Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 52, pp. 1–14.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.52

EXISTENCE, MULTIPLICITY AND ASYMPTOTIC BEHAVIOR OF POSITIVE SOLUTIONS TO (p,q)-LAPLACIAN ELLIPTIC SYSTEMS

XIANKUI MENG, MEIQIANG FENG

Abstract. In this article, we study elliptic systems consisting of p-Laplacian equations and q-Laplacian equations (for short (p,q)-Laplacian elliptic system). By employing topological methods, we discuss the existence, nonexistence, and multiplicity of positive solutions depending on multiple parameters. We also study the asymptotic behavior of the solutions.

1. Introduction

Differential equations involving p-Laplacian appear in the study on population biology, nonlinear elasticity, nonlinear flow laws, and non-Newtonian mechanics; see [3, 7, 19, 33]. Owing to this profusion of applications and theory, many authors have paid much attention to single p-Laplacian equations, use different methods to study them in the past few decades; see [2, 4, 5, 8, 14, 16, 22, 25, 32, 34, 35, 38, 40, 39, 41] and the references cited therein. Particularly, He [30] used topological methods to analyze the multiplicity of positive solutions to the p-Laplacian problem

$$\begin{gathered} -\Delta_p u = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ denotes a ball of radius b.

Elliptic systems have become an important research field in recent years; see for example [6, 10, 11, 12, 13, 15, 17, 18, 24, 27, 28, 29, 31, 36]. Hai [23] showed the existence and uniqueness of positive solutions for the semilinear elliptic system

$$\begin{gathered} \Delta u = -\lambda f(v) \quad \text{in } B, \\ \Delta v = -\mu g(u) \quad \text{in } B, \\ u = v = 0 \quad \text{on } \partial B, \end{gathered}$$

where $\lambda, \mu > 0$ are parameters and B denotes the open unit ball in $\mathbb{R}^N$. Hai and Shivaji [26] considered the p-Laplacian system

$$\begin{gathered} -\Delta_p u = \lambda f(v) \quad \text{in } \Omega, \\ -\Delta_p v = \lambda g(u) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p > 1$, $\lambda > 0$ is a parameter, and Ω denotes a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. They proved the existence of positive solutions to (1.2) by using the method of sub- and super- solutions. Recently, Zhang and Feng [42] investigated system (1.2) and demonstrated uniqueness, existence and asymptotic behavior of positive solutions, using the eigenvalue theory of completely continuous operators.

2020 *Mathematics Subject Classification.* 35J62, 35J66, 35J92.
Key words and phrases. (p,q)-Laplacian elliptic system; positive solution; topological method; existence; uniqueness; multiplicity; asymptotic behavior.

Submitted March 10, 2026. Published July 9, 2026.

Inspired by the above mentioned studies, in this paper, we are interested in the existence, nonexistence, and multiplicity of positive solutions to the (p,q)-Laplacian elliptic system

$$\begin{gathered} -\Delta_p z_1 = \lambda_1 g_1(|x|) f_1(z_1, z_2) \quad \text{in } \Omega, \\ -\Delta_q z_2 = \lambda_2 g_2(|x|) f_2(z_1, z_2) \quad \text{in } \Omega, \\ z_1 = z_2 = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $1 < p, q < \infty$, $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2}\nabla w)$ denotes the p-Laplacian; $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$, $N \geq 2$; $g_i(t)$ is continuous on $\bar{\Omega}$ and $f_i(z_1, z_2)$ is continuous on $[0, +\infty) \times [0, +\infty)$ for each $i \in \{1, 2\}$.

Letting $r = |x|$ and $u_i(r) = z_i(x)$ for $i \in \{1, 2\}$, we derive

$$\Delta_p z_i(x) = r^{1-N}(r^{N-1}|u_i{}'(r)|^{p-2} u_i{}')'.$$

Hence, the positive solutions to the system

$$\begin{gathered} -(r^{N-1}\varphi_p(u_1'))' = \lambda_1 r^{N-1} g_1(r) f_1(u_1, u_2) \quad \text{with } 0 < r < 1, \\ -(r^{N-1}\varphi_q(u_2'))' = \lambda_2 r^{N-1} g_2(r) f_2(u_1, u_2) \quad \text{with } 0 < r < 1, \\ u_1'(0) = u_2'(0) = u_1(1) = u_2(1) = 0 \end{gathered} \tag{1.4}$$

are positive radial solutions to system (1.3), where

$$\varphi_p(s) = |s|^{p-2}s, \quad (\varphi_p)^{-1} = \varphi_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Some special cases of system (1.3) have been studied. For example, Cerd, Souto and Ubilla [9] considered (1.3) when $\lambda_1 = \lambda_2 = 1$. Using a priori bound method, they demonstrated the existence of positive solutions to (1.3).

In this article, we shall analyze the existence and multiplicity of positive solutions to (1.3) by using a topological method, which is completely different from that used in Hai and Shivaji [26], Zhang and Feng [42], and Cerd, Souto and Ubilla [9]. Here, we also extend the study of Hai and Shivaji [26] and Zhang and Feng [42] from a p-Laplacian elliptic system to a (p,q)-Laplacian elliptic systems. Moreover, comparing with Hai and Shivaji [26], Zhang and Feng [42] and Cerd, Souto and Ubilla [9], the multiplicity and nonexistence of positive solutions to (1.3) are also considered. In addition, we analyze the asymptotic behavior of positive solutions to (1.3) depending on the parameter λ_i.

In this article, for $i \in \{1, 2\}$, we use the following assumptions:

(A1) $f_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, and $f_i(u_1, u_2)$ is a nondecreasing function of u_1 and u_2, that is

$$f_1(u_1, u_2) \leq f_1(u_1^*, u_2^*), \ f_2(u_1, u_2) \leq f_2(u_1^*, u_2^*),$$

when $u_1 < u_1^*$, $u_2 < u_2^*$, where $\mathbb{R}_+ = [0, +\infty)$;

(A2) $g_i \in C([0,1], \mathbb{R}_+)$ is a nonincreasing function, $g_i(t) \not\equiv 0$ on any subinterval of $[0,1]$, and

$$\int_0^1 g_i(t)dt < +\infty, \quad \text{for } i \in \{1, 2\};$$

(A3) $f_i(u_1, u_2) > 0$ for $\mathbf{u} = (u_1, u_2) \in \mathbb{R}_+^2$ and $|\mathbf{u}| > 0$, where $|\cdot|$ denotes the maximum norm in $\mathbb{R}^2$ defined by

$$|\mathbf{u}| = \max\{|u_1|, |u_2|\}.$$

Let $X = E^2$ denote the real Banach space of continuous functions with the norm

$$\|\mathbf{u}\| = \max\{\|u_1\|_0, \|u_2\|_0\},$$

where $E = C[0,1]$, which is a Banach space with the supremum norm

$$\|y\|_0 = \sup_{t \in J = [0,1]} |y(t)|, \quad \forall y \in E,$$

and $\mathbf{u}(t) = (u_1(t), u_2(t))$ for $t \in J$. Similarly, one can define $\mathbf{u}_R(t) = (u_{1R}(t), u_{2R}(t))$ for $t \in J$. To state our main results, we introduce the notation

$$f_1^0 = \lim_{|\mathbf{u}| \to 0^+} \frac{f_1(\mathbf{u})}{\varphi_p(|\mathbf{u}|)}, \quad f_2^0 = \lim_{|\mathbf{u}| \to 0^+} \frac{f_2(\mathbf{u})}{\varphi_q(|\mathbf{u}|)},$$

$$f_1^\infty = \lim_{|\mathbf{u}|\to+\infty} \frac{f_1(\mathbf{u})}{\varphi_p(|\mathbf{u}|)}, \quad f_2^\infty = \lim_{|\mathbf{u}|\to+\infty} \frac{f_2(\mathbf{u})}{\varphi_q(|\mathbf{u}|)}.$$

Let us now describe the main results. The existence of positive solutions are considered in the following theorems.

