psi(\xi)\Big) - c\psi'(\xi) - D\psi + H_2(\phi,\psi) = 0.
\end{gathered}
$$

Take $0 < \omega < \min\big\{\frac{D+d}{c}, \frac{D+1}{c}\big\}$ and define

$$
B_\omega(\mathbb{R},\mathbb{R}^2) := \Big\{u(\xi) : u(\xi) \in C(\mathbb{R},\mathbb{R}^2) \text{ and } \sup_{\xi\in\mathbb{R}} |u(\xi)|e^{-\omega|\xi|} < \infty\Big\}
$$

with $|\cdot|$ denotes the supper norm in $\mathbb{R}^2$. Naturally, $B_\omega(\mathbb{R},\mathbb{R}^2)$ is a Banach space equipped with the norm $|\cdot|_\omega$ defined by

$$
|u|_\omega = \sup_{\xi\in\mathbb{R}} |u(\xi)|e^{-\omega|\xi|} \text{ for } u \in B_\omega(\mathbb{R},\mathbb{R}^2).
$$

Set $\Pi := [0,1] \times [0, q(1)]$. For any $(\phi,\psi) \in \Pi$, define the operator $P(\phi,\psi) := (P_1(\phi,\psi), P_2(\phi,\psi))$ by

$$
\begin{aligned}
P_1(\phi,\psi) &= \frac{1}{c}\int_{-\infty}^{\xi} [J * \phi + D\phi + f(\phi)(p(\phi) - \psi)]\, e^{\frac{D+1}{c}(\tau-\xi)}d\tau, \\
P_2(\phi,\psi) &= \frac{1}{c}\int_{-\infty}^{\xi} \Big[dJ * \psi + D\psi + s\psi\Big(1 - \frac{\psi}{q(\phi)}\Big)\Big]e^{\frac{D+d}{c}(\tau-\xi)}d\tau.
\end{aligned}
$$

Simultaneously, let

$$
\Gamma = \big\{(\phi(\cdot),\psi(\cdot)) \in C(\mathbb{R},\mathbb{R}^2) : \underline{u}(\xi) \le \phi(\xi) \le \overline{u}(\xi), \underline{v}(\xi) \le \psi(\xi) \le \overline{v}(\xi)\big\}.
$$

It is easy to obtain that Γ is a bounded, closed, nonempty and convex subset of $C(\mathbb{R},\mathbb{R}^2)$ with respect to the decay norm $|\cdot|_\omega$.

Lemma 2.6. *The operator P maps Γ into Γ.*

Proof. For any $(\phi,\psi) \in \Gamma$, we need to prove that

$$
\underline{u} \le P_1(\phi,\psi) \le \overline{u} \quad \text{and} \quad \underline{v} \le P_2(\phi,\psi) \le \overline{v}. \tag{2.12}
$$

To obtain the first inequality of (2.12), it is sufficient to show that

$$
\underline{u} \le P_1(\phi,\overline{v}) \quad \text{and} \quad P_1(\phi,\underline{v}) \le \overline{u}.
$$

By direct calculations, one has

$$\begin{aligned}
P_1(\phi,\overline{v}) &= \frac{1}{c}\int_{-\infty}^{\xi}\left[J*\phi+D\phi+f(\phi)(p(\phi)-\overline{v})\right]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\\
&\ge \frac{1}{c}\int_{-\infty}^{\xi}\left[J*\underline{u}+D\underline{u}+f(\underline{u})(p(\underline{u})-\overline{v})\right]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\\
&\ge \frac{1}{c}\int_{-\infty}^{\xi}\left[c\underline{u}'(\tau)+\underline{u}(\tau)+D\underline{u}(\tau)\right]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\\
&= \underline{u}(\xi)
\end{aligned}$$

and

$$\begin{aligned}
P_1(\phi,\underline{v}) &= \frac{1}{c}\int_{-\infty}^{\xi}\left[J*\phi+D\phi+f(\phi)(p(\phi)-\underline{v})\right]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\\
&\le \frac{1}{c}\int_{-\infty}^{\xi}\left[J*\overline{u}+D\overline{u}+f(\overline{u})(p(\overline{u})-\underline{v})\right]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\\
&\le \frac{1}{c}\int_{-\infty}^{\xi}\left[c\overline{u}'(\tau)+\overline{u}(\tau)+D\overline{u}(\tau)\right]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\\
&= \overline{u}(\xi).
\end{aligned}$$

Similarly, we can get

$$\underline{v}\le P_2(\underline{u},\psi)\quad\text{and}\quad P_2(\overline{u},\psi)\le\overline{v}.$$

Consequently, (2.12) holds. □

Lemma 2.7. *$P: C(\Pi,\mathbb{R}^2)\to C(\mathbb{R},\mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\omega$.*

