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EXISTENCE OF NONTRIVIAL SOLUTIONS FOR ANISOTROPIC P-LAPLACIAN SYSTEMS VIA MORSE THEORY

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Abstract. We study the existence of nontrivial weak solutions for anisotropic p-Laplacian systems. A sequence of eigenvalues is characterized through the $\mathbb{Z}_2$-cohomological index on a suitable constraint manifold of Finsler type. By combining cohomological local splitting with Morse theoretic tools, we prove the existence of nontrivial solutions to the anisotropic p-Laplacian system under local assumptions on the nonlinear terms, both in the nonresonant and resonant cases.

1. Introduction

In this article, we study the existence of nontrivial weak solutions for anisotropic p-Laplacian systems within a Morse-theoretic framework. Our aim is to extend the variational and topological methods developed for scalar p-Laplacian equations to vector-valued systems with componentwise nonlinear diffusion, where different components may exhibit distinct growth behaviors.

The classical p-Laplacian operator is defined by

$$-\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad p > 1. \tag{1.1}$$

It plays a fundamental role in nonlinear analysis, mathematical physics, continuum mechanics, and nonlinear diffusion phenomena. The p-Laplacian equation has been extensively studied in the past decades, leading to a rich theory concerning the existence, regularity, multiplicity, and qualitative properties of solutions [5, 10, 20]. When $p = 2$, problem (1.1) reduces to a semilinear equation driven by the Laplacian. In this case, the linear operator $-\Delta$ possesses a complete orthogonal system of eigenfunctions. Consequently, the underlying Hilbert space admits an orthogonal decomposition into eigenspaces, which allows classical minimax methods, such as the Mountain Pass Theorem, to be applied effectively; see [1, 3, 4, 21, 22, 23]. For $p > 1$, the existence of a nontrivial solution follows directly from the Mountain Pass Theorem if λ, the parameter in the equation we studied, is below the first variational eigenvalue associated with the p-Laplacian; see [2, 11]. These classical variational approaches rely on the global geometric structure of the associated energy functional, typically requiring superlinear growth or the mountain-pass type geometry. For the case $\lambda \geq \lambda_1$, the trivial solution ceases to be a strict local minimizer, and the standard mountain-pass geometry may fail. As a result, standard minimax and linking arguments cannot be applied directly. To deal with this difficulty, several nonlinear linking techniques based on cone constructions have been developed; see [8, 9]. However, these methods still rely on strong global assumptions on the energy functional.

By contrast, Morse-theoretic methods based on critical groups and local splitting near critical points allow one to relax these global requirements and to derive existence results under substantially weaker local assumptions. For a general account of Morse theory and critical point theory in infinite-dimensional settings, we refer to [6, 13, 15]. These methods have been successfully applied to p-Laplacian problems, especially in the cases where standard variational techniques become ineffective. Perera, Agarwal, and O'Regan developed a comprehensive Morse-theoretic framework

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[17], where nonlinearities are classified as p-sublinear, asymptotically p-linear, or p-superlinear, and existence and multiplicity results are obtained mainly in the non-resonant case $\lambda \neq \lambda_k$. In [7], the existence of nontrivial solutions for p-superlinear p-Laplacian problems was established for all values of λ by introducing the notion of cohomological local splitting and combining it with Morse-theoretic critical group techniques under local sign conditions.

A natural extension of the scalar p-Laplacian operator is the study of p-Laplacian systems. Such systems arise naturally in models describing multi-component processes, including reaction–diffusion systems, nonlinear elasticity, and phase separation phenomena. Compared with the scalar case, p-Laplacian systems present additional analytical difficulties due to the interaction between components, the possible lack of homogeneity when $p_1, \dots, p_m$ are distinct, and the more intricate variational structure of the associated energy functional. Specifically, anisotropic systems, where each component is governed by a different exponent p_i, break the standard scaling invariance and require refined analytical tools. Perera et al. [16] considered an anisotropic p-Laplacian system with nonlinearities of the form

$$F(x,u) = \lambda V(x)|u_1|^{r_1} \cdots |u_m|^{r_m} + G(x,u),$$

where the remainder term $G(x,u)$ satisfies suitable higher-order growth conditions.

Motivated by the above contributions, the present work investigates the existence of nontrivial solutions for anisotropic p-Laplacian systems

$$\begin{gathered} -\Delta_{p_i} u_i = \frac{\partial F}{\partial u_i}(x,u) \quad \text{in } \Omega, \quad i = 1, \dots, m, \\ u_1 = \cdots = u_m = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where the expression is $F(x,u) = \lambda V(x)|u_1|^{r_1} \dots |u_m|^{r_m} + G(x,u)$. Moreover, we have $u = (u_1, \dots, u_m)$, $p = (p_1, \dots, p_m)$ with each $p_i \in (1,\infty)$, and $F \in C^1(\Omega \times \mathbb{R}^m)$ satisfies $F(x,0) \equiv 0$. We impose the following assumption with respect to the perturbation term $G(x,u)$:

(A1) Assume that $G \in C^1(\Omega \times \mathbb{R}^m)$, $G(x,0) \equiv 0$, and there exists a constant $C > 0$ such that

$$\Big|\frac{\partial G}{\partial u_i}(x,u)\Big| \le C\Big(1 + \sum_{j=1}^{m} |u_j|^{r_{ij}-1}\Big) \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}^m, \tag{1.3}$$

where $r_{ij} \in (1, p_i^*)$ for $i,j = 1, \dots, m$.

(A2) Assume that the Ambrosetti-Rabinowitz *(AR)* condition holds for G in the anisotropic system setting, namely,

$$0 < G(x,u) \le \sum_{i=1}^{m} \frac{u_i}{\mu_i} \frac{\partial G}{\partial u_i}(x,u) \quad \text{for all } x \in \Omega \text{ and } |u| \ge T, \tag{1.4}$$

where $\mu_i > p_i$ for $i = 1, \dots, m$, and $T > 0$ is a constant.

(A3) For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\partial_{u_i} G(x,u)| \le \varepsilon |u_i|^{p_i - 1} \tag{1.5}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}^m$ with $|u| \le \delta$, and all $i = 1, \dots, m$.

Under assumptions (A1)–(A3), we establish existence results for problem 1.2, depending on the position of the parameter λ with respect to the variational eigenvalues of the associated anisotropic operator. Our main results are summarized below.

Theorem 1.1. *Assume that* (A1)–(A3) *hold. If $\lambda < \lambda_1$, then problem 1.2 has a nontrivial solution.*

When $\lambda < \lambda_1$, the origin is a strict local minimizer of the associated functional Φ. This local variational structure plays a key role in the existence result. When $\lambda > \lambda_1$, however, this property is lost and the variational structure of the problem changes significantly.

Theorem 1.2. *Let $\{\lambda_k\}_{k \ge 1}$ be the sequence of variational eigenvalues associated with the corresponding eigenvalue problem. If*

$$\lambda_k < \lambda < \lambda_{k+1}$$

for some $k \ge 1$, then (1.2) *admits a nontrivial weak solution under the conditions* (A1)–(A3).

The situation becomes more delicate when the parameter λ coincides with a variational eigenvalue of the associated anisotropic operator. In this resonant case, the principal part of the functional becomes degenerate along the corresponding eigenspace. Consequently, standard critical point arguments cannot be applied directly. Motivated by the approach developed in [14], we employ Morse theoretic techniques combined with suitable local sign conditions on the nonlinear term. These conditions allow us to analyze the behavior of the functional near the origin and establish the existence of nontrivial critical points.

Theorem 1.3. *Assume that* (A1)–(A3) *hold. Let $\lambda = \lambda_k$ be a variational eigenvalue of the anisotropic operator. Suppose that one of the following local sign conditions hold:*

(1) *There exists $\delta > 0$ such that*
$$G(x,u) \geq 0 \quad \text{for a.e. } x \in \Omega \text{ and all } u \in \mathbb{R}^m \text{ with } |u| \leq \delta.$$
(2) *There exists $\delta > 0$ such that*
$$G(x,u) \leq 0 \quad \text{for a.e. } x \in \Omega \text{ and all } u \in \mathbb{R}^m \text{ with } |u| \leq \delta.$$

Then problem (1.2) *admits at least one nontrivial weak solution.*

This paper is organized as follows. In Section 2, we introduce the functional setting and recall several preliminary results from variational analysis and Morse theory that will be used throughout the paper. In Section 3, we study the associated eigenvalue problem and construct a sequence of variational eigenvalues via the $\mathbb{Z}_2$–cohomological index. Section 4 is devoted to the proof of the main existence results, where cohomological local splitting and critical group arguments are employed.