Theorem 1.1. *Under conditions* (A1) *and* (A2)*, if $f_i^\infty = +\infty$ for $i \in \{1,2\}$, then there exists $\beta_0 > 0$ such that, for every $R > \beta_0$, system* (1.4) *admits a positive solution $\mathbf{u}_R = (\mathbf{u}_{1R}, \mathbf{u}_{2R})$ satisfying*

$$\|\mathbf{u}_R\| = \max\{\|u_{1R}\|_0,\ \|u_{2R}\|_0\} = R$$

when

$$\lambda_i = \lambda_{iR} \in (0, \lambda_0], \tag{1.5}$$

where λ_0 is a positive finite number.

Theorem 1.2. *Under conditions* (A1) *and* (A2)*, if $f_i^0 = +\infty$ for $i \in \{1,2\}$, then there exists $\beta_1 > 0$ such that, for every $0 < R^* < \beta_1$, system* (1.4) *admits a positive solution $\mathbf{u}_{R^*} = (\mathbf{u}_{1R^*}, \mathbf{u}_{2R^*})$ satisfying*

$$\|\mathbf{u}_{R^*}\| = \max\{\|u_{1R^*}\|_0,\ \|u_{2R^*}\|_0\} = R^*$$

when $\lambda_i = \lambda_{iR^*} \in (0, \lambda^0]$, where λ^0 is a positive finite number.

For $i \in \{1,2\}$, we define

$$m_1(l) = \min_{u_i \in [\frac{1}{6}l, l]} \Big\{\frac{f_1(u_1,u_2)}{\varphi_p(l)}\Big\}, \quad m_2(l) = \min_{u_i \in [\frac{1}{6}l, l]} \Big\{\frac{f_2(u_1,u_2)}{\varphi_q(l)}\Big\},$$

where $l > 0$ is a constant.

Theorem 1.3. *Under conditions* (A1) *and* (A2)*, if there are $r^* > 0$ and $\beta_{r^*} > 0$ so that $m_i(r^*) \geq \beta_{r^*}$ for $i \in \{1,2\}$, then system* (1.4) *admits a positive solution $\mathbf{u}_{r^*} = (\mathbf{u}_{1r^*}, \mathbf{u}_{2r^*})$ satisfying*

$$\|\mathbf{u}_{r^*}\| = \max\{\|u_{1r^*}\|_0,\ \|u_{2r^*}\|_0\} = r^*$$

when $\lambda_i = \lambda_{ir^*} \in (0, \lambda^*]$, where λ^* is a positive finite number.

The following theorem considers the existence and asymptotic behavior of positive solutions to (1.4).

Theorem 1.4. *Under assumptions* (A1)*and* (A2)*, the following results hold:*

(a) *If $f_i^0 = 0$ and $f_i^\infty = \infty$ for $i \in \{1,2\}$, then for every $\lambda_i > 0$, system* (1.4) *admits a positive solution $(u_{1\lambda_1}, u_{2\lambda_2})$ satisfying $\lim_{\lambda_i\to0^+} \|u_{i\lambda_i}\| = +\infty$;*
(b) *If $f_i^0 = \infty$ and $f_i^\infty = 0$ for $i \in \{1,2\}$, then for every $\lambda_i > 0$ system* (1.4) *admits a positive solution $(u_{1\lambda_1}, u_{2\lambda_2})$ satisfying $\lim_{\lambda_i\to0^+} \|u_{i\lambda_i}\| = 0$.*

In the next theorem, we discuss the multiplicity of positive solutions to (1.4). For $i \in \{1,2\}$, let

$$f_\infty^i := \lim_{|\mathbf{u}|\to+\infty} f_i(\mathbf{u}).$$

Theorem 1.5. *Under conditions* (A1)–(A3)*, if $f_i^0 = 0, f_i^\infty = 0$ and $f_\infty^i \in (0,+\infty]$ for $i \in \{1,2\}$, then for every $D > 0$ there exists $\lambda_i^* > 0$ such that* (1.4) *admits at least two positive solutions $(u_{1\lambda_1}^{(1)}, u_{2\lambda_2}^{(1)})$ and $(u_{1\lambda_1}^{(2)}, u_{2\lambda_2}^{(2)})$ with*

$$\max_{t\in J} \big\{u_{1\lambda_1}^{(1)}(t), u_{2\lambda_2(t)}^{(1)}\big\} > D$$

for $\lambda_i \in (\lambda_i^*, +\infty)$.

In the next theorem, we investigate the nonexistence of positive solutions to (1.4).

Theorem 1.6. *Under assumptions* (A1)–(A3)*, the following two results hold.*

(a) *If there is $\xi_i > 0$ for $i \in \{1,2\}$ such that*
$$f_1(u_1,u_2) \geq \xi_1\varphi_p(|\mathbf{u}|) \text{ or } f_2(u_1,u_2) \geq \xi_2\varphi_q(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}_+^2, \tag{1.6}$$
then there is $\lambda_i^{**} > 0$ such (1.4) *admits no positive solutions when $\lambda_i > \lambda_i^{**}$;*

(b) *If there is* $\zeta_i > 0$ *for* $i \in \{1,2\}$ *such that*
$$f_1(u_1,u_2) \le \zeta_1\varphi_p(|\mathbf{u}|) \text{ and } f_2(u_1,u_2) \le \zeta_2\varphi_q(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}^2_+, \tag{1.7}$$
then there is $\lambda_i^{***} > 0$ *such that* (1.4) *admits no positive solutions when* $0 < \lambda_i < \lambda_i^{***}$.

The rest of this article is organized as follows. In Section 2, we shall collect some known conclusions to be employed in the subsequent section. In Section 3, we prove Theorems 1.1-1.6.

2. Preliminaries

First we review several definitions and conclusions for later use.

Lemma 2.1 (Zhang and Feng [42, Theorem 2.1])**.** *Let* $p > 1$ *and* $q > 1$*, where* $\frac{1}{p} + \frac{1}{q} = 1$*. Then* $\varphi_p(s) = |s|^{p-2}s$ *is odd, and*
$$s\varphi_p(s) > 0 \text{ if } s \neq 0, \quad \varphi_p(st) = \varphi_p(s)\varphi_p(t), \varphi_p(0) = 0, \quad \varphi_p(1) = 1, \quad \varphi_p(-1) = -1,$$
$$\varphi_p(s+t) = \begin{cases} 2^{p-1}(\varphi_p(s) + \varphi_p(t)), & \text{if } p \ge 2,\ s,t > 0,\\ \varphi_p(s) + \varphi_p(t), & \text{if } 1 < p < 2,\ s,t > 0. \end{cases}$$

Additionally, $\varphi_p(s^a) = \varphi_p^a(s)$ on $[0,\infty)$ for $a \ge 0$, and $\varphi_p(s)$ is increasing on $[0,\infty)$.

Definition 2.2 (Guo and Lakshmikantham [21, Def. 1.1.1])**.** Let $\mathbb{E}$ be a real Banach space over $\mathbb{R}$. A nonempty closed set $P \subset \mathbb{E}$ is said to be a cone provided that

(i) $x \in P$ and $\lambda \ge 0$ imply $\lambda x \in P$;
(ii) $x \in P$ and $-x \in P$ imply $x = 0$.

Every cone $P \subset \mathbb{E}$ induces a partial ordering in $\mathbb{E}$ given by $x \le y$ if and only if $y - x \in P$. Let P be a cone in real Banach space $\mathbb{E}$ and $\le$ be the partial ordering defined by P. Let D be a subset of $\mathbb{E}$.

Definition 2.3 (Guo and Lakshmikantham [21, Def. 2.1.2])**.** An operator $A : D \to \mathbb{E}$ is said to be completely continuous if it is continuous and compact.

Notice that the compactness means that the set $A(S)$ is relatively compact for any bounded set $S \subset D$.

Lemma 2.4 (Amann [1, Lemma 12.1])**.** *Let* P *be a cone in a real Banach space* $\mathbb{E}$ *and* Ω *be a bounded open set of* $\mathbb{E}$*. Suppose that operator* $A : P \cap \bar{\Omega} \to P$ *is completely continuous. If there is a* $x_0 > 0$ *such that*
$$x - Ax \neq tx_0, \quad \forall x \in P \cap \partial\Omega,\ t \ge 0,$$
then $i(A, P \cap \Omega, \mathbb{K}) = 0$.

Remark 2.5. Corollary of Lemma 4.2 in Guo [20] states that $x_0 > 0$ implies that $x_0 \in K$ and $x_0 \neq 0$.

