Electronic Journal of Differential Equations, Vol. 2026 (2026), No. 55, pp. 1–22.
ISSN: 1072-6691. URL: https://ejde.math.txstate.edu, DOI: 10.58997/ejde.2026.55

ILL-POSEDNESS FOR KELVIN-HELMHOLTZ PROBLEMS OF COMPRESSIBLE EULER FLUIDS WITH A RADIATION FIELD

JIAKAI MA, BINQIANG XIE

Abstract. We prove the nonlinear ill-posedness of Kelvin-Helmholtz problems for compressible Euler fluids with a radiation field. For this we assume that the magnitude of the Mach number is between some small enough constant ϵ_0 and $\sqrt{2(1+\ell)}$ in which ℓ is the radiation parameter.

1. Introduction

1.1. **Eulerian formulation.** The Kelvin-Helmholtz (KH) instability is a fundamental fluid phenomenon that occurs when two adjacent layers of a fluid (or plasma) are in relative motion, such as in shear flows [13, 14]. It plays a crucial role in various astrophysical systems, including the stability of the interface between the solar wind and the magnetosphere [8, 12, 22], interaction between adjacent streams of different velocities in the solar wind [28] and the dynamic structure of cometary tails [10].

While early studies focused on the effects of compressibility and magnetic fields, recent analytical work has extended the classical linear theory to include the dynamical role of a radiation field [23, 24]. In many high-energy astrophysical environments-such as radiation-dominated accretion disks around black holes, H II regions around massive stars (e.g., the Trapezium cluster in the Orion nebula), and galactic outflows-radiation pressure can be comparable to or even exceed gas pressure.

The presence of a dynamically important radiation field generally enhances the growth rate of KH instability. Because the total effective pressure (gas + radiation) increases, leading to a higher effective sound speed and thus reduced compressibility of the system.

In this paper, we will give a rigorous confirmation that a radiation field has a destabilizing effect on the Kelvin-Helmholtz instability. More specifically, we examine two distinct, inviscid, compressible, and immiscible fluids that evolve within the domain $\mathbb{R}^2$ for all time $t \geq 0$. These two fluids are separated by a moving free interface $\Gamma(t)$, and this interface partitions the plane $\mathbb{R}^2$ into two time-varying, disjoint open subsets $\Omega^{\pm}(t)$, such that $\mathbb{R}^2 = \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t)$ and $\Gamma(t) = \bar{\Omega}^+(t) \cap \bar{\Omega}^-(t)$. The fluid residing in $\Omega^+(t)$ is denoted as the upper fluid, while the fluid occupying $\Omega^-(t)$ is referred to as the lower fluid. Both fluids exhibit sufficient smoothness to satisfy the system of compressible Euler equations with a radiation field:

$$
\begin{gathered}
\partial_t \rho^{\pm} + \operatorname{div}(\rho^{\pm} u^{\pm}) = 0, \\
\partial_t(\rho^{\pm} u^{\pm}) + \operatorname{div}(\rho^{\pm} u^{\pm} \otimes u^{\pm}) + \nabla p^{\pm} = 0, \\
\partial_t E^{\pm} + u^{\pm} \cdot \nabla E^{\pm} + (E^{\pm} + p^{\pm}) \operatorname{div} u^{\pm} = -\operatorname{div} F^{\pm}, \\
F^{\pm} = -\frac{c}{3\kappa \rho^{\pm}} \nabla e_r^{\pm},
\end{gathered}
\tag{1.1}
$$

where $u^{\pm} = (u_1^{\pm}, u_2^{\pm})$ denotes the velocity field corresponding to the two fluids, and $\rho^{\pm}$ represents the density of these two fluids. The pressure $p^{\pm}$ can be decomposed into $p^{\pm} = p_g^{\pm} + p_r^{\pm}$, where $p_g^{\pm}$ and $p_r^{\pm}$ stand for the gas pressure and radiation pressure, respectively. The total internal energy

2020 *Mathematics Subject Classification.* 76W05, 35Q35, 35D05, 76X05.
Key words and phrases. Free surface; Kelvin-Helmholtz instability; radiation field.

Submitted April 17, 2026. Published July 14, 2026.

is defined as $E^{\pm} = e_g^{\pm} + e_r^{\pm}$, in which $e_g^{\pm} = p_g^{\pm}/(\gamma - 1)$, $e_r^{\pm} = 3p_r^{\pm}$, and γ denotes the adiabatic index of the gas. In addition, $F^{\pm}$ refers to the radiative flux in which κ is the mean opacity of the gas within the interface layer and c is the speed of light. For the sake of simplification, the diffusion effect of radiation is ignored. The sound speed c_s in the fluid is defined through the following relation:

$$c_s = \sqrt{p_g'} \quad \forall \rho > 0. \tag{1.2}$$

In the process of investigating the existence of weak solutions for (1.1) by the Rankine-Hugoniot jump relations associated with the hyperbolic system of equations, a conventional presumption is that both the pressure and the normal component of the velocity ought to be continuous across the free boundary $\Gamma(t) = \{x_2 = f(t, x_1)\}$. In this context, the function f that characterizes the discontinuity front constitutes a component of the unknown quantities in the problem, namely, this is a free boundary problem. Consequently, such piecewise smooth solutions are required to satisfy the subsequent boundary conditions on $\Gamma(t)$:

$$\partial_t f = u^+ \cdot n = u^- \cdot n, \quad p_g^+ + p_r^+ = p_g^- + p_r^-, \quad \text{on } \Gamma(t), \tag{1.3}$$

in which $n = (-\partial_{x_1} f, 1)$ denotes the normal vector corresponding to $\Gamma(t)$.

To fulfill the formulation of the problem, it is necessary to define initial conditions. We introduce the initial interface Γ_0, which generates the open sets $\Omega_0^{\pm}$; on these sets, we define the initial density and velocity field as $\rho^{\pm}(0, x) = \rho_0^{\pm}(x) : \Omega_0^{\pm} \to \mathbb{R}^+$ and $u^{\pm}(0, x) = u_0^{\pm}(x) : \Omega_0^{\pm} \to \mathbb{R}^2$, respectively.

1.2. **Rectilinear solution.** It is straightforward to observe that the system (1.1)-(1.3) possesses a rectilinear solution $U = (\bar{f}, \bar{\rho}^{\pm}, \bar{u}^{\pm},)$ which is defined as follows, with the interface specified by $\{x_2 = 0\}$ for all $t \geq 0$. Consequently, $\Omega^+ = \Omega^+(t) = \mathbb{R} \times (0, \infty)$ and $\Omega^- = \Omega^-(t) = \mathbb{R} \times (-\infty, 0)$ hold for all $t \geq 0$. In more detail, the front is planar, that is, $\bar{f} = 0$. To ensure that the constant density $\bar{\rho}^{\pm}$ satisfies the jump condition (1.3), it is necessary to impose that

$$\bar{\rho}^+ = \bar{\rho}^- := \bar{\rho}, \tag{1.4}$$

in which $\bar{\rho}$ denotes a positive constant. It can also be observed that the upper fluid travels in the horizontal direction with a certain constant velocity, while the lower fluid moves with the same constant velocity in the reverse direction; that is, the constant velocity field $\bar{u}^{\pm}$ takes the form

$$\bar{u} = \begin{cases} (\bar{u}_1^+, 0) & x_2 \geq 0, \\ (\bar{u}_1^-, 0) & x_2 < 0, \end{cases} \tag{1.5}$$

in which the constants $\bar{u}_1^+, \bar{u}_1^-$ satisfy

$$\bar{u}_1^+ = -\bar{u}_1^-. \tag{1.6}$$

1.3. **New reformulation.** The analysis conducted in this paper is based on the rephrasing of problem (1.1)-(1.3) within new coordinate systems. To start with, we introduce the fixed domains $\Omega^{\pm}$ as

$$\Omega^+ := \left\{x \in \mathbb{R}^2 : x_2 > 0\right\}, \quad \Omega^- := \left\{x \in \mathbb{R}^2 : x_2 < 0\right\}. \tag{1.7}$$

Define the fixed boundary Γ as

$$\Gamma := \left\{x \in \mathbb{R}^2 : x_2 = 0\right\}.$$

To convert our free boundary problem into the fixed domain $\Omega^{\pm}$, we adopt a variable transformation over the entire space that maps $\Omega^{\pm}$ back to the original domains $\Omega^{\pm}(t)$ via $(t, x_1, x_2) \mapsto (t, x_1, x_2 + \psi(t, x))$. We establish such a ψ by means of multiplying the front f by a smooth cut-off function that depends on x_2:

$$\psi(t, x_1, x_2) = \theta\Big(\frac{x_2}{3(1 + a)}\Big) f(t, x_1), \quad a = \|f_0\|_{L^\infty(\mathbb{R})}, \tag{1.8}$$

in which $\theta \in C_c^\infty(\mathbb{R})$ denotes a smooth cut-off function satisfying $0 \leq \theta \leq 1$, $\theta(x_2) = 1$ when $|x_2| \leq 1$, $\theta(x_2) = 0$ for $|x_2| \geq 3$, and $|\partial_2 \theta(x_2)| \leq 1$ for all $x_2 \in \mathbb{R}$, with $\partial_j = \partial/\partial x_j$ denoted as such. We further assume that

$$\|f_0\|_{L^\infty(\mathbb{R})} \leq 1. \tag{1.9}$$

Moreover, we have

$$
\begin{aligned}
&\psi(t,x_1,0)=f(t,x_1),\\
&\partial_2\psi(t,x_1,0)=0,\\
&|\partial_2\psi|\le \frac{1}{3(1+a)}|f|.
\end{aligned}
\tag{1.10}
$$

Next we present a change of variable that converts the free boundary problem (1.1) into the fixed domain $\Omega^{\pm}$.

Lemma 1.1. *Define the function*

$$
\Psi(t,x):=(x_1,x_2+\psi(t,x)),\quad (t,x)\in[0,T]\times\Omega. \tag{1.11}
$$

Subsequently, $\Psi:(x_1,x_2)\mapsto(x_1,x_2+\psi(t,x))$ serves as a diffeomorphism transformation from $\Omega^{\pm}$ to $\Omega^{\pm}(t)$ for all $t\in[0,T]$. Specifically, we emphasize that $\Psi|_{t=0}:\bar\Omega^{\pm}\mapsto\bar\Omega^{\pm}(0)$ constitutes a diffeomorphism transformation.

Proof. Given that $\|f_0\|_{L^\infty(\mathbb{R})}\le 1$, it can be demonstrated that there exists a certain $T>0$ satisfying $\sup_{[0,T]}\|f\|_{L^\infty}<2$, and the free interface remains a graph during the time interval$[0,T]$ and

$$
\partial_2\Psi_2(t,x)=1+\partial_2\psi(t,x)\ge 1-\frac{1}{3}\times 2=\frac{1}{3},
$$

which guarantee that $\Psi:(t,x_1,x_2)\mapsto(t,x_1,x_2+\psi(t,x))$ are diffeomorphism of Ω for all $t\in[0,T]$. $\square$

We adopt the following operator notation

$$
A=[D\Psi]^{-1}=\begin{pmatrix}1 & 0\\ -\partial_1\psi/J & 1/J\end{pmatrix},
$$

and $J=\det[D\Psi]=1+\partial_2\psi$. At this point, we can convert the free boundary problem (1.1)-(1.3) into a problem within the fixed domain $\Omega^{\pm}$ by the map Ψ defined in Lemma 1.1. Let us set

$$
\begin{aligned}
&v^{\pm}(t,x):=u^{\pm}(t,\Psi(t,x)),\ \varrho^{\pm}(t,x):=\rho^{\pm}(t,\Psi(t,x)),\\
&q_g^{\pm}(t,x):=p_g^{\pm}(t,\Psi(t,x)),\ q_r^{\pm}(t,x):=p_r^{\pm}(t,\Psi(t,x)).
\end{aligned}
\tag{1.12}
$$

Throughout the remainder of this paper, an equation defined on Ω indicates that the equation holds in both Ω^+ and Ω^-. For the sake of convenience, we unify the notation by employing ϱ, v, q_g, q_r, h to denote $\varrho^{\pm},v^{\pm},q_g^{\pm},q_r^{\pm},h^{\pm}$, except when it is necessary to distinguish between the two. Subsequently, the system (1.1) and the boundary conditions (1.3) can be re-expressed on the fixed reference domain $\Omega^{\pm}$ as

$$
\begin{gathered}
\partial_t q_g+(\breve v\cdot\nabla)q_g+\varrho c_s^2(A^T\nabla)\cdot v=0\quad\text{in }\Omega,\\
\rho(\partial_t v+(\breve v\cdot\nabla)v)+A^T\nabla(q_g+q_r)=0\quad\text{in }\Omega,\\
\partial_t q_r+(\breve v\cdot\nabla)q_r+\frac{1}{3}(q_g+4q_r)(A^T\nabla)\cdot v=0\quad\text{in }\Omega,\\
\partial_t f=v\cdot n\quad\text{on }\Gamma,\\
[v\cdot n]=0,\quad [q_g+q_r]=0,\quad\text{on }\Gamma,\\
v_{|t=0}=v_0,\quad q_{r|t=0}=q_{r_0},\quad q_{g|t=0}=q_{g_0}\quad\text{in }\Omega\\
f_{|t=0}=f_0\quad\text{on }\Gamma,
\end{gathered}
\tag{1.13}
$$

where we define

$$
\breve v:=Av-(0,\partial_t\psi/J)=(v_1,(v\cdot n-\partial_t\psi)/J),
$$

and $[h]=h^+|_\Gamma-h^-|_\Gamma$ denotes the jump of a physical quantity h across the interface Γ.