Proof. For any (ϕ,ψ) and $(\tilde\phi,\tilde\psi)$ in Π, it holds

$$\begin{aligned}
&|P_1(\phi,\psi)-P_1(\tilde\phi,\tilde\psi)|e^{-\omega|\xi|}\\
&= \frac{1}{c}\Big|\int_{-\infty}^{\xi}\big[J*\phi+D\phi+f(\phi)(p(\phi)-\psi)\big]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\\
&\quad -\int_{-\infty}^{\xi}\big[J*\tilde\phi+D\tilde\phi+f(\tilde\phi)(p(\tilde\phi)-\tilde\psi)\big]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|}\\
&\le \frac{1}{c}\Big|\int_{-\infty}^{\xi}\Big(J*\phi-J*\tilde\phi\Big)e^{\frac{D+1}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|}+\frac{D}{c}\Big|\int_{-\infty}^{\xi}(\phi-\tilde\phi)e^{\frac{D+1}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|}\\
&\quad +\frac{1}{c}\Big|\int_{-\infty}^{\xi}\Big[f(\phi)p(\phi)-f(\tilde\phi)p(\tilde\phi)+f(\tilde\phi)\tilde\psi-f(\phi)\psi\Big]e^{\frac{D+1}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|}\\
&\le \frac{2(D+1)}{(D+1+c\omega)(D+1-c\omega)}\int_{-\infty}^{\xi}J(y)e^{-\omega y}dy|\phi-\tilde\phi|_\omega+\frac{2(D+1)}{(D+1+c\omega)(D+1-c\omega)}|\phi-\tilde\phi|_\omega\\
&\quad +\frac{2(D+1)\Big[\max_{\tilde\phi\in[0,1]}\{f(1)|p'(\tilde\phi)|+|p(\tilde\phi)f'(\tilde\phi)|\}+\widehat{M}\Big]}{(D+1+c\omega)(D+1-c\omega)}|\phi-\tilde\phi|_\omega\\
&\quad +\frac{2(D+1)\Big[\max_{\tilde\phi\in[0,1]}\{q(1)f'(\tilde\phi)\}+\overline{M}\Big]}{(D+1+c\omega)(D+1-c\omega)}|\phi-\tilde\phi|_\omega+\frac{2(D+1)f(1)}{(D+1+c\omega)(D+1-c\omega)}|\psi-\tilde\psi|_\omega\\
&= C_1|\phi-\tilde\phi|_\omega+C_2|\psi-\tilde\psi|_\omega,
\end{aligned}$$

where

$$C_1:=A\Big[\int_{-\infty}^{\xi}J(y)e^{-\omega y}dy+D+\max_{\tilde\phi\in[0,1]}\{f(1)|p'(\tilde\phi)|+|p(\tilde\phi)f'(\tilde\phi)|+q(1)f'(\tilde\phi)\}+\widehat{M}+\overline{M}\Big],$$

$$C_2:=Af(1),A:=\frac{2(D+1)}{(D+1+c\omega)(D+1-c\omega)},$$

and the positive constants $\widehat{M}$ and $\overline{M}$ satisfy

$$|f(1)\cdot o(\phi-\tilde{\phi})| + |\max_{\tilde{\phi}\in[0,1]} p(\tilde{\phi})\cdot o(\phi-\tilde{\phi})| \le \widehat{M}|\phi-\tilde{\phi}|, \quad |q(1)\cdot o(\phi-\tilde{\phi})| \le \overline{M}|\phi-\tilde{\phi}|.$$

Similarly,

$$\begin{aligned}
&|P_2(\phi,\psi)-P_2(\tilde{\phi},\tilde{\psi})|e^{-\omega|\xi|}\\
&= \frac{1}{c}\Big|\int_{-\infty}^{\xi}\Big[dJ*\psi + D\psi + s\psi\Big(1-\frac{\psi}{q(\phi)}\Big)\Big]e^{\frac{D+d}{c}(\tau-\xi)}d\tau\\
&\quad - \int_{-\infty}^{\xi}\Big[J*\tilde{\psi} + D\tilde{\psi} + s\tilde{\psi}\Big(1-\frac{\tilde{\psi}}{q(\tilde{\phi})}\Big)\Big]e^{\frac{D+d}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|}\\
&\le \frac{d}{c}\Big|\int_{-\infty}^{\xi}\Big(J*\psi - J*\tilde{\psi}\Big)e^{\frac{D+d}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|} + \frac{D+s}{c}\Big|\int_{-\infty}^{\xi}(\psi-\tilde{\psi})e^{\frac{D+d}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|}\\
&\quad + \frac{s}{c}\Big|\int_{-\infty}^{\xi}\Big(\frac{(\psi)^2}{q(\phi)} - \frac{(\tilde{\psi})^2}{q(\tilde{\phi})}\Big)e^{\frac{D+d}{c}(\tau-\xi)}d\tau\Big|e^{-\omega|\xi|}\\
&\le \frac{2d(D+1)}{(D+d+c\omega)(D+d-c\omega)}\int_{-\infty}^{\xi}J(y)e^{-\omega y}dy|\psi-\tilde{\psi}|_\omega + \frac{2(D+s)(D+1)}{(D+d+c\omega)(D+d-c\omega)}|\psi-\tilde{\psi}|_\omega\\
&\quad + \frac{4sq(1)(D+1)}{q(0)(D+d+c\omega)(D+d-c\omega)}|\psi-\tilde{\psi}|_\omega + \frac{2s(D+1)\big(\frac{q(1)}{q(0)}\big)^2\big[\max_{\tilde{\phi}\in[0,1]}\{|q'(\tilde{\phi})| + \widehat{M}\big]}{(D+d+c\omega)(D+d-c\omega)}|\phi-\tilde{\phi}|_\omega\\
&= \tilde{C}_1|\phi-\tilde{\phi}|_\omega + \tilde{C}_2|\psi-\tilde{\psi}|_\omega,
\end{aligned}$$

where

$$\begin{gathered}
\tilde{C}_1 := \frac{2s(D+1)\big(\frac{q(1)}{q(0)}\big)^2\big[\max_{\tilde{\phi}\in[0,1]}\{|q'(\tilde{\phi})| + \widehat{M}\big]}{(D+d+c\omega)(D+d-c\omega)},\\
C_2 := \frac{2(D+1)}{(D+d+c\omega)(D+d-c\omega)}\Big[d\int_{-\infty}^{\xi}J(y)e^{-\omega y}dy + D + s + \frac{2sq(1)}{q(0)}\Big].
\end{gathered}$$