Lemma 2.6. *Let* $\mathbb{E}$ *be a real Banach space and* P *be a cone in* $\mathbb{E}$*. For* $r > 0$*, define* $P_r = \{x \in P : \|x\| < r\}$*. Assume that* $T : \bar{P}_r \to P$ *is completely continuous such that* $Tx \neq x$ *for* $x \in \partial P_r = \{x \in P : \|x\| = r\}$*.*

(i) *If* $\|Tx\| \ge \|x\|$ *for* $x \in \partial P_r$*, then* $i(T, P_r, P) = 0$*.*
(ii) *If* $\|Tx\| \le \|x\|$ *for* $x \in \partial P_r$*, then* $i(T, P_r, P) = 1$*.*

The results of Lemma 2.6 come from the proof in Guo and Lakshmikantham [21, Theorem 2.3.4].

Lemma 2.7 (Guo and Lakshmikantham [21, Theorem 2.3.4])**.** *Let* P *be a cone in a real Banach space* $\mathbb{E}$*. Assume* Ω_1 *and* Ω_2 *are bounded open sets in* $\mathbb{E}$ *with* $0 \in \Omega_1$*,* $\bar{\Omega}_1 \subset \Omega_2$*. If* $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ *is completely continuous such that either*

(i) $\|Ax\| \le \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \ge \|x\|$ *for all* $x \in P \cap \partial\Omega_2$*; or*
(ii) $\|Ax\| \ge \|x\|$ *for all* $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \le \|x\|$ *for all* $x \in P \cap \partial\Omega_2$*,*

then A *has at least one fixed point in* $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$*.*

3. Proof of main results

Lemma 3.1. *Suppose that* (A1) *and* (A2) *hold. Then* (u_1,u_2) *is a solution of* (1.4) *if and only if* $(u_1,u_2)\in X$ *is a solution to the system*

$$
\begin{aligned}
u_1(r) &= \int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s \lambda_1 g_1(\tau)\tau^{N-1} f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds,\\
u_2(r) &= \int_r^1 \varphi_p\Big(\frac{1}{s^{N-1}}\int_0^s \lambda_2 g_2(\tau)\tau^{N-1} f_2\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds,
\end{aligned}
\tag{3.1}
$$

and

$$
\min_{r\in[1/6,5/6]} u_i(r) \ge \frac{1}{6}\|u_i\|_0 \tag{3.2}
$$

for $i\in\{1,2\}$.

Proof. By computations we can show that (3.1) holds. In addition, it is obvious that $u_i(r)\ge 0$ for $r\in J$ and $i\in\{1,2\}$.

Next, we verify that (3.2) is correct. In fact, from (3.1) it follows that

$$
u_1(r) = \int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s \lambda_1 g_1(\tau)\tau^{N-1} f_1(u_1(\tau),u_2(\tau))d\tau\Big)ds.
$$

Differentiating with respect to r,

$$
u_1'(r) = -\varphi_q\Big(\frac{1}{r^{N-1}}\int_0^r g_1(\tau)\tau^{N-1}\lambda_1 f_1(u_1(\tau),u_2(\tau))d\tau\Big).
$$

We define

$$
I(r) = \int_0^r g(\tau)\tau^{N-1}\lambda_1 f_1(u_1(\tau),u_2(\tau))d\tau.
$$

Then

$$
u_1'(r) = -\varphi_q\Big(\frac{I(r)}{r^{N-1}}\Big).
$$

On the other hand, under conditions (A1) and (A2), we derive $I(r)\ge 0$. So, it follows from φ_q is increasing and $\varphi_q(0)=0$ that $\varphi_q(I(r)/r^{N-1})\ge 0$. So $u_1'(r)\le 0$. This shows $u_1(r)$ is non-increasing on J.

Similarly, one can demonstrate that $u_2(r)$ is also non-increasing on J. Furthermore,

$$
\begin{aligned}
-u_1''(r) &= \varphi_q'\Big(\frac{I(r)}{r^{N-1}}\Big)\Big(\frac{I(r)}{r^{N-1}}\Big)'\\
&= \varphi_q'\Big(\frac{I(r)}{r^{N-1}}\Big)\Big(\frac{1-N}{r^N}\int_0^r \lambda_1 g_1(\tau)\tau^{N-1} f_1(u_1(\tau),u_2(\tau))d\tau + \lambda_1 g_1(r) f_1(u_1(r),u_2(r))\Big)\\
&\ge \varphi_q'\Big(\frac{I(r)}{r^{N-1}}\Big)\Big(\frac{1-N}{r^N}\frac{r^N}{N}\lambda_1 g_1(r) f_1(u_1(r),u_2(r)) + \lambda_1 g_1(r) f_1(u_1(r),u_2(r))\Big)\\
&= \frac{1}{q-1}\Big(\frac{I(r)}{r^{N-1}}\Big)^{q-2}\Big(\frac{1}{N}\lambda_1 g_1(r) f_1(u_1(r),u_2(r))\Big) \ge 0,
\end{aligned}
$$

which shows that $u_1''(t)\le 0$ on J. Hence we derive that

$$
\frac{u_1(r)-u_1(1)}{u_1(0)-u_1(1)} \ge \frac{1-r}{1-0} = 1-r
$$

for $r\in[1/6,5/6]$, which indicates that

$$
u_1(r) \ge (1-r)u_1(0) \ge \frac{1}{6}u_1(0).
$$

Similarly, one can verify

$$
u_2(r) \ge \frac{1}{6}u_2(0) = \frac{1}{6}\|u_2\|_0
$$

for $r\in[1/6,5/6]$. So we have

$$
\min_{r\in[1/6,5/6]}\{u_1(r),u_2(r)\} \ge \frac{1}{6}\|(u_1,u_2)\|_0.
$$

This completes the proof. □

Suppose that K_0 is the cone of all nonnegative functions in X defined by

$$K_0 = \{\mathbf{u} \in E : u_i(t) \geq 0,\ \forall t \in J,\ i \in \{1,2\}\}.$$

Let K be a cone in X as

$$K := \Big\{\mathbf{u} \in K_0 : \min_{t\in[1/6,5/6]} \{u_1(t), u_2(t)\} \geq \frac{1}{6}\|\mathbf{u}\|\Big\}. \tag{3.3}$$

For $\varrho > 0$, we also define

$$K_\varrho = \{\mathbf{u} : \mathbf{u} \in K, \|\mathbf{u}\| < \varrho\}, \quad \partial K_\varrho = \{\mathbf{u} : \mathbf{u} \in K, \|\mathbf{u}\| = \varrho\},$$
$$\bar{K}_\varrho = \{\mathbf{u} \in K : \|\mathbf{u}\| \leq \varrho\}.$$

Similarly, we can define $K_{0\varrho}$, $\partial K_{0\varrho}$, and $\bar{K}_{0\varrho}$.

Let K_1 be a cone in E as

$$K_1 := \Big\{y \in E : y(t) \geq 0 \text{ for } t \in J, \min_{t\in[1/6,5/6]} y(t) \geq \frac{1}{6}\|y\|_0\Big\}.$$

For $i \in \{1,2\}$ and $r \in J$, let $T_i : K_1 \to E$ be defined as

$$\begin{aligned}
T_{1\lambda_1}u_1(r) &= \int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau), u_2(\tau)\big)d\tau\Big)ds,\\
T_{2\lambda_2}u_2(r) &= \int_r^1 \varphi_p\Big(\frac{1}{s^{N-1}}\int_0^s g_2(\tau)\tau^{N-1}\lambda_2 f_2\big(u_1(\tau), u_2(\tau)\big)d\tau\Big)ds.
\end{aligned} \tag{3.4}$$

For each $i \in \{1,2\}$, Lemma 3.1 yields that $T_iK_1 \subset K_1$ is completely continuous . It follows from the above analysis that system (1.4) is equivalent to the fixed point equation

$$\mathbf{u}(r) = (T_{1\lambda_1}u_1(r), T_{2\lambda_2}u_2(r)) := T_\lambda \mathbf{u}(r) \quad \text{for } r \in J. \tag{3.5}$$

Because $T_{i\lambda_i} : K_1 \to K_1$ is completely continuous for each $i \in \{1,2,\}$, $T_\lambda : K \to K$ is completely continuous. We now prove our main results.

