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INFINITE MANY NORMALIZED SOLUTIONS FOR SCHRÖDINGER-BOPP-PODOLSKY SYSTEMS WITH SINGULAR COEFFICIENT AND CRITICAL GROWTH

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Abstract. This article studies normalized solutions for the critical Schrödinger-Bopp-Podolsky system with singular coefficient and critical growth,

$$\begin{gathered} -\Delta u+\lambda u=\phi u+\frac{|u|^{s-2}u}{|x|^q}+(I_\alpha*|u|^{2_\alpha^*})|u|^{2_\alpha^*-2}u, \quad x\in\mathbb{R}^3, \\ -\Delta\phi+d^2\Delta^2\phi=4\pi u^2, \quad x\in\mathbb{R}^3 \end{gathered}$$

under the mass constraint

$$\int_{\mathbb{R}^3}|u|^2dx=a^2,$$

where $a>0$ is a constant, $0<\alpha<3$, $0<q<2<s<10/3$ satisfying $q+\frac{s}{2}<3<q+\frac{3s}{2}<5$, and

$$I_\alpha(x)=\frac{A_\alpha}{|x|^{3-\alpha}}, \quad \text{with } A_\alpha=\frac{\Gamma\big(\frac{3-\alpha}{2}\big)}{2^\alpha\pi^{3/2}\Gamma(\frac{\alpha}{2})}$$

which is the Riesz potential. Here $2_\alpha^*=3+\alpha$ is the Hardy Littlewood Sobolev upper critical exponent. We establish the existence of multiple normalized solutions by using truncation techniques and topological genus theory. To address the non-compactness of the energy functional caused by singular and critical growth, we use the concentration-compactness principle. This paper also presents results regarding the stabilization of the orbit and the asymptotic behavior of the ground-state solution as $d\to 0$.

1. Introduction and main results

The divergence of the self-energy of a point charge is a well-known drawback of classical Maxwell electrodynamics. To overcome this difficulty, Bopp [8] and Podolsky [20] independently proposed a higher-derivative generalization of Maxwell's Lagrangian in which the photon propagator is regularized at short distances while keeping the long-range behavior unchanged. The resulting Bopp-Podolsky electrodynamics introduces a new parameter $d>0$ with the dimension of inverse mass; when $d\to 0$ one recovers the classical Maxwell theory. Quantization of the BP model leads to a Schrödinger-type equation for the matter field coupled to the BP potential, which, in the electrostatic approximation, gives rise to the Schrödinger-Bopp-Podolsky system. From the point of view of the electromagnetic field, the Bopp-Podolsky theory and the Maxwell theory are experimentally indistinguishable, and can be interpreted as effective theories for large and short distances, see [1]-[12].

This article aims to prove that the following Schrödinger-Bopp-Podolsky system possesses infinitely many normalized solutions.

$$\begin{gathered} -\Delta u+\lambda u=\phi u+\frac{|u|^{s-2}u}{|x|^q}+\Big(I_\alpha*|u|^{2_\alpha^*}\Big)|u|^{2_\alpha^*-2}u, \quad x\in\mathbb{R}^3, \\ -\Delta\phi+d^2\Delta^2\phi=4\pi u^2, \quad x\in\mathbb{R}^3 \end{gathered} \tag{1.1}$$

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with the prescribed L^2-norm

$$\int_{\mathbb{R}^3} u^2\,dx = a^2. \tag{1.2}$$

Since D'Avenia and Siciliano's 2019 publication [4], the Schrödinger-Bopp-Podolsky system has become the object of research in a number of recent works. They studied the following nonlinear Schrödinger-Bopp-Podolsky system

$$\begin{gathered}-\Delta u+\omega u+q^2\phi u=|u|^{p-2}u,\quad \text{in } \mathbb{R}^3,\\ -\Delta\phi+a^2\Delta^2\phi=4\pi u^2,\quad \text{in } \mathbb{R}^3\end{gathered}$$

with $\omega, a>0$. They proved existence and nonexistence results depending on the parameters q, p. Moreover, the solutions they found tend to solutions of the classical Schrödinger-Poisson system as $a\to 0$ in the radial case. Li et al. [17] extended the previous results to the critical case