The proof is complete. □

Lemma 2.8. *P is compact on Γ with respect to the norm $|\cdot|_\omega$.*

Proof. For $(\phi,\psi)\in\Gamma$ and $n\in N$, define

$$P^n(\phi,\psi)(\xi) = \begin{cases} P(\phi,\psi)(-n), & \xi < -n,\\ P(\phi,\psi)(\xi), & \xi\in[-n,n],\\ P(\phi,\psi)(n), & \xi > n.\end{cases}$$

First, we show that each P^n is compact. Since $P(\Gamma)\subset\Gamma$ and Γ is uniformly bounded in the sup-norm. Consequently, the set $\{P(\phi,\psi)|_{[-n,n]} : (\phi,\psi)\in\Gamma\}$ is uniformly bounded and equicontinuous on the interval $[-n,n]$. The Arzela-Ascoli theorem yields that the set is precompact in $C([-n,n])$. Since for $|\xi|>n$, the function $P^n(\phi,\psi)(\xi)$ is constant, it follows that $\{P^n(\phi,\psi) : (\phi,\psi)\in\Gamma\}$ remains uniformly bounded and equicontinuous in $B_\omega(\mathbb{R},\mathbb{R}^2)$. Moreover, it satisfies the uniform decay condition

$$\lim_{R\to\infty}\sup_{(\phi,\psi)\in\Gamma}\sup_{|\xi|>R}|P^n(\phi,\psi)(\xi)|e^{-\omega|\xi|} = 0.$$

Therefore, by the weighted Arzela-Ascoli theorem, $\{P^n(\phi,\psi) : (\phi,\psi)\in\Gamma\}$ is precompact in $B_\omega(\mathbb{R},\mathbb{R}^2)$, that is, each P^n is a compact operator.

Next, we prove that P^n converges to P in the operator norm. Because Γ is sup-bounded, there exists a constant $M>0$ (depending only on the parameters d,D,s) such that

$$|P(\phi,\psi)(\xi)| \le M \quad \text{for all } (\phi,\psi)\in\Gamma,\ \xi\in\mathbb{R}.$$

For $|\xi|\le n$, we have $P^n(\phi,\psi)(\xi) = P(\phi,\psi)(\xi)$. For $|\xi|>n$,

$$|P^n(\phi,\psi)(\xi) - P(\phi,\psi)(\xi)| \le |P(\phi,\psi)(\pm n)| + |P(\phi,\psi)(\xi)| \le 2M.$$

Multiplying by the weight factor $e^{-\omega|\xi|}$ yields
$$|P^n(\phi,\psi)(\xi) - P(\phi,\psi)(\xi)|e^{-\omega|\xi|} \le 2Me^{-\omega n}.$$
Hence,
$$\|P^n - P\|_\omega := \sup_{(\phi,\psi)\in\Gamma} |P^n(\phi,\psi) - P(\phi,\psi)|_\omega \le 2Me^{-\omega n} \to 0 \quad \text{as } n \to \infty.$$
Finally, since each P^n is compact and P^n converges to P in operator norm, the limit operator P is also compact. This completes the proof. □

By applying Schauder fixed-point theorem on Γ, it holds $P(\tilde u(\xi), \tilde v(\xi)) = (\tilde u(\xi), \tilde v(\xi))$, that is, $(\tilde u(\xi), \tilde v(\xi))$ is a fixed point of P in Γ. Whence, we have the following result.

Theorem 2.9. *Assume* (H1)–(H4) *hold. Then for any $c \ge c_*$, there exists a solution $(\tilde u(\xi), \tilde v(\xi))$ of system* (2.1) *satisfying $(\tilde u, \tilde v)(-\infty) = (1, 0)$.*

2.3. **Asymptotic behavior.** To study the asymptotic behavior of the solution obtained in Theorem 2.9, we suppose $h(u) = 1$, that is $q(u) = u + \mu$. Note that condition $p(0) > q(1) = 1 + \mu$ in (H_2), together with the monotonicity of $p(u)$ and $q(u)$, implies that the equations $p(u) = q(1)$ and $p(u) = q(0)$ admit unique positive solutions, denoted by l_1 and l_2, respectively. Set
$$\tilde u^- := \liminf_{\xi\to+\infty} \tilde u(\xi), \quad \tilde u^+ := \limsup_{\xi\to+\infty} \tilde u(\xi),$$
$$\tilde v^- := \liminf_{\xi\to+\infty} \tilde v(\xi), \quad \tilde v^+ := \limsup_{\xi\to+\infty} \tilde v(\xi).$$

Lemma 2.10. *Assume* (H3) *holds. Then $\tilde u^- \ge l_1 > 0$, $\tilde v^- \ge \mu > 0$.*

Proof. Considering (1.2) with $(u, v)(x, t) = (\tilde u, \tilde v)(x + ct)$. Because $\tilde v \le q(1)$, it follows that $\tilde u$ satisfies
$$\frac{\partial u(x,t)}{\partial t} \ge (J * u(x,t) - u(x,t)) + f(u(x,t))\,(p(u(x,t)) - q(1))\,, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x, 0) = \tilde u(x), \quad x \in \mathbb{R}.$$
The comparison principle yields that
$$\tilde u^- := \liminf_{\xi\to+\infty} \tilde u(\xi) = \liminf_{\xi\to+\infty} u(0, \frac{\xi}{c}) \ge l_1 > 0.$$
In the same manner, one can obtain $\tilde v^- \ge \mu > 0$. □