2. Preliminaries

For the convenience of the reader, this section recalls the basic notations and functional-analytic tools that will be used throughout the paper. Throughout this work, we follow the abstract variational and Morse-theoretic framework developed in [17]. Let $(X, \|\cdot\|)$ be a Banach space and let $\Phi \in C^1(X,\mathbb{R})$. For $c \in \mathbb{R}$, $u_0 \in X$, and $\rho > 0$, we define
$$\Phi^c := \{u \in X : \Phi(u) \leq c\}, \quad \bar{B}_\rho(u_0) := \{u \in X : \|u - u_0\| \leq \rho\}.$$
Moreover, $\Omega \subset \mathbb{R}^N$ denotes a bounded domain with smooth boundary, $m \in \mathbb{N}$ is fixed, and $p = (p_1, \dots, p_m)$ with $p_i \in (1,\infty)$ for $i = 1, \dots, m$. For each component, we consider the Sobolev space
$$X_i := W_0^{1,p_i}(\Omega),$$
endowed with the norm
$$\|u_i\|_i := \Big(\int_\Omega |\nabla u_i|^{p_i}\,dx\Big)^{1/p_i}.$$
We define the product space
$$X := X_1 \times \dots \times X_m, \quad u = (u_1, \dots, u_m) \in X,$$
endowed with the norm
$$\|u\| := \Big(\sum_{i=1}^m \|u_i\|_i^2\Big)^{1/2}.$$
Since each space X_i is uniformly convex and reflexive, X is a uniformly convex and reflexive Banach space. For each i, the Sobolev embedding $W_0^{1,p_i}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $1 \leq q \leq p_i^*$ and compact for all $1 \leq q < p_i^*$, where
$$p_i^* = \begin{cases} \frac{Np_i}{N-p_i}, & \text{if } p_i < N, \\ \infty, & \text{if } p_i \geq N. \end{cases}$$
For notational convenience, we write
$$\frac{u}{p} := \Big(\frac{u_1}{p_1}, \dots, \frac{u_m}{p_m}\Big).$$

We denote the energy functional as I_p, and A_p stands for the corresponding operator. We have

$$\langle I'_p(u), u\rangle = \langle A_p(u), u\rangle = \sum_{i=1}^m \int_\Omega |\nabla u_i|^{p_i}\,dx,$$

and consequently

$$\Big\langle I'_p(u), \frac{u}{p}\Big\rangle = \sum_{i=1}^m \frac{1}{p_i}\int_\Omega |\nabla u_i|^{p_i}\,dx = I_p(u). \tag{2.1}$$

According to [17], each operator A_{p_i} is odd and (p_i-1)-homogeneous, coercive, strictly monotone, and of potential type. Moreover, for all $u_i \in X_i$,

$$\langle A_{p_i}(u_i), u_i\rangle_i = \int_\Omega |\nabla u_i|^{p_i}\,dx = \|u_i\|_i^{p_i}, \quad \|A_{p_i}(u_i)\|_{X_i^*} \le \|u_i\|_i^{p_i-1}.$$

Next, we show that the anisotropic operator A_p is of type (S). Recall that an operator $A : X \to X^*$ is said to be of type (S) if every sequence $(u^n) \subset X$ is such that $u^n \rightharpoonup u$ in X and $\langle A(u^n), u^n - u\rangle \to 0$, then the sequence admits a subsequence strongly converging to u in X.

Lemma 2.1. *The operator A_p is of type (S).*

Proof. Using Hölder's inequality, for any $u_i, v_i \in X_i$ we obtain

$$\begin{aligned}\langle A_{p_i}(u_i), v_i\rangle_i &= \int_\Omega |\nabla u_i|^{p_i-2}\nabla u_i\cdot\nabla v_i\,dx\\ &\le \Big(\int_\Omega |\nabla u_i|^{p_i}\,dx\Big)^{\frac{p_i-1}{p_i}}\Big(\int_\Omega |\nabla v_i|^{p_i}\,dx\Big)^{\frac{1}{p_i}}\\ &= \|u_i\|_i^{p_i-1}\,\|v_i\|_i.\end{aligned}$$

Thus,

$$\langle A_{p_i}(u_i), u_i\rangle_i = \|u_i\|_i^{p_i}.$$

Now suppose that $u^j = (u_1^j, \dots, u_m^j) \rightharpoonup u = (u_1, \dots, u_m)$ in X and

$$\langle A_p(u^j), u^j - u\rangle \to 0.$$

Hence we obtain

$$\begin{aligned}0 &\le \sum_{i=1}^m \big(\|u_i^j\|_i^{p_i-1} - \|u_i\|_i^{p_i-1}\big)\big(\|u_i^j\|_i - \|u_i\|_i\big)\\ &\le \sum_{i=1}^m \Big(\langle A_{p_i}(u_i^j), u_i^j - u_i\rangle_i + \langle A_{p_i}(u_i), u_i^j - u_i\rangle_i\Big)\\ &= \langle A_p(u^j), u^j - u\rangle - \langle A_p(u), u^j - u\rangle \;\to\; 0.\end{aligned}$$

Consequently, for each $i = 1, \dots, m$,

$$\|u_i^j\|_i \to \|u_i\|_i.$$

Since each space X_i is uniformly convex, it follows that $u_i^j \to u_i$ strongly in X_i for every i. Therefore, $u^j \to u$ strongly in X, and the proof is complete. $\square$

Next, we consider the potential functional associated with A_p, which provides the variational structure for the anisotropic system. Invoking [17, Proposition 1.2], he potential I_p associated with A_p and satisfying $I_p(0) = 0$ is given by

$$I_p(u) = \sum_{i=1}^m \frac{1}{p_i}\langle A_{p_i}(u_i), u_i\rangle_i = \sum_{i=1}^m \frac{1}{p_i}\int_\Omega |\nabla u_i|^{p_i}\,dx. \tag{2.2}$$

Unlike the scalar case, the functional I_p is not homogeneous unless $p_1 = \dots = p_m$. Nevertheless, it has a weaker form of scaling invariance induced by a suitable continuous flow on the product space X. For $\alpha \in \mathbb{R}$, we define the scaling transformations

$$u_\alpha = \rho_{1/p}(\alpha)u = \big(|\alpha|^{\frac{1}{p_1}-1}\alpha u_1, \dots, |\alpha|^{\frac{1}{p_m}-1}\alpha u_m\big). \tag{2.3}$$

Lemma 2.2. *For the mapping $\mathbb{R}\times X\to X$ defined by $(\alpha,u)\mapsto u_\alpha$ as in (2.3), the functional I_p satisfies*

$$I_p(u_\alpha)=|\alpha|\,I_p(u)\quad\forall\alpha\in\mathbb{R},\ u\in X. \tag{2.4}$$

Proof. By definition of the scaling (2.3), for each $i=1,\dots,m$ we have

$$u_{\alpha,i}=|\alpha|^{\frac{1}{p_i}-1}\alpha u_i,\quad \nabla u_{\alpha,i}=|\alpha|^{\frac{1}{p_i}-1}\alpha\nabla u_i.$$

Consequently,

$$|\nabla u_{\alpha,i}|^{p_i}=|\alpha|^{p_i\left(\frac{1}{p_i}-1\right)}|\alpha|^{p_i}|\nabla u_i|^{p_i}=|\alpha|\,|\nabla u_i|^{p_i}.$$

Substituting this identity into the definition of I_p, we obtain

$$\begin{aligned} I_p(u_\alpha)&=\sum_{i=1}^m\frac{1}{p_i}\int_\Omega|\nabla u_{\alpha,i}|^{p_i}\,dx\\ &=\sum_{i=1}^m\frac{1}{p_i}\int_\Omega|\alpha|\,|\nabla u_i|^{p_i}\,dx\\ &=|\alpha|\sum_{i=1}^m\frac{1}{p_i}\int_\Omega|\nabla u_i|^{p_i}\,dx=|\alpha|\,I_p(u), \end{aligned}$$

which completes the proof. □

Next, we prove a lemma that helps us to establish the (PS) condition for Φ in the system version.