Proof of Theorem 1.1. Since $f_i^\infty = +\infty$ for $i \in \{1,2\}$, there are $l_i > 0$ and $\mu > 0$ such that

$$f_1(\mathbf{u}) > l_1\varphi_p(|\mathbf{u}|), \quad f_2(\mathbf{u}) > l_2\varphi_q(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}_+^2 \text{ with } |\mathbf{u}| \geq \mu. \tag{3.6}$$

Next, we verify that $\beta_0 = 6\mu$ is required. If $\mathbf{u} \in \partial K_R$, we derive

$$\min_{t\in[1/6,5/6]} \{u_1(t), u_2(t)\} \geq \frac{1}{6}\|\mathbf{u}\| = \frac{1}{6}R.$$

Because $R > \beta_0$, we derive

$$\min_{t\in[1/6,5/6]} \{u_1(t), u_2(t)\} \geq \frac{1}{6}R > \frac{1}{6}\beta_0 = \mu.$$

We define $\lambda_0 = \{\lambda_{10}, \lambda_{20}\}$, where

$$\lambda_{10} = \frac{6^{2p-2}6^{N-1}}{l_1\eta_1}, \quad \lambda_{20} = \frac{6^{2q-2}6^{N-1}}{l_2\eta_2}.$$

Here

$$\eta_1 = \int_{1/6}^{5/6} g_1(\tau)d\tau, \quad \eta_2 = \int_{1/6}^{5/6} g_2(\tau)d\tau.$$

Similarly, we can define

$$\lambda_R = \{\lambda_{1R}, \lambda_{2R}\}.$$

We note that

$$\mathbf{u} - T_{\lambda_0}\mathbf{u} \neq 0 \quad \forall \mathbf{u} \in \partial K_R, \tag{3.7}$$

where $T_{\lambda_0}\mathbf{u} = (T_{1\lambda_{10}}u_1, T_{2\lambda_{20}}u_2)$.

Similarly, we define

$$T_{\lambda_0}\mathbf{u}_R = (T_{1\lambda_{10}}u_{1R}, T_{2\lambda_{20}}u_{2R}).$$

If not, then there is $\mathbf{u}_R \in \partial K_R$ so that $T_{\lambda_0}\mathbf{u}_R = \mathbf{u}_R$ and hence (1.5) already holds for $\lambda_R = \lambda_0$, which shows $\lambda_{1R} = \lambda_{10}$ and $\lambda_{2R} = \lambda_{20}$.

We define $\psi(t) = (\psi_1(t), \psi_2(t))$, where $\psi_i(t) \equiv 1$ for $i \in \{1,2\}$ and $t \in J$. Then $\psi \in K$ with $\|\psi\| = 1$.

Next now show that

$$\mathbf{u} - T_{\lambda_0}\mathbf{u} \neq \zeta\psi, \quad \forall \mathbf{u} \in \partial K_R,\ \zeta \geq 0. \tag{3.8}$$

Indeed, if there are $\mathbf{u} \in \partial K_R$ and $\zeta \geq 0$ such that $\mathbf{u} - T_{\lambda_0}\mathbf{u} = \zeta\psi$, then (3.7) implies that $\zeta > 0$, and

$$u_1 = \zeta\psi_1 + T_{1\lambda_{10}}u_1 \geq \zeta\psi_1. \tag{3.9}$$

Let

$$\zeta^* = \sup\{\zeta | u_1 \geq \zeta\psi_1\}.$$

Then $\zeta \leq \zeta^* < +\infty$, $u_1 \geq \zeta^*\psi_1$. Consequently, for $r \in [\frac{1}{6}, \frac{5}{6}]$, (3.4), (3.5), (3.6), and (3.9) yield

$$\begin{aligned}
u_1(r) &= T_{1\lambda_{10}}u_1(r) + \zeta\psi_1(r) \\
&\geq \frac{1}{6}\|T_{1\lambda_{10}}u_1\|_0 + \zeta\psi_1(r) \\
&\geq \frac{1}{6}\int_{5/6}^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_{1/6}^{5/6} g_1(\tau)\tau^{N-1}\lambda_{10}f_1(u_1(\tau), u_2(\tau))d\tau\Big)ds + \zeta\psi_1(r) \\
&> \frac{1}{6}\int_{5/6}^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_{1/6}^{5/6} g_1(\tau)\tau^{N-1}\lambda_{10}l_1\varphi_p(\zeta^*)d\tau\Big)ds + \zeta\psi_1(r) \\
&\geq (\frac{1}{6})^2\varphi_q(\lambda_{10})\varphi_q(l_1)\zeta^*\varphi_q\Big(\int_{1/6}^{5/6}\tau^{N-1}g_1(\tau)d\tau\Big) + \zeta\psi_1(r) \\
&\geq (\frac{1}{6})^2\varphi_q(\lambda_{10})\varphi_q(l_1)\zeta^*\varphi_q\Big(\frac{1}{6^{N-1}}\int_{1/6}^{5/6} g_1(\tau)d\tau\Big) + \zeta\psi_1(r) \\
&= \zeta^* + \zeta\psi_1(r).
\end{aligned}$$

This indicates that $u_1(r) \geq (\zeta^* + \zeta_1)\psi(r)$ for $r \in J$, which contradicts the definition of ζ^*. So (3.8) is true and, by Lemma 2.4, the fixed point index is

$$i(T_{\lambda_0}, K_R, K) = 0. \tag{3.10}$$

On the other hand, it is easy to see that

$$i(\theta, K_R, K) = 1, \tag{3.11}$$

where θ is the zero operator. So from (3.10), (3.11) and the homotopy invariance property that there are $\mathbf{u}_R \in \partial K_R$ and $0 < \nu_R < 1$ such that $\nu_R T_{\lambda_0}\mathbf{u}_R = \mathbf{u}_R$, which indicates

$$\lambda_{1R} = \lambda_{10}\nu_R^{p-1} < \lambda_{10}, \quad \lambda_{2R} = \lambda_{20}\nu_R^{q-1} < \lambda_{20}.$$

From the demonstration above, for $R > \beta_0$, there is a positive solution $\mathbf{u}_R \in \partial K_R$. This indicates that

$$\begin{gathered}
u_{1R}(r) = \int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s \lambda_{1R}g_1(\tau)\tau^{N-1}f_1\big(u_{1R}(\tau), u_{2R}(\tau)\big)d\tau\Big)ds, \\
u_{2R}(r) = \int_r^1 \varphi_p\Big(\frac{1}{s^{N-1}}\int_0^s \lambda_{2R}g_2(\tau)\tau^{N-1}f_2\big(u_{1R}(\tau), u_{2R}(\tau)\big)d\tau\Big)ds, \\
\|\mathbf{u}_R\| = \max\{\|u_{1R}\|_0, \|u_{2R}\|_0\} = R.
\end{gathered}$$

□

The proof of Theorem 1.2 is similar to that of Theorem 1.1. So we omit it here.

Proof of Theorem 1.3. For $\mathbf{u} \in \partial K_{r^*}$, we derive that

$$\frac{1}{6}r^* \leq \min\{u_1(t), u_2(t)\} \leq r^*, \quad t \in [1/6, 5/6].$$

Since $m_i(r^*) \ge \beta_{r^*} > 0$ for $i \in \{1,2\}$, we have

$$f_1(u_1,u_2) \ge m_1(r^*)\varphi_p(r^*) \ge \varphi_p(r^*)\beta_{r^*} \ge \beta_{r^*}\varphi_p(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}^2_+ \text{ with } |\mathbf{u}| \in [\frac{1}{6}r^*, r^*],$$

$$f_2(u_1,u_2) \ge m_2(r^*)\varphi_q(r^*) \ge \varphi_q(r^*)\beta_{r^*} \ge \beta_{r^*}\varphi_q(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}^2_+ \text{ wit h} |\mathbf{u}| \in [\frac{1}{6}r^*, r^*].$$

We define

$$\lambda^*_{10} = \frac{6^{2p-2}6^{N-1}}{\eta_1\beta_{r^*}}, \quad \lambda^*_{20} = \frac{6^{2q-2}6^{N-1}}{\eta_2\beta_{r^*}}.$$

Following the same arguments as in the proof of Theorem 1.1 we obtain the desired result. $\square$

Proof of Theorem 1.4. We only need to verify this theorem under the conditions $f_i^0 = 0$ and $f_i^\infty = \infty$ for $i \in \{1,2\}$ because the proofs of the other cases are similar to theses cases..