Next, we obtain the second part of (2.2) by constructing the sequence of shrinking rectangles. For $\theta \in [0, 1]$, we define
$$k_1(\theta) = \theta u^* + (1-\theta)(l_1 - \epsilon\nu_1), \quad K_1(\theta) = \theta u^* + (1-\theta)(1 + \epsilon\nu_2),$$
$$k_2(\theta) = \theta v^* + (1-\theta)(\mu - \epsilon), \quad K_2(\theta) = \theta v^* + (1-\theta)(q(1) + \epsilon\nu_3),$$
where ϵ is an arbitrary small positive constant and
$$\nu_1 = \frac{1}{\min\{|p'(\tilde u)|^2\}}, \quad \nu_2 = \frac{1}{|p'(u^*)|}, \quad \frac{1}{|p'(u^*)|} < \nu_3 < \frac{1}{\min\{|p'(\tilde u)|\}}.$$
Notice that $l_1 < u^*$ and $\mu < v^*$, thus k_i and K_i are increasing and decreasing functions in $\theta \in [0,1]$ respectively, and $k_i(1) = K_i(1) = (u^*, v^*)$ $(i = 1, 2)$.

Next, we need to state that the set
$$\Theta = \{\theta \in [0, 1] : k_1(\theta) < \tilde u^- \le \tilde u^+ < K_1(\theta),\ k_2(\theta) < \tilde v^- \le \tilde v^+ < K_2(\theta)\}$$
is nonempty and $\sup\Theta = 1$. In fact, from the definitions of upper-lower solutions and Lemma 2.10, we have
$$k_1(0) = l_1 - \epsilon\nu_1 < l_1 \le \tilde u^- \le \tilde u^+ \le 1 < 1 + \epsilon\nu_2 = K_1(0),$$
$$k_2(0) = \mu - \epsilon < \mu \le \tilde v^- \le \tilde v^+ \le q(1) < q(1) + \epsilon\nu_3 = K_2(0).$$

This indicates $0 \in \Theta$ and $\Theta \neq \emptyset$. We next show $\sup \Theta = 1$. Conversely, suppose that $\sup \Theta = \bar{\theta} \in (0,1)$ and let $\theta \to \bar{\theta}$, then one has

$$k_1(\bar{\theta}) \leq \tilde{u}^- \leq \tilde{u}^+ \leq K_1(\bar{\theta}), \quad k_2(\bar{\theta}) \leq \tilde{v}^- \leq \tilde{v}^+ \leq K_2(\bar{\theta}).$$

Therefore, one of the following equalities is true

$$\tilde{u}^- = k_1(\bar{\theta}), \tilde{u}^+ = K_1(\bar{\theta}), \quad \tilde{v}^- = k_2(\bar{\theta}), \tilde{v}^+ = K_2(\bar{\theta}).$$

We first show $\tilde{u}^- = k_1(\bar{\theta})$. If $\tilde{u}$ is ultimately monotone as $\xi \to +\infty$, there is $\tilde{u}(+\infty) = k_1(\bar{\theta})$. It can be obtained by integrating the first equation of (2.1) from 0 to n for any $n \in N$ that

$$c[\tilde{u}(n) - \tilde{u}(0)] = \int_0^n (J * \tilde{u} - \tilde{u})(\xi) d\xi + \int_0^n f(\tilde{u})\,(p(\tilde{u} - \tilde{v})\,d\xi, \tag{2.13}$$

because

$$\begin{aligned}\int_0^n (J * \tilde{u} - \tilde{u})(\xi) d\xi &= \int_{-\infty}^{+\infty} J(y) \int_0^n (\tilde{u})(\xi - y) - \tilde{u})(\xi)) d\xi dy \\ &= \int_{-\infty}^{+\infty} J(y)(-y) \int_0^1 [(\tilde{u})(n - \varsigma y) - \tilde{u}(-\varsigma y)] d\varsigma dy.\end{aligned}$$

According to (2.13), we have

$$c[\tilde{u}(n) - \tilde{u}(0)] + \int_{-\infty}^{+\infty} J(y) y \int_0^1 [(\tilde{u})(n - \varsigma y) - \tilde{u}(-\varsigma y)] d\varsigma dy = \int_0^n f(\tilde{u})\,(p(\tilde{u} - \tilde{v})\,d\xi. \tag{2.14}$$

Note that

$$\begin{aligned}&\liminf_{\xi \to +\infty} f(\tilde{u}(\xi))\,(p(\tilde{u}(\xi)) - \tilde{v}(\xi)) \\ &\geq f(k_1(\bar{\theta}))\,\big(p(k_1(\bar{\theta})) - K_2(\bar{\theta})\big) \\ &= f(k_1(\bar{\theta}))\,\big(p(k_1(\bar{\theta})) - \bar{\theta} v^* - (1 - \bar{\theta})(q(1) + \epsilon \nu_3)\big) \\ &= f(k_1(\bar{\theta}))\,\big[p(u^*) + p'(u^*)(1 - \bar{\theta})(l_1 - \epsilon \nu_1 - u^*) + o((1 - \bar{\theta})(l_1 - \epsilon \nu_1 - u^*)) \\ &-(1 - \bar{\theta})[p(u^*) + p'(u^*)(l_1 - u^*) + o(l_1 - u^*) - \bar{\theta} v^* - (1 - \bar{\theta})\epsilon \nu_3\big] \\ &= f(k_1(\bar{\theta}))(1 - \bar{\theta})\epsilon\,[-\nu_1 p'(u^*) - \nu_3 - o(1)] > 0\end{aligned}$$

by the choice of ν_1, ν_3 and $\bar{\theta} < 1$, which yields that

$$\int_0^n f(\tilde{u}(\xi))\,(p(\tilde{u}(\xi) - \tilde{v}(\xi))\,d\xi \to +\infty \quad \text{as } n \to +\infty,$$

this contradicts to the boundedness of the left side of (2.14).