Lemma 2.3. *Assume that* (A1) *holds. Let $\{u_n\}\subset X$ be a bounded sequence such that*

$$\Phi'(u_n)\to 0\quad\textit{in } X^*.$$

Then $\{u_n\}$ possesses a strongly convergent subsequence in X.

Proof. Since $\{u_n\}$ is bounded in X, passing to a subsequence if necessary, we may assume that

$$u_n\rightharpoonup u\quad\text{in } X,$$

and, by the compact Sobolev embeddings,

$$u_n\to u\quad\text{in } L^{r_{ij}}(\Omega)$$

for all exponents r_{ij} appearing in (A1). Let $A:X\to X^*$ denote the operator corresponding to the principal part of Φ. Then

$$\langle A(u_n),u_n-u\rangle=\langle\Phi'(u_n),u_n-u\rangle+\lambda\langle J'(u_n),u_n-u\rangle+\sum_{i=1}^m\int_\Omega\frac{\partial G}{\partial u_i}(x,u_n)(u_{n,i}-u_i)\,dx.$$

Since $\Phi'(u_n)\to 0$ in X^* and $\{u_n-u\}$ is bounded in X, we have

$$\langle\Phi'(u_n),u_n-u\rangle\to 0.$$

Next, by the compactness of the embedding corresponding to the exponents involved in J,

$$\langle J'(u_n),u_n-u\rangle\to 0.$$

Moreover, by (A1), the boundedness of $\{u_n\}$ in X, and the compact Sobolev embeddings, we have

$$\int_\Omega\frac{\partial G}{\partial u_i}(x,u_n)(u_{n,i}-u_i)\,dx\to 0\quad\text{for each } i=1,\dots,m.$$

Therefore, $\langle A(u_n),u_n-u\rangle\to 0$. Since A is of type (S), we conclude that

$$u_n\to u\quad\text{strongly in } X.$$

This completes the proof. □

To conclude this section, we recall the notion of cohomological local splitting.

Definition 2.4. *Let $\Phi \in C^1(X,\mathbb{R})$ with $\Phi(0)=0$. We say that Φ admits a cohomological local splitting at 0 in dimension k if there exist symmetric cones W_- and W_+ in X such that*

$$W_- \cap W_+ = \{0\}, \quad i(W_- \setminus \{0\}) = k,$$

and there exists $\rho > 0$ satisfying

$$\Phi(u) \le 0 \quad \text{for all } u \in W_- \cap \bar{B}_\rho, \quad \Phi(u) \ge 0 \quad \text{for all } u \in W_+ \cap \bar{B}_\rho.$$

3. Eigenvalue problem

We consider the eigenvalue problem for the system of p-Laplacians:

$$\begin{gathered} -\Delta_{p_i} u_i = \lambda V(x)|u_1|^{r_1}\cdots|u_i|^{r_i-2}\cdots|u_m|^{r_m}u_i \quad \text{in } \Omega, i=1,\dots,m \\ u_1=\dots=u_m=0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

where $\sum_{i=1}^m \frac{r_i}{p_i}=1$. We define the C^1-Finsler manifold as

$$\mathcal{M} = \{u \in X : I_p(u)=1\}. \tag{3.2}$$

Next we verify that the nonlinear term in the eigenvalue problem has the weak homogeneity property defined above.

Lemma 3.1. *Let $J(u)=\int_\Omega V(x)|u_1|^{r_1}\cdots|u_m|^{r_m}dx$ where $V\in L^\infty(\Omega)$ is a bounded function on Ω. If $\sum_{i=1}^m \frac{r_i}{p_i}=1$, then $J(u)$ satisfies the homogeneity property:*

$$J(u_\alpha) = |\alpha| J(u) \quad \forall u \in X,\ \alpha \in R$$

Proof. Substituting u_α into J, we obtain

$$J(u_\alpha) = \int_\Omega V(x) \prod_{i=1}^m \big||\alpha|^{\frac{1}{p_i}-1}\alpha u_i\big|^{r_i}\,dx.$$

For each i, we compute

$$\big||\alpha|^{\frac{1}{p_i}-1}\alpha u_i\big|^{r_i} = |\alpha|^{r_i\left(\frac{1}{p_i}-1\right)}|\alpha|^{r_i}|u_i|^{r_i} = |\alpha|^{\frac{r_i}{p_i}}|u_i|^{r_i}.$$

Therefore,

$$J(u_\alpha) = \int_\Omega V(x)\prod_{i=1}^m |\alpha|^{\frac{r_i}{p_i}}|u_i|^{r_i}\,dx = |\alpha|^{\sum_{i=1}^m \frac{r_i}{p_i}} \int_\Omega V(x)\prod_{i=1}^m |u_i|^{r_i}\,dx.$$

Using the condition $\sum_{i=1}^m \frac{r_i}{p_i}=1$, we obtain $J(u_\alpha)=|\alpha|J(u)$. □

Let

$$M^+ := \{u \in M : J(u) > 0\}.$$

Define the functional $\Psi^+ : M^+ \to \mathbb{R}$ by

$$\Psi^+(u) := \frac{1}{J(u)}, \quad u \in M^+.$$

Lemma 3.2. *The positive eigenvalues of system 3.2 coincide with the critical values of $\widetilde{\Psi}^+$.*

Proof. By [17, Proposition 3.54], we have that $(\widetilde{\Psi}^+)'(u)=0$ if and only if there exists $\mu\in\mathbb{R}$ such that

$$\mu A_p(u) + \widetilde{\Psi}^+(u)^2 J'(u) = 0. \tag{3.3}$$

Testing (3.3) with u/p and using (2.1) together with

$$\big\langle J'(u), \frac{u}{p}\big\rangle = \sum_{i=1}^m \frac{r_i}{p_i}\int_\Omega V(x)\prod_{j=1}^m |u_j|^{r_j}\,dx = J(u),$$

we obtain

$$\mu = -\widetilde{\Psi}^+(u)^2 \frac{\langle J'(u), u/p\rangle}{\langle A_p(u), u/p\rangle} = -\widetilde{\Psi}^+(u)^2\frac{J(u)}{I_p(u)} = -\widetilde{\Psi}^+(u) < 0.$$

Hence (3.3) reduces to the eigenvalue equation with $\lambda = \widetilde{\Psi}^+(u)$. Conversely, if $\lambda > 0$ is an eigenvalue of (3.2) and $u \in \mathcal{M}^+$ is an associated eigenfunction, then

$$\lambda = \frac{\langle A_p(u), u/p\rangle}{\langle J'(u), u/p\rangle} = \frac{I_p(u)}{J(u)} = \widetilde{\Psi}^+(u),$$

which implies that u is a critical point of $\widetilde{\Psi}^+$. $\square$

Next, we employ the $\mathbb{Z}_2$-cohomological index introduced by Fadell and Rabinowitz [12] to compute nontrivial critical groups and to construct linking sets associated with the variational eigenvalues. It is well known from variational spectral theory that the positive spectrum $\sigma^+(A_p, J')$, consisting of all positive eigenvalues, is a closed subset of $\mathbb{R}$. We can use a cohomological index to define a non-decreasing and unbounded sequence of positive eigenvalues of the eigenvalue problem, see [19]. We begin by recalling the definition of the $\mathbb{Z}_2$-cohomological index. Let M be a class of closed symmetric subsets of $\mathcal{M}$. We define:

- $\bar{M} = M/\mathbb{Z}_2$ as the quotient space where each pair of points $u \in B$ and $-u$ is identified,
- $f : \bar{M} \to \mathbb{R}P^\infty$ as the classifying map of $\bar{M}$,
- $f^* : H^*(\mathbb{R}P^\infty) \to H^*(\bar{M})$ as the induced homomorphism on the cohomology rings.

The cohomological index of M is defined as

$$i(M) = \begin{cases} \sup\{p \ge 1 : f^*(\omega^{p-1}) \ne 0\}, & \text{if } M \ne \emptyset, \\ 0, & \text{if } M = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{R}P^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega]$. Let $i(\mathcal{M}^+)$ be the cohomological index of $\mathcal{M}^+$. Define

$$\mathcal{F}_k = \{M \in \mathcal{F} : i(M) \ge k\}$$

and set

$$\lambda_k := \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \widetilde{\Psi}(u).$$

Using the variational framework developed above and the properties of the $\mathbb{Z}_2$-cohomological index, one obtains a well-ordered sequence of variational eigenvalues associated with the functional $\widetilde{\Psi}^+$. The following result summarizes the main spectral properties of this sequence.