The initial data are assumed to satisfy

$$
q_{g_0}^++q_{r_0}^+=q_{g_0}^-+q_{r_0}^-,v_0^+\cdot n_0=v_0^-\cdot n_0\text{ on }\Gamma. \tag{1.14}
$$

Notice that

$$
J=1,\quad \breve v_2=0\quad\text{ on }\Gamma. \tag{1.15}
$$

Given that our study focuses on Kelvin-Helmholtz instability, the instability phenomenon initially occurs on the boundary as a result of the discontinuity in the tangential velocity across the boundary. To illustrate this point, we intend to derive a second-order evolution equation for the front f on the fixed boundary Γ. By utilizing the momentum equation associated with (1.13), we infer that the following equation on the boundary Γ holds

$$\begin{aligned}\partial_t^2 f &= \partial_t v^+ \cdot n + v^+ \cdot \partial_t n\\ &= -((\breve{v}^+ \cdot \nabla) v^+ + \frac{1}{\varrho^+} A^T \nabla (q_g^+ + q_r^+) \cdot n - v^+ \cdot (\partial_1 \partial_t f, 0)\\ &= -v_1^+ \partial_1 v^+ \cdot n - \frac{1}{\varrho^+} A^T \nabla (q_g^+ + q_r^+) \cdot n - v_1^+ \partial_1 \partial_t f\\ &= v_1^+ \partial_1 n \cdot v^+ - v_1^+ \partial_1 \partial_t f + \frac{1}{\varrho^+} A^T \nabla (q_g^+ + q_r^+) \cdot n - v_1^+ \partial_1 \partial_t f\\ &= -2 v_1^+ \partial_1 \partial_t f - \frac{1}{\varrho^+} A^T \nabla (q_g^+ + q_r^+) \cdot n - (v_1^+)^2 \partial_1^2 f.\end{aligned} \tag{1.16}$$

Likewise, we deduce an evolution equation for the front f from the negative component:

$$\partial_t^2 f = -2 v_1^- \partial_1 \partial_t f - \frac{1}{\varrho^-} A^T \nabla (q_g^- + q_r^-) \cdot n - (v_1^-)^2 \partial_1^2 f \quad \text{on } \Gamma. \tag{1.17}$$

Consequently, we sum the " + " equation (1.16) and the " − " equation (1.17) to obtain

$$\begin{aligned}&\partial_t^2 f + (v_1^+ + v_1^-) \partial_1 \partial_t f + \frac{1}{2}(\frac{1}{\varrho^+} A^T \nabla (q_g^+ + q_r^+) \cdot n + \frac{1}{\varrho^-} A^T \nabla (q_g^- + q_r^-) \cdot n)\\ &+ \frac{1}{2}((v_1^+)^2 + (v_1^-)^2) \partial_1^2 f = 0 \quad \text{on } \Gamma.\end{aligned} \tag{1.18}$$

1.4. **Wave equation for q_g and q_r.** Applying the operator $\partial_t + \breve{v} \cdot \nabla$ to the first equation of (1.13) and the operator $A^T \nabla \cdot$ to the second equation yields

$$\begin{gathered}(\partial_t + \breve{v} \cdot \nabla)^2 q_g + (\partial_t + \breve{v} \cdot \nabla)(\varrho c_s^2 (A^T \nabla) \cdot v) = 0,\\ (A^T \nabla) \cdot (\varrho c_s^2 (\partial_t + \breve{v} \cdot \nabla) v + c_s^2 (A^T \nabla) \cdot (q_g + q_r) = 0.\end{gathered} \tag{1.19}$$

Subsequently, we compute the difference between the two equations in (1.19) so as to derive a wave-type equation,

$$(\partial_t + \breve{v} \cdot \nabla)^2 q_g - (A^T \nabla) \cdot (c_s^2 A^T \nabla (q_g + q_r)) = \mathcal{F}. \tag{1.20}$$

where the term $\mathcal{F} = -\varrho c_s^2 [\partial_t + \breve{v} \cdot \nabla, A^T \nabla \cdot] v - (A^T \nabla) \cdot (\varrho c_s^2)(\partial_t + \breve{v} \nabla) v - (\partial_t + \breve{v} \cdot \nabla)(\varrho c_s^2)(A^T \nabla) \cdot v$ serves as a lower-order term in the second-order differential equation corresponding to $q_g + q_r$.

Based on the boundary conditions specified in (1.13), we have already ascertained that

$$[q_g + q_r] = 0 \text{ on } \Gamma. \tag{1.21}$$

To determine the value of h, we introduce an additional condition that involves the normal derivatives of h on the boundary Γ. Specifically, by computing the difference between the two equations (1.16) and (1.17), we are able to acquire the jump of $\frac{1}{\varrho}(A^T \nabla)(q_g + q_r) \cdot n$,

$$[\frac{1}{\varrho}(A^T \nabla)(q_g + q_r) \cdot n] = [-2 v_1 \partial_1 \partial_t f - (v_1)^2 \partial_1^2 f] \text{ on } \Gamma. \tag{1.22}$$

Integrating (1.20), (1.21) and (1.22) yields a nonlinear system corresponding to h,

$$\begin{gathered}(\partial_t + \breve{v} \cdot \nabla)^2 q_g - (A^T \nabla) \cdot (c_s^2 (A^T \nabla)(q_g + q_r)) = \mathcal{F} \quad \text{in } \Omega,\\ [q_g + q_r] = 0 \quad \text{on } \Gamma,\\ [\frac{1}{\varrho}(A^T \nabla)(q_g + q_r) \cdot n] = [-2 v_1 \partial_1 \partial_t f - (v_1)^2 \partial_1^2 f] \quad \text{on } \Gamma.\end{gathered} \tag{1.23}$$

1.5. **Previous results.** Significant progress has been made regarding the well-posedness of solutions for the Kelvin-Helmholtz problem in ideal fluids. The Kelvin-Helmholtz instability configuration is also referred to in the literature as "vortex sheets", since the vorticity distribution is described by a δ-function supported on a discontinuity in the velocity field at the sheet location. In the pioneering works Coulombel [5] and Secchi [6] proved the nonlinear stability of vortex sheets for ideal compressible flows using microlocal analysis and the Nash-Moser method. Subsequently, Morando, Trebeschi, and Wang [18, 19] extended this result to two-dimensional ideal non-isentropic compressible flows. The method developed in [5] was also employed to study two-dimensional magnetohydrodynamics (MHD) flows; a necessary and sufficient condition for the linear stability of rectilinear vortex sheets was obtained by Wang and Yu [32]. Moreover, for three-dimensional compressible MHD flows, Trakhinin [30, 31] and Chen-Wang [1] adopted a different symmetrization approach to prove the linear and nonlinear stability of compressible vortex sheets. These results indicate the stabilizing effect of magnetic fields on current-vortex sheets.

On the other hand, the stabilizing effect of magnetic fields on current-vortex sheets has also been demonstrated in the incompressible case. Under the Syrovatskij stability condition [26], Morando, Trakhinin, and Trebeschi [17] proved an a priori estimate with a loss of three derivatives for the linearized system. Trakhinin [31] established an a priori estimate without loss of derivatives from the data for the linearized system with variable coefficients under a strong stability condition. In a recent work [7], Coulombel, Morando, Secchi, and Trebeschi proved an a priori estimate without loss of derivatives for the nonlinear current-vortex sheet problem under the strong stability condition. Nonlinear stability of the incompressible current-vortex sheet problem was proved by Sun, Wang, and Zhang under the Syrovatskij stability condition in [29]. For vortex sheets in elastic flows, Chen, Hu, and Wang ([2], [3]) showed that strong elasticity can inhibit Kelvin-Helmholtz instability. Furthermore, Chen, Huang, Wang, and Yuan proved the stabilizing effect of elasticity on three-dimensional compressible vortex sheets in [4].

Meanwhile, progress has also been made on the ill-posedness of solutions for the Kelvin-Helmholtz problem in ideal fluids. For the Kelvin-Helmholtz instability in incompressible Euler flows, Ebin in [9] proved the linear and nonlinear ill-posedness of the well-known Kelvin-Helmholtz problem. Recently, we proved the linear and nonlinear ill-posedness of the Kelvin-Helmholtz problem for incompressible MHD fluids under conditions that violate the Syrovatskij stability condition [35], [37]. We also prove the ill-posedness of the Kelvin-Helmholtz problem for three-dimensional in compressible elastodynamic fluids where the destabilizing effect of velocity shear dominates the stabilizing influence of the elastic force [36].

For the Kelvin-Helmholtz instability in compressible Euler flows, it has been well established since Landau [15] that the instability is suppressed in compressible supersonic flows. Through normal mode analysis, it was further shown in [11] and [21] that the linear Kelvin-Helmholtz instability can be inhibited when the Mach number $M := \frac{|\bar{v}_1^+|}{\bar{c}_s} > \sqrt{2}$, while the solutions of the linear equation are violently unstable when $M < \sqrt{2}$. In our work [38], we proved that the ill-posedness of the Kelvin-Helmholtz problem for nonlinear Euler fluids exhibits the same behavior as its linearized counterparts in [11, 21] under the condition $\epsilon_0 \le M \le \sqrt{2} - \epsilon_0$, where ϵ_0 is a small but fixed constant. Recently, we also showed that weak magnetic fields and elasticity have a destabilizing effect on the Kelvin-Helmholtz instability [39, 40, 41].

1.6. **Definitions and terminology.** Prior to presenting the core conclusion, we introduce the notation that will be utilized tin this paper. $C > 0$ will stand for a general constant which may rely on the parameters of the problem, yet is independent of the data and other relevant quantities. We term such constants "universal constants". These constants are permitted to vary from one inequality to another. We will adopt the notation $a \lesssim b$ to indicate that $a \le Cb$ for a universal constant $C > 0$. Additionally, the notation $a \gtrsim b$ signifies $a \ge Cb$. In the meantime, we will employ $\Re$ to represent the real part of a complex number or a complex function.

Given that we investigate two mutually disjoint fluids, for a function ψ defined on Ω, we denote ψ_+ as the restriction of ψ to Ω_+ and ψ_- as the restriction of ψ to Ω_-. For every $j \in \mathbb{R}$, we define the piecewise Sobolev space as follows:

$$H^j(\Omega) := \{\psi | \psi^+ \in H^j(\Omega^+), \psi^- \in H^j(\Omega^-)\}, \tag{1.24}$$

endowed with the norm $\|\psi\|_{H^j}^2 = \|\psi^+\|_{H^j(\Omega^+)}^2 + \|\psi^-\|_{H^j(\Omega^-)}^2$. The conventional Sobolev norm $\|\psi\|_{H^j(\Omega^\pm)}^2$ is furnished with the norm

$$\begin{aligned} \|\psi\|_{H^j(\Omega^\pm)}^2 &:= \sum_{s=0}^{j} \int_{\mathbb{R}\times I_\pm} (1+k^2)^{j-s} |\partial_2^s \hat{\psi}^\pm(k,x_2)|^2 dk dx_2 \\ &= \sum_{s=0}^{j} \int_{\mathbb{R}} (1+k^2)^{j-s} \|\partial_2^s \hat{\psi}^\pm(k,\cdot)\|_{L^2(I_\pm)}^2 dk, \end{aligned} \tag{1.25}$$

in which $I_- = (-\infty, 0)$ and $I_+ = (0,\infty)$, and $\hat{\psi}$ denotes the Fourier transform corresponding to ψ by means of

$$\hat{\psi}(k) = \int_{\mathbb{R}} \psi e^{-ix_1 k} dx_1, \tag{1.26}$$

for a function ψ defined on Γ, we introduce the standard Sobolev space through

$$\|\psi\|_{H^j(\Gamma)}^2 := \int_{\mathbb{R}} (1+k^2)^j |\hat{\psi}(k)|^2 dk. \tag{1.27}$$

To simplify the notation, for $j \geq 0$ we introduce

$$\|(f, q_g, q_r, v)(t)\|_{H^j} = \|f(t)\|_{H^j(\Gamma)} + \|q_g(t)\|_{H^j(\Omega)} + \|q_r(t)\|_{H^j(\Omega)} + \|v(t)\|_{H^j(\Omega)}. \tag{1.28}$$

1.7. **Main result.** This paper is dedicated to establishing the ill-posedness of the Kelvin-Helmholtz problem for the compressible Euler system with a radiation field under the destabilizing influence of velocity shear that violates the supersonic stability condition:

$$\epsilon_0 \leq M := \frac{|\bar{v}_1^+|}{\bar{c}_s} < \sqrt{2(1+\ell)}, \tag{1.29}$$

in which ϵ_0 denotes a small yet fixed constant, $\bar{c}_s := \sqrt{p_g'(\bar{\rho})}$ and ℓ is the radiation parameter which is defined in (2.14).

Definition 1.2. We say that the perturbed problem (5.6) possesses property $EE(k)$ for a certain $k \geq 3$ if there exist $\delta, t_0, C > 0$ as well as a function $F : [0,\delta) \to \mathbb{R}^+$ that satisfies $F(z) \leq Cz$ for all $z \in [0,\delta)$, such that for any initial data $(\tilde{f}_0, \tilde{h}_0, \tilde{v}_0)$ fulfilling

$$\|\tilde{f}_0, \tilde{q}_{g,0}, \tilde{q}_{r,0}, \tilde{v}_0\|_{H^k} < \delta, \tag{1.30}$$

there exist a unique solution $(\tilde{f}, \tilde{q}_g, \tilde{q}_r, \tilde{v})$ to the perturbed problem (5.6) with the initial data $(\tilde{f}, \tilde{q}_g, \tilde{q}_r, \tilde{v})|_{t=0} = (\tilde{f}_0, \tilde{q}_{g,0}, \tilde{q}_{r,0}, \tilde{v}_0)$, and the following inequality holds

$$\sup_{0\leq t\leq t_0} \|(\tilde{f}, \tilde{q}_g, \tilde{q}_r, \tilde{v})(t)\|_{H^3} \leq F(\|(\tilde{f}_0, \tilde{q}_{g,0}, \tilde{q}_{r,0}, \tilde{v}_0)\|_{H^k}). \tag{1.31}$$

Theorem 1.3. *Assume that the initial data satisfy conditions* (1.9) *and* (1.14)*; furthermore, suppose that the rectilinear solution satisfies the instability condition* (1.29)*. Then the perturbed problem* (5.6) *fails to possess property $EE(k)$ in accordance with Definition 1.2.*

The remainder of this paper is structured as follows. In Section 2, we obtain a linearized system within a new coordinate and construct a growing solution for this linear system. In Section 3, we conduct an analysis on the roots of the symbol associated with the linearized system for the Kelvin-Helmholtz problem in two-dimensional Euler fluids with a radiation field, and determine the instability condition that the rectilinear solutions are required to satisfy. Sections 4 and 5 are dedicated to establishing the linear and nonlinear ill-posedness, respectively, of the Kelvin-Helmholtz problem for Euler fluids with a radiation field.

2. Linearized equations in new coordinates

In the current section, we examine a linearized system within the new coordinate system and aim to establish a growing normal mode solution for this linearized system. By performing the Fourier transform on the linearized system and solving the linear equation associated with $\hat{s}$, we ultimately deduce a second-order ordinary differential equation for $\hat{g}$ and obtain the symbol corresponding to $\hat{g}$. Through analyzing the symbol of $\hat{g}$, we derive the instability criterion for the linear system of the Kelvin-Helmholtz problem involving compressible Euler fluids with radiation field.