$$\begin{gathered}-\Delta u+V(x)u+\phi u=\mu|u|^{p-2}u+|u|^4u,\quad \text{in } \mathbb{R}^3,\\ -\Delta\phi+a^2\Delta^2\phi=4\pi u^2,\quad \text{in } \mathbb{R}^3,\end{gathered}$$

where $\mu>0$, $p\in(2,5)$. They proved the existence of ground state solutions by coupling the Pohozaev-Nehari manifold with the monotonicity trick. In fact, Zhu et al. [27] studied the nonlinear Schrödinger-Bopp-Podolsky system

$$\begin{gathered}-\Delta u+V(x)u+q\phi u=f(u),\quad \text{in } \mathbb{R}^3,\\ -\Delta\phi+a^2\Delta^2\phi=4\pi u^2,\end{gathered}$$

where $a>0$, $q>0$ and $V\in C(\mathbb{R}^3,\mathbb{R})$. By using the variational methods, they established the existence of infinitely many nontrivial solutions. Moreover, they obtained the existence of a ground state and at least one positive solution for the two classes of nonlinearities under appropriate assumptions.

In recent years, Liu [18] looked for solutions to the Schr?dinger-Bopp-Podolsky system with prescribed L^2-norm constraint,

$$\begin{gathered}-\Delta u+\omega u+q\phi u=u^{p-2}u\quad \text{in } \mathbb{R}^3,\\ -\Delta\phi+a^2\Delta^2\phi=4\pi u^2\quad \text{in } \mathbb{R}^3,\\ \int_{\mathbb{R}^3}|u|^2\,dx=\rho,\end{gathered}$$

where q, a and p are positive constants. The parameter ω arises as a Lagrange multiplier. By the classical minimizing argument, the author obtained a ground-state solution for sufficiently small ρ when $p\in(2,\frac{10}{3})$ and the non-existence of positive solutions via a Liouville-type result when $p=6$. This article also investigated the orbital stability of standing waves for $p\in(2,\frac{10}{3})$ by a contradiction argument. In the article [7], Bellazzini and Siciliano proved the existence of orbitally stable standing waves with prescribed L^2-norm for the following Schrödinger-Poisson type equation

$$i\psi_t+\Delta\psi-(|x|^{-1}*|\psi|^2)\psi+|\psi|^{p-2}\psi=0\quad \text{in}\quad \mathbb{R}^3$$

when $p\in\left\{\frac{8}{3}\right\}\cup(3,\frac{10}{3})$. For more details, please refer to [21] and [6].

From the physical viewpoint, the mass of particles $\int_{\mathbb{R}^3}|u|^2\,dx$ is conserved along the time evolution. It is therefore natural to search for normalized solutions, i.e. pairs (u,λ) solving the stationary system together with the prescribed mass constraint. Such solutions correspond to standing waves of the evolutionary model and their variational properties are closely related to orbital stability or instability. So it is meaningful to look for solutions of (1.1) whose L^2-norm is prescribed. In this paper we therefore pursue normalized solutions, i.e. couples $(u,\lambda)\in E\times\mathbb{R}$ that solve (1.1) while fulfilling the fixed-mass constraint.

Our multiplicity results for the Schrödinger-Bopp-Podolsky system with $|u|^{s-2}u/|x|^q$ appear to be the first of this type as far as we know. The main purpose of this paper is to formulate the normalization problem, precisely, to constructively demonstrate multiple normalized solution through variational methods. Concretely speaking, we shall study the Schrödinger-Bopp-Podolsky system (1.1) with critical growth. Moreover, the variational properties of these solutions aid in examining whether they are orbitally stable or unstable. Meanwhile, we try to obtain the asymptotic behaviour of the normalized solution as $d \to 0$.