Proposition 3.3 ([17, Theorem 4.6]). *Let $\{\lambda_k\}_{k\ge1}$ be the sequence of variational eigenvalues generated by the functional $\widetilde{\Psi}^+$ on the manifold $\mathcal{M}^+$. Then the following properties hold:*

(i) *The sequence $\{\lambda_k\}$ is positive and nondecreasing, namely*
$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots .$$
(ii) *The eigenvalues are unbounded from above, that is,*
$$\lambda_k \to +\infty \quad \text{as } k \to \infty.$$
(iii) *When $\lambda_k < \lambda_{k+1}$, the $\mathbb{Z}_2$–cohomological index satisfies*
$$i\big(\{u \in \mathcal{M}^+ : \widetilde{\Psi}^+(u) \le \lambda_k\}\big) = k.$$

4. Main results

In this section, we study the problem in the form of

$$\begin{gathered} -\Delta_{p_i} u_i = \lambda V(x)|u_1|^{r_1}\cdots|u_i|^{r_i-2}\cdots|u_m|^{r_m}u_i + \frac{\partial G}{\partial u_i}(x,u) \quad \text{in } \Omega, \\ u_1 = u_2 = \cdots = u_m = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

where $\lambda \in \mathbb{R}$ is a real parameter. Here, we define nonlinear term as

$$F(x,u) = \lambda V(x)|u_1|^{r_1}\cdots|u_m|^{r_m} + G(x,u). \tag{4.2}$$

In addition to the assumptions on the nonlinear term (A1)–(A3). We first prove the theorem 1.1.

Lemma 4.1. *Assume that* (A3) *holds and $\lambda < \lambda_1$. Then $u = 0$ is a strict local minimizer of Φ.*

Proof. Recall that

$$\Phi(u)=I_p(u)-\lambda J(u)-\int_\Omega G(x,u)\,dx,\quad I_p(u)=\sum_{i=1}^m\frac{1}{p_i}\int_\Omega|\nabla u_i|^{p_i}\,dx.$$

By *(A3)*, for every $\varepsilon>0$ there exists $\delta>0$ such that

$$|\partial_{u_i}G(x,z)|\le\varepsilon|z_i|^{p_i-1}$$

for a.e. $x\in\Omega$ and all $z\in\mathbb{R}^m$ satisfying $|z|\le\delta$, $i=1,\dots,m$. Since $G(x,0)=0$, by the fundamental theorem of calculus we have

$$G(x,u)=\int_0^1\sum_{i=1}^m\partial_{u_i}G(x,tu)\,u_i\,dt.$$

Hence

$$|G(x,u)|\le\sum_{i=1}^m\int_0^1|\partial_{u_i}G(x,tu)|\,|u_i|\,dt\le\varepsilon\sum_{i=1}^m\int_0^1|tu_i|^{p_i-1}|u_i|\,dt\le C\varepsilon\sum_{i=1}^m|u_i|^{p_i}.$$

Integrating over Ω and using the Poincaré inequality, we obtain

$$\int_\Omega|G(x,u)|\,dx\le C\varepsilon\sum_{i=1}^m\int_\Omega|u_i|^{p_i}\,dx\le C\varepsilon\sum_{i=1}^m\int_\Omega|\nabla u_i|^{p_i}\,dx\le C\varepsilon I_p(u).$$

Since $\varepsilon>0$ is arbitrary, it follows that

$$\int_\Omega G(x,u)\,dx=o(I_p(u))\quad\text{as }\|u\|\to0.$$

Therefore,

$$\Phi(u)=I_p(u)-\lambda J(u)+o(I_p(u))\quad\text{as }\|u\|\to0.$$

By the variational characterization of the first eigenvalue λ_1, we have

$$I_p(u)\ge\lambda_1J(u)\quad\text{for all }u\in X.$$

Hence

$$I_p(u)-\lambda J(u)\ge\Big(1-\frac{\lambda}{\lambda_1}\Big)I_p(u).$$

Since $\lambda<\lambda_1$, it follows that

$$\Phi(u)\ge\Big(1-\frac{\lambda}{\lambda_1}\Big)I_p(u)+o(I_p(u)).$$

Thus there exists $\rho>0$ such that

$$\Phi(u)\ge\frac12\Big(1-\frac{\lambda}{\lambda_1}\Big)I_p(u)>0\quad\text{for all }0<\|u\|\le\rho.$$

Therefore, $u=0$ is a strict local minimizer of Φ. □

Proof of Theorem 1.1. First, by Lemma 4.1, $u=0$ is a strict local minimizer of Φ. Hence, by [17, Proposition 3.18],

$$C^0(\Phi,0)\approx\mathbb{Z}_2.$$

Next, we show that Φ is unbounded from below. Fix $u\in X\setminus\{0\}$ and define

$$u_\alpha=(\alpha^{1/p_1}u_1,\dots,\alpha^{1/p_m}u_m),\quad\alpha>0.$$

Set

$$\psi_x(\alpha)=G(x,u_\alpha)\quad\text{for a.e. }x\in\Omega.$$

By the chain rule,

$$\frac{d}{d\alpha}G(x,u_\alpha)=\sum_{i=1}^m\partial_{u_i}G(x,u_\alpha)\frac{d}{d\alpha}(\alpha^{1/p_i}u_i)=\sum_{i=1}^m\frac{1}{p_i}\alpha^{1/p_i-1}u_i\,\partial_{u_i}G(x,u_\alpha).$$

Hence

$$\alpha\psi_x'(\alpha) = \sum_{i=1}^{m} \frac{1}{p_i} u_{\alpha,i}\partial_{u_i} G(x,u_\alpha),$$

where $u_{\alpha,i} = \alpha^{1/p_i} u_i$. Let

$$p_* = \max_{1\le i\le m} p_i, \quad \mu_* = \min_{1\le i\le m} \mu_i.$$

Then

$$\alpha\psi_x'(\alpha) \ge \frac{1}{p_*} \sum_{i=1}^{m} u_{\alpha,i}\partial_{u_i} G(x,u_\alpha).$$

By *(G2)*,

$$\sum_{i=1}^{m} u_{\alpha,i}\partial_{u_i} G(x,u_\alpha) \ge \mu_*\, G(x,u_\alpha) = \mu_*\, \psi_x(\alpha)$$

whenever $|u_\alpha(x)| \ge T$. Therefore, for a.e. $x \in \Omega$ such that $u(x) \neq 0$, there exists $\alpha_x > 0$ such that

$$\frac{\psi_x'(\alpha)}{\psi_x(\alpha)} \ge \frac{\mu_*}{p_*}\frac{1}{\alpha} \quad \text{for all } \alpha \ge \alpha_x.$$

Integrating over $[1,\alpha]$, we obtain

$$G(x,u_\alpha) = \psi_x(\alpha) \ge G(x,u)\alpha^{\mu_*/p_*} \quad \text{for a.e. } x \in \Omega \text{ with } u(x) \neq 0.$$

Then, by the above pointwise estimate,

$$\int_\Omega G(x,u_\alpha)\, dx \ge \alpha^{\mu_*/p_*} \int_\Omega G(x,u)\, dx = C_u\, \alpha^{\mu_*/p_*}$$

for sufficiently large α. On the other hand, by the homogeneity of I_p and J, we have

$$I_p(u_\alpha) = \alpha I_p(u), \quad J(u_\alpha) = \alpha J(u).$$

Therefore,

$$\Phi(u_\alpha) = \alpha I_p(u) - \lambda\alpha J(u) - \int_\Omega G(x,u_\alpha)\, dx \le \alpha\big(I_p(u) - \lambda J(u)\big) - C_u\alpha^{\mu_*/p_*}.$$

Since $\mu_*/p_* > 1$, it follows that

$$\Phi(u_\alpha) \to -\infty \quad \text{as } \alpha \to +\infty.$$

Thus Φ is unbounded from below. Therefore, by [17, Proposition 3.15(ii)],

$$C^0(\Phi,\infty) = 0.$$

Moreover, Φ satisfies the (C)-condition. Using [17, Corollary 3.31], we obtain a critical point $u \neq 0$ such that

$$C^1(\Phi,u) \neq 0.$$

Hence u is a nontrivial critical point of Φ, and therefore a nontrivial weak solution of (4.1). □

Having proved Theorem 1.1, we now turn to the proofs of Theorems 1.2 and 1.3. To apply Morse theoretic arguments, we first verify that the functional Φ admits a cohomological local splitting at the origin.