2.1. **Construction of a growing solution of the linearized system.** It can be easily verified that the particular solution in Euler coordinates is also a particular solution within the new coordinate system, such that

$$\bar{v}^{\pm} = \bar{u}^{\pm} = \begin{cases} (\bar{v}_1^{+}, 0) & x_2 \geq 0, \\ (\bar{v}_1^{-}, 0) & x_2 < 0, \end{cases} \tag{2.1}$$

with $\bar{v}_1^{+} = -\bar{v}_1^{-}$ and

$$\bar{\varrho}^{+} = \bar{\varrho}^{-} := \bar{\varrho}. \tag{2.2}$$

At this stage, we examine a linearized equation with constant coefficients, which is derived by linearizing the equation (1.13) around the constant velocity $\bar{v}^{\pm} = (\bar{v}_1^{\pm}, 0)$, constant pressure $\bar{q}_g^{+} + \bar{q}_r^{+} = \bar{q}_g^{-} + \bar{q}_r^{-}$, and flat interface $\Gamma = \{x_2 = 0\}$ (i.e., $\bar{f} = 0$), where the outer normal vector $\bar{n} = (0, 1) := e_2$. The linearized equations obtained in this manner are

$$\begin{gathered} \partial_t q_g + \bar{v}_1 \partial_1 q_g + \bar{\varrho} \bar{c}_s^2 \operatorname{div} v = 0, \quad \text{in } \Omega, \\ \bar{\varrho}(\partial_t v + \bar{v}_1 \partial_1 v) + \nabla(q_g + q_r) = 0, \quad \text{in } \Omega, \\ \partial_t q_r + \bar{v}_1 \partial_1 q_r + \frac{1}{3}(\bar{q}_g + 4\bar{q}_r) \nabla \cdot v = 0 \quad \text{in } \Omega, \\ \partial_t f = v_2 - \bar{v}_1 \partial_1 f, \quad \text{on } \Gamma. \end{gathered} \tag{2.3}$$

To linearize the jump conditions contained in (1.13), we set $v = \bar{v} + \tilde{v}$ and $n = e_2 + \tilde{n}$, and then linearize the original boundary condition $[v \cdot n] = 0$ in the following manner:

$$[(\bar{v} + \tilde{v}) \cdot (e_2 + \tilde{n})] = [\tilde{v} \cdot e_2] + [\bar{v} \cdot \tilde{n}] + [\tilde{v} \cdot \tilde{n}] = 0,$$

in which $\tilde{n} = (-\partial_1 \tilde{f}, 0)$. Obviously, the third term corresponds to a nonlinear term, from which it can be deduced that

$$[\tilde{v} \cdot e_2] = -[\bar{v} \cdot \tilde{n}] = 2\bar{v}_1^{+} \partial_1 \tilde{f}.$$

Accordingly, the jump conditions imposed on the boundary may be subjected to linearization to form

$$[q_g + q_r] = 0, \quad [v \cdot e_2] = 2\bar{v}_1^{+} \partial_1 f. \tag{2.4}$$

By carrying out linearization on the evolution equation (1.18) around the rectilinear solution, we derive a linearized equation associated with the front f

$$\partial_t^2 f + (\bar{v}_1^{+})^2 \partial_1^2 f + \frac{1}{2\bar{\varrho}} \partial_2 (q_g^{+} + q_r^{+} + q_g^{-} + q_r^{-}) = 0 \quad \text{on } \Gamma. \tag{2.5}$$

Meanwhile, by conducting linearization on the nonlinear wave equations with boundary values (1.23), we derive the linear system associated with the pressure h as follows

$$\begin{gathered} (\partial_t + \bar{v}_1 \partial_1)^2 q_g - \bar{c}_s^2 \Delta(q_g + q_r) = 0 \quad \text{in } \Omega, \\ [q_g + q_r] = 0 \quad \text{on } \Gamma, \\ [\frac{1}{\bar{\varrho}} \partial_2 (q_g + q_r)] = -4\bar{v}_1^{+} \partial_1 \partial_t f \quad \text{on } \Gamma. \end{gathered} \tag{2.6}$$

Considering that we aim to construct a solution for the linear system (2.3)-(2.6) that has an increasing H^k norm for any k, in other words, we assume that the solution adopts the normal mode form

$$\begin{gathered} q_g(t,x_1,x_2)=e^{\lambda t}s(x_1,x_2), \quad q_r(t,x_1,x_2)=e^{\lambda t}t(x_1,x_2),\\ v(t,x_1,x_2)=e^{\lambda t}w(x_1,x_2), \quad f(t,x_1)=e^{\lambda t}g(x_1). \end{gathered} \tag{2.7}$$

In the current setting, we assume that $\lambda=\tau+i\delta\in\mathbb{C}\backslash\{0\}$ remains uniform on both the upper and lower interfaces. A solution that satisfies the condition $\Re(\lambda)>0$ is correlated with a growing mode. After substituting the ansatz (2.7) into (2.3)-(2.6), we deduce that

$$\begin{gathered} \lambda s+\bar v_1\partial_1 s+\bar\varrho\bar c_s^2\operatorname{div} w=0 \quad \text{in } \Omega,\\ \bar\varrho(\lambda w+\bar v_1\partial_1 w)+\nabla(s+t)=0 \quad \text{in } \Omega,\\ \lambda t+\bar v_1\partial_1 t+\frac{1}{3}(\bar q_g+4\bar q_r)\operatorname{div} w=0 \quad \text{in } \Omega,\\ \lambda g=w_2-\bar v_1\partial_1 g, \quad [w\cdot e_2]=2\bar v_1^+\partial_1 g, \quad [s+t]=0 \quad \text{on } \Gamma, \end{gathered} \tag{2.8}$$

and

$$\lambda^2 g+(\bar v_1^+)^2\partial_1^2 g+\frac{1}{2\bar\rho}\partial_2(s^++t^++s^-+t^-)=0 \quad \text{on } \Gamma, \tag{2.9}$$

and

$$\begin{gathered} (\lambda+\bar v_1\partial_1)^2 s-\bar c_s^2\Delta(s+t)=0 \quad \text{in } \Omega,\\ [s+t]=0 \quad \text{on } \Gamma,\\ [\frac{1}{\bar\varrho}\partial_2(s+t)]=-4\bar v_1^+\lambda\partial_1 g \quad \text{on } \Gamma. \end{gathered} \tag{2.10}$$

2.2. **Formula for $\partial_2\hat s^++\partial_2\hat s^-$.** We conduct the horizontal Fourier transform on the equations (2.8), (2.9) and (2.10), and deduce the expression of $\partial_2\hat s^++\partial_2\hat s^-$ on Γ; afterwards, we substitute this expression into (2.9), thus acquiring a second-order equation for g that is not coupled with any other physical quantities. First of all, we define the Fourier transforms of s,t and g,w as follows:

$$\begin{gathered} \hat s(k,x_2)=\int_{\mathbb{R}} s(x_1,x_2)e^{-ix_1k}dx_1, \quad \hat t(k,x_2)=\int_{\mathbb{R}} t(x_1,x_2)e^{-ix_1k}dx_1,\\ \hat g(k,x_2)=\int_{\mathbb{R}} g(x_1,x_2)e^{-ix_1k}dx_1, \quad \hat w(k,x_2)=\int_{\mathbb{R}} w(x_1,x_2)e^{-ix_1k}dx_1. \end{gathered}$$

Computing the Fourier transform on the equations (2.8), (2.9), (2.10) with respect to the horizontal variable x_1 gives rise to

$$\begin{gathered} (\lambda+i\bar v_1k)\hat s+\bar\varrho\bar c_s^2(ik\hat w_1+\partial_2\hat w_2)=0 \quad \text{in } \Omega,\\ \bar\varrho(\lambda+i\bar v_1k)\widehat{w}_1+ik(\hat s+\hat t)=0 \quad \text{in } \Omega,\\ \bar\varrho(\lambda+i\bar v_1k)\widehat{w}_2+\partial_2(\hat s+\hat t)=0 \quad \text{in } \Omega,\\ (\lambda+i\bar v_1k)\hat t+\frac{1}{3}(\bar q_g+4\bar q_r)(ik\hat w_1+\partial_2\hat w_2)=0 \quad \text{in } \Omega,\\ (\lambda+i\bar v_1k)\hat g=\hat w_2, \quad [\hat w_2]=2i\bar v_1k\hat g, \quad [\hat s+\hat t]=0 \quad \text{on } \Gamma, \end{gathered} \tag{2.11}$$

and

$$\lambda^2\hat g-(\bar v_1^+)^2k^2\hat g+\frac{1}{2\bar\varrho}\partial_2(\hat s^++\hat t^++\hat s^-+\hat t^-)=0 \quad \text{on } \Gamma, \tag{2.12}$$

and

$$\begin{gathered} (\lambda+i\bar v_1k)^2\hat s+\bar c_s^2k^2(\hat s+\hat t)-\bar c_s^2\partial_2^2(\hat s+\hat t)=0 \quad \text{in } \Omega,\\ [\hat s+\hat t]=0 \quad \text{on } \Gamma,\\ [\frac{1}{\bar\varrho}\partial_2(\hat s+\hat t)]=-4i\bar v_1^+k\lambda\hat g \quad \text{on } \Gamma. \end{gathered} \tag{2.13}$$

On the basis of the first equation and the fourth equation in (2.11), we can deduce the relationship between the gas pressure $\hat s$ and the radiation pressure

$$\hat t=\ell\hat s, \ \ell:=(1+4\bar q_r/\bar q_g)/3. \tag{2.14}$$

In the current setting, we assume that the radiation pressure is lower than the gas pressure, which implies $\frac{1}{3} \le \ell \le \frac{5}{3}$. Afterwards, we substitute this relationship (2.14) into (2.12) and (2.13) so as to obtain

$$\lambda^2 \hat{g} - (\bar{v}_1^+)^2 k^2 \hat{g} + \frac{1+\ell}{2\bar{\varrho}} \partial_2(\hat{s}^+ + \hat{s}^-) = 0 \text{ on } \Gamma, \tag{2.15}$$

and

$$\begin{gathered} (\lambda + i\bar{v}_1 k)^2 \hat{s} + (1+\ell)\bar{c}_s^2 k^2 \hat{n} - (1+\ell)\bar{c}_s^2 \partial_2^2 \hat{s} = 0 \quad \text{in } \Omega, \\ [\hat{s}] = 0 \quad \text{on } \Gamma, \\ \big[\frac{1+\ell}{\bar{\varrho}} \partial_2 \hat{s}\big] = -4i\bar{v}_1^+ \lambda k \hat{g} \quad \text{on } \Gamma. \end{gathered} \tag{2.16}$$

By solving this system, we deduce that

$$\hat{s}(k, x_2) = \begin{cases} \frac{\bar{\varrho} 4i\bar{v}_1^+ k\lambda \hat{g}}{(1+\ell)(\mu^+ + \mu^-)} e^{-\bar{\mu}^+ x_2} & x_2 \ge 0, \\ \frac{\bar{\varrho} 4i\bar{v}_1^+ k\lambda \hat{g}}{(1+\ell)(\mu^+ + \mu^-)} e^{\bar{\mu}^- x_2} & x_2 < 0, \end{cases} \tag{2.17}$$

in which $\mu^\pm = \sqrt{\frac{(\lambda \pm i\bar{v}_1^+ k)^2}{(1+\ell)\bar{c}_s^2} + k^2}$ serve as the roots of the equation

$$(1+\ell)\bar{c}_s^2 X^2 - (\lambda \pm i\bar{v}_1^+ k)^2 - (1+\ell)\bar{c}_s^2 k^2 = 0. \tag{2.18}$$

In the current setting, we notice that $\Re\mu^\pm > 0$ on the condition that $\Re\lambda > 0$.

By direct computations, taking into account (2.17), we arrive at

$$\frac{1+\ell}{\bar{\varrho}} (\partial_2 \hat{s}^+ + \partial_2 \hat{s}^-) = -4i\bar{v}_1^+ \lambda k \hat{g} \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-}. \tag{2.19}$$

Finally, by substituting (2.19) into (2.15), we acquire a second-order differential equation associated with $\hat{g}$,

$$(\lambda^2 - (\bar{v}_1^+)^2 k^2 - 2i\bar{v}_1^+ k\lambda \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-}) \hat{g} = 0 \quad \text{on } \Gamma, \tag{2.20}$$

in which the symbol associated with (2.20) is defined as

$$\Sigma := \lambda^2 - (\bar{v}_1^+)^2 k^2 - 2i\bar{v}_1^+ k\lambda \frac{\mu^+ - \mu^-}{\mu^+ + \mu^-}. \tag{2.21}$$

3. Analysis of the symbol Σ

In this section, we analyze the symbol (2.21) constructed in line with Morando-Secchi-Trebeschi [20]. First of all, we introduce a set of "frequencies"

$$\Xi = \{(\lambda, k) \in \mathbb{C} \times \mathbb{R} : \Re\lambda > 0, (\lambda, k) \ne (0, 0)\}. \tag{3.1}$$

As we have already stated $\Re\mu^\pm > 0$ at all points where $\Re\lambda > 0$. It can be deduced that $\Re(\mu^+ + \mu^-) > 0$ and consequently $\mu^+ + \mu^- \ne 0$ at all such points. On the basis of (2.21), the notation Σ is defined at points $(\lambda, k) \in \Xi$. We also need to determine whether the difference $\mu^+ - \mu^-$ vanishes in the subsequent lemma.

Lemma 3.1. *Let $(\lambda, k) \in \Xi$. Then $\mu^+ = \mu^-$ if and only if $(\lambda, k) = (\lambda, 0)$.*

Proof. On the basis of (2.18), it can be inferred that $(\mu^+)^2 = (\mu^-)^2$ if and only if $k = 0$ or $\lambda = 0$. Given that $(\lambda, k) \in \Xi$, only the case of $k = 0$ requires investigation. When $k = 0$, it can be deduced that $\mu^+ = \mu^- = \frac{\lambda}{\sqrt{1+\ell}\bar{c}_s}$. $\square$

Next we shall analyze the roots of the symbol (2.21) under the instability scenario.

Lemma 3.2. *Let $\Sigma(\lambda, k)$ denote the symbol defined in (2.21), where $(\lambda, k) \in \Xi$. Provided that $|\bar{v}_1^+| < \sqrt{2(1+\ell)}\bar{c}_s$, then $\Sigma(\lambda, k) = 0$ if and only if*

$$\lambda = X_1 k, \tag{3.2}$$

where $X_1^2 = \sqrt{4(1+\ell)\bar{c}_s^2 \bar{v}_1^2 + (1+\ell)^2 \bar{c}_s^4} - (\bar{v}_1^+)^2 - (1+\ell)\bar{c}_s^2 > 0$.