Considering $f_i^0 = 0$ for $i \in \{1,2\}$, there exists $R_1 > 0$ such that

$$f_1(\mathbf{u}) \le \varphi_p(\varepsilon_1)\varphi_p(|\mathbf{u}|), \quad f_2(\mathbf{u}) \le \varphi_q(\varepsilon_2)\varphi_q(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}^2_+ \text{ with } |\mathbf{u}| \in [0, R_1], \tag{3.12}$$

where ε_1 and ε_2 satisfy

$$\varphi_q(\lambda_1\eta_1')\varepsilon_1 < 1, \quad \text{and} \quad \varphi_p(\lambda_2\eta_2')\varepsilon_2 < 1,$$

respectively, where

$$\eta_1' = \int_0^1 g_1(\tau)d\tau, \quad \eta_2' = \int_0^1 g_2(\tau)d\tau.$$

So, for $\mathbf{u} \in \partial K_{R_1}$, we have taht (3.4), (3.5) and (3.12) yield

$$\begin{aligned}
\|T_{1\lambda_1}u_1\|_0 &= \max_{r\in J}\int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds \\
&\le \max_{r\in J}\int_r^1 \varphi_q\Big(\int_0^s g_1(\tau)\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds \\
&\le \int_0^1 \varphi_q\Big(\int_0^1 g_1(\tau)\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\ bigr)d\tau\Big)ds \\
&\le \varphi_q\Big(\int_0^1 g_1(\tau)\lambda_1\varphi_p(\varepsilon_1)\varphi_p(\|\mathbf{u}\|))d\tau\Big) \\
&\le \varphi_q(\lambda_1)\varepsilon_1\|\mathbf{u}\|\varphi_q\Big(\int_0^1 g_1(\tau)d\tau\Big) \\
&= \varphi_q(\lambda_1\eta_1')\varepsilon_1\|\mathbf{u}\| \\
&< \|\mathbf{u}\|.
\end{aligned}$$

Similarly, one can prove $\|T_{2\lambda_2}u_2\|_0 < \|\mathbf{u}\|$. Hence we have

$$\|T_\lambda\mathbf{u}\| < \|\mathbf{u}\|. \tag{3.13}$$

Turning to the case $f_i^\infty = \infty$, there is $\bar{R} > 0$ such that

$$\begin{aligned}
&f_1(\mathbf{u}) \ge \varphi_p(\varepsilon_3)\varphi_p(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}^2_+ \text{ with } |\mathbf{u}| \in [\bar{R}, +\infty), \\
&f_2(\mathbf{u}) \ge \varphi_q(\varepsilon_4)\varphi_q(|\mathbf{u}|), \quad \forall \mathbf{u} \in \mathbb{R}^2_+ \text{ with } |\mathbf{u}| \in [\bar{R}, +\infty),
\end{aligned} \tag{3.14}$$

where ε_3 and ε_4 satisfy

$$(\frac{1}{6})^2\varphi_q((\frac{1}{6})^N\lambda_1\eta_1)\varepsilon_3 > 1, \quad (\frac{1}{6})^2\varphi_p((\frac{1}{6})^N\lambda_2\eta_2)\varepsilon_4 > 1.$$

Let $R_2 = \max\{2R_1, 6\bar{R}\}$. If $\mathbf{u} \in \partial K_{R_2}$, then

$$\min_{r\in[\frac{1}{6},\frac{5}{6}]}\{u_1(r),u_2(r)\} \ge \frac{1}{6}\|\mathbf{u}\| = \frac{1}{6}R_2 > \bar{R}.$$

For $\mathbf{u} \in \partial K_{R_2}$, we have that (3.4), (3.5) and (3.14) yield

$$\|T_{1\lambda_1}u_1\|_0 = \max_{r\in J}\int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds$$

$$
\begin{aligned}
&\geq \min_{r\in[1/6,5/6]}\int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds\\
&\geq \int_{5/6}^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_{1/6}^{5/6} g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds\\
&\geq \frac{1}{6}\varphi_q((\frac{1}{6})^N\lambda_1)\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)\varphi_p(\varepsilon_3)\varphi_p(|\mathbf{u}|))d\tau\Big)\\
&\geq \frac{1}{6}\varphi_q((\frac{1}{6})^N\lambda_1)\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)\varphi_p(\varepsilon_3)\varphi_p(u_1(\tau)))d\tau\Big)\\
&\geq \frac{1}{6}\varphi_q((\frac{1}{6})^N\lambda_1)\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)\varphi_p(\varepsilon_3)\varphi_p(\frac{1}{6}\|u_1\|_0)d\tau\Big)\\
&= (\frac{1}{6})^2\varphi_q((\frac{1}{6})^N\lambda_1)\varepsilon_3\|u_1\|_0\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)d\tau\Big)\\
&= (\frac{1}{6})^2\varphi_q((\frac{1}{6})^N\lambda_1\eta_1)\varepsilon_3\|u_1\|_0\\
&> \|u_1\|_0.
\end{aligned}
$$

Similarly, one can prove that $\|T_{2\lambda_2}u_2\|_0 > \|u_2\|_0$. Hence we have

$$
\|T_\lambda \mathbf{u}\| > \|\mathbf{u}\|. \tag{3.15}
$$

Therefore, Lemma 2.7(i) yields that operator T admits a fixed point

$$
(u_{1\lambda_1}, u_{2\lambda_2}) \in \bar{K}_{R_2}\backslash K_{R_1},
$$

which is a positive solution to (1.4) for $\lambda_1>0$ and $\lambda_2>0$.

Next, we verify $\|u_{i\lambda_i}\| = +\infty$ as $\lambda_i\to+\infty$ for $i\in\{1,2\}$. If not, then there is a number $\xi>0$ and a sequence $\lambda_{in}\to+\infty$ such that

$$
\|u_{i\lambda_{in}}\| \leq \xi \quad (n\in\{1,2,3,\dots\}).
$$

Moreover, for $i\in\{1,2\}$, the sequence $\{\|u_{i\lambda_{in}}\|\}$ contains a subsequence which converges to a number δ $(0\leq\delta\leq\xi)$. For convenience, assume that $\{\|u_{i\lambda_{in}}\|\}$ converges to δ.

If $\delta>0$, then $\|u_{i\lambda_{in}}\| > \frac{\delta}{2}$ for sufficiently large n $(n>\mathbb{N})$, and hence

$$
\begin{aligned}
\frac{1}{\lambda_{1n}} &\leq \frac{\|\int_0^1 g_1(\tau)f_1(u_1(\tau),u_2(\tau))d\tau\|_0}{\|u_{1\lambda_{1n}}\|_0^{p-1}}\\
&\leq \frac{\mathbb{M}\|\int_0^1 g_1(\tau)d\tau\|_0}{\|u_{1\lambda_{1n}}\|_0^{p-1}}\\
&= \frac{\mathbb{M}\eta_3}{\|u_{1\lambda_{1n}}\|_0^{p-1}}\\
&< \frac{\mathbb{M}\eta_3 2^{p-1}}{\delta^{p-1}} \quad (n>\mathbb{N}),
\end{aligned}
$$

where,

$$
\mathbb{M} = \max_{\|\mathbf{u}\|\in[0,\xi]} f_1(\mathbf{u}).
$$

This contradicts $\lambda_{1n}\to 0^+$.