On the other hand, if $\tilde{u}$ is oscillates as $\xi \to +\infty$, one can choose a sequence of local minimal points $\{\xi_n\}$ of $\tilde{u}$ satisfying $\{\xi_n\} \to +\infty$ and $\tilde{u}(\xi_n) \to k_1(\bar{\theta})$ as $n \to +\infty$. The first equation of (2.1) gives that

$$0 = c\tilde{u}'(\xi_n) = \int_{\mathbb{R}} J(y) \tilde{u}(\xi_n - y) dy - \tilde{u}(\xi_n) + f(\tilde{u}(\xi_n))\,(p(\tilde{u}(\xi_n)) - \tilde{v}(\xi_n))\,.$$

Moreover,

$$\begin{gathered}\lim_{n \to +\infty} \int_{\mathbb{R}} J(y)(\tilde{u}(\xi_n - y) - \tilde{u}(\xi_n)) dy > 0, \\ \liminf_{n \to +\infty} f(\tilde{u}(\xi_n))\,(p(\tilde{u}(\xi_n) - \tilde{v}(\xi_n)) \geq f(k_1(\bar{\theta}))\,\big(p(k_1(\bar{\theta})) - K_2(\bar{\theta})\big) > 0,\end{gathered}$$

this is a contradiction.

The other cases can be treated similarly, we are currently only providing some key calculations that are necessary for the proof.

(a) For $\tilde{u}^+ = K_1(\bar{\theta})$, we have

$$\begin{aligned}\limsup_{\xi \to +\infty} f(\tilde{u})\,(p(\tilde{u}) - \tilde{v}) &\leq f(K_1(\bar{\theta}))\,\big(p(K_1(\bar{\theta})) - k_2(\bar{\theta})\big) \\ &= f(K_1(\bar{\theta}))\,\big(p(K_1(\bar{\theta})) - \bar{\theta} v^* - (1 - \bar{\theta})(\mu - \epsilon)\big)\end{aligned}$$

$$
\begin{aligned}
&= f(k_1(\bar\theta))[p(u^*) + p'(u^*)(1-\bar\theta)(1+\epsilon\nu_2 - u^*) + o((1-\bar\theta)(1+\epsilon\nu_2-u^*))] \\
&\quad - (1-\bar\theta)[p(u^*) + p'(u^*)(l_2-u^*) + o(l_2-u^*) - \bar\theta v^* - (1-\bar\theta)(-\epsilon)] \\
&= f(k_1(\bar\theta))(1-\bar\theta)[p'(u^*)(1+\epsilon\nu_2 - l_2) + \epsilon + o(1)] < 0.
\end{aligned}
$$

(b) For $\tilde v^- = k_2(\bar\theta)$, we have

$$
\begin{aligned}
\liminf_{\xi\to+\infty} s\tilde v(\xi)\Big(1-\frac{\tilde v(\xi)}{\tilde u(\xi)+\mu}\Big) &\ge sk_2(\bar\theta)\Big(1-\frac{k_2(\bar\theta)}{k_1(\bar\theta)+\mu}\Big) \\
&= sk_2(\bar\theta)\Big(\frac{\bar\theta u^* + (1-\bar\theta)(l_1-\epsilon\nu_1) + \mu - \bar\theta v^* - (1-\bar\theta)(\mu-\epsilon)}{\bar\theta u^* + (1-\bar\theta)(l_1-\epsilon\nu_1)+\mu}\Big) \\
&= sk_2(\bar\theta)\Big(\frac{(1-\bar\theta)(l_1-\epsilon(\nu_1-1))}{\bar\theta u^* + (1-\bar\theta)(l_1-\epsilon\nu_1)+\mu}\Big) > 0
\end{aligned}
$$

(c) For $\tilde v^+ = K_2(\bar\theta)$, we have

$$
\begin{aligned}
\liminf_{\xi\to+\infty} s\tilde v(\xi)\Big(1-\frac{\tilde v(\xi)}{\tilde u(\xi)+\mu}\Big) &\le sK_2(\bar\theta)\Big(1-\frac{K_2(\bar\theta)}{K_1(\bar\theta)+\mu}\Big) \\
&= sK_2(\bar\theta)\Big(\frac{\theta u^* + (1-\bar\theta)(1+\epsilon\nu_2) + \mu - \theta v^* - (1-\bar\theta)(q(1)+\epsilon\nu_3)}{\bar\theta u^* + (1-\bar\theta)(1+\epsilon\nu_2)+\mu}\Big) \\
&= sK_2(\bar\theta)\Big(\frac{(1-\bar\theta)\epsilon(\nu_2-\nu_3)}{\bar\theta u^* + (1-\bar\theta)(1+\epsilon\nu_2)+\mu}\Big) < 0.
\end{aligned}
$$

Hence, $\sup\Theta = 1$ and $(\tilde u(+\infty), \tilde v(+\infty)) = (u^*, v^*)$. Now, we are in a position to state the main result of this section.