Proof. From the definition of Σ and Lemma 3.1, we can confirming that $\Sigma(\lambda, 0) = \lambda^2 \neq 0$ for $(\lambda, 0) \in \Xi$. Similarly, it is straightforward to verify that $\Sigma(\lambda, k) = \Sigma(\lambda, -k)$. Hence, without any loss of generality, we suppose that $\lambda \neq 0$, $k \neq 0$ and $k > 0$; besides, on the ground of Lemma 3.1, we can deduce that $\mu^+ - \mu^- \neq 0$. For this purpose, we perform the computation

$$\frac{\bar{\mu}^+ - \bar{\mu}^-}{\bar{\mu}^+ + \bar{\mu}^-} = \frac{(\bar{\mu}^+ - \bar{\mu}^-)^2}{(\bar{\mu}^+)^2 - (\bar{\mu}^-)^2} = \frac{(1+\ell)\bar{c}_s^2(\bar{\mu}^+ - \bar{\mu}^-)^2}{4i\bar{v}_1^+ k\lambda}, \tag{3.3}$$

$$(\mu^+ - \mu^-)^2 = 2\Big(\frac{\lambda^2}{(1+\ell)\bar{c}_s^2} - \frac{(\bar{v}_1^+)^2 k^2}{(1+\ell)\bar{c}_s^2} + k^2 - \mu^+\mu^-\Big). \tag{3.4}$$

Hence, we can deduce that

$$\frac{\mu^+ - \mu^-}{\mu^+ + \mu^-} = \frac{\lambda^2 - (\bar{v}_1^+ k)^2 + (1+\ell)\bar{c}_s^2(k^2 - \mu^+\mu^-)}{2i\bar{v}_1^+ k\lambda}, \tag{3.5}$$

and by substituting the formula (3.5) into (2.20), we are able to rewrite it as

$$(1+\ell)\bar{c}_s^2(\mu^+\mu^- - k^2)\hat{g} = 0 \text{ on } \Gamma, \tag{3.6}$$

the symbol Σ may be rewritten as

$$\Sigma = (1+\ell)\bar{c}_s^2(\mu^+\mu^- - k^2). \tag{3.7}$$

Let us define $\mu^+\mu^- - k^2 = 0$ and introduce the relevant quantities:

$$X = \frac{\lambda}{k}, \quad \tilde{\mu}^\pm = \frac{\mu^\pm}{k}. \tag{3.8}$$

For this reason, we can infer that

$$\tilde{\mu}^+\tilde{\mu}^- = 1, \tag{3.9}$$

$$(\tilde{\mu}^+)^2(\tilde{\mu}^-)^2 = 1. \tag{3.10}$$

From the formula for the roots $\mu^\pm$, it can be deduced that

$$(\tilde{\mu}^+)^2 = \frac{(X + i\bar{v}_1^+)^2}{(1+\ell)\bar{c}_s^2} + 1, \tag{3.11}$$

$$(\tilde{\mu}^-)^2 = \frac{(X - i\bar{v}_1^+)^2}{(1+\ell)\bar{c}_s^2} + 1, \tag{3.12}$$

hence by substituting (3.11) and (3.12) into (3.10), we obtain

$$[(X + i\bar{v}_1^+)^2 + (1+\ell)\bar{c}_s^2][(X - i\bar{v}_1^+)^2 + (1+\ell)\bar{c}_s^2] = (1+\ell)^2\bar{c}_s^4, \tag{3.13}$$

which gives rise to a quadratic equation with respect to X^2:

$$X^4 + 2((\bar{v}_1^+)^2 + (1+\ell)\bar{c}_s^2)X^2 + (\bar{v}_1^+)^4 - 2(1+\ell)\bar{c}_s^2(\bar{v}_1^+)^2 = 0. \tag{3.14}$$

By applying the quadratic root formula, the two roots of the equation (3.14) are

$$X_1^2 = -(\bar{v}_1^+)^2 - (1+\ell)\bar{c}_s^2 + \sqrt{4(1+\ell)\bar{c}_s^2\bar{v}_1^2 + (1+\ell)^2\bar{c}_s^4}, \tag{3.15}$$

$$X_2^2 = -(\bar{v}_1^+)^2 - (1+\ell)\bar{c}_s^2 - \sqrt{4(1+\ell)\bar{c}_s^2\bar{v}_1^2 + (1+\ell)^2\bar{c}_s^4}. \tag{3.16}$$

We assert that the points $(\lambda, k) \in \Sigma$ satisfying $\lambda = \pm X_2 k$ are not the roots of $\mu^+\mu^- = k^2$. Without loss of generality, we may assume that Y_2 is positive. On the basis of (3.16), we infer that

$$X_2 = iY_2, Y_2 = \sqrt{(\bar{v}_1^+)^2 + (1+\ell)\bar{c}_s^2 + \sqrt{(1+\ell)^2\bar{c}_s^4 + 4(1+\ell)\bar{c}_s^2(\bar{v}_1^+)^2}}. \tag{3.17}$$

Then we infer that $Y_2 \pm \bar{v}_1^+ > \sqrt{1+\ell}\,\bar{c}_s$. From (3.11) and (3.12), we deduce that

$$\tilde{\mu}^+ = i\sqrt{\frac{(Y_2 + \bar{v}_1^+)^2}{(1+\ell)\bar{c}_s^2} - 1}, \quad \tilde{\mu}^- = i\sqrt{\frac{(Y_2 - \bar{v}_1^+)^2}{(1+\ell)\bar{c}_s^2} - 1},$$

from which we can conclude that $\tilde{\mu}^+\tilde{\mu}^- = 1$ does not hold. Similarly, we may demonstrate that the point $(\lambda, k) \in \Sigma$ satisfying $\lambda = -X_2 k$ is not a root of $\mu^+\mu^- = k^2$. On the other hand, on the ground of (3.17), we can conclude that $\lambda = iY_2k$ is an imaginary root, which thereby implies that $\Re\lambda = 0$ and $(\pm X_2k, k) \notin \Xi$.

Now we concentrate on the root X_1^2. If $\bar{v}_1^+ < \sqrt{2(1+\ell)}\bar{c}_s$, on the basis of (3.15), we can conclude that X_1^2 is positive, from which it follows that $\lambda = \pm X_1 k$ are real-valued. The point $(-X_1k, k) \nsubseteq \Xi$, so we discard this point and only investigate the root $\lambda = +X_1k$. By utilizing the fact that the square roots of the complex number $a + ib$ are

$$\pm\{\sqrt{\frac{r+a}{2}} + i\,\mathrm{sign}(b)\sqrt{\frac{r-a}{2}}\}, \quad r = |a+ib|, \tag{3.18}$$

in our research scenario, we may calculate

$$\mu^+ = \sqrt{\frac{r+a}{2}} + i\sqrt{\frac{r-a}{2}}, \quad \mu^- = \sqrt{\frac{r+a}{2}} - i\sqrt{\frac{r-a}{2}}, \tag{3.19}$$

where

$$a = \frac{X_1^2 - (\bar{v}_1^+)^2 + (1+\ell)\bar{c}_s^2}{(1+\ell)\bar{c}_s^2}k^2, \quad b = \frac{2X_1\bar{v}^+}{(1+\ell)\bar{c}_s^2}k^2, \tag{3.20}$$

such that $\mu^+\mu^- = r > 0$, consequently we can deduce that under the condition of $\bar{v}_1^+ < \sqrt{2(1+\ell)}\bar{c}_s$, the root of the symbol Σ is the point $(+X_1k, k)$. To sum up, we are able to obtain a root (λ, k) satisfying $\Re\lambda > 0$, which corresponds to an unstable solution. □

4. Ill-posedness of solutions to the linear problem

4.1. **Uniqueness for the linearized equations** (2.3)-(2.4)**.** First, we establish a uniqueness conclusion for the linearized equations (2.3)-(2.4).

Lemma 4.1. *Let f, q_g, q_r, v denote a solution to the linearized equations* (2.3)-(2.4) *subject to the initial condition $(f, q_g, q_r, v)|_{t=0} = 0$. It then follows that $(f, q_g, q_r, v) \equiv 0$.*

Proof. Taking the standard inner product of the first, second and third equations in (2.3) with q_g^+, v^+, q_r^+ and integrating over Ω^+, we obtain

$$\frac{1}{2}\partial_t \int_{\Omega^+} \frac{1}{\bar{\varrho}\bar{c}_s^2}|q_g^+|^2 + \frac{1}{2}\int_{\Omega^+} \bar{v}_1\partial_1(\frac{1}{\bar{\varrho}\bar{c}_s^2}|q_g^+|^2) + \int_{\Omega^+} q_g^+ \operatorname{div} v^+ = 0, \tag{4.1}$$

$$\frac{1}{2}\partial_t \int_{\Omega^+} \bar{\varrho}|v^+|^2 + \frac{1}{2}\int_{\Omega^+} \bar{v}_1^+\partial_1(\bar{\varrho}|v^+|^2) + \int_{\Omega^+} \nabla(q_g^+ + q_r^+)v^+ = 0, \tag{4.2}$$

$$\frac{1}{2}\partial_t \int_{\Omega^+} \frac{3}{\bar{q}_g^+ + 4\bar{q}_r^+}|q_r^+|^2 + \frac{1}{2}\int_{\Omega^+} \bar{v}_1\partial_1(\frac{3}{\bar{q}_g^+ + 4\bar{q}_r^+}|q_r^+|^2) + \int_{\Omega^+} q_r^+ \operatorname{div} v^+ = 0. \tag{4.3}$$

The second terms on the left-hand side of (4.1), (4.2) and (4.3) vanish; therefore, by summing up (4.1), (4.2) and (4.3) and conducting integration by parts, we can obtain

$$\frac{1}{2}\partial_t \int_{\Omega^+} (\frac{1}{\bar{\varrho}\bar{c}_s^2}|q_g^+|^2 + \bar{\varrho}|v^+|^2 + \frac{3}{\bar{q}_g^+ + 4\bar{q}_r^+}|q_r^+|^2) = \int_{\Gamma} (q_g^+ + q_r^+)v^+ \cdot e_2. \tag{4.4}$$

An analogous conclusion holds true on Ω_- with the opposite sign appearing on the right-hand side,

$$\frac{1}{2}\partial_t \int_{\Omega^-} (\frac{1}{\bar{\varrho}\bar{c}_s^2}|q_g^-|^2 + \bar{\varrho}|v^-|^2 + \frac{3}{\bar{q}_g^+ + 4\bar{q}_r^+}|q_r^-|^2) = -\int_{\Gamma} (q_g^- + q_r^-)v^- e^2. \tag{4.5}$$

Summing (4.4) and (4.5) leads to

$$\begin{aligned} &\frac{1}{2}\partial_t \int_{\Omega} (\frac{1}{\bar{\varrho}\bar{c}_s^2}|q_g|^2 + \bar{\varrho}|v|^2 + \frac{3}{\bar{q}_g + 4\bar{q}_r}|q_r|^2) \\ &= \int_{\Gamma} [(q_g + q_r)v \cdot e_2] = 2\int_{\Gamma} (q_g + q_r)\bar{v}_1^+\partial_1 f. \end{aligned} \tag{4.6}$$

Furthermore, by multiplying the third equation in (2.3) by f, we can obtain

$$\frac{1}{2}\partial_t \int_{\Gamma} |f|^2 = \int_{\Gamma} v_2 f. \tag{4.7}$$

Summing (4.6) and (4.7) and applying the Hölder inequality yields

$$\begin{aligned}&\frac{1}{2}\partial_t\int_\Omega(\frac{1}{\bar\varrho\bar c_s^2}|q_g|^2+\bar\varrho|v|^2+\frac{3}{\bar q_g+4\bar q_r}|q_r|^2)+\frac{1}{2}\partial_t\int_\Gamma|f|^2\\&=2\int_\Gamma(q_g+q_r)\bar v_1^+\partial_1 f+\int_\Gamma v_2 f\\&\le 2|\bar v_1^+|(\|q_g\|_{L^2(\Gamma)}+\|q_r\|_{L^2(\Gamma)})\|\partial_1 f\|_{L^2(\Gamma)}+\|v_2\|_{L^2(\Gamma)}\|f\|_{L^2(\Gamma)}:=J.\end{aligned}\tag{4.8}$$

To preclude the loss of derivatives, we assume that the solutions are band-limited with a radius of $R>0$, that is to say, that

$$\underset{x_2\in\mathbb{R}}{\cup}\operatorname{supp}(|\hat f(\cdot)|+|\hat q_g(\cdot,x_2)|+|\hat q_r(\cdot,x_2)|+|\hat v(\cdot,x_2)|)\subset B(0,R).$$

Furthermore, we present an anisotropic trace estimate in [33, Lemma B.1]:

$$\|\phi\|^2_{L^2(\Gamma)}\le C(\|\bar v\cdot\nabla\phi\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}+\|\phi\|^2_{L^2(\Omega)}),\tag{4.9}$$

where $|\bar v|=\bar v_1^+\ge\varepsilon_0\sqrt{1+\ell}\bar c_s>0$. At this point, we carry out an estimation of J in the following manner:

$$\begin{aligned}J&\lesssim(\|\bar v_1^+\partial_1 q_g\|_{L^2(\Omega)}\|q_g\|_{L^2(\Omega)}+\|q_g\|^2_{L^2(\Omega)})^{1/2}\|k\hat f\|_{L^2(\Gamma)}\\&\quad+(\|\bar v_1^+\partial_1 q_r\|_{L^2(\Omega)}\|q_r\|_{L^2(\Omega)}+\|q_r\|^2_{L^2(\Omega)})^{1/2}\|k\hat f\|_{L^2(\Gamma)}\\&\quad+(\|\bar v_1^+\partial_1 v_2\|_{L^2(\Omega)}\|v_2\|_{L^2(\Omega)}+\|v_2\|^2_{L^2(\Omega)})^{1/2}\|f\|_{L^2(\Gamma)}\\&\lesssim((\bar v_1^+R+1)R\|q_g\|^2_{L^2(\Omega)})^{1/2}\|f\|_{L^2(\Gamma)}+((\bar v_1^+R+1)R\|q_r\|^2_{L^2(\Omega)})^{1/2}\|f\|_{L^2(\Gamma)}\\&\quad+((\bar v_1^+R+1)\|v_2\|^2_{L^2(\Omega)})^{1/2}\|f\|_{L^2(\Gamma)}.\end{aligned}\tag{4.10}$$

Substituting (4.10) into (4.8) and using Gronwall's inequality, we obtain that for ay arbitrary R,

$$\begin{aligned}&\|f\|^2_{L^2(\Gamma)}+\|q_g\|^2_{L^2(\Omega)}+\|q_r\|^2_{L^2(\Omega)}+\|v\|^2_{L^2(\Omega)}\\&\le C(\|f_0\|^2_{L^2(\Gamma)}+\|q_{g_0}\|^2_{L^2(\Omega)}+\|q_{r_0}\|^2_{L^2(\Omega)}+\|v_0\|^2_{L^2(\Omega)}).\end{aligned}\tag{4.11}$$

in which C depends on R From this, we infer that if $(f,q_g,q_r,v)|_{t=0}=0$, then it can be concluded $(f,q_g,q_r,v)\equiv 0$. □

4.2. **Discontinuous dependence on the initial data.** From Lemma 3.2 and (3.15), if $0<|\bar v_1^+|<\sqrt{2(1+\ell)}\bar c_s$, we can deduce that X_1^2 is positive, from which it can be concluded that $\lambda=X_1k$ are real. Furthermore, we deduce that the equation (2.5) is reduced to

$$\partial_t^2 f+\kappa\partial_1^2 f=0,\tag{4.12}$$

in which λ must be positive when $0<|\bar v_1^+|<\sqrt{2(1+\ell)}\bar c_s$. To begin with, we assert that the equation (2.5) is essentially in the form of (4.12). As a matter of fact, substituting $f=e^{\lambda t}g$ into the equation (4.12), we can obtain $\lambda^2 g+\kappa\partial_1^2 g=0$; thus, the corresponding Fourier transform form is

$$(\lambda^2-\kappa k^2)\hat g=0,\tag{4.13}$$

which can yield $\kappa=\lambda^2/k^2$. On the basis of Lemma 3.2, we can conclude that $X_1^2=\frac{\lambda^2}{k^2}>0$ when $0<|\bar v_1^+|<\sqrt{2(1+\ell)}\bar c_s$. Hence, we can obtain $\lambda=X_1^2>0$. Consequently, (4.12) is actually an elliptic equation. We are now able to demonstrate the ill-posedness of this linear problem.