Lemma 4.2. *Assume that* (A1)–(A3) *hold. Suppose that*

$$\lambda_k < \lambda < \lambda_{k+1}.$$

Then the associated energy functional Φ admits a cohomological local splitting at 0 *in dimension* k.

Proof. By (A3), arguing as in the proof of Lemma 4.1, we obtain

$$\int_\Omega G(x,u)\,dx = o\big(I_p(u)\big) \quad \text{as } \|u\| \to 0.$$

We set

$$X_+ := \{u \in X : \ I_p(u) \ge \lambda_{k+1} J(u)\}, \quad X_- := \{u \in X : \ I_p(u) \le \lambda_k J(u)\}.$$

Clearly, $X_\pm$ are symmetric closed cones in X and $X_+ \cap X_- = \{0\}$, since $\lambda_k < \lambda_{k+1}$. Let

$$M = \{u \in X : \ I_p(u) = 1\}.$$

By Proposition 3.3, we have

$$i(M \cap X_-) = i(M \setminus X_+) = k.$$

Moreover, arguing as in [14], one sees that $M \cap X_-$ is a deformation retract of $X_- \setminus \{0\}$, and similarly $M \setminus X_+$ is a deformation retract of $X \setminus X_+$. Hence

$$i(X_- \setminus \{0\}) = i(X \setminus X_+) = k.$$

Now we show that there exists $\rho > 0$ such that

$$\Phi(u) \le 0 \quad \text{for all } u \in \overline{B}_\rho(0) \cap X_-,$$

and

$$\Phi(u) \ge 0 \quad \text{for all } u \in \overline{B}_\rho(0) \cap X_+.$$

For every $u \in X_- \setminus \{0\}$, we have

$$I_p(u) - \lambda J(u) \le \Big(1 - \frac{\lambda}{\lambda_k}\Big) I_p(u),$$

so

$$\Phi(u) = I_p(u) - \lambda J(u) - \int_\Omega G(x,u)\,dx \le \Big(1 - \frac{\lambda}{\lambda_k}\Big) I_p(u) + o\big(I_p(u)\big)$$

as $\|u\| \to 0$. Since $\lambda > \lambda_k$, the coefficient $1 - \lambda/\lambda_k$ is negative, hence

$$\Phi(u) \le 0$$

for all $u \in X_- \cap \overline{B}_\rho(0)$, provided $\rho > 0$ is small enough. On the other hand, for every $u \in X_+ \setminus \{0\}$, we have

$$I_p(u) - \lambda J(u) \ge \Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big) I_p(u),$$

and therefore

$$\Phi(u) \ge \Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big) I_p(u) + o\big(I_p(u)\big)$$

as $\|u\| \to 0$. Since $\lambda < \lambda_{k+1}$, the coefficient $1 - \lambda/\lambda_{k+1}$ is positive, so

$$\Phi(u) \ge 0$$

for all $u \in X_+ \cap \overline{B}_\rho(0)$, for $\rho > 0$ sufficiently small. Therefore, by Definition 2.4, Φ admits a cohomological local splitting at 0 in dimension k. $\square$

The above lemma shows the the origin has a local splitting near the origin under the condition of the theorem 1.2. Next, we aim to prove that the same result holds under the condition of the theorem 1.3. The following L^∞ estimate is inspired by the [14, theorem 3.1] where the equation is scalar.

Lemma 4.3. *Let $u = (u_1, \dots, u_m) \in X$ be a weak solution of the problem 1.2. Assume that there exist constants $a > 0$ and*

$$r \in \Big[\max_{1 \le i \le m} p_i, \ \min_{1 \le i \le m} p_i^*\Big)$$

such that

$$|f_i(x,u)| \le a\Big(|u_i|^{p_i - 1} + |u|^{r-1}\Big) \tag{4.3}$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}^m$, and $i = 1, \dots, m$. Then for every $0 < \varepsilon < 1$ there exists $K = K(\varepsilon) > 0$ such that if

$$\|u\|_r < K,$$

then $u \in L^\infty(\Omega, \mathbb{R}^m)$ *and*

$$\|u\|_\infty \le \varepsilon^{-1}\|u\|_r.$$

Proof. Fix $0 < \varepsilon < 1$ and set

$$\rho := \varepsilon^{-1} > 1.$$

Let $u = (u_1, \dots, u_m) \in X$ be a weak solution such that

$$\|u\|_r < K.$$

Define

$$v = (v_1, \dots, v_m) := \frac{u}{\rho\|u\|_r}.$$

Then

$$\|v\|_r = \rho^{-1} = \varepsilon < 1.$$

For each $i = 1, \dots, m$, the function v_i satisfies

$$-\Delta_{p_i} v_i = g_i(x, v_i) \quad \text{in } \Omega,$$

where

$$g_i(x,t) := (\rho\|u\|_r)^{1-p_i} f_i(x, \rho\|u\|_r\, t).$$

By the growth assumption 4.3, for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$,

$$|g_i(x,t)| \le a\Big(|t|^{p_i-1} + (\rho\|u\|_r)^{r-p_i}|t|^{r-1}\Big). \tag{4.4}$$

Since $r \ge p_i$ and $\rho\|u\|_r < 1$ provided $K < \rho^{-1} = \varepsilon$, we obtain

$$|g_i(x,t)| \le a\big(|t|^{p_i-1} + |t|^{r-1}\big).$$

Fix $i \in \{1, \dots, m\}$. We show that $\|v_i\|_{L^\infty(\Omega)} \le 1$. If $v_i^+ \equiv 0$, the result is obvious. Otherwise, for all $n \in \mathbb{N}_0$ set

$$v_{i,n} := (v_i - 1 + 2^{-n})^+, \quad R_{i,n} := \|v_{i,n}\|^r_{L^r(\Omega)}.$$

Then $v_{i,n} \in W_0^{1,p_i}(\Omega)$, $v_{i,n+1} \le v_{i,n}$ a.e. in Ω, and

$$R_{i,0} = \|v_i^+\|^r_{L^r(\Omega)} \le \|v_i\|^r_{L^r(\Omega)} \le \|v\|_r^r = \varepsilon^r.$$

Moreover, arguing exactly as in the scalar case, we have the inclusion

$$\{v_{i,n+1} > 0\} \subset \{0 < v_i < (2^{n+1}-1)v_{i,n}\} \cap \{v_{i,n} > 2^{-(n+1)}\} \quad \text{a.e. in } \Omega.$$

Hence, by Chebyshev's inequality,

$$|\{v_{i,n+1} > 0\}| \le |\{v_{i,n} > 2^{-(n+1)}\}| \le 2^{r(n+1)} R_{i,n}. \tag{4.5}$$

Using Hölder's inequality and the Sobolev embedding

$$W_0^{1,p_i}(\Omega) \hookrightarrow L^{p_i^*}(\Omega),$$

we obtain

$$\begin{aligned} R_{i,n+1} &= \|v_{i,n+1}\|^r_{L^r(\Omega)} \\ &\le |\{v_{i,n+1} > 0\}|^{1-\frac{r}{p_i^*}} \|v_{i,n+1}\|^r_{L^{p_i^*}(\Omega)} \\ &\le C_i\, |\{v_{i,n+1} > 0\}|^{1-\frac{r}{p_i^*}} \|v_{i,n+1}\|^r_{W_0^{1,p_i}(\Omega)}. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we obtain

$$R_{i,n+1} \le C_i\, 2^{r(1-\frac{r}{p_i^*})(n+1)} R_{i,n}^{1-\frac{r}{p_i^*}} \|v_{i,n+1}\|^r_{W_0^{1,p_i}(\Omega)}. \tag{4.7}$$