If $\delta=0$, then $\|u_{i\lambda_{in}}\|_0\to 0$ for sufficiently large n $(n>\mathbb{N})$, and so it yields from $f_i^0=0$ that for every $\varepsilon^*>0$ there is $R^*>0$ so that

$$
f_1(u_{1\lambda_{1n}}, u_{2\lambda_{2n}}) \leq \varepsilon^*\varphi_p(\mathbf{u}), \quad \forall \mathbf{u}\in\mathbb{R}_+^2 \text{ with } \|\mathbf{u}\|\in[0,R^*].
$$

Hence, for $\mathbf{u}\in K_{R^*}$, we derive that

$$
u_{i\lambda_{in}}(t) \leq \|u_{i\lambda_{in}}\|_0 \leq \|\mathbf{u}\| = R^*,
$$

and hence

$$\begin{aligned}
\frac{1}{\lambda_{1n}} &\le \frac{\|\int_0^1 g_1(\tau)f_1(u_1(\tau),u_2(\tau))d\tau\|_0}{\|u_{1\lambda_{1n}}\|_0^{p-1}}\\
&\le \frac{\varepsilon^*\|\mathbf{u}\|^{p-1}\|\int_0^1 g_1(\tau)d\tau\|_0}{\|u_{1\lambda_{1n}}\|_0^{p-1}}\\
&= \frac{\varepsilon^*\|\mathbf{u}\|^{p-1}\|\eta_3}{\|u_{1\lambda_{1n}}\|_0^{p-1}}\\
&\le \varepsilon^*\eta_3.
\end{aligned}$$

Because ε^* is arbitrary, $\lambda_{1n}\to+\infty$ (as $n\to+\infty$) which contradicts $\lambda_{1n}\to 0^+$. So, $\|u_{i\lambda_i}\|\to+\infty$ as $\lambda_i\to 0^+$. This completes the proof. □

Proof of Theorem 1.5. For arbitrary $D>0$, from $f_\infty^i\in(0,\infty]$, there exists $\delta_i>0$ and $l>N$ such that

$$f_1(u_1,u_2)\ge\delta_1,\quad f_2(u_1,u_2)\ge\delta_2,\quad \forall\mathbf{u}\in\mathbb{R}_+^2 \text{ with } |\mathbf{u}|\in[l,+\infty) \tag{3.16}$$

for $i\in\{1,2\}$. Let

$$\lambda_1^*=\frac{6^{p+N-2}l^{p-1}}{\delta_1\eta_1},\quad \lambda_2^*=\frac{6^{q+N-2}l^{q-1}}{\delta_2\eta_2}.$$

Then for $\lambda_i\in(\lambda_i^*,+\infty)$, one can prove that $T_\lambda:K_0\to K_0$ is completely continuous.

Because $f_i^0=0$ for $i\in\{1,2\}$, there is $0<R_3<l$ such that

$$\begin{gathered}
f_1(u_1,u_2)\le\varphi_p(\varepsilon_5)\varphi_p(|\mathbf{u}|),\quad \forall\mathbf{u}\in\mathbb{R}_+^2 \text{ with } |\mathbf{u}|\in[0,R_3],\\
f_2(u_1,u_2)\le\varphi_q(\varepsilon_6)\varphi_q(|\mathbf{u}|),\quad \forall\mathbf{u}\in\mathbb{R}_+^2 \text{ with } |\mathbf{u}|\in[0,R_3],
\end{gathered}$$

where $\varepsilon_5>0$ and $\varepsilon_6>0$ satisfy

$$\varphi_q(\lambda_1\eta_1')\varepsilon_5<1,\quad \varphi_p(\lambda_2\eta_2')\varepsilon_6<1.$$

Similar to the proof of (3.13), we can verify that $\|T_\lambda\mathbf{u}\|<\|\mathbf{u}\|$ for $\mathbf{u}\in\partial K_{0R_3}$. Hence, Lemma 2.6(ii) yields that

$$i(T_\lambda,K_{0R_3},K_0)=1. \tag{3.17}$$

Considering $f_i^\infty=0$ for $i\in\{1,2\}$, there is $\mathcal{L}>0$ such that

$$\begin{gathered}
f_1(u_1,u_2)\le\varphi_p(\varepsilon_7)\varphi_p(|\mathbf{u}|),\quad \forall\mathbf{u}\in\mathbb{R}_+^2 \text{ with } |\mathbf{u}|\in[\mathcal{L},+\infty),\\
f_2(u_1,u_2)\le\varphi_q(\varepsilon_8)\varphi_q(|\mathbf{u}|),\quad \forall\mathbf{u}\in\mathbb{R}_+^2 \text{ with } |\mathbf{u}|\in[\mathcal{L},+\infty),
\end{gathered}$$

where $\varepsilon_7>0$ and $\varepsilon_8>0$ respectively satisfy

$$\begin{gathered}
2L\varepsilon_7(\lambda_1\eta_1')^{q-1}\le 1,\\
2L\varepsilon_8(\lambda_2\eta_2')^{p-1}\le 1,\\
L=\max\{2^{q-2},2^{p-2},1\}.
\end{gathered}$$

So,

$$\begin{gathered}
0\le f_1(u_1,u_2)\le\varphi_p(\varepsilon_7)\varphi_p(|\mathbf{u}|)+\mathfrak{M}_{1\mathcal{L}},\quad \forall\mathbf{u}\in\mathbb{R}_+^2,\\
0\le f_2(u_1,u_2)\le\varphi_q(\varepsilon_8)\varphi_q(|\mathbf{u}|)+\mathfrak{M}_{2\mathcal{L}},\quad \forall\mathbf{u}\in\mathbb{R}_+^2,
\end{gathered} \tag{3.18}$$

where

$$\mathfrak{M}_{1\mathcal{L}}=\max_{|\mathbf{u}|\in[0,\mathcal{L}]}f_1(u_1,u_2)>0,\quad \mathfrak{M}_{2\mathcal{L}}=\max_{|\mathbf{u}|\in[0,\mathcal{L}]}f_2(u_1,u_2)>0.$$

Let

$$R_4>\max\{l,\ 2L(\lambda_1\eta_1'\mathfrak{M}_{1\mathcal{L}},\ 2L(\lambda_2\eta_2'\mathfrak{M}_{2\mathcal{L}})^{q-1}\}.$$

So, for $\mathbf{u}\in\partial K_{0R_4}$, from (3.4), (3.5), (3.18) and Lemma **??** we have that

$$\begin{aligned}
\|T_{1\lambda_1}u_1\|_0 &= \max_{r\in J}\int_r^1\varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds\\
&\le \max_{r\in J}\int_r^1\varphi_q\Big(\int_0^s g_1(\tau)\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \varphi_q\Big(\int_0^1 g_1(\tau)\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds\\
&\le \varphi_q\Big(\int_0^1 g_1(\tau)\lambda_1[\varphi_p(\varepsilon_7)\varphi_p(\|\mathbf{u}\|)+\mathfrak{M}_{1\mathcal{L}}]d\tau\Big)\\
&\le L\Big[\varphi_q\Big(\int_0^1 g_1(\tau)\lambda_1\varphi_p(\varepsilon_7)\varphi_p(\|\mathbf{u}\|)d\tau\Big)+\varphi_q\Big(\int_0^1 g_1(\tau)\lambda_1\mathfrak{M}_{1\mathcal{L}}d\tau\Big)\Big]\\
&\le \frac{\|\mathbf{u}\|}{2}+\frac{R_4}{2}\\
&< R_4.
\end{aligned}
$$

Similarly, we can verify that $\|T_{2\lambda_2}u_2\|_0 < R_4$. This indicates that

$$
\|T_\lambda \mathbf{u}\| < R_4, \tag{3.19}
$$

and hence

$$
i(T_\lambda, K_{0R_4}, K_0) = 1. \tag{3.20}
$$

We define

$$
\mathbf{u}\in K_{0l}^{R_4}=\Big\{\mathbf{u}\in K_0:\|\mathbf{u}\|<R_4,\ \min_{t\in[1/6,5/6]}\{u_(t),u_2(t)\}>l\Big\}.
$$

So, for $\mathbf{u}\in \bar K_{0l}^{R_4}=\big\{\mathbf{u}\in K_0:\|\mathbf{u}\|\le R_4,\ \min_{t\in[1/6,5/6]}\{u_(t),u_2(t)\}\ge l\big\}$, it follows from (3.19) that $\|T_\lambda\mathbf{u}\|<R_4$. Secondly, for every $\mathbf{u}\in\bar K_{0l}^{R_4}$, we derive

$$
\min_{t\in[1/6,5/6]}\{u_(t),u_2(t)\}\ge l,
$$

so from (3.16) it follows that

$$
\begin{aligned}
\min_{r\in[\frac16,\frac56]} T_{1\lambda_1}u_1(r) &= \min_{r\in[\frac16,\frac56]}\int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds\\
&\ge \int_{5/6}^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_{1/6}^{5/6} g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_1(\tau),u_2(\tau)\big)d\tau\Big)ds\\
&\ge \frac16\varphi_q((\frac16)^{N-1}\lambda_1)\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)\delta_1 d\tau\Big)\\
&> \frac16\varphi_q((\frac16)^{N-1}\lambda_1^*\delta_1)\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)d\tau\Big)\\
&= \frac16\varphi_q((\frac16)^{N-1}\lambda_1^*\delta_1\eta_1)=l.
\end{aligned}
$$