Lemma 4.2. *When $\epsilon_0\le M:=\frac{|\bar v_1^+|}{\bar c_s}<\sqrt{2(1+\ell)}$, the linear problem (2.3) associated with the corresponding jump boundary conditions (2.4) is ill-posed in the Hadamard sense within $H^k(\Omega)$ for all k. More specifically, for any $k,j\in\mathbb{N}$ with $j\ge k$, as well as for any $T_0>0$ and $\alpha>0$, there exists a sequence $\{(f_n,q_{g,n},q_{r,n},v_n)\}_{n=1}^\infty$ corresponding to (2.3) that satisfies the jump boundary conditions (2.4), such that*

$$\|(f_n(0),q_{g,n}(0),q_{r,n}(0),v_n(0))\|_{H^j}\lesssim\frac{1}{n},\tag{4.14}$$

but

$$\|(f_n(t), q_{g,n}(t), q_{r,n}(t), v_n(t))\|_{H^k} \geq \alpha, \ for\ all\ t \geq T_0. \tag{4.15}$$

Proof. For each $j \in \mathbb{N}$, we define $\chi_n(k) \in C_c^\infty(\mathbb{R})$ as a real-valued function such that $\operatorname{supp}(\chi_n) \subset B(0, n+1) \backslash B(0, n)$ and

$$\int_{\mathbb{R}} (1 + |k|^2)^{j+1} |\chi_n(k)|^2 dk = \frac{1}{\bar{C}_j^{\,2} n^2}. \tag{4.16}$$

Frist we select $\hat{g}_n = \chi_n(k)$ in (4.13) as the solution sequence. Subsequently, we introduce

$$f_n(t, x_1) = e^{\lambda t} g_n(x_1) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{X_1 k t} \chi_n(k) e^{ikx_1} dk, \tag{4.17}$$

which satisfies (4.12). Herein, we employ $\lambda = X_1 k$ on the basis of Lemma 3.2. From this observation, we conclude that the linearized front equation is qualitatively more unstable for large values of frequency k. As $k \to \infty$, the solutions (4.12) with higher frequencies grow more rapidly over time, which supplies a mechanism for Kelvin-Helmholtz instability. By the selection of χ_n and the Plancherel theorem, we obtain the estimate

$$\|f_n(t = 0, x_1)\|_{H^j(\Gamma)} = \|g_n(x_1)\|_{H^j(\Gamma)} = (\int_{\mathbb{R}} (1 + |k|^2)^j |\chi_n(k)|^2 dk)^{1/2} \lesssim \frac{1}{n}, \tag{4.18}$$

similarly, for $n + 1 \geq k \geq n$ and $t \geq T_0$, we obtain

$$\begin{aligned} \|f_n(t, x_1)\|_{H^k(\Gamma)}^2 &\geq e^{2X_1 n T_0} \int_{\mathbb{R}} (1 + |k|^2)^k |\chi_n(k)|^2 dk \\ &\geq \frac{e^{2X_1 n T_0}}{(1 + (n+1)^2)^{j-k+1}} \int_{\mathbb{R}} (1 + k^2)^{j+1} |\chi_n(k)|^2 dk. \end{aligned} \tag{4.19}$$

Let n be adequately large such that

$$\frac{e^{2X_1 n T_0}}{(1 + (n+1)^2)^{j-k+1}} \geq \alpha^2 \bar{C}_j^{\,2} n^2, \tag{4.20}$$

therefore, we can evaluate

$$\|f_n(t)\|_{H^k(\Gamma)} \geq \alpha. \tag{4.21}$$

Based on the calculation in Section 2.2, by recalling the expression formula (2.17) for the solution $\hat{s}_n(k, x_2)$, we can conclude that

$$\hat{s}_n(k, x_2) = \begin{cases} \frac{4i\bar{\varrho}\bar{v}_1^+ k\lambda}{(1+\ell)(\mu^+ + \mu^-)} \hat{g}_n(k) e^{-\mu^+ x_2} & x_2 \geq 0, \\ \frac{4i\bar{\varrho}\bar{v}_1^+ k\lambda}{(1+\ell)(\mu^+ + \mu^-)} \hat{g}_n(k) e^{\mu^- x_2} & x_2 < 0. \end{cases} \tag{4.22}$$

Since $\lambda = X_1 k > 0$ and $k > 0$, from Lemma 3.1 we know that $\mu^+ - \mu^- \neq 0$, then (4.22) can be rewritten as

$$\hat{s}_n(k, x_2) = \begin{cases} \bar{\varrho}\bar{c}_s^2 (\mu^+ - \mu^-) \hat{g}_n(k) e^{-\mu^+ x_2} & x_2 \geq 0, \\ \bar{\varrho}\bar{c}_s^2 (\mu^+ - \mu^-) \hat{g}_n(k) e^{\mu^- x_2} & x_2 < 0, \end{cases} \tag{4.23}$$

herein we observe that $\mu^\pm$ solely rely on k, as we have $\lambda = X_1 k$, hence it indicates that $\mu(\lambda, k) = \mu(X_1 k, k)$.

By recalling the formulas (3.19) and (3.20), we deduce that

$$\Big|\frac{\mu^+ - \mu^-}{\mu^+}\Big|^2 = \frac{|2i\sqrt{\frac{r-a}{2}}|^2}{|\sqrt{\frac{r+a}{2}} + i\sqrt{\frac{r-a}{2}}|^2} = 2\frac{r - a}{r}. \tag{4.24}$$

Employing the Mach number $\epsilon_0 \leq M := \frac{|\bar{v}_1^+|}{\bar{c}_s} < \sqrt{2(1 + \ell)}$ and the root

$$X_1^2 = \sqrt{(1 + \ell)^2 \bar{c}_s^4 + 4(1 + \ell)\bar{c}_s^2 (\bar{v}_1^+)^2} - (\bar{v}_1^+)^2 - (1 + \ell)\bar{c}_s^2,$$

we obtain

$$a = \frac{X_1^2 - (\bar{v}_1^+)^2 + (1 + \ell)\bar{c}_s^2}{(1 + \ell)\bar{c}_s^2} k^2 = (\sqrt{1 + 4M^2} - 2M^2) k^2, \tag{4.25}$$

in which we estimate a in as

$$(\sqrt{1+8(1+\ell)}-4(1+\ell))k^2 < a \le (\sqrt{1+4\epsilon_0^2}-2\epsilon_0^2)k^2, \tag{4.26}$$

herein the lower bound is attained when M assumes the value $\sqrt{2(1+\ell)}$. Furthermore, we calculate

$$\begin{aligned} |\mu^+| &= |\sqrt{\frac{r+a}{2}}+i\sqrt{\frac{r-a}{2}}| = \sqrt{r}, \\ |\mu^-| &= |\sqrt{\frac{r+a}{2}}-i\sqrt{\frac{r-a}{2}}| = \sqrt{r}. \end{aligned} \tag{4.27}$$

From (3.15) and (3.20), it follows that

$$\begin{aligned} r^2 &= a^2+b^2 \\ &= \frac{X_1^4+2((\bar{v}_1^+)^2+(1+\ell)\bar{c}_s^2)X_1^2+(\bar{v}_1^+)^4-2(1+\ell)\bar{c}_s^2(\bar{v}_1^+)^2+(1+\ell)^2\bar{c}_s^4}{(1+\ell)^2\bar{c}_s^4}k^4 \\ &= k^4. \end{aligned} \tag{4.28}$$

Integrating with (4.24), (4.25), (4.26) and (4.28) indicates that

$$\tilde{C} := 2-2(\sqrt{1+4\epsilon_0^2}-2\epsilon_0^2) \le \Big|\frac{\mu^+-\mu^-}{\mu^\pm}\Big|^2 < 10+8\ell-\sqrt{36+32\ell}, \tag{4.29}$$

herein we note that $\epsilon_0 \le M$ must hold, where ϵ_0 is a small yet fixed constant. Since if the Mach number M approaches zero, this lower bound approaches zero.

We evaluate the terms $|\frac{\mu^\pm}{\mu^++\mu^-}|$ by taking account (4.27) and (4.28). We obtain

$$\Big|\frac{\mu^\pm}{\mu^++\mu^-}\Big| = \frac{1}{2(1+\frac{a}{r})} = \frac{1}{\sqrt{2}\sqrt{1+\sqrt{1+4M^2}-2M^2}} \tag{4.30}$$

As $0<M<\sqrt{2(1+\ell)}$, the aforementioned equality possesses the subsequent uniform lower and upper bounds,

$$\frac{1}{2} < \Big|\frac{\mu^\pm}{\mu^++\mu^-}\Big| < C^*, \quad C^* := \frac{1}{\sqrt{18+16\ell}-(3+4\ell)\sqrt{2}}. \tag{4.31}$$

Since we have

$$|\partial_2^m e^{-\mu^+x_2}|^2 = |\mu^+|^{2m}|e^{-\mu^+x_2}|^2 = |\mu^+|^{2m}e^{-2(\Re\mu^+)x_2} = |\mu^+|^{2m}e^{-2\sqrt{\frac{r+a}{2}}x_2}, \tag{4.32}$$

the identical equality applies to μ^-. Consequently, by utilizing (4.29), (4.23), (4.27) and (4.32), we evaluate $\|h_n(0)\|_{H^k(\Omega)}$ as

$$\begin{aligned} &\|q_{g,n}(t=0,x_1,x_2)\|_{H^j(\Omega)}^2 = \|s_n(x_1,x_2)\|_{H^j(\Omega)}^2 \\ &\le \sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\bar{\varrho}^2\bar{c}_s^4|\mu^+-\mu^-|^2|\hat{g}_n(k)|^2\int_0^\infty|\partial_2^m e^{-\mu^+x_2}|^2dx_2dk \\ &\quad+\sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\bar{\varrho}^2\bar{c}_s^4|\mu^+-\mu^-|^2|\hat{g}_n(k)|^2\int_{-\infty}^0|\partial_2^m e^{\mu^-x_2}|^2dx_2dk \\ &\le \sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\bar{\varrho}^2\bar{c}_s^4\Big|\frac{\mu^+-\mu^-}{\mu^+}\Big|^2\Big|\frac{\mu^+}{\mu^++\mu^-}\Big||\mu^+|^{2m+1}|\chi_n(k)|^2dk \\ &\quad+\sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\bar{\varrho}^2\bar{c}_s^4\Big|\frac{\mu^+-\mu^-}{\mu^-}\Big|^2\Big|\frac{\mu^-}{\mu^++\mu^-}\Big||\mu^-|^{2m+1}|\chi_n(k)|^2dk \\ &< 8C^*(j+1)\bar{\varrho}^2\bar{c}_s^4\int_{\mathbb{R}}(1+k^2)^{j+1}|\chi_n(k)|^2dk \\ &\lesssim \frac{1}{n^2}. \end{aligned} \tag{4.33}$$

Meanwhile, by the Plancherel theorem, (4.22), and (4.32) with $s=0$, we obtain

$$\begin{aligned}
\|q_{g,n}(t,x_1,x_2)\|^2_{H^k(\Omega)} &= \|e^{\lambda t} m_n(x_1,x_2)\|^2_{H^k(\Omega)}\\
&\geq \int_{\mathbb{R}} (1+k^2)^k \bar\varrho^2 \bar c_s^4 |\mu^+-\mu^-|^2 |e^{\lambda t}\hat g_n(k)|^2 \int_0^\infty |e^{-\mu^+ x_2}|^2 dx_2 dk\\
&\quad + \int_{\mathbb{R}} (1+k^2)^k \bar\varrho^2 \bar c_s^4 |\mu^+-\mu^-|^2 |e^{\lambda t}\hat g_n(k)|^2 \int_{-\infty}^0 |e^{\mu^- x_2}|^2 dx_2 dk\\
&\geq \int_{\mathbb{R}} (1+k^2)^k \bar\varrho^2 \bar c_s^4 \Big|\frac{\mu^+-\mu^-}{\mu^+}\Big|^2 \Big|\frac{\mu^+}{\mu^+ + \mu^-}\Big| |\mu^+| e^{2X_1 kt} |\chi_n(k)|^2 dk\\
&\quad + \int_{\mathbb{R}} (1+k^2)^k \bar\varrho^2 \bar c_s^4 \Big|\frac{\mu^+-\mu^-}{\mu^-}\Big|^2 \Big|\frac{\mu^-}{\mu^+ + \mu^-}\Big| |\mu^-| e^{2X_1 kt} |\chi_n(k)|^2 dk.
\end{aligned} \tag{4.34}$$

Since $\operatorname{supp}(\chi_n) \subset B(0,n+1)\backslash B(0,n)$, it follows that for $k \geq n \geq 1$ and $t \geq T_0$, we can evaluate (4.34) as

$$\|q_{g,n}(t)\|^2_{H^k(\Omega)} \geq \tilde C \frac{e^{2X_1 n T_0} \bar\varrho^2 \bar c_s^4}{(1+(n+1)^2)^{j-k+1}} \int_{\mathbb{R}} (1+k^2)^{j+1} |\chi_n(k)|^2 dk, \tag{4.35}$$

Let n be sufficiently large such that

$$\tilde C \frac{e^{2X_1 n T_0} \bar\varrho^2 \bar c_s^4}{(1+(n+1)^2)^{j-k+1}} \geq \alpha^2 n^2 \bar C_j^2. \tag{4.36}$$