Next we estimate $\|v_{i,n+1}\|_{W_0^{1,p_i}(\Omega)}$. Using the elementary inequality

$$|\xi^+ - \eta^+|^{p_i} \le |\xi - \eta|^{p_i-2}(\xi - \eta)(\xi^+ - \eta^+) \quad (\xi, \eta \in \mathbb{R}),$$

and testing the i-th equation with $v_{i,n+1}$, we obtain

$$\|v_{i,n+1}\|^{p_i}_{W_0^{1,p_i}(\Omega)} \le \int_\Omega g_i(x, v_i)\, v_{i,n+1}\, dx.$$

By (4.4),

$$\|v_{i,n+1}\|^{p_i}_{W_0^{1,p_i}(\Omega)} \le a \int_{\{v_{i,n+1}>0\}} \big(|v_i|^{p_i-1} + |v_i|^{r-1}\big) v_{i,n+1}\, dx. \tag{4.8}$$

On the set $\{v_{i,n+1} > 0\}$, (4.5) implies

$$0 < v_i < (2^{n+1}-1)v_{i,n}, \quad v_{i,n} > 2^{-(n+1)}.$$

Since $r \ge p_i$, we have

$$v_{i,n}^{p_i} \le 2^{(r-p_i)(n+1)} v_{i,n}^r \quad \text{on } \{v_{i,n+1} > 0\}.$$

Therefore,

$$\begin{aligned}\int_{\{v_{i,n+1}>0\}} |v_i|^{p_i-1} v_{i,n+1}\, dx &\le (2^{n+1}-1)^{p_i-1} \int_{\{v_{i,n+1}>0\}} v_{i,n}^{p_i}\, dx \\ &\le 2^{(p_i-1)(n+1)} 2^{(r-p_i)(n+1)} \int_\Omega v_{i,n}^r\, dx \\ &\le 2^{(r-1)(n+1)} R_{i,n}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\{v_{i,n+1}>0\}} |v_i|^{r-1} v_{i,n+1}\, dx &\le (2^{n+1}-1)^{r-1} \int_{\{v_{i,n+1}>0\}} v_{i,n}^r\, dx \\ &\le 2^{(r-1)(n+1)} R_{i,n}.\end{aligned}$$

Substituting these estimates into (4.8), we obtain

$$\|v_{i,n+1}\|^{p_i}_{W_0^{1,p_i}(\Omega)} \le C_i\, 2^{(r-1)(n+1)} R_{i,n},$$

hence

$$\|v_{i,n+1}\|^{r}_{W_0^{1,p_i}(\Omega)} \le C_i\, 2^{\frac{r(r-1)}{p_i}(n+1)} R_{i,n}^{\frac{r}{p_i}}. \tag{4.9}$$

Combining (4.7) and (4.9), we obtain

$$R_{i,n+1} \le H_i^n\, R_{i,n}^{1+\beta_i} \quad \text{for all } n \in \mathbb{N}_0, \tag{4.10}$$

where

$$H_i > 1, \quad \beta_i := \frac{r}{p_i} - \frac{r}{p_i^*} > 0.$$

Now set

$$\eta_i := H_i^{-1/\beta_i} \in (0,1), \quad \delta_i := \eta_i^{1/(\beta_i r)} \in (0,1).$$

We claim that if

$$R_{i,0} \le \delta_i^r, \tag{4.11}$$

then

$$R_{i,n} \le \delta_i^r \eta_i^n \quad \text{for all } n \in \mathbb{N}_0. \tag{4.12}$$

Indeed, (4.12) is true for $n = 0$ by (4.11). Then by (4.10),

$$R_{i,n+1} \le H_i^n (\delta_i^r \eta_i^n)^{1+\beta_i} = \delta_i^r \eta_i^{n+1},$$

because $\delta_i^{r\beta_i} = \eta_i = H_i^{-1/\beta_i}$. Thus (4.12) follows by induction. By (4.12), we have $R_{i,n} \to 0$ as $n \to \infty$, hence

$$(v_i - 1)^+ = 0 \quad \text{a.e. in } \Omega,$$

that is,

$$v_i \le 1 \quad \text{a.e. in } \Omega.$$

Applying the same argument to $-v_i$, we obtain

$$-v_i \le 1 \quad \text{a.e. in } \Omega.$$

Therefore,

$$\|v_i\|_{L^\infty(\Omega)} \le 1.$$

Since this holds for every $i = 1, \dots, m$, setting

$$\delta := \min_{1 \le i \le m} \delta_i,$$

Then it is easy to obtain $\|v\|_\infty \le 1$ provided

$$R_{i,0} \le \delta^r \quad \text{for all } i = 1, \dots, m.$$

Finally, rescaling back, we obtain

$$\|u\|_\infty = \rho \|u\|_r \, \|v\|_\infty \le \rho \|u\|_r = \varepsilon^{-1} \|u\|_r.$$

This completes the proof. $\square$

Lemma 4.4. *Assume that* (A1)–(A3) *hold and that* 0 *is an isolated critical point of* Φ*. Let* $\theta \in C^1(\mathbb{R})$ *be a nondecreasing function such that*

$$\theta(t) = \begin{cases} t, & |t| \le \frac{\delta}{2}, \\ \delta \operatorname{sgn}(t), & |t| \ge \delta. \end{cases}$$

For each $\tau \in [0,1]$*, define*

$$\Phi_\tau(u) := I_p(u) - \lambda J(u) - \int_\Omega G\big(x, (1-\tau)u + \tau\theta(u)\big)\, dx, \quad u \in X.$$

Then 0 *is an isolated critical point of* Φ_τ *uniformly with respect to* $\tau \in [0,1]$*, and*

$$C^k(\Phi, 0) = C^k(\Phi_1, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

Proof. Since 0 is an isolated critical point of Φ, there exists $\rho > 0$ such that

$$K(\Phi) \cap \overline{B}_\rho(0) = \{0\}.$$

We claim that

$$K(\Phi_\tau) \cap \overline{B}_\rho(0) = \{0\} \quad \text{for all } \tau \in [0,1] \tag{4.13}$$

for sufficiently small ρ. We argue by contradiction. Suppose that (4.13) fails. Then there exist sequences

$$\tau_n \in [0,1], \quad u_n = (u_{1,n}, \dots, u_{m,n}) \in X \setminus \{0\},$$

such that

$$u_n \to 0 \quad \text{in } X, \quad \Phi'_{\tau_n}(u_n) = 0 \quad \text{for all } n \in \mathbb{N}.$$

For each $i = 1, \dots, m$, u_n satisfies

$$\langle A_{p_i}(u_{i,n}), v_i \rangle = \lambda \int_\Omega \partial_{u_i} J(u_n)\, v_i \, dx + \int_\Omega g_{i,n}(x, u_n)\, v_i \, dx$$

for every $v_i \in X_i$, where

$$g_{i,n}(x,t) := \big(1 - \tau_n + \tau_n \theta'(t_i)\big)\, \partial_{u_i} G\big(x, (1-\tau_n)t + \tau_n \theta(t)\big), \quad t = (t_1, \dots, t_m) \in \mathbb{R}^m.$$

Hence u_n is a weak solution of the auxiliary system

$$\begin{gathered} -\Delta_{p_i} u_{i,n} = \lambda\, \partial_{u_i} J(u_n) + g_{i,n}(x, u_n) \quad \text{in } \Omega, \quad i = 1, \dots, m, \\ u_{i,n} = 0 \quad \text{on } \partial\Omega, \quad i = 1, \dots, m. \end{gathered} \tag{4.14}$$

By (A1) and the boundedness of θ and θ', there exists a constant $C > 0$ such that

$$\big|\lambda\, \partial_{u_i} J(t) + g_{i,n}(x,t)\big| \le C\Big(1 + \sum_{j=1}^m |t_j|^{r_{ij}-1}\Big)$$

for a.e. $x \in \Omega$, all $t = (t_1, \dots, t_m) \in \mathbb{R}^m$, and all $i = 1, \dots, m$. Therefore, by Lemma 4.3, there exists $K > 0$ such that every weak solution u of (4.14) with $\|u\|_r < K$ satisfies

$$u \in L^\infty(\Omega, \mathbb{R}^m) \quad \text{and} \quad \|u\|_\infty \le \varepsilon^{-1} \|u\|_r.$$