Theorem 2.11. *Suppose that* (H1)–(H3) *and* $h(u) = 1$ *hold. Then for any* $c \ge c_*$, *system* (2.1) *admits traveling wave solutions connecting* $(1,0)$ *and* (u^*, v^*).

3. Nonexistence of traveling waves

In this section we discuss the nonexistence of bounded positive traveling wave solutions connecting $(1,0)$ and (u^*,v^*) of system (2.1) with speed $c < c_*$. For this, we introduce the following known results.

Lemma 3.1 ([34])**.** *Assume* $c > 0$ *and* $B(\cdot)$ *is a continuous function with* $B(\pm\infty) := \lim_{\xi\to\pm\infty} B(\xi)$. *Let* $z(\cdot)$ *be a measurable function satisfying*

$$
cz(\xi) = \int_{\mathbb{R}} J_i(y) e^{\int_\xi^{\xi-y} z(s)ds} dy + B(\xi) \quad \text{in } \mathbb{R}.
$$

Then, z *is uniformly continuous and bounded. Moreover,* $\rho^\pm := \lim_{\xi\to\pm\infty} z(\xi)$ *exist and are real roots of the characteristic equation*

$$
c\rho = \int_{\mathbb{R}} J_i(y) e^{-\rho y} dy + B(\pm\infty)(i = 1,2).
$$

Lemma 3.2 ([31])**.** *Let* $z \in C^1(\mathbb{R})$ *satisfy*

$$
z'(\xi) \ge \int_{\mathbb{R}} J(y) z(\xi - y) dy + b(\xi) z(\xi) \quad \text{in } \mathbb{R},
$$

where $b(\cdot)$ *is continuous and* $b(\xi) \ge -M$ *on* $\mathbb{R}$ *for some* $M > 0$. *Then there exists a constant* $C = C(M) > 0$ *such that*

$$
C^{-1} < \int_{\mathbb{R}} J(y) \frac{z(\xi-y)}{z(\xi)} dy < C \quad \text{in } \mathbb{R}.
$$

Theorem 3.3. *For any* $0 < c < c_*$, *there are no bounded positive traveling wave solutions connecting* $(1,0)$ *and* (u^*,v^*) *of system* (2.1).

Proof. We will prove this theorem by contradiction. Assume that there is a bounded traveling wave solution $(u(\xi), v(\xi))$ of system (2.1) satisfying $(u, v)(-\infty) = (1, 0)$ with speed $c < c_*$. From the second equation of (2.1), we have

$$c\frac{v'(\xi)}{v(\xi)} \geq d\Big(\int_{\mathbb{R}} J(y)\frac{v(\xi-y)}{v(\xi)}dy - 1\Big) + s \quad \text{for all } \xi \in \mathbb{R}.$$

Lemma 3.2 states that $\left|\frac{v'(\xi)}{v(\xi)}\right|$ is bounded in $\mathbb{R}$. Now, we consider the sequence $\{\xi_n\}$ converging to $-\infty$ as $n \to +\infty$. We define functions

$$u_n(\xi) := u(\xi + \xi_n), v_n(\xi) := \frac{v(\xi + \xi_n)}{v(\xi_n)}.$$

Since $\lim_{\xi\to-\infty} u(\xi) = 1$ and $\lim_{\xi\to-\infty} v(\xi) = 0$, it follows that

$$\lim_{n\to+\infty} u_n(\xi) = 1, \quad \lim_{n\to+\infty} v(\xi + \xi_n) = 0$$

locally uniformly in $\mathbb{R}$ and they satisfy

$$cv_n'(\xi) = d\Big(\int_{\mathbb{R}} J(y)v_n(\xi - y)dy - v_n(\xi)\Big) + sv_n(\xi)\Big(1 - \frac{v((\xi + \xi_n))}{u((\xi + \xi_n))h(u((\xi + \xi_n))) + \mu}\Big).$$

Since

$$\int_{\mathbb{R}} J(y)v_n(\xi - y)dy = \int_{\mathbb{R}} J(y)e^{\int_{\xi_n}^{\xi_n+\xi-y} \frac{v'(\zeta)}{v(\zeta)}d\zeta}dy \quad \text{and} \quad v_n(\xi) = e^{\int_{\xi_n}^{\xi_n+\xi} \frac{v'(\zeta)}{v(\zeta)}d\zeta}.$$

we have $\int_{\mathbb{R}} J(y)v_n(\xi - y)dy$ and $v_n(\xi)$ are locally uniformly bounded in $\mathbb{R}$, this gives that $v_n'(\xi)$ and $v_n''(\xi)$ are locally uniformly bounded in $\mathbb{R}$. By the Arzela-Ascoli theorem, the functions v_n converge in $C^1_{loc}(\mathbb{R})$, up to extraction of a subsequence, to a function v_∞ satisfying

$$cv_\infty'(\xi) = d\Big(\int_{\mathbb{R}} J(y)v_\infty(\xi - y)dy - v_\infty(\xi)\Big) + sv_\infty(\xi). \tag{3.1}$$

Actually, $v_\infty(\xi) > 0$ in $\mathbb{R}$. Indeed, if there exists ξ_0 such that $v_\infty(\xi_0) = 0$, thus $v_\infty'(\xi_0) = 0$ and $\int_{\mathbb{R}} J(y)v_\infty(\xi_0 - y)dy = 0$, which indicates that $v_\infty(\xi_0 - y) = 0$ in $\mathbb{R}$ and contradicts to the fact $v_\infty(0) = 1$.