Throughout this article, for any $1 \le t < \infty$, we denote by $L^t(\mathbb{R}^3)$ the usual Lebesgue space with norm

$$\|u\|_t := \Big(\int_{\mathbb{R}^3} |u|^t \, dx\Big)^{1/t}.$$

We use $C_0^\infty(\mathbb{R}^3)$ to denote the space of the functions infinitely differentiable with compact support in $\mathbb{R}^3$. $C_1, C_2, \dots$ are the positive constants whose values possibly vary from line to line. Let $H^1(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{H^1} := \Big(\int_{\mathbb{R}^3} |\nabla u|^2 + |u|^2 dx\Big)^{1/2}.$$

Moreover, $D^{1,2}(\mathbb{R}^3)$ is defined by

$$D^{1,2}(\mathbb{R}^3) = \big\{u \in L^6(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty\big\}$$

endowed with the norm

$$\|u\|_{D^{1,2}(\mathbb{R}^3)} = \Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{1/2},$$

and we introduce here the space $\mathcal{D} := \{\phi \in D^{1,2}(\mathbb{R}^3) | \Delta\phi \in L^2(\mathbb{R}^3)\}$ equipped with the norm

$$\|u\|_{\mathcal{D}} = \big(\|\nabla \phi\|_2^2 + d^2 \|\Delta \phi\|_2^2\big)^{1/2}.$$

From [4], we know that $C_0^\infty(\mathbb{R}^3)$ is dense in the Hilbert space $\mathcal{D}$ and $\mathcal{D} \hookrightarrow L^6(\mathbb{R}^3)$.

We will work in the space

$$E := H^1_{\rm rad}(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u \text{ is radial}\}$$

and the assumption

(A1) $0 < \alpha < 3$, $0 < q < 2 < s < 10/3$ satisfying $q + \frac{s}{2} < 3 < q + \frac{3s}{2} < 5$.

Definition 1.1. Let

$$S(a) = \Big\{u \in E : \int_{\mathbb{R}^3} |u|^2 dx = a^2\Big\}.$$

We say $u \in S(a)$ is a normalized ground state solution to (1.1), if it is a solution having minimal energy among all the solutions which belong to $S(a)$, i.e.

$$I(u) =: \inf\{I(u) : u \in S(a), I'|_{S(a)}(u) = 0\}.$$

We define

$$a_* := \big(\frac{1}{\mathcal{K}}\big)^{\frac{5+2\alpha}{12+6\alpha}},$$

where

$$\mathcal{K} := C_*^{\frac{4+2\alpha}{5+2\alpha}} S_{h,l}^{-\frac{3+\alpha}{5+2\alpha}} \Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{5+2\alpha}} \frac{5+2\alpha}{2(4+2\alpha)}.$$

Then primary research findings of this paper are stated as follows.

Theorem 1.2. *Under condition* (A1), $a \in (0, a_*)$, *for any given* $k \in \mathbb{N}$, *problem* (1.1)-(1.2) *possesses at least* k *couples* $(u_j, \lambda_j) \in E \times \mathbb{R}$ *of distinct weak solutions with* $\int_{\mathbb{R}^3} u_j^2 \, dx = a^2, \lambda_j > 0$ *for any* $j = 1, \dots, k$. *In addition, the system has a couple ground state solution* (u_a, λ_a) *with* $\|u_a\|_E \to 0$ *as* $a \to 0$.

To establish Theorem 1.2, we work within a constrained variational framework. Technically, problem (1.1)-(1.2) simultaneously exhibits Hardy-Littlewood-Sobolev upper-critical growth, singular and a prescribed-mass constraint, so that certain (PS) sequences may fail to converge. Furthermore, the corresponding energy functional restricted to the constraint manifold $S(a)$ ceases to be bounded below. To overcome these difficulties, we employ the concentration-compactness principle and a truncation method to restore the compactness lost to critical growth. And we work in the radial subspace to address the lack of compactness in the full space. Our innovation also lies in the fact that we have obtained an accurate value of a_*, which is different from the situation where many previous studies only indicated the existence of a_*. Furthermore, this paper establishes the orbital stability.