Since $u_n \to 0$ in X, by the Sobolev embedding we have

$$u_n \to 0 \quad \text{in } L^r(\Omega, \mathbb{R}^m).$$

Hence, for n sufficiently large, we have

$$\|u_n\|_\infty < \delta/2.$$

By the definition of θ, this implies

$$\theta(u_n) = u_n \quad \text{for all sufficiently large } n.$$

Therefore,

$$\Phi'_{\tau_n}(u_n) = \Phi'(u_n) = 0$$

for all sufficiently large n, which yields

$$u_n \in K(\Phi) \cap \overline{B}_\rho(0) \setminus \{0\}.$$

This proves (4.13). Moreover, for every $\tau \in [0,1]$, the functional Φ_τ is of class C^1, and the map

$$[0,1] \to C^1(\overline{B}_\rho(0), \mathbb{R}), \quad \tau \mapsto \Phi_\tau$$

is continuous. Since 0 remains an isolated critical point of Φ_τ uniformly in τ, the homotopy invariance of critical groups yields

$$C^k(\Phi, 0) = C^k(\Phi_1, 0) \quad \text{for all } k \in \mathbb{N}_0.$$

The proof is complete. □

Lemma 4.5. *If* (A1) *and* (G2) *hold, then the functional* $\Phi \in C^1(X(\Omega))$ *satisfies the* (PS) *condition.*

Proof. Let (u^j) be a (PS) sequence, that is,

$$\Phi(u^j) \to c \in \mathbb{R}, \quad \Phi'(u^j) \to 0 \quad \text{in } X(\Omega)^*.$$

It suffices to show that (u^j) is bounded in $X(\Omega)$. Define

$$\frac{u^j}{\mu} := \Big(\frac{u_1^j}{\mu_1}, \dots, \frac{u_m^j}{\mu_m}\Big).$$

Using the expression

$$\Phi(u) = I_p(u) - \lambda J(u) - \int_\Omega G(x,u)\,dx,$$

we compute

$$\Phi(u^j) - \big\langle \Phi'(u^j), \frac{u^j}{\mu}\big\rangle = \sum_{i=1}^m \Big(\frac{1}{p_i} - \frac{1}{\mu_i}\Big)\|u_i^j\|^{p_i}_{W_0^{1,p_i}(\Omega)} + \int_\Omega H_\mu\left(x, u^j\right)\,dx,$$

where

$$H_\mu(x,u) := \sum_{i=1}^m \frac{u_i}{\mu_i}\frac{\partial G}{\partial u_i}(x,u) - G(x,u).$$

By (A2), there exists $\delta > 0$ such that

$$H_\mu(x,u) \ge 0 \quad \text{for a.e. } x \in \Omega \text{ and all } |u| \ge \delta.$$

Since H_μ is continuous in u, there exists a constant $C_\delta > 0$ such that

$$H_\mu(x,u) \ge -C_\delta \quad \text{for a.e. } x \in \Omega \text{ and all } u \in \mathbb{R}^m.$$

Consequently,

$$\int_\Omega H_\mu(x,u^j)\,dx \ge -C_\delta|\Omega|.$$

Moreover, since $\Phi'(u^j) \to 0$ in $X(\Omega)^*$, we have

$$\big|\big\langle \Phi'(u^j), \frac{u^j}{\mu}\big\rangle\big| \le \|\Phi'(u^j)\|_{X(\Omega)^*}\|\frac{u^j}{\mu}\|_{X(\Omega)} = o(1)\|\frac{u^j}{\mu}\|_{X(\Omega)}.$$

Therefore,

$$\sum_{i=1}^m \Big(\frac{1}{p_i} - \frac{1}{\mu_i}\Big)\|u_i^j\|^{p_i}_{W_0^{1,p_i}(\Omega)} \le o(1)\|\frac{u^j}{\mu}\|_{X(\Omega)} + C.$$

Since $\mu_i > p_i$ for all $i = 1, \dots, m$, each coefficient

$$a_i := \frac{1}{p_i} - \frac{1}{\mu_i}$$

is positive. Moreover, since

$$\|\frac{u^j}{\mu}\|_{X(\Omega)} \le C \sum_{i=1}^{m} \|u_i^j\|_{W_0^{1,p_i}(\Omega)},$$

it follows from Young's inequality that

$$o(1)\|\frac{u^j}{\mu}\|_{X(\Omega)} \le \varepsilon \sum_{i=1}^{m} \|u_i^j\|^{p_i}_{W_0^{1,p_i}(\Omega)} + C_\varepsilon$$

for any $\varepsilon > 0$ and all sufficiently large j. Therefore,

$$\sum_{i=1}^{m} a_i \|u_i^j\|^{p_i}_{W_0^{1,p_i}(\Omega)} \le \varepsilon \sum_{i=1}^{m} \|u_i^j\|^{p_i}_{W_0^{1,p_i}(\Omega)} + C.$$

Let

$$a_* := \min_{1 \le i \le m} a_i > 0.$$

Choosing $\varepsilon \in (0, a_*)$, we obtain

$$(a_* - \varepsilon) \sum_{i=1}^{m} \|u_i^j\|^{p_i}_{W_0^{1,p_i}(\Omega)} \le C$$

for all sufficiently large j. Hence each sequence $\{u_i^j\}_j$ is bounded in $W_0^{1,p_i}(\Omega)$, and therefore $\{u^j\}_j$ is bounded in $X(\Omega)$. Therefore, by Lemma 2.3, (u^j) possesses a strongly convergent subsequence in $X(\Omega)$. Hence Φ satisfies the (PS) condition. □

Lemma 4.6. *Assume that* (A1)–(A3) *hold and that* 0 *is an isolated critical point of* Φ. *Suppose that one of the following conditions is satisfied for some* $k \in \mathbb{N}$:

(i) $\lambda = \lambda_k < \lambda_{k+1}$, *and there exists* $\delta > 0$ *such that*

$$G(x, u) \ge 0 \quad \text{for a.e. } x \in \Omega \text{ and all } |u| \le \delta;$$

(ii) $\lambda_k < \lambda = \lambda_{k+1}$, *and there exists* $\delta > 0$ *such that*

$$G(x, u) \le 0 \quad \text{for a.e. } x \in \Omega \text{ and all } |u| \le \delta.$$

Then Φ_1 *admits a cohomological local splitting at* 0 *in dimension* k. *In particular,*

$$C^k(\Phi, 0) \ne 0.$$

Proof. By Lemma 4.4, we have

$$C^k(\Phi, 0) = C^k(\Phi_1, 0), \quad \forall k \in \mathbb{N}_0.$$

Hence it is enough to prove that Φ_1 admits a cohomological local splitting at 0 in dimension k. Set

$$X_+ := \{u \in X : I_p(u) \ge \lambda_{k+1} J(u)\}, \quad X_- := \{u \in X : I_p(u) \le \lambda_k J(u)\}.$$

Since $\lambda_k < \lambda_{k+1}$, $X_\pm$ are symmetric closed cones in X and

$$X_+ \cap X_- = \{0\}.$$

Let

$$M := \{u \in X : I_p(u) = 1\}.$$

By Proposition 3.3,

$$i(M \cap X_-) = i(M \setminus X_+) = k.$$

Arguing as in the proof of Lemma 4.2, one sees that $M \cap X_-$ is a deformation retract of $X_- \setminus \{0\}$, while $M \setminus X_+$ is a deformation retract of $X \setminus X_+$. Therefore,

$$i(X_- \setminus \{0\}) = i(X \setminus X_+) = k.$$

By (A3), arguing as in the proof of Lemma 4.1, we have

$$\int_\Omega G(x,v)\,dx = o(I_p(v)) \quad \text{as } \|v\| \to 0.$$

Since $\theta(t) = t$ for $|t| \le \delta/2$, it follows that

$$\int_\Omega G(x,\theta(u))\,dx = o(I_p(u)) \quad \text{as } \|u\| \to 0.$$

We now prove that there exists $\rho > 0$ such that

$$\begin{gathered}\Phi_1(u) \le 0 \quad \text{for all } u \in \overline{B}_\rho(0) \cap X_-,\\ \Phi_1(u) \ge 0 \quad \text{for all } u \in \overline{B}_\rho(0) \cap X_+.\end{gathered}$$