Taking $w(\xi) = \frac{v_\infty'(\xi)}{v_\infty(\xi)}$, equation (3.1) implies that $w(\xi)$ satisfies

$$cw(\xi) = d\Big(\int_{\mathbb{R}} J(y)e^{\int_\xi^{\xi-y} w(\zeta)d\zeta}dy - 1\Big) + s.$$

Lemma 3.1 gives that the limits $w(\pm\infty) = \lambda^{\pm}$ exist and are roots of the equation

$$c\lambda = d\Big(\int_{\mathbb{R}} J(y)e^{-\lambda y}dy - 1\Big) + s.$$

Since $0 < c < c_*$, Lemma 2.1 gives

$$c\lambda < d\Big(\int_{\mathbb{R}} J(y)e^{-\lambda y}dy - 1\Big) + s$$

for all $\lambda > 0$. This is a contradiction and the proof is complete. □

4. Conclusion

This paper investigates traveling wave solutions for a generalized Leslie-Gower predator-prey system with nonlocal dispersal, which generalizes the classical local diffusion Leslie-Gower models and covers a wide class of functional responses including Holling type I, Holling type II and Ivlev type. Owing to the intrinsic non-monotonicity of predator-prey interactions combined with integral nonlocal dispersal operators, conventional Lyapunov-function-based methods cannot characterize the asymptotic profiles of traveling waves, which brings essential technical difficulties for analyzing heteroclinic traveling waves connecting the prey-present equilibrium $(1, 0)$ and the positive coexistence equilibrium (u^*, v^*).

We first linearize the system at the semi-trivial equilibrium and derive the characteristic equation to determine the minimal wave speed c_*. The qualitative analysis of the characteristic function reveals a dichotomy of wave existence: for all wave speeds $c \geq c_*$, we establish the existence of nonnegative bounded traveling waves, while no such traveling waves exist when $0 < c < c_*$. Furthermore, we employ the shrinking rectangle technique to rigorously prove that the obtained wave profiles converge to the coexistence equilibrium as $\xi \to +\infty$ under the assumption $h(u) = 1$. Compared with previous literature focusing on local diffusion Leslie-Gower systems or nonlocal dispersal competitive models, our work extends the traveling wave theory to generalized predator-prey systems with nonlocal dispersal and provides a unified analytical approach to handle both critical and supercritical wave speeds. The shrinking rectangle method used in this paper offers an alternative tool to analyze the far-field limits of wave solutions for non-monotone nonlocal reaction-diffusion systems, which may be adapted to other predator-prey or ecological interaction models.

Nevertheless, the current study still has several limitations. The convergence of wave profiles at $+\infty$ is only fully verified for $h(u) = 1$, and the asymptotic behavior for the general increasing function $h(u)$ remains unresolved. In addition, we only consider one-dimensional spatial domains and symmetric convolution kernels in this paper, without discussing multi-dimensional cases or asymmetric dispersal kernels. Future research directions include extending our upper lower solution and shrinking rectangle strategies to the general $h(u)$ case, investigating traveling waves in multi-dimensional spaces, exploring periodic traveling waves, and incorporating stage structures or time delays into the generalized nonlocal dispersal Leslie-Gower model.

In summary, this work systematically establishes the threshold dynamics of invasion traveling waves for generalized nonlocal dispersal Leslie-Gower predator-prey systems, enriches the qualitative theory of nonlocal dispersal evolution equations, and provides theoretical support for understanding spatial propagation phenomena in population ecology.

Acknowledgments. Y.-X. Hao was partially supported by the NSF of China (No. 12501225) and by the NSF of Gansu Province (No. 26JRRA491).

References

[1] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi, J. Toledo-Melero; *Nonlocal Diffusion Problems*, Mathematical Surveys and Monographs, AMS, Providence, Rhode Island, 2010.

[2] S. Ai, Y. Du, R. Peng; Traveling waves for a generalized Holling-Tanner predator-prey model, *J. Differ. Equ.*, **263** (2017) 7782-7814.

[3] M .A. Aziz-Alaoui, M. D. Okiye; Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes, *Appl. Math. Lett.*, **16** (2003), 1069-1075.

[4] M. A. Aziz-Alaoui; Study of a Leslie-Gower-type tritrophic population model, *Chaos Solitons Fractals*, **14** (2002), 1275-1293.

[5] Y. Y. Chen, J. S. Guo, C. H. Yao; Traveling wave solutions for a continuous and discrete diffusive predator-prey model, *J. Math. Anal. Appl.*, **445** (2017), 212-239.

[6] S. R. Dunbar; Travelling wave solutions of diffusive Lotka-Volterra equations, *J. Math. Biol.* **17** (1983) 11-32.

[7] S. R. Dunbar; Traveling wave solutions of diffusive Lotka-Volterra equations: a heteroclinic connection in $\mathbb{R}^4$, *Trans. Amer. Math. Soc.*, **286** (1984), 557-594.

[8] S. R. Dunbar; Traveling waves in diffusive predator-prey equations: periodic orbits and point-to-periodic heteroclinic orbits, *SIAM J. Appl. Math.*, **46** (1986), 1057-1078.

[9] Z. Du, Z. Feng, X. Zhang; Traveling wave phenomena of n-dimensional diffusive predator-prey systems, *Nonlinear Anal. Real World Appl.*, **41** (2018), 288-312.