Similarly, we prove $\min_{r\in[\frac16,\frac56]}T_{2\lambda_2}u_2(r)>l$. So that

$$
\min_{r\in[1/6,5/6]}\{T_{1\lambda_1}u_1(r),T_{2\lambda_2}u_2(r)\}>l.
$$

Let

$$
\begin{gathered}
\mathbf{u}_0=(u_{10},u_{20})\equiv\Big(\frac{l+R_4}{2},\frac{l+R_4}{2}\Big),\\
U(t,\mathbf{u})=(1-t)T_\lambda\mathbf{u}+t\mathbf{u}_0,
\end{gathered}
$$

where

$$
\begin{gathered}
U(t,\mathbf{u})=(U_1(t,u_1),U_2(t,u_2)),\\
U_1(t,u_1)=(1-t)T_{1\lambda_1}+tu_{10},\\
U_2(t,u_2)=(1-t)T_{2\lambda_2}+tu_{20}.
\end{gathered}
$$

Then $U:J\times\bar K_{0l}^{R_4}\to K_0$ is completely continuous. So it follows from the analysis above that $U(t,\mathbf{u})\in K_{0l}^{R_4}$ for $(t,\mathbf{u})\in J\times\bar K_{0l}^{R_4}$.

So, for $t \in J$ and $\mathbf{u} \in \partial K_{0l}^{R_4}$, we have $U(t,\mathbf{u}) \neq \mathbf{u}$. Therefore, from the normality property and the homotopy invariance property of the fixed point index it follows that

$$i(T_\lambda, K_{0l}^{R_4}, K_0) = i(\mathbf{u}_0, K_{0l}^{R_4}, K_0) = 1. \tag{3.21}$$

From the solution property of the fixed point index it follows that T_λ admits a fixed point $\mathbf{u}_\lambda^{(1)}$ so that $\mathbf{u}_\lambda^{(1)} \in K_{0l}^{R_4}$, and

$$\max_{t\in J} \mathbf{u}_\lambda^{(1)}(t) = \max_{t\in J}\{u_{1\lambda_1}^{(1)}, u_{2\lambda_2}^{(1)}\} \geq \min_{t\in[1/6,5/6]}\{u_{1\lambda_1}^{(1)}, u_{2\lambda_2}^{(1)}\} > l > D.$$

Moreover, by (3.17), (3.20), (3.21) and the additivity of the fixed point index, we derive

$$i(T_\lambda, K_{0R_4}\backslash(\bar{K}_{0R_3} \cup \bar{K}_{0l}^{R_4}), K_0) = i(T_\lambda, K_{0R_4}, K_0) - i(T_\lambda, K_{0l}^{R_4}, K_0) - i(T_\lambda, K_{0R_3}, K_0) \quad = 1 - 1 - 1 = -1.$$

From the solution property of the fixed point index it follows that T_λ admits a fixed point $\mathbf{u}_\lambda^{(2)}$ so that $\mathbf{u}_\lambda^{(2)} \in K_{0R_4}\backslash(\bar{K}_{0R_3} \cup \bar{K}_{0l}^{R_4})$ and $\mathbf{u}_\lambda^{(1)} \neq \mathbf{u}_\lambda^{(2)}$. □

Proof of Theorem 1.6. Part (a). Suppose that $\mathbf{u}_\lambda \in K$ is a positive solution to (1.4). Then for every $r \in [0,1)$, we have $u_{i\lambda_i} > 0$ for $i \in \{1,2\}$. Next, we show that this will lead to a contradiction for

$$\lambda_1 > \lambda_1^{**} = \frac{6^{p+N-2}}{\xi_1\eta_1} \quad \text{or} \quad \lambda_2 > \lambda_2^{**} = \frac{6^{q+N-2}}{\xi_2\eta_2}.$$

Since $T_\lambda\mathbf{u}(r) = \mathbf{u}(r)$ for $\lambda_i > \lambda_i^{**}$ and $r \in J$, it yields from (1.6), (3.4) and (3.5) that

$$\begin{aligned}
\|u_{1\lambda_1}\|_0 &= \|T_{1\lambda_1}u_{1\lambda_1}\|_0 \\
&= \max_{r\in J}\int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_{1\lambda_1}(\tau), u_{2\lambda_2}(\tau)\big)d\tau\Big)ds \\
&= \min_{r\in[1/6,5/6]}\int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_{1\lambda_1}(\tau), u_{2\lambda_2}(\tau)\big)d\tau\Big)ds \\
&\geq \int_{5/6}^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_{1/6}^{5/6} g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_{1\lambda_1}(\tau), u_{2\lambda_2}(\tau)\big)d\tau\Big)ds \\
&\geq \frac{1}{6}\varphi_q((\frac{1}{6})^{N-1}\lambda_1)\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)\xi_1\varphi_p(\|\mathbf{u}\|)d\tau\Big) \\
&> \frac{1}{6}\varphi_q((\frac{1}{6})^{N-1}\lambda_1^{**}\xi_1)\|\mathbf{u}\|\varphi_q\Big(\int_{1/6}^{5/6} g_1(\tau)d\tau\Big) \\
&= \frac{1}{6}\varphi_q((\frac{1}{6})^{N-1}\lambda_1^{**}\xi_1\eta_1)\|\mathbf{u}\| \\
&= \|\mathbf{u}\|.
\end{aligned}$$

Similarly, one can verify that $\|u_{2\lambda_2}\|_0 > \|\mathbf{u}\|$. This is a contradiction.

Part (b). Assume that $\mathbf{u}_\lambda \in K$ is a positive solution of (1.4). Then for each $r \in [0,1)$, we have $u_{i\lambda_i} > 0$ for $i \in \{1,2\}$. Next we show that this will lead to a contradiction to

$$0 < \lambda_1 < \lambda_1^{***} = \frac{1}{\zeta_1\eta_1'} \quad \text{and} \quad 0 < \lambda_2 < \lambda_2^{***} = \frac{1}{\zeta_2\eta_2'}.$$

Because $T_\lambda\mathbf{u}(r) = \mathbf{u}(r)$ for $0 < \lambda_i < \lambda_i^{***}$ and $r \in J$, it follows from (1.7), (3.4) and (3.5) that

$$\begin{aligned}
\|u_{1\lambda_1}\|_0 &= \|T_{1\lambda_1}u_{1\lambda_1}\|_0 \\
&= \max_{r\in J}\int_r^1 \varphi_q\Big(\frac{1}{s^{N-1}}\int_0^s g_1(\tau)\tau^{N-1}\lambda_1 f_1\big(u_{1\lambda_1}(\tau), u_{2\lambda_2}(\tau)\big)d\tau\Big)ds \\
&\leq \int_0^1 \varphi_q\Big(\int_0^1 g_1(\tau)\lambda_1 f_1\big(u_{1\lambda_1}(\tau), u_{2\lambda_2}(\tau)\big)d\tau\Big)ds \\
&\leq \varphi_q(\lambda_1)\varphi_q\Big(\int_0^1 g_1(\tau)\zeta_1\varphi_p(\|\mathbf{u}\|)d\tau\Big)
\end{aligned}$$

$$
\begin{aligned}
&< \varphi_q(\lambda_1^{***}\zeta_1)\|\mathbf{u}\|\varphi_q\Big(\int_0^1 g_1(\tau)d\tau\Big)\\
&= \varphi_q(\lambda_1^{***}\zeta_1\eta_1')\|\mathbf{u}\|\\
&= \|\mathbf{u}\|.
\end{aligned}
$$

Similarly, one can verify that $\|u_{2\lambda_2}\|_0 < \|\mathbf{u}\|$. So we derive $\|\mathbf{u}\| < \|\mathbf{u}\|$. This is a contradiction. $\square$

Remark 3.2. It would be very interesting to analyze the existence of infinitely many positive solutions to (1.4) by using topological approaches, but we can not give a demonstration right now.