Theorem 1.3. *Under condition* (A1), $a \in (0, a_*)$. *The ground-state standing waves of* (1.1) *are orbitally stable.*

Theorem 1.4. *Under condition* (A1), *fix* $a \in (0, a_*)$ *For every sequence* $d \to 0$, *let* (u_d, ϕ^d) *be the radial ground-state solution of* (1.3)-(1.4)*:*

$$\begin{gathered}\Delta u+\lambda_d u=\phi u+\frac{|u|^{s-2}u}{|x|^q}+\big(I_\alpha * |u|^{2^*_\alpha}\big)|u|^{2^*_\alpha-2}u, \quad x\in\mathbb{R}^3,\\ \Delta\phi+d^2\Delta^2\phi=4\pi u^2, \quad x\in\mathbb{R}^3\end{gathered}\tag{1.3}$$

under the mass constraint

$$\int_{\mathbb{R}^3}|u|^2dx=a^2.\tag{1.4}$$

Then there exists a subsequence (still denoted by d) and a pair (u_0, ϕ^0) *such that*

$$\begin{gathered}u_d\to u_0 \quad in \quad E,\\ \phi^d\to\phi^0 \quad in \quad D^{1,2}(\mathbb{R}^3),\end{gathered}$$

and the pair (u_0, ϕ^0) *serves as a radial ground-state solution to the limit Schrödinger-Poisson system*

$$\begin{gathered}\Delta u+\lambda_0 u=\phi u+\frac{|u|^{s-2}u}{|x|^q}+\big(I_\alpha * |u|^{2^*_\alpha}\big)|u|^{2^*_\alpha-2}u, \quad x\in\mathbb{R}^3,\\ \Delta\phi=4\pi u^2, \quad x\in\mathbb{R}^3\end{gathered}\tag{1.5}$$

with $\int_{\mathbb{R}^3}|u_0|^2dx=a^2$ *and* $\lambda_0>0$.

The remainder of this paper is structured as follows. In Section 2, we provide some preliminary results that will be useful throughout this paper. Theorem 1.2 is proved in Section 3 and Section 4. We give the proof of Theorem 1.3 in Section 5, and Theorem 1.4 is addressed in Section 6.

2. Preliminaries

Let X be a Banach space and $A \subseteq X$. The set A is called symmetric if $u \in A$ entails $-u \in A$. Now recall the notion of genus. We denote

$$\Sigma := \{A \subset X \setminus \{0\} : A \text{ is closed and symmetric with respect to the origin}\}.$$

For $A \in \Sigma$, set

$$\gamma(A)=\begin{cases}0 & \text{if } A=\emptyset,\\ \inf\{k\in\mathbb{N}: \exists \text{ an odd } \varphi\in C(A,\mathbb{R}^k\setminus\{0\})\}, & \\ +\infty & \text{if no such odd map exists,}\end{cases}$$

and $\Sigma_k = \{A \in \Sigma : \gamma(A) \geqslant k\}$. Next we give some lemmas which will be used to prove Theorems 1.2–1.4.

Lemma 2.1 ([14, Lemma 2.1])**.** *Let* $u \in H^1(\mathbb{R}^3)$, *and* $1 < \sigma < 3$. *Then*

$$\int_{\mathbb{R}^3}\frac{|u|^\sigma}{|x|^\sigma}dx\le C_\sigma\int_{\mathbb{R}^3}|\nabla u|^\sigma\,dx,$$

where $C_\sigma = \big(\frac{\sigma}{3-\sigma}\big)^\sigma$.

Lemma 2.2 (Gagliardo-Nirenberg-type inequality [25, Theorem B])**.** *Let* C_t *be the best constant in the Gagliardo-Nirenberg inequality*

$$\|u\|_t^t\le C_t^t\|u\|_2^{t(1-\gamma_t)}\|\nabla u\|_2^{t\gamma_t} \quad for\ 2<t<6,$$

where $\gamma_t = 3(\frac{1}{2} - \frac{1}{t})$.