[10] F. D. Dong, W. T. Li, G. B. Zhang; Invasion traveling wave solutions of a predator-prey model with nonlocal dispersal, *Commun. Nonlinear Sci. Numer. Simul.*, **79** (2019), 17 pp.

[11] Q. Fang, H. Cheng, R. Yuan; Spatial dynamics of a nonlocal dispersal Leslie-Gower predator-prey model with some shifting habitats, *Discrete Contin. Dyn. Syst.*, **43** (2023), 2985-3007.

[12] Q. Fang, Z. Yu, Forced wave of modified Leslie-Gower predator-prey model with nonlocal dispersal under shifting environment, *J. Math. Anal. Appl* **553** (2026) 31 pp.

[13] R. A. Fisher; The wave of advance of advantageous genes, *Ann. Eugenice.*, **7** (1937), 355-369.

[14] E. Gonaález-Olivares, C. Arancibia-Ibarra, A. Rojas-Palma, B. González-Yañez; Bifurcations and multistability on the May-Holling-Tanner predation model considering alternative food for the predators, *Math. Biosci. Eng.*, **16** (2019), 4274-4298.

[15] J. S. Guo, K.I . Nakamura, T. Ogiwara, C. C. Wu; Traveling wave solutions for a predator-prey system with two predators and one prey, *Nonlinear Anal. Real World Appl.*, **54** (2020), 103-111.

[16] Y.-X. Hao, W.-T. Li, F.-Y. Yang; Traveling waves in a nonlocal dispersal predator-prey model, *Discrete Contin. Dyn. Syst. Ser. S*, **14** (2021), 3113-3139.

[17] Y. L. Huang, G. Lin; Traveling wave solutions in a diffusive system with two preys and one predator, *J. Math. Anal. Appl.*, **418** (2014), 163-184.
[18] A. N. Kolmogorov, I. G. Petrovskii, N. S. Piskunov; Study of a diffusion equation that is related to the growth of a quality of matter, and its application to a biological problem, *Byul. Mosk. Gos. Univ. Ser. A: Mat. Mekh.*, **1** (1937), 1-26.
[19] P. H. Leslie, J. C. Gower; The properties of a stochastic model for the predator- prey type of interaction between two species, *Biometrika*, **47** (1960), 219-234.
[20] P. H. Leslie; Some further notes on the use of matrices in population mathematics, *Biometrika*, **35** (1948), 213-245.
[21] H. Li, M. Zhao, R. Yuan; Traveling waves of modified Leslie-Gower predator-prey systems, *Int. J. Biomath.*, **18** (2025), 34 pp.
[22] W.-T. Li, S. L. Wu; Traveling waves in a diffusive predator-prey model with Holling type-III functional response, *Chaos Solitons Fractals*, **37** (2008), 476-486.
[23] X. Lin, P. Weng, C. Wu; Traveling wave solutions for a predator-prey system with sigmoidal response function, *J. Dynam. Differential Equations*, **23** (2011), 903-921.
[24] A. J. Lotka;
emphElements of Physicals Biology, Williams and Wilkins Company, Baltimore, 1925.
[25] J. D. Murray; *Mathematical Biology: I. An Introduction*, third ed., Springer, New York, 2002.
[26] J. A. Sherratt; Invasion generates periodic traveling waves (wavetrains) in predator-prey models with nonlocal dispersal, *SIAM J. Appl. Math.*, **76** (2016), 293-313.
[27] V. Volterra; Fluctuations in the abundance of a species considered mathematically, *Nature*, **118** (1926), 558-560.
[28] C. C. Wu; The spreading speed for a predator-prey model with one predator and two preys, *Appl. Math. Lett.*, **91** (2019), 9-14.
[29] S.-L. Wu, L.Y. Pang, S. G. Ruan; Propagation dynamics in periodic predator-prey systems with nonlocal dispersal, *J. Math. Pures Appl.*, **170** (2023), 57-95.
[30] F.-Y. Yang, W.-T. Li, J.-B. Wang; Wave propagation for a class of non-local dispersal non-cooperative systems, *Proc. Roy. Soc. Edinbuegh Sect. A*, **150** (2020), 1965-1997.
[31] F.-Y. Yang, W.-T. Li, R.-H. Wang; Invasion waves for a nonlocal dispersal predator-prey model with two predators and one prey, *Commun. Pure Appl. Anal.*, **20** (2021), 4083-4105.
[32] F.-Y. Yang, Q. Zhao, J. Wang; Forced waves of Leslie-Gower predator-prey system in shifting habitats with nonlocal dispersal, *Discrete Contin. Dyn. Syst. Ser. B*, **30** (2025), 790-816.
[33] G. B. Zhang, W.-T. Li, G. Lin; Traveling waves in delayed predator-prey systems with nonlocal diffusion and stage structure, *Math. Comput. Modelling*, **49** (2009), 1021-1029.
[34] G.-B. Zhang, W.-T. Li, Z.-C. Wang; Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity, *J. Differential Equations*, **252** (2012), 5096-5124.
[35] R. Zhou, S.-L. Wu; Spreading properties for a predator-prey system with nonlocal dispersal and climate change, *J. Differential Equations*, **415** (2025) 791-828.

Yu-Xia Hao (corresponding author)
College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China.
Gansu Provincial Research Center for Basic Disciplines of Mathematics and Statistics, Lanzhou, Gansu 730070, China
Email address: haoyx23@nwnu.edu.cn

Zi-Ming Liu
College of Mathematics and Statistics, Northwest Normal University, Lanzhou 730070, China
Email address: 166473086@qq.com