We only prove $\lambda = \lambda_k < \lambda_{k+1}$ and

$$G(x,u) \ge 0 \quad \text{for a.e. } x \in \Omega \text{ and all } |u| \le \delta,$$

the other cases can be proved similarly. Since $|\theta(u)| \le \delta$, we have

$$G(x,\theta(u)) \ge 0 \quad \text{for a.e. } x \in \Omega,$$

and hence

$$\int_\Omega G(x,\theta(u))\,dx \ge 0.$$

Thus, for every $u \in X_- \setminus \{0\}$,

$$I_p(u) - \lambda J(u) = I_p(u) - \lambda_k J(u) \le 0,$$

so that

$$\Phi_1(u) = I_p(u) - \lambda_k J(u) - \int_\Omega G(x,\theta(u))\,dx \le 0.$$

On the other hand, for every $u \in X_+ \setminus \{0\}$,

$$I_p(u) - \lambda J(u) = I_p(u) - \lambda_k J(u) \ge \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big) I_p(u),$$

and therefore

$$\Phi_1(u) \ge \Big(1 - \frac{\lambda_k}{\lambda_{k+1}}\Big) I_p(u) + o(I_p(u))$$

as $\|u\| \to 0$. Since $\lambda_k < \lambda_{k+1}$, the coefficient $1 - \lambda_k/\lambda_{k+1}$ is positive, hence

$$\Phi_1(u) \ge 0$$

for all $u \in X_+ \cap \overline{B}_\rho(0)$, provided $\rho > 0$ is small enough. Since $|\theta(u)| \le \delta$, we have

$$G(x,\theta(u)) \le 0 \quad \text{for a.e. } x \in \Omega,$$

and hence

$$-\int_\Omega G(x,\theta(u))\,dx \ge 0.$$

Thus, for every $u \in X_+ \setminus \{0\}$,

$$I_p(u) - \lambda J(u) = I_p(u) - \lambda_{k+1} J(u) \ge 0,$$

which yields

$$\Phi_1(u) = I_p(u) - \lambda_{k+1} J(u) - \int_\Omega G(x,\theta(u))\,dx \ge 0.$$

On the other hand, for every $u \in X_- \setminus \{0\}$,

$$I_p(u) - \lambda J(u) = I_p(u) - \lambda_{k+1} J(u) \le \Big(1 - \frac{\lambda_{k+1}}{\lambda_k}\Big) I_p(u),$$

and therefore

$$\Phi_1(u) \le \Big(1 - \frac{\lambda_{k+1}}{\lambda_k}\Big) I_p(u) + o(I_p(u))$$

as $\|u\| \to 0$. Since $\lambda_k < \lambda_{k+1}$, the coefficient $1 - \lambda_{k+1}/\lambda_k$ is negative, hence

$$\Phi_1(u) \le 0$$

for all $u \in X_- \cap \overline{B}_\rho(0)$, provided $\rho > 0$ is sufficiently small. Therefore, Φ_1 admits a cohomological local splitting at 0 in dimension k. Since 0 is an isolated critical point of Φ_1 and Φ_1 admits a cohomological local splitting at 0 in dimension k, it follows from [17, Proposition 3.34] that

$$C^k(\Phi_1, 0) \ne 0.$$

Hence, by Lemma 4.4,

$$C^k(\Phi, 0) \ne 0.$$

□

Proof of Theorems 1.2 and 1.3. We argue by contradiction. Assuming that

$$K(\Phi) = \{0\}.$$

Since Φ satisfies the (PS) condition, there exists $\eta < 0$ such that η is not a critical value of Φ. Moreover, under the assumption $K(\Phi) = \{0\}$, there is no critical value of Φ in $[\eta, 0)$ or in $(0, \infty)$. Therefore, by the second deformation theorem, Φ^η is a deformation retract of $\Phi^0 \setminus \{0\}$, while Φ^0 is a deformation retract of X. Hence, for every $q \in \mathbb{N}_0$,

$$C^q(\Phi, 0) = H^q(\Phi^0, \Phi^0 \setminus \{0\}) \cong H^q(X, \Phi^\eta) = C^q(\Phi, \infty).$$

Since Φ^η is a deformation retract of X, we further obtain

$$H^q(X, \Phi^\eta) = 0, \quad \forall q \in \mathbb{N}_0.$$

Consequently, $C^q(\Phi, 0) = 0$ for all $q \in \mathbb{N}_0$.

The case of theorem 1.2: $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \ge 1$. By Lemma 4.2, the functional Φ admits a cohomological local splitting at 0 in dimension k. Since 0 is isolated, it follows from the standard result on cohomological local splitting that

$$C^k(\Phi, 0) \ne 0.$$

This contradicts the fact that $C^q(\Phi, 0) = 0$ for all $q \in \mathbb{N}_0$. Therefore, $K(\Phi) \ne \{0\}$, and hence Φ possesses a nonzero critical point.

The case of theorem 1.3: $\lambda = \lambda_k$ and one of the sign conditions (G^+) or (G^-) holds. By Lemma 4.6, we have

$$C^k(\Phi, 0) \ne 0$$

for some $k \in \mathbb{N}$. Again this contradicts the fact that $C^q(\Phi, 0) = 0$ for all $q \in \mathbb{N}_0$. Hence $K(\Phi) \ne \{0\}$, so that Φ has a nontrivial critical point. This proves Theorem 1.3. □

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References

[1] A. Ambrosetti, H. Brezis, G. Cerami; *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. 122 (1994), 519–543.

[2] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Anal. 14 (1973), 349–381.

[3] H. Brezis, L. Nirenberg; *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), 437–477.

[4] X. Cabré, Y. Sire; *Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014), 23–53.

[5] K. C. Chang; *Methods in Nonlinear Analysis*, Springer Monographs in Mathematics, Springer, Berlin, 2005.

[6] K. C. Chang; *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, 1993.

[7] M. Degiovanni, S. Lancelotti, K. Perera; *Nontrivial solutions of p-superlinear p-Laplacian problems via a cohomological local splitting*, Commun. Contemp. Math. 12 (2010), no. 3, 475–486.

[8] M. Degiovanni, M. Marzocchi; *A critical point theory for nonsmooth functionals*, Ann. Mat. Pura Appl., (4) 167 (1994), 73–100.

[9] M. Degiovanni, S. Lancelotti; *Linking over cones and nontrivial solutions for p-Laplacian equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), 907–919.

[10] J. I. Díaz; *Nonlinear Partial Differential Equations and Free Boundaries*, Vol. I, Elliptic Equations, Pitman Research Notes in Mathematics Series, Longman, London, 1985.
[11] G. Dinca, P. Jebelean, J. Mawhin; *Variational and topological methods for Dirichlet problems with p-Laplacian*, Portugaliae Math. 58 (2001), 339–378.
[12] E. R. Fadell, P. H. Rabinowitz; *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. 45 (1978), 139–174.
[13] N. Ghoussoub; *Duality and Perturbation Methods in Critical Point Theory*, Cambridge University Press, Cambridge, 1993.
[14] A. Iannizzotto, S. Liu, K. Perera, M. Squassina; *Existence results for fractional p-Laplacian problems via Morse theory*, Adv. Calc. Var. **9** (2016), no. 2, 101–125.
[15] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.
[16] K. Perera, R. P. Agarwal, D. O'Regan; *Nontrivial solutions of p-superlinear anisotropic p-Laplacian systems via Morse theory*, Topol. Methods Nonlinear Anal., 35 (2010), no. 2, 367–378.
[17] K. Perera, R. P. Agarwal, D. O'Regan; *Morse Theoretic Aspects of p-Laplacian Type Operators*, Mathematical Surveys and Monographs, Vol. 161, American Mathematical Society, Providence, RI, 2010.
[18] K. Perera, R. P. Agarwal, D. O'Regan; *Multiplicity results for p-sublinear p-Laplacian problems involving indefinite eigenvalue problems via Morse theory*, Electron. J. Differential Equations 2010 (2010), no. 105, 1–15.
[19] K. Perera; *Nontrivial critical groups in p-Laplacian problems via the Yang index*, Topol. Methods Nonlinear Anal. 21 (2003), 301–309.
[20] P. Pucci, J. Serrin; *The Maximum Principle*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 73, Birkhäuser, Basel, 2007.
[21] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics, Vol. 65, American Mathematical Society, Providence, RI, 1986.
[22] R. Servadei, E. Valdinoci; *Mountain pass solutions for non-local elliptic operators*, J. Math. Anal. Appl. 389 (2012), 887–898.
[23] R. Servadei, E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst. 33 (2013), 2105–2137.

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