By Hölder inequality and Lemmas 2.1-2.2, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q} dx &= \int_{\mathbb{R}^3} \frac{|u|^q}{|x|^q} \cdot |u|^{s-q} dx \\
&\le \Big(\int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} dx \Big)^{q/2} \Big(\int_{\mathbb{R}^3} |u|^{\frac{2(s-q)}{2-q}} dx \Big)^{\frac{2-q}{2}} \\
&\le \Big(4 \int_{\mathbb{R}^3} |\nabla u|^2 dx \Big)^{q/2} \Big[C_{q,s} \|u\|_2^{\frac{6-2q-s}{2-q}} \|\nabla u\|_2^{\frac{3s-6}{2-q}} \Big]^{\frac{2-q}{2}} \\
&\le 2^q C_{q,s}^{\frac{2-q}{2}} \|\nabla u\|_2^{q+\frac{3s}{2}-3} \|u\|_2^{3-q-\frac{s}{2}}.
\end{aligned} \tag{2.1}$$

Let us recall the following Hardy-Littlewood-Sobolev inequality [17],

$$\Big| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)\, g(y)}{|x-y|^\lambda} \, dx\, dy \Big| \le C(N, \lambda, r, s) \, \|f\|_{L^r(\mathbb{R}^N)} \|g\|_{L^s(\mathbb{R}^N)},$$

where $f \in L^r(\mathbb{R}^N)$, $g \in L^s(\mathbb{R}^N)$, $r, s > 1$, $0 < \lambda < N$, and

$$\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2.$$

From Hardy-Littlewood-Sobolev inequality, we can define the best constant and the following lemma, which can be found in [16]:

$$S_{h,l} := \inf_{u \in D^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{[\int_{\mathbb{R}^3} (I_\alpha * |u|^{3+\alpha}) \, |u|^{3+\alpha} dx]^{\frac{1}{3+\alpha}}}.$$

Lemma 2.3. *Let $u \in S(a)$. Then there exists constant $C_* > 0$ such that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} \, dx\, dy \le C_* \|\nabla u\|_2 \, a^3. \tag{2.2}$$

Indeed, the Lax-Milgram theorem guarantees that for every fixed $u \in H^1(\mathbb{R}^3)$, the second equation of (1.1) admits a unique solution ϕ_u, explicitly given by

$$\phi_u = \int_{\mathbb{R}^3} \frac{1 - e^{-|x-y|/d}}{|x-y|} \, |u(y)|^2 \, \mathrm{d}y,$$

then system (1.1) can be transformed into a single Schrödinger equation as follows,

$$-\Delta u + \lambda u = \phi_u u + \frac{|u|^{s-2} u}{|x|^q} + \big(I_\alpha * |u|^{3+\alpha} \big) \, |u|^{1+\alpha} u \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

And the solutions of (2.3) with $\|u\|_2^2 = a^2 > 0$ can be obtained as critical points of the energy functional

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{1}{s} \int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q} dx - \frac{1}{6+2\alpha} \int_{\mathbb{R}^3} \big(I_\alpha * |u|^{3+\alpha} \big) \, |u|^{3+\alpha} dx$$

on the constraint set

$$S(a) = \Big\{ u \in E : \int_{\mathbb{R}^3} |u|^2 dx = a^2 \Big\}.$$

The properties of the function ϕ_u are given as follows.

Lemma 2.4 ([4])**.** *For every $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, we have*

(i) $\phi_u \ge 0$;
(ii) $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$ *for any* $y \in \mathbb{R}^3$;
(iii) $\phi_u \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$ *for every* $s \in (3, +\infty]$;
(iv) $\nabla \phi_u \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$ *for every* $s \in (3/2, +\infty]$;
(v) ϕ_u *is the unique minimizer of the functional*

$$\mathcal{E}_{u,a}(\phi) := \frac{1}{2} |\nabla \phi|_2^2 + \frac{d^2}{2} |\Delta \phi|_2^2 - \int_{\mathbb{R}^3} \phi u^2 \, dx, \quad \phi \in \mathcal{D}.$$

Moreover, if u is radial, then ϕ_u is also radial; and if $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, then

(vi) $\phi_{u_n} \to \phi_u$ *in* $\mathcal{D}$;
(vii) $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx \to \int_{\mathbb{R}^3} \phi_u u^2\,dx$;
(viii) $\int_{\mathbb{R}^3} \phi_{u_n} u_n v\,dx \to \int_{\mathbb{R}^3} \phi_u uv\,dx$ *for any* $v \in H_r^1(\mathbb{R}^3)$.