Therefore, we can evaluate

$$\|q_{g,n}(t)\|_{H^k(\Omega)} \geq \alpha. \tag{4.37}$$

By utilizing the correlation between the gas pressure $\hat q_g$ and the radiation pressure $\hat q_r$: $\hat q_r = \ell \hat q_g$, $\ell := (1+4\bar q_r/\bar q_g)/3$, we deduce that:

$$\|q_{r,n}(t=0,x_1,x_2)\|^2_{H^j(\Omega)} \lesssim \frac{1}{n^2}, \tag{4.38}$$

$$\|q_{r,n}(t)\|_{H^k(\Omega)} \geq \alpha. \tag{4.39}$$

Taking the horizontal Fourier transform of the second equation in (2.8), we arrive at

$$(\lambda + i\bar v_1 k)\hat w_1 + \frac{1}{\bar\varrho} ik(1+\ell)\hat s = 0, \tag{4.40}$$

$$(\lambda + i\bar v_1 k)\hat w_2 + \frac{1}{\bar\varrho} \partial_2(1+\ell)\hat s = 0, \tag{4.41}$$

solving the above equations, by the formula (4.23), we find that

$$\hat w_1(k,x_2) = \begin{cases} \frac{(\mu^+-\mu^-)\bar c_s^2(1+\ell)ik}{\lambda + i\bar v_1^+ k} \hat g(k) e^{-\mu^+ x_2} & x_2 \geq 0,\\ \frac{(\mu^+-\mu^-)\bar c_s^2(1+\ell)ik}{\lambda - i\bar v_1^+ k} \hat g(k) e^{\mu^- x_2} & x_2 < 0, \end{cases} \tag{4.42}$$

and

$$\hat w_2(k,x_2) = \begin{cases} \frac{(\mu^+-\mu^-)\bar c_s^2(1+\ell)\mu^+}{(\lambda + i\bar v_1^+ k)} \hat g(k) e^{-\mu^+ x_2} & x_2 \geq 0,\\ -\frac{(\mu^+-\mu^-)\bar c_s^2(1+\ell)\mu^-}{(\lambda - i\bar v_1^+ k)} \hat g(k) e^{\mu^- x_2} & x_2 < 0. \end{cases} \tag{4.43}$$

Next we evaluate $|\frac{\bar c_s^2(1+\ell)ik}{\lambda \pm i\bar v_1^+ k}|^2$ and $|\frac{\bar c_s^2(1+\ell)\mu^\pm}{\lambda \pm i\bar v_1^+ k}|^2$ in the following manner:

$$\begin{aligned}
\Big|\frac{\bar c_s^2(1+\ell)ik}{\lambda \pm i\bar v_1^+ k}\Big|^2 &= \frac{\bar c_s^4(1+\ell)^2}{X_1^2 + (\bar v_1^+)^2}\\
&= \frac{\bar c_s^4(1+\ell)^2}{\sqrt{(1+\ell)^2\bar c_s^4 + 4(1+\ell)\bar c_s^2(\bar v_1^+)^2} - (1+\ell)\bar c_s^2}\\
&= \frac{(1+\ell)\bar c_s^2}{\sqrt{1+4M^2}-1}.
\end{aligned} \tag{4.44}$$

Given that $\epsilon_0 \le M < \sqrt{2(1+\ell)}$, we conclude that

$$\frac{(1+\ell)\bar{c}_s^2}{\sqrt{9+8\ell}-1} \le \Big|\frac{\bar{c}_s^2(1+\ell)ik}{\lambda \pm i\bar{v}_1^+ k}\Big|^2 \le \frac{(1+\ell)\bar{c}_s^2}{\sqrt{1+4\epsilon_0^2}-1}. \tag{4.45}$$

Integrating (4.27), by estimation (4.45), we infer that

$$\frac{(1+\ell)\bar{c}_s^2}{\sqrt{9+8\ell}-1} < |\frac{\bar{c}_s^2(1+\ell)\mu^\pm}{\lambda \pm i\bar{v}_1^+ k}|^2 \le \frac{(1+\ell)\bar{c}_s^2}{\sqrt{1+4\epsilon_0^2}-1}. \tag{4.46}$$

Thus, by utilizing (2.7), (4.45), (4.42) and (4.32), we infer that

$$\begin{aligned}
&\|v_{n,1}(t=0,x_1,x_2)\|_{H^j(\Omega)}^2 = \|w_n(x_1,x_2)\|_{H^j(\Omega)}^2 \\
&\le \sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\Big|\frac{(\mu^+-\mu^-)\bar{c}_s^2(1+\ell)ik}{\lambda+i\bar{v}_1^+k}\Big|^2|\hat{g}_n(k)|^2\int_0^\infty|\partial_2^m e^{-\mu^+x_2}|^2dx_2dk \\
&\quad + \sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\Big|\frac{(\mu^+-\mu^-)\bar{c}_s^2(1+\ell)ik}{\lambda-i\bar{v}_1^+k}\Big|^2|\hat{g}_n(k)|^2\int_{-\infty}^0|\partial_2^m e^{\mu^-x_2}|^2dx_2dk \\
&\le \frac{\bar{c}_s^2(1+\ell)}{\sqrt{1+4\epsilon_0^2}-1}\sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\Big|\frac{\mu^+-\mu^-}{\mu^+}\Big|^2\Big|\frac{\mu^+}{\mu^++\mu^-}\Big||\mu^+|^{2m+1}|\chi_n(k)|^2dk \\
&\quad + \frac{\bar{c}_s^2(1+\ell)}{\sqrt{1+4\epsilon_0^2}-1}\sum_{m=0}^{j}\int_{\mathbb{R}}(1+k^2)^{j-m}\Big|\frac{\mu^+-\mu^-}{\mu^-}\Big|^2\Big|\frac{\mu^-}{\mu^++\mu^-}\Big||\mu^-|^{2m+1}|\chi_n(k)|^2dk \\
&\le \frac{8C^*\bar{c}_s^2(1+\ell)(j+1)}{\sqrt{1+4\epsilon_0^2}-1}\int_{\mathbb{R}}(1+k^2)^{j+1}|\chi_n(k)|^2dk \lesssim \frac{1}{n^2}.
\end{aligned} \tag{4.47}$$

Similarly, we have

$$\|v_{n,2}(t=0,x_1,x_2)\|_{H^j(\Omega)}^2 \lesssim \frac{1}{n^2}. \tag{4.48}$$

Taking into account again that $\operatorname{supp}(\chi_n) \subset B(0,n+1)\backslash B(0,n)$, by (1.25), (2.7) and (4.42), while for $k \ge n$ and $t \ge T_0$, $\lambda = X_1 k$, we infer that

$$\begin{aligned}
\|v_{n,1}\|_{H^k(\Omega)}^2 &\ge \int_{\mathbb{R}}(1+k^2)^k\|\hat{v}_{n,1}^\pm(k,\cdot)\|_{L^2(I_\pm)}^2dk \\
&\ge \int_{\mathbb{R}}(1+k^2)^k\Big|\frac{(\mu^+-\mu^-)\bar{c}_s^2(1+\ell)ik}{\lambda+i\bar{v}_1^+k}\Big|^2|e^{\lambda t}|^2|\hat{g}_n|^2\int_0^\infty|e^{-\mu^+x_2}|^2dx_2dk \\
&\quad + \int_{\mathbb{R}}(1+k^2)^k\left|\frac{(\mu^+-\mu^-)c^2ik}{\lambda-i\bar{v}_1^+k}\right|^2|e^{\lambda t}|^2|\hat{g}_n|^2\int_{-\infty}^0|e^{\mu^-x_2}|^2dx_2dk \\
&\ge \int_{\mathbb{R}}(1+k^2)^k|\frac{\mu^+-\mu^-}{\mu^+}|^2|\frac{\bar{c}_s^2(1+\ell)ik}{\lambda+i\bar{v}_1^+k}|^2e^{2X_1kt}|\chi_n(k)|^2|\frac{\mu^+}{\mu^++\mu^-}||\mu^+|dk \\
&\quad + \int_{\mathbb{R}}(1+k^2)^k\Big|\frac{\mu^+-\mu^-}{\mu^-}\Big|^2\Big|\frac{\bar{c}_s^2(1+\ell)ik}{\lambda-i\bar{v}_1^+k}\Big|^2e^{2X_1kt}|\chi_n(k)|^2|\frac{\mu^-}{\mu^++\mu^-}||\mu^-|dk \\
&> \frac{(1+\ell)\tilde{C}\bar{c}_s^2}{\sqrt{9+8\ell}-1}\frac{e^{2X_1nT_0}}{(1+(n+1)^2)^{j-k+1}}n\int_{\mathbb{R}}(1+k^2)^{j+1}|\chi_n(k)|^2dk,
\end{aligned} \tag{4.49}$$

where we adopt the bounded results (4.29), (4.31), (4.47) to verify the final inequality within the aforesaid deductions.

Let n be adequately large such that

$$\frac{(1+\ell)\tilde{C}\bar{c}_s^2}{\sqrt{9+8\ell}-1}\frac{e^{2X_1nT_0}}{(1+(n+1)^2)^{j-k+1}} \ge \alpha^2n^2\bar{C}_j^2. \tag{4.50}$$

Therefore,

$$\|v_{n,1}(t)\|_{H^k(\Omega)} \ge \alpha. \tag{4.51}$$

Similarly we have

$$\|v_{n,2}\|_{H^k(\Omega)} \geq \alpha. \tag{4.52}$$

Gathering the estimates (4.18), (4.33), (4.38),(4.47) and (4.48) yields

$$\|f_n(0)\|_{H^j(\Omega)} + \|q_{g,n}(0)\|_{H^j((\Omega)} + \|q_{r,n}(0)\|_{H^j((\Omega)} + \|v_n(0)\|_{H^j(\Omega)} \lesssim \frac{1}{n}, \tag{4.53}$$

but the estimates (4.21), (4.37), (4.39), (4.51) and (4.52) yield

$$\|f_n\|_{H^k(\Gamma)} + \|q_{g,n}\|_{H^k(\Omega)} + \|q_{r,n}\|_{H^k(\Omega)} + \|v_n\|_{H^k(\Omega)} \geq \alpha, \ for\ all\ t \geq T_0. \tag{4.54}$$

This completes the proof. □

5. Ill-posedness for the nonlinear problem

Now we prove nonlinear ill-posedness for problem (1.13). First, we restate the nonlinear system (1.13) in a perturbation formulation around the rectilinear solution. Let

$$\begin{gathered} f = 0 + \tilde{f}, \quad v = \bar{v} + \tilde{v}, \quad q_g = \bar{q}_g + \tilde{q}_g, q_r = \bar{q}_r + \tilde{q}_r, \quad \Psi = Id + \tilde{\Psi}, \\ \varrho = \bar{\varrho} + \tilde{\varrho}, \quad \psi = 0 + \tilde{\psi}, \quad n = e_2 + \tilde{n}, \quad A = I - B, \end{gathered} \tag{5.1}$$

where Id denotes an identity transformation and I stands for a 2×2 identity matrix, while B is defined as

$$B = \sum_{n=1}^{\infty} (-1)^{n-1} (D\tilde{\Psi})^n. \tag{5.2}$$

We are able to restate the term $\breve{v}$ as

$$\begin{aligned} \breve{v} &= (I - B)(\bar{v} + \tilde{v}) - (0, \frac{\partial_t \tilde{\psi}}{1 + \partial_2 \tilde{\psi}}) \\ &= \bar{v} + \tilde{v} - B(\bar{v} + \tilde{v}) - (0, \frac{\partial_t \tilde{\psi}}{1 + \partial_2 \tilde{\psi}}) := \bar{v} + M, \end{aligned} \tag{5.3}$$

where

$$M = \tilde{v} - B(\bar{v} + \tilde{v}) - (0, \frac{\partial_t \tilde{\psi}}{1 + \partial_2 \tilde{\psi}}). \tag{5.4}$$

With respect to the term $v \cdot n$, we are able to rewrite it as

$$v \cdot n = (\bar{v} + \tilde{v}) \cdot (e_2 + \tilde{n}) = \tilde{v}_2 - \bar{v}_1 \partial_1 \tilde{f} + \tilde{v} \cdot \tilde{n}. \tag{5.5}$$

Subsequently, the nonlinear system (1.13) can be transformed into a perturbed system centered around $(\bar{f}, \bar{q}_g, \bar{q}_r, \bar{v})$ as follows

$$\begin{aligned} &\partial_t \tilde{q}_g + (\bar{v} \cdot \nabla)\tilde{q}_g + \bar{\varrho}\bar{c}_s^2 \nabla \cdot \tilde{v} = -(M \cdot \nabla)\tilde{q}_g + \bar{q}_g B^T \nabla \cdot \tilde{v} - \tilde{q}_g \nabla \cdot \tilde{v} + \tilde{q}_g B^T \nabla \cdot \tilde{v}, \quad \text{in } \Omega, \\ &\quad \bar{\varrho}(\partial_t \tilde{v} + (\bar{v} \cdot \nabla)\tilde{v}) + \nabla(\tilde{q}_g + \tilde{q}_r) \\ &\quad = -\tilde{\varrho}\partial_t \tilde{v} - \bar{\varrho}(M \cdot \nabla)\tilde{v} - \tilde{\varrho}((\bar{v} + M) \cdot \nabla)\tilde{v} + B^T \nabla(\tilde{q}_g + \tilde{q}_r), \quad \text{in } \Omega, \\ &\partial_t \tilde{q}_r + (\bar{v} \cdot \nabla)\tilde{q}_r + \frac{1}{3}(\bar{q}_g + 4\bar{q}_r)\nabla \cdot \tilde{v} \\ &= -(M \cdot \nabla)\tilde{q}_r + \frac{1}{3}(\bar{q}_g + 4\bar{q}_r)B^T \nabla \cdot \tilde{v} - \frac{1}{3}(\tilde{q}_g + 4\tilde{q}_r)\nabla \cdot \tilde{v} + \frac{1}{3}(\tilde{q}_g + 4\tilde{q}_r)B^T \nabla \cdot \tilde{v}, \quad \text{in } \Omega, \\ &\qquad \partial_t \tilde{f} + \bar{v}_1 \partial_1 \tilde{f} - \tilde{v}_2 = \tilde{v} \cdot \tilde{n}, \quad \text{on } \Gamma. \end{aligned} \tag{5.6}$$

The jump conditions adopt a new form in terms of $\tilde{q}_g, \tilde{q}_r, \tilde{v}$ on the boundary Γ

$$\begin{gathered} (\tilde{v}^+ - \tilde{v}^-) \cdot e_2 + (\bar{v}^+ - \bar{v}^-) \cdot \tilde{n} = -(\tilde{v}^+ - \tilde{v}^-) \cdot \tilde{n}, \\ \tilde{q}_g^+ + \tilde{q}_r^+ = \tilde{q}_g^- + \tilde{q}_r^-. \end{gathered} \tag{5.7}$$

Proof of Theorem 1.3. We use the contradiction method. Assume that the perturbed problem (5.6) possesses the property $EE(k)$ for a certain $k \geq 3$. Let $\delta, t_0, C > 0$ denote the constants furnished by Definition 1.2. Select a fixed $n \in \mathbb{N}$ such that $n > C_*$. By applying Lemma 4.2 with this $n, T_0 = t_0/2, k \geq 3$, and $\alpha = 2$, we are able to identify f^L, h^L, v^L that satisfy (2.3) such that

$$\|(f_0^L, q_{g,0}^L, q_{r,0}^L, v_0^L)\|_{H^k} \lesssim \frac{1}{n}, \tag{5.8}$$

but

$$\|(f^L(t), q_g^L(t), q_r^L(t), v^L(t))\|_{H^3} \geq 2 \quad \text{for } t \geq t_0/2. \tag{5.9}$$

We introduce the definitions $\tilde{f}_0^\varepsilon = f_0^\varepsilon - \bar{f} := \varepsilon f_0^L$, $\tilde{q}_{g,0}^\varepsilon = q_{g,0}^\varepsilon - \bar{q}_g := \varepsilon q_{g,0}^L$, $\tilde{q}_{r,0}^\varepsilon = q_{r,0}^\varepsilon - \bar{q}_r := \varepsilon q_{r,0}^L$, and $\tilde{v}_0^\varepsilon = v_0^\varepsilon - \bar{v} := \varepsilon v_0^L$. For $\varepsilon < \delta n$, it follows that $\|(\tilde{f}_0^\varepsilon, \tilde{q}_{g,0}^\varepsilon, \tilde{q}_{r,0}^\varepsilon, \tilde{v}_0^\varepsilon)\|_{H^k} < \delta$, whence by Definition 1.2, there exists

$$\big(\tilde{f}^\varepsilon := f^\varepsilon - \bar{f}, \tilde{q}_g^\varepsilon := q_g^\varepsilon - \bar{q}_g, \tilde{q}_r^\varepsilon := q_r^\varepsilon - \bar{q}_r, \tilde{v}^\varepsilon := v^\varepsilon - \bar{v}\big) \in L^\infty([0, t_0]; H^3(\Omega))$$

which satisfies (5.6)-(5.7) by taking $(\tilde{f}_0^\varepsilon, \tilde{q}_{g,0}^\varepsilon, \tilde{q}_{r,0}^\varepsilon, \tilde{v}_0^\varepsilon)$ as the initial data and fulfills the inequality

$$\sup_{0 \leq t \leq t_0} \|(\tilde{f}^\varepsilon, \tilde{q}_g^\varepsilon, \tilde{q}_r^\varepsilon, \tilde{v}^\varepsilon)(t)\|_{H^3} \leq C(\|(\tilde{f}_0^\varepsilon, \tilde{q}_{g,0}^\varepsilon, \tilde{q}_{r,0}^\varepsilon, \tilde{v}_0^\varepsilon)\|_{H^k}) \leq C_* \varepsilon \frac{1}{n} < \varepsilon, \tag{5.10}$$

where we use (5.8) to derive the penultimate inequality.