Acknowledgments. This work was sponsored by the National Natural Science Foundation of China under Grant No. 12371112, and by the Beijing Natural Science Foundation of China under Grant No. 1212003. The authors want to express their gratitude to the reviewers for their careful reading and valuable suggestions which led to an improvement of the original manuscript.

References

[1] H. Amann; Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, **18** (1976), 620-709.
[2] A. Ambrosetti, J. Azorero, I. Alonso; Multiplicity results for some nonlinear elliptic equations, *J. Funct. Anal.*, **137** (1996), 219-242.
[3] C. Atkinson, K. El-Ali; Some boundary value problems for the Bingham model, *J. Non-Newtonian Fluid Mech.*, **41** (1992), 339-363.
[4] G. A. Afrouzi, S. H. Rasouli; A remark on the existence and multiplicity result for a nonlinear elliptic problem involving the p-Laplacian, *Nonlinear Differ. Equ. Appl.*, **16** (2009), 717-730.
[5] T. Bartsch, Z. Liu, T. Weth; Nodal solutions of a p-Laplacian equation, *P. Lond. Math. Soc.*, **91** (2005), 129-152.
[6] M. Benrhouma; Existence and uniqueness of solutions for a singular semilinear elliptic system, *Nonlinear Anal.*, **107** (2014), 134-146.
[7] G. Bonannao, R. Livrea; Multiplicity theorems for the Dirichlet problem involving the p-Laplacian, *Nonlinear Anal.* **54** (2003), 1-7.
[8] A. Cañada, P. Drábek, J. Gámez; Existence of positive solutions for some problems with nonlinear diffusion, *Trans. Am. Math. Soc.*, **349** (1997), 4231-4249.
[9] P. Cerda, M. Souto, P. Ubilla; Existence of solution of some (p,q)-Laplacian system under local superlinear conditions, *Math. Nachr.* **295** (2022), 44-57.
[10] Y. Chen, Y. Zhu; Large solutions for a cooperative elliptic system of p-Laplacian equations, *Nonlinear Anal.*, **73** (2010), 450-457.
[11] C.-M. Chu, C.-L. Tang; Existence and multiplicity of positive solutions for semilinear elliptic systems with Sobolev critical exponents, *Nonlinear Anal.*, **71** (2009), 5118-5130.
[12] R. Cui, P. Li, J. Shi, Y. Wang; Existence, uniqueness and stability of positive solutions for a class of semilinear elliptic systems, *Topol. Methods Nonlinear Anal.*, **42** (2013), 91-104.
[13] R. Dalmasso; Existence and uniqueness of positive solutions of semilinear elliptic systems, *Nonlinear Anal.*, **39** (2000), 559-568.
[14] M. Feng, X. Zhang, W. Ge; Exact number of solutions for a class of two-point boundary value problems with one-dimensional p-Laplacian, *J. Math. Anal. Appl.*, **338** (2008), 784-792.
[15] M. Feng, Y. Zhang; Positive solutions of singular multiparameter p-Laplacian elliptic systems, *Discrete Contin. Dyn. Syst. Ser. B*, **27** (2022), 1121-1147.
[16] B. Franchi, E. Lanconelli; Existence and uniqueness of nonnegative solutions of quasilinear equations in R^n, *Adv. Math.*, **118** (1996), 177-243.
[17] J. García-Melián; Large solutions for an elliptic system of quasilinear equations, *J. Differential Equations* **245** (2008), 3735-3752.
[18] M. Ghergu; Lane-Emden systems with negative exponents, *J. Funct. Anal.*, **258** (2010), 3295-3318.
[19] R. Glowinski, J. Rappaz; Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology, *Math. Model. Numer. Anal.*, **37** (2003), 175šC186.
[20] D. Guo; Nonlinear Functional Analysis, Shandong Science and Technology Press, Jinan, **1985** (in Chinese).
[21] D. Guo, V. Lakshmikantham; Nonlinear problems in abstract cones, Academic Press, New York, 1988.
[22] Z. Guo, J. Webb; Structure of boundary blow-up solutions for quasi-linear elliptic problems II: small and intermediate solutions, *J. Differential Equations*, **211** (2005), 187-217.
[23] D. D. Hai; Uniqueness of positive solutions for a class of semilinear elliptic systems, *Nonlinear Anal.*, **52** (2003), 595-603.
[24] D. D. Hai; Existence and uniqueness of solutions for quasilinear elliptic systems, *Proc. Amer. Math. Soc.*, **133** (2005), 223-228.

[25] D. D. Hai, R. Shivaji; Existence and uniqueness for a class of quasilinear elliptic boundary value problems, *J. Differential Equations*, **193** (2003), 500-510.
[26] D. D. Hai, R. Shivaji; An existence result on positive solutions for a class of p-Laplacian systems, *Nonlinear Anal.*, **56** (2004), 1007-1010.
[27] D. D. Hai, R. Shivaji; Existence and multiplicity of positive radial solutions for singular superlinear elliptic systems in the exterior of a ball, *J. Differential Equations*, **266** (2019), 2232-2243.
[28] D. D. Hai, H. Wang; Nontrivial solutions for p-Laplacian systems, *J. Math. Anal. Appl.*, **330** (2007), 186-194.
[29] P. Han; Multiple positive solutions of nonhomogeneous elliptic systems involving critical Sobolev exponents, *Nonlinear Anal.*, **64** (2006), 869-886
[30] X. He; Multiple radial solutions for a class of quasilinear elliptic problems, *Appl. Math. Lett.*, **23** (2010), 110-114.
[31] K. Lan, Z. Zhang; Nonzero positive weak solutions of systems of p-Laplace equations, *J. Math. Anal. Appl.*, **394** (2012), 581-591,
[32] S. Kichenassamy, L. Verlon; Singular solutions of the p-Laplacian equation, *Math. Ann.*, **275** (1986), 599-615.
[33] S. Oruganti, J. Shi, R. Shivaji; Logistic equation with the p-Laplacian and constant yeild harvesting, *Abstract. Appl. Anal.*, **9** (2004), 723-727.
[34] K. Perera, R. Shivaji, I. Sim; A class of semipositone p-Laplacian problems with a critical growth reaction term, *Adv. Nonlinear Anal.*, **9** (2020), 516-525.
[35] J. Sánchez; Multiple positive solutions of singular eigenvalue type problems involving the one-dimensional p-Laplacian, *J. Math. Anal. Appl.*, **292** (2004), 401-414.
[36] B. Son, P. Wang; Analysis of positive radial solutions for singular superlinear p-Laplacian systems on the exterior of a ball, *Nonlinear Anal.*, **192** (2020), 111657.
[37] P. Tolksdorf; Regularity for a more general class of quasilinear elliptic equations, *J. Differential Equations*, **51** (1984), 126-150.
[38] M. Xiang, B. Zhang, V. D. Radulescu; Existence of solutions for perturbed fractional p-Laplacian equations, *J. Differential Equations*, **260** (2016), 1392-1413.
[39] X. Zhang, M. Feng; Existence of a positive solution for one-dimensional singular p-Laplacian problems and its parameter dependence, *J. Math. Anal. Appl.*, **413** (2014), 566-582.
[40] X. Zhang, S. Kan; Sufficient and necessary conditions on the existence and estimates of boundary blow-up solutions for singular p-Laplacian equations, *Acta. Math. Sci. B*, **43** (2023), 1175-1194.
[41] Z. Zhang, S. Li; On sign-changing andmultiple solutions of the p-Laplacian, *J. Funct. Anal.*, **197** (2003), 447-468.
[42] Y. Zhang, M. Feng; A coupled p-Laplacian elliptic system: Existence, uniqueness and asymptotic behavior, *Electron. Res. Arch.*, **28** (2020), 1419-1438.

Xiankui Meng
College of Applied Science, Beijing Information Science and Technology University, Beijing 102206, China
Email address: `chihiroshinja@foxmail.com`

Meiqiang Feng
College of Applied Science, Beijing Information Science and Technology University, Beijing 102206, China
Email address: `meiqiangfeng@sina.com`