We need the concentration-compactness principles to solve the non-compactness of the energy functional caused by critical growth.

Lemma 2.5 ([10]). *Let $\{u_n\}$ be a bounded sequence in $D^{1,2}(\mathbb{R}^3)$ converging weakly and almost everywhere to some $u \in D^{1,2}(\mathbb{R}^3)$, there exists a subsequence which is still denoted as $\{u_n\}$ for convenience:*

$$|\nabla u_n|^2 \rightharpoonup \mu,$$
$$\left(I_\alpha * |u_n|^{3+\alpha}\right)|u_n|^{3+\alpha} \rightharpoonup \nu$$

weakly in the measure space. Using the concentration compactness principle due to Lions, we obtain the existence of a set $\mathcal{J}$ (at most countable) and positive numbers $\{\mu_j\}_{j\in\mathcal{J}}$, $\{\nu_j\}_{j\in\mathcal{J}} \subset [0,\infty)$ such that there exist two positive measures $\mu,\nu \in \mathcal{M}(\mathbb{R}^3)$,

$$\mu \ge |\nabla u|^2 + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j},$$
$$\nu = \left(I_\alpha * |u|^{3+\alpha}\right)|u|^{3+\alpha} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}, \sum_{j\in\mathcal{J}} \nu_j^{\frac{1}{3+\alpha}} < \infty,$$

and

$$\mu_j \ge S_{h,l} \nu_j^{\frac{1}{3+\alpha}}, \tag{2.4}$$

where δ_{x_j} is the Dirac-mass of mass 1 concentrated at x_j and the subsequence $\{x_j\} \subset \mathbb{R}^3$.

Lemma 2.6 ([13]). *Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^3)$ be a sequence in Lemma 2.5 and define*

$$\mu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geqslant R} |\nabla u_n|^2 dx,$$
$$\nu_\infty := \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|\geqslant R} \left(I_\alpha * |u_n|^{3+\alpha}\right)|u_n|^{3+\alpha} dx.$$

Then it follows that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} d\mu + \mu_\infty, \tag{2.5}$$
$$\limsup_{n\to\infty} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right)|u_n|^{3+\alpha} dx = \int_{\mathbb{R}^3} d\nu + \nu_\infty, \tag{2.6}$$
$$S_{h,l}^2 \nu_\infty^{\frac{2}{3+\alpha}} \le \mu_\infty \Big(\int_{\mathbb{R}^3} d\mu + \mu_\infty\Big). \tag{2.7}$$

The following result deals with the asymptotic behavior of the solutions of (1.3) when $d \to 0$, which can see [4, Lemma 6.1].

Lemma 2.7. *Consider $f^0 \in L^{6/5}(\mathbb{R}^3)$, $\{f^d\}_{d\in(0,1)} \subset L^{6/5}(\mathbb{R}^3)$ and let*

$$\phi^0 \in D^{1,2}(\mathbb{R}^3) \text{ be the unique solution of } -\Delta\phi = f^0 \text{ in } \mathbb{R}^3,$$
$$\phi^d \in \mathcal{D} \text{ be the unique solution of } -\Delta\phi + d^2\Delta^2\phi = f^d \text{ in } \mathbb{R}^3.$$

As $d \to 0$ we have:

(i) *if $f^d \rightharpoonup f^0$ in $L^{6/5}(\mathbb{R}^3)$, then $\phi^d \rightharpoonup \phi^0$ in $D^{1,2}(\mathbb{R}^3)$;*
(ii) *if $f^d \to f^0$ in $L^{6/5}(\mathbb{R}^3)$, then $\phi^d \to \phi^0$ in $D^{1,2}(\mathbb{R}^3)$ and $d\Delta\phi^d \to 0$ in $L^2(\mathbb{R}^3)$.*

3. Existence of multiple solutions

We always require condition (A1) to be satisfied. For $u \in S(a)$, in view of the Sobolev inequality, (2.1) and (2.2), one has