We now introduce the rescaled functions $\bar{f}^\varepsilon = \tilde{f}^\varepsilon/\varepsilon$, $\bar{q}_g^\varepsilon = \tilde{q}_g^\varepsilon/\varepsilon$, $\bar{q}_r^\varepsilon = \tilde{q}_r^\varepsilon/\varepsilon$, $\bar{v}^\varepsilon = \tilde{v}^\varepsilon/\varepsilon$. A subsequent rescaling of (5.10) demonstrates that

$$\sup_{0 \leq t \leq t_0} \|(\bar{f}^\varepsilon, \bar{q}_g^\varepsilon, \bar{q}_r^\varepsilon, \bar{v}^\varepsilon)(t)\|_{H^3} < 1. \tag{5.11}$$

By construction, we conclude that

$$(\bar{f}_0^\varepsilon, \bar{q}_{g,0}^\varepsilon, \bar{q}_{r,0}^\varepsilon, \bar{v}_0^\varepsilon) = (f_0^L, q_{g,0}^L, q_{r,0}^L, v_0^L).$$

Next, we show that the rescaled functions $(\bar{f}^\varepsilon, \bar{q}_g^\varepsilon, \bar{q}_r^\varepsilon, \bar{v}^\varepsilon)$ converge to the solutions (f^L, q_g^L, q_r^L, v^L) of the linearized equations (3.1) as $\varepsilon \to 0$.

We intend to reexpress (5.6)-(5.7) with respect to the rescaled functions $(\bar{f}^\varepsilon, \bar{q}_g^\varepsilon, \bar{q}_r^\varepsilon, \bar{v}^\varepsilon)$ and demonstrate certain convergence conclusions. The fourth equation within (5.6) can be restated based on the rescaled functions $(\bar{f}^\varepsilon, \bar{q}_g^\varepsilon, \bar{q}_r^\varepsilon, \bar{v}^\varepsilon)$ as follows:

$$\partial_t \bar{f}^\varepsilon + \bar{v}_1 \partial_1 \bar{f}^\varepsilon - \bar{v}_2^\varepsilon = \varepsilon \bar{v}^\varepsilon \cdot n^\varepsilon, \tag{5.12}$$

where $n^\varepsilon = \frac{(-\varepsilon \partial_1 \bar{f}^\varepsilon, 0)}{\varepsilon} = (-\partial_1 \bar{f}^\varepsilon, 0)$ is properly defined and uniformly bounded in $L^\infty([0, t_0]; H^2(\Gamma))$ because

$$\|n^\varepsilon\|_{H^2(\Gamma)} \leq \|\bar{f}^\varepsilon\|_{H^3(\Gamma)} < 1, \tag{5.13}$$

by means of (5.11) within the aforementioned inequality. Consequently, from (5.11) and (5.13), we derive from (5.12) that

$$\lim_{\varepsilon \to 0} \sup_{0 \leq t \leq t_0} \|\partial_t \bar{f}^\varepsilon + \bar{v}_1 \partial_1 \bar{f}^\varepsilon - \bar{v}_2^\varepsilon\|_{H^2} = 0, \tag{5.14}$$

and

$$\sup_{0 \leq t \leq t_0} \|\partial_t \bar{f}^\varepsilon(t)\|_{H^2} \leq |\bar{v}_1| \sup_{0 \leq t \leq t_0} \|\partial_1 \bar{f}^\varepsilon(t)\|_{H^2} + \sup_{0 \leq t \leq t_0} \|\bar{v}_2^\varepsilon\|_{H^2} \leq C. \tag{5.15}$$

Upon expanding the first equation in (5.6) and dividing both sides of the resulting equation by ε, we deduce that

$$\partial_t \bar{q}_g^\varepsilon + (\bar{v} \cdot \nabla) \bar{q}_g^\varepsilon + \bar{\varrho} \bar{c}_s^2 \nabla \cdot \bar{v}^\varepsilon = -\varepsilon (\bar{M}^\varepsilon \cdot \nabla) \bar{q}_g^\varepsilon + \varepsilon \bar{q}_g (\bar{B}^\varepsilon)^T \nabla \cdot \bar{v}^\varepsilon - \varepsilon \bar{q}_g^\varepsilon \nabla \cdot \bar{v}^\varepsilon + \varepsilon^2 \bar{q}_g^\varepsilon (\bar{B}^\varepsilon)^T \nabla \cdot \bar{v}^\varepsilon, \tag{5.16}$$

in which

$$\bar{M}^\varepsilon = \bar{v}^\varepsilon - \bar{B}^\varepsilon (\bar{v} + \varepsilon \bar{v}^\varepsilon) - (0, \frac{\partial_t \bar{\psi}^\varepsilon}{1 + \varepsilon \partial_2 \bar{\psi}^\varepsilon}), \bar{\psi}^\varepsilon = \theta \bar{f}^\varepsilon, \tag{5.17}$$

and

$$\bar{B}^\varepsilon := (I - (I + \varepsilon \nabla \bar{\Psi}^\varepsilon)^{-1})/\varepsilon, \tag{5.18}$$

with $\bar{\Psi}^\varepsilon = (0, \bar{\psi}^\varepsilon)$.

To evaluate the bound of $\bar{M}^\varepsilon$, we first assess the bound of $\bar{B}^\varepsilon$. We suppose that ε is adequately small such that $\varepsilon < 1/(2C_1)$, where $C_1 > 0$ denotes the optimal constant in the inequality $\|UV\|_{H^2} \le C_1\|U\|_{H^2}\|V\|_{H^2}$ for 2×2 matrix-valued functions U, V. This supposition guarantees that $\bar{B}^\varepsilon$ is properly defined and uniformly bounded in $L^\infty([0,t_0];H^2(\Omega))$ since

$$\begin{aligned}
\|\bar{B}^\varepsilon\|_{H^2} &= \|\sum_{n=1}^{\infty}(-\varepsilon)^{n-1}(\nabla\bar{\Psi}^\varepsilon)^n\|_{H^2} \le \sum_{n=1}^{\infty}\varepsilon^{n-1}\|(\nabla\bar{\Psi}^\varepsilon)^n\|_{H^2} \\
&\le \sum_{n=1}^{\infty}(\varepsilon C_1)^{n-1}\|\nabla\bar{\Psi}^\varepsilon\|_{H^2}^n \le \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\|\bar{\psi}^\varepsilon\|_{H^3}^n \\
&\le \sum_{n=1}^{\infty}\frac{1}{2^{n-1}}\|\bar{f}^\varepsilon\|_{H^3}^n < \sum_{n=1}^{\infty}\frac{1}{2^{n-1}} = 2,
\end{aligned} \tag{5.19}$$

where we used (5.11) in the final inequality mentioned above. Meanwhile, we demonstrate by (5.11) and (5.15) that

$$\begin{aligned}
\|\bar{M}^\varepsilon\|_{H^2} &\lesssim \|\bar{v}^\varepsilon\|_{H^2} + |\bar{v}|\|\bar{B}^\varepsilon\|_{H^2} + \varepsilon\|\bar{B}^\varepsilon\|_{H^2}\|\bar{v}^\varepsilon\|_{H^2} + \|\partial_t\bar{\psi}^\varepsilon\|_{H^2} \\
&\lesssim \|\bar{v}^\varepsilon\|_{H^2} + |\bar{v}|\|\bar{B}^\varepsilon\|_{H^2} + \varepsilon\|\bar{B}^\varepsilon\|_{H^2}\|\bar{v}^\varepsilon\|_{H^2} + \|\partial_t\bar{f}^\varepsilon\|_{H^2} \le C.
\end{aligned} \tag{5.20}$$

Thus, from (5.11),(5.19) and (5.20), we derive by means of (5.16) that

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le t_0}\|\partial_t\bar{q}_g^\varepsilon + (\bar{v}\cdot\nabla)\bar{q}_g^\varepsilon + \bar{\varrho}\bar{c}_s^2\nabla\cdot\bar{v}^\varepsilon\|_{H^2} = 0, \tag{5.21}$$

$$\sup_{0\le t\le t_0}\|\partial_t\bar{q}_g^\varepsilon(t)\|_{H^2} < C. \tag{5.22}$$

By expanding the second equation in (5.6), we observe that

$$\begin{aligned}
&\bar{\varrho}(\partial_t\bar{v}^\varepsilon + (\bar{v}\cdot\nabla)\bar{v}^\varepsilon) + \nabla(\bar{q}_g^\varepsilon + \bar{q}_r^\varepsilon) \\
&= -\varepsilon\bar{\varrho}^\varepsilon\partial_t\bar{v}^\varepsilon - \varepsilon\bar{\varrho}(\bar{M}^\varepsilon\cdot\nabla)\bar{v}^\varepsilon - \varepsilon\bar{\varrho}^\varepsilon((\bar{v}+\varepsilon\bar{M}^\varepsilon)\cdot\nabla)\bar{v}^\varepsilon + \varepsilon(\bar{B}^\varepsilon)^T\nabla(\bar{q}_g^\varepsilon + \bar{q}_r^\varepsilon),
\end{aligned} \tag{5.23}$$

in which $\bar{M}^\varepsilon$ and $\bar{B}^\varepsilon$ are defined by (5.17) and (5.18), respectively.

Consequently, from (5.11), (5.19) and (5.20), we infer with the assistance of (5.23) that

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le t_0}\|\bar{\varrho}(\partial_t\bar{v}^\varepsilon + (\bar{v}\cdot\nabla)\bar{v}^\varepsilon) + \nabla(\bar{q}_g^\varepsilon + \bar{q}_r^\varepsilon)\|_{H^2} = 0, \tag{5.24}$$

$$\sup_{0\le t\le t_0}\|\partial_t\bar{v}^\varepsilon(t)\|_{H^2} < C. \tag{5.25}$$

By expanding the third equation in (5.6) and dividing by ε on each side of the resulting equation, we deduce that

$$\begin{aligned}
&\partial_t\bar{q}_r^\varepsilon + (\bar{v}\cdot\nabla)\bar{q}_r^\varepsilon + \frac{1}{3}(\bar{q}_g + 4\bar{q}_r)\nabla\cdot\bar{v}^\varepsilon \\
&= -\varepsilon(\bar{M}^\varepsilon\cdot\nabla)\bar{q}_r^\varepsilon + \varepsilon\frac{1}{3}(\bar{q}_g + 4\bar{q}_r)(\bar{B}^\varepsilon)^T\nabla\cdot\bar{v}^\varepsilon - \varepsilon\frac{1}{3}(\bar{q}_g^\varepsilon + 4\bar{q}_r^\varepsilon)\nabla\cdot\bar{v}^\varepsilon \\
&\quad + \varepsilon^2\frac{1}{3}(\bar{q}_g^\varepsilon + 4\bar{q}_r^\varepsilon)(\bar{B}^\varepsilon)^T\nabla\cdot\bar{v}^\varepsilon,
\end{aligned} \tag{5.26}$$

Consequently, from (5.11), (5.19) and (5.20), we infer with the aid of (5.26) that

$$\lim_{\varepsilon\to 0}\sup_{0\le t\le t_0}\|\partial_t\bar{q}_r^\varepsilon + (\bar{v}\cdot\nabla)\bar{q}_r^\varepsilon + \frac{1}{3}(\bar{q}_g + 4\bar{q}_r)\nabla\cdot\bar{v}^\varepsilon\|_{H^2} = 0, \tag{5.27}$$

$$\sup_{0\le t\le t_0}\|\partial_t\bar{q}_r^\varepsilon(t)\|_{H^2} < C. \tag{5.28}$$

Next we address several convergence questions about the jump conditions. With respect to the first equation in (5.7), we rewrite the normal vector n^ε as

$$n^\varepsilon = e_2 + \tilde{n}^\varepsilon := e_2 + \varepsilon\bar{n}^\varepsilon, \quad \bar{n}^\varepsilon = (-\partial_1\bar{f}^\varepsilon, 0),$$

thus we rewrite the first equation in (5.7) as

$$(\bar{v}^+ + \varepsilon\bar{v}^{+,\varepsilon} - \bar{v}^- - \varepsilon\bar{v}^{-,\varepsilon})\cdot(e_2 + \varepsilon\bar{n}^\varepsilon) = 0, \tag{5.29}$$

which is tantamount to

$$(\bar{v}^{+,\varepsilon} - \bar{v}^{-,\varepsilon}) \cdot e_2 + (\bar{v}^+ - \bar{v}^-) \cdot \bar{n}^\varepsilon = -\varepsilon(\bar{v}^{+,\varepsilon} - \bar{v}^{-,\varepsilon}) \cdot \bar{n}^\varepsilon. \tag{5.30}$$

Given that $\sup_{0\le t\le t_0} \|\bar{n}^\varepsilon(t)\|_{L^\infty} \lesssim \|\bar{n}^\varepsilon\|_{H^2(\Gamma)} < C$ and $\|\bar{v}^\varepsilon\|_{L^\infty} \lesssim \|\bar{v}^\varepsilon\|_2 \le C$ are uniformly bounded, we obtain with the help of (5.30) that

$$\sup_{0\le t\le t_0} \|e_2 \cdot (\bar{v}^{+,\varepsilon}(t) - \bar{v}^{-,\varepsilon}(t) + (\bar{v}^+ - \bar{v}^-) \cdot \bar{n}^\varepsilon)\|_{L^\infty} \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.31}$$

Consequently, by (5.31), we obtain

$$[\bar{v}^\varepsilon \cdot e_2] = 2\bar{v}_1^+ \partial_1 f^\varepsilon \quad \text{on } \Gamma. \tag{5.32}$$

We expand the second equation in (5.7) as

$$\varepsilon \bar{q}_g^{+,\varepsilon} + \varepsilon \bar{q}_r^{+,\varepsilon} = \varepsilon \bar{q}_g^{-,\varepsilon} + \varepsilon \bar{q}_r^{-,\varepsilon} \quad \text{on } \Gamma. \tag{5.33}$$

We perform division by ε on both sides of (5.33) so as to obtain

$$\bar{q}_g^{+,\varepsilon} + \bar{q}_r^{+,\varepsilon} = \bar{q}_g^{-,\varepsilon} + \bar{q}_r^{-,\varepsilon} \quad \text{on } \Gamma. \tag{5.34}$$

From (5.11) and the sequential weak$*$ compactness, we conclude that upon extracting a subsequence (which we continue to denote merely by ε),

$$(\bar{f}^\varepsilon, \bar{q}_g^\varepsilon, \bar{q}_r^\varepsilon, \bar{v}^\varepsilon) \overset{*}{\rightharpoonup} (f^\star, q_g^\star, q_r^\star, v^\star) \quad \text{weakly } * \text{ in } L^\infty(H^3(\Omega)). \tag{5.35}$$

By lower semicontinuity, we conclude that

$$\sup_{0\le t\le t_0} \|(f^\star, q_g^\star, q_r^\star, v^\star)(t)\|_{H^3} \le 1. \tag{5.36}$$

In conformity with (5.15), (5.22), (5.25) and (5.28), we obtain

$$\limsup_{\varepsilon\to 0} \sup_{0\le t\le t_0} \|(\partial_t \bar{f}^\varepsilon, \partial_t \bar{q}_g^\varepsilon, \partial_t \bar{q}_r^\varepsilon, \partial_t \bar{v}^\varepsilon)(t)\|_{H^2} < \infty. \tag{5.37}$$

By the Lions-Aubin lemma [27], we thereby conclude that the sequence $\{(f^\varepsilon, q_g^\varepsilon, q_r^\varepsilon, v^\varepsilon)\}$ is strongly precompact within the space $L^\infty([0,t_0]\,; H^{8/3}(\Omega))$, hence

$$(\bar{f}^\varepsilon, \bar{q}_g^\varepsilon, \bar{q}_r^\varepsilon, \bar{v}^\varepsilon) \to (f^\star, q_g^\star, q_r^\star, v^\star) \quad \text{strongly in } L^\infty(H^{8/3}(\Omega)). \tag{5.38}$$

This strong convergence, in conjunction with (5.15), (5.22), (5.25) and (5.28), leads to the conclusion that

$$(\partial_t \bar{f}^\varepsilon, \partial_t \bar{q}_g^\varepsilon, \partial_t \bar{q}_r^\varepsilon, \partial_t \bar{v}^\varepsilon) \to (\partial_t f^\star, \partial_t q_g^\star, \partial_t q_r^\star, \partial_t v^\star) \quad \text{strongly in } L^\infty(H^{5/3}(\Omega)). \tag{5.39}$$

The indices $\frac{8}{3}$ and $\frac{5}{3}$ are sufficiently large to yield the $L^\infty(L^\infty(\Omega))$ convergence of the sequence $\{(f^\varepsilon, q_g^\varepsilon, q_r^\varepsilon, v^\varepsilon)\}$, thereby we obtain the following convergence system as ε tends to zero

$$\begin{gathered}
\partial_t q_g^\star + (\bar{v} \cdot \nabla) q_g^\star + \bar{\varrho} \bar{c}_s^2 \nabla \cdot v^\star = 0, \quad \text{in } \Omega, \\
\bar{\varrho}(\partial_t v^\star + (\bar{v} \cdot \nabla) v^\star) + \nabla (q_g^\star + q_r^\star) = 0, \quad \text{in } \Omega, \\
\partial_t q_r^\star + (\bar{v} \cdot \nabla) q_r^\star + \frac{1}{3}(\bar{q}_g + 4\bar{q}_r) \nabla \cdot v^\star = 0, \quad \text{in } \Omega, \\
\partial_t f^\star + \bar{v}_1 \partial_1 f^\star - v_2^\star = 0 \quad \text{on } \Gamma,
\end{gathered} \tag{5.40}$$

and

$$\begin{gathered}
q_g^{+,\star} + q_r^{+,\star} = q_g^{-,\star} + q_r^{-,\star} \quad \text{on } \Gamma, \\
(v_+^\star - v_-^\star) \cdot e_2 = 2\bar{v}_1^+ \partial_1 f^\star \quad \text{on } \Gamma.
\end{gathered} \tag{5.41}$$

We also take the limit $(\bar{f}_0^\varepsilon, \bar{q}_{g,0}^\varepsilon, \bar{q}_{r,0}^\varepsilon, \bar{v}_0^\varepsilon) = (f_0^L, q_{g,0}^L, q_{r,0}^L, v_0^L)$ so as to have

$$(f_0^\star, q_{g,0}^\star, q_{r,0}^\star, v_0^\star) = (f_0^L, q_{g,0}^L, q_{r,0}^L, v_0^L).$$

At this point, we observe that $(f^\star, q_g^\star, q_r^\star, v^\star)(t)$ constitutes the solution to equations (2.3) and the boundary conditions (2.4) with the identical initial data. In conformity with the uniqueness conclusion in Lemma 4.1, we obtain

$$(f^\star, q_g^\star, q_r^\star, v^\star)(t) = (f^L, q_g^L, q_r^L, v^L)(t). \tag{5.42}$$

Consequently, we integrate (5.36), (5.42) and (5.9) to obtain

$$2=\alpha<\sup_{0\le t\le t_0}\|(f^\star,q_g^\star,q_r^\star,v^\star)(t)\|_{H^3}\le 1, \tag{5.43}$$

which constitutes a contradiction. Consequently, the proof of Theorem 1.3 is complete. □

Acknowledgements. Binqiang Xie was supported by the NSF-China under grant No. 11901207, and by the Guangdong Association for Science and Technology under the Young Scientific Talent Training Program SKXRC2026488.

References

[1] G. Q. Chen, Y. G. Wang, Existence and Stability of Compressible Current-Vortex Sheets in Three-Dimensional Magnetohydrodynamics. Arch. Ration. Mech. Anal., 187 (2008), 369-408.

[2] R. M. Chen, J. Hu, D. Wang; *Linear stability of compressible vortex sheets in two-dimensional elastodynamics*, Adv. Math., 311 (2017), 18-60.

[3] R. M. Chen, J. Hu, D. Wang; *Linear stability of compressible vortex sheets in 2D elastodynamics: variable coefficients*. Math. Ann., 376 (2020), 863-912.

[4] R. M. Chen, F. M. Huang, D. Wang, D. F. Yuan; *Stabilization effect of elasticity on three-dimensional compressible vortex sheets*, J. Math.Pures Appl., 172(2023), 105-138.

[5] J.-F. Coulombel, P. Secchi; *The stability of compressible vortex sheets in two space dimensions*, Indiana Univ. Math. J., 53 (2004), no. 4, 941-1012.

[6] J.-F. Coulombel, P. Secchi; *Nonlinear compressible vortex sheets in two space dimensions*, Ann. Sci.cole Norm. Supr., (4) 41 (2008), no. 1, 85-139.

[7] J.-F. Coulombel, A. Morando, P. Secchi, P. A. Trebeschi; *priori estimates for 3D incompressible current-vortex sheets*. Comm. Math. Phys., 311 (2012), 247-275.

[8] J. W. Dungey; *Electrodynamics of the outer atmosphere*, in Proceedings of the Ionosphere, The Physical Society of London, London, 255-265, 1955.

[9] D. G. Ebin; *Ill-posedness of the rayleigh-taylor and helmholtz problems for incompressible fluids*. Communications in Partial Differential Equations, 13 (1988), 1265-1295.

[10] A. I. Ershkovich; *Solar wind interaction with the tail of comet Kohoutek*, Planet. Space Sci., 24, 287-290, 1976.

[11] J. A. Fejer, J. W. Miles; *On the stability of a plane vortex sheet with respect to three-dimensional disturbances*, J. Fluid Mech. 15 (1963), 335-336.

[12] H. Hasegawa, M. Fujimoto, T. D. Phan, H. Reme, A. Balogh, M. W. Dunlop, C. Hashimoto, R. TanDokoro; *Transport of solar wind into Earth's magnetosphere through rolled-up Kelvin-Helmholtz vortices*, Nature, 430(2004), 755-758.

[13] H. L. F. Helmholtz; *On the discontinuous movements of fluids*. Sitz. ber. Preuss. Akad.Wiss. Berl. Philos.-Hist. Kl., 23 (1868), 215-228.

[14] L. Kelvin (W.T. Thomson); *Hydrokinetic solutions and observations*. Philos. Mag. 42, 362-377, 1871

[15] L. D. Landau; *On the stability of tangential discontinuities in a compressible fluid*, Akad. Kauk. L. S. S. S. R. Comptes Rendus (Doklndy) 44, 139, 1944.

[16] G. Lebeau; *Régularité du probléme de Kelvin-Helmholtz pour l'equation d'Euler 2d. Séminaire: Êquations aux Dérivées Partielles*, 2000-2001, Exp. No. II, 12 pp. Séminaire: Equations aux Dérivéss Partielles. Ecole Polytechnique, Palaiseau, France, 2001.

[17] A. Morando, Y. Trakhinin, P. Trebeschi; *Stability of incompressible current-vortex sheets*. J. Math. Anal. Appl., 347 (2008), 502-520.

[18] A. Morando, P. Trebeschi; *Two-dimensional vortex sheets for the nonisentropic Euler equations: linear stability*, J. Hyperbolic Differ. Equ., 5 (2008), no. 3, 487-518.

[19] A. Morando, P. Trebeschi, T.Wang; *Two-dimensional vortex sheets for the nonisentropic Euler equations: nonlinear stability*, J. Differential Equations, 266 (2019), no. 9, 5397-5430.

[20] A. Morando, P. Secchi, P. Trebeschi; *On the evolution equation of compressible vortex sheets*, Mathematische Nachrichten, 293 (2020) 945-969.

[21] J. W. Miles; *On the disturbed motion of a plane vortex sheet*, J. Fluid Mech. 4 (1958), 538-552.

[22] E. N. Parker; *Dynamics of the interplanetary gas and magnetic fields*, Astrophys. J., 128, 664-678, 1958.

[23] H. Peng, F. Yu, Y. Huliuta1, L. Wei, Z. X. Wang, Y. Liu; *Effect of Transverse Magnetic Field on Kelvin-Helmholtz Instability in the Presence of a Radiation Field*, The Astrophysical Journal, 97 (2024) 8-16.

[24] M. Shadmehri, Z. Enayati, M. Khajavi; *Magnetized Kelvin-Helmholtz instability in the presence of a radiation field*, Astrophys Space Sci., 341 (2012), 369-374.

[25] De Sterck; *Hyperbolic theory of the "shallow water" magnetohydodynamics equations*. Phys. Plasmas 8 (2001), 3293-3304.

[26] S. I. Syrovatskij; *The stability of tangential discontinuities in a magnetohydrodynamic medium*. Z. Eksperim. Teoret. Fiz. 24 (1953), 622-629.

[27] J. Simon; *Comact sets in the space* $L^p(0,T);B$, Ann. Mat. Pura Appl., (4)146 (1987), 65-96.

[28] P. A. Sturrock, R. E. Hartle; *Two-fluid model of the solar wind*, Phys. Rev. Lett., 16, 628-636, 1966.

[29] Y. Z. Sun, W. Wang, Z. F. Zhang; *Nonlinear Stability of the Current-Vortex Sheet to the Incompressible MHD Equations*. Communications on Pure and Applied Mathematics, 71 (2018), 0356-0403.
[30] Y. Trakhinin; *Existence of compressible current-vortex sheets: variable coefficients linear analysis*. Arch. Ration. Mech. Anal., 177 (2005), 331-366.
[31] Y. Trakhinin; *The existence of current-vortex sheets in ideal compressible magnetohydrodynamics*. Arch. Ration. Mech. Anal., 191 (2009), 245-310.
[32] Y.-G. Wang, F. Yu; *Stabilization effect of magnetic fields on two-dimensional compressible current-vortex sheets*, Arch. Ration. Mech. Anal. 208 (2013), no. 2, 341-389.
[33] Y. J. Wang, Z.P. Xin; *Existence of multi-dimensional contact discontinuities for the ideal compressible magnetohydrodynamics*, Comm. Pure. Appl. Math, 77 (2024) , no. 1, 583-629.
[34] S. J. Wu; *Mathematical Analysis of Vortex Sheets*, Comm. Pure. Appl. Math, 59 (2006), no. 8, 1065-1206.
[35] B. Q. Xie, B. Zhao, D. W. Huang; *Ill-posedness of the Kelvin-Helmholtz problem for incompressible MHD fluids*, SIAM J. Math. Anal. (to appear).
[36] B. Q. Xie, B. Zhao; *Ill-posedness of the Kelvin-Helmholtz problem for three-dimensional incompressible elastodynamic fluids*, Journal of Differential Equations, 476 (2026) 114530.
[37] B. Q. Xie, B. L. Guo, B. Zhao; *Ill-posedness of incompressible Kelvin-Helmholtz problem with transverse magnetic field*, J. Math. Phys., 66, 121506 (2025).
[38] B. Q. Xie, B. Zhao; *Kelvin-Helmholtz Instability for Compressible Euler Fluids*, Mathematical Methods in the Applied Sciences, 49 (10), 11448-11472 (2026).
[39] B. Q. Xie, J. K. Ma; *Effect of parallel magnetic field on the compressible Kelvin-Helmholtz problem*, Communications in Analysis and Mechanics, 18, 366-399 (2026).
[40] B.Q. Xie, B. Zhao; *Effect of transverse magnetic field on the compressible Kelvin-Helmholtz problem*, The Journal of Geometric Analysis (2026) 36:7.
[41] B. Q. Xie, B. L. Guo, B. Zhao; *Effect of weak elasticity on the Kelvin-Helmholtz instability*, J. Math. Fluid Mech., (2026) 28:20.

Jiakai Ma
School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510006, China
Email address: majiakai51@163.com

Binqiang Xie (corresponding author)
School of Mathematics and Statistics, Guangdong University of Technology, Guangzhou 510006, China
Email address: xbqmath@gdut.edu.cn