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{4}C_* a^3\|\nabla u\|_2 - \frac{2^q}{s}C_{q,s}^{\frac{2-q}{2}}\|\nabla u\|_2^{q+\frac{3s}{2}-3}\|\nabla u\|_2^{3-q-\frac{s}{2}} \\ &\quad - \frac{1}{6+2\alpha}S_{h,l}^{-(3+\alpha)}\|u\|_2^{6+2\alpha} \\ &:= h(\|\nabla u\|_2), \end{aligned} \tag{3.1}$$

where

$$h(t) = \frac{1}{2}t^2 - \frac{1}{4}C_* a^3 t - \frac{2^q}{s}C_{q,s}^{\frac{2-q}{2}}a^{3-q-\frac{s}{2}}t^{q+\frac{3s}{2}-3} - \frac{1}{6+2\alpha}S_{h,l}^{-(3+\alpha)}t^{6+2\alpha}.$$

Let

$$g_a(t) = \frac{1}{2} - \frac{1}{4}C_* a^3 t^{-1} - \frac{1}{6+2\alpha}S_{h,l}^{-(3+\alpha)}t^{4+2\alpha}.$$

Then $g_a(t) \to -\infty$ as $t \to 0^+$ and $g_a(t) \to -\infty$ as $t \to +\infty$. Note that

$$g_a'(t) = \frac{1}{4}C_* a^3 t^{-2} - \frac{4+2\alpha}{6+2\alpha}S_{h,l}^{-(3+\alpha)}t^{3+2\alpha},$$

thus $g_a(t)$ has only one maximum

$$t_a = \Big(\frac{\frac{1}{4}C_* a^3\cdot(6+2\alpha)}{4+2\alpha}\cdot S_{h,l}^{3+\alpha}\Big)^{\frac{1}{5+2\alpha}} = \Big(\frac{C_* a^3(3+\alpha)}{2(4+2\alpha)}\cdot S_{h,l}^{3+\alpha}\Big)^{\frac{1}{5+2\alpha}}.$$

Then we have

$$\begin{aligned} \max_{t>0} g_a(t) &= g_a(t_a) \\ &= \frac{1}{2} - \frac{1}{4}C_* a^3\Big(\frac{C_* a^3(3+\alpha)}{2(4+2\alpha)}S_{h,l}^{3+\alpha}\Big)^{-\frac{1}{5+2\alpha}} - \frac{1}{6+2\alpha}S_{h,l}^{-(3+\alpha)}\Big(\frac{C_* a^3(3+\alpha)}{2(4+2\alpha)}S_{h,l}^{3+\alpha}\Big)^{\frac{4+2\alpha}{5+2\alpha}} \\ &= \frac{1}{2} - \frac{1}{2}C_*^{\frac{4+2\alpha}{5+2\alpha}}a^{\frac{12+6\alpha}{5+2\alpha}}S_{h,l}^{-\frac{3+\alpha}{5+2\alpha}}\Big[\frac{1}{2}\Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{5+2\alpha}} + \frac{1}{3+\alpha}\Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{\frac{4+2\alpha}{5+2\alpha}}\Big] \\ &= \frac{1}{2} - \frac{1}{2}C_*^{\frac{4+2\alpha}{5+2\alpha}}a^{\frac{12+6\alpha}{5+2\alpha}}S_{h,l}^{-\frac{3+\alpha}{5+2\alpha}}\Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{5+2\alpha}}\frac{5+4\alpha}{2(4+2\alpha)}. \end{aligned}$$

Let

$$\mathcal{K} :\overset{\triangle}{=} C_*^{\frac{4+2\alpha}{5+2\alpha}}S_{h,l}^{-\frac{3+\alpha}{5+2\alpha}}\Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{5+2\alpha}}\frac{5+2\alpha}{2(4+2\alpha)}.$$

Then

$$g_a(t_a) = 0 \implies \frac{1}{2} - \frac{1}{2}\mathcal{K}\cdot a^{\frac{12+6\alpha}{5+2\alpha}} = 0 \implies a = \big(\frac{1}{\mathcal{K}}\big)^{\frac{5+2\alpha}{12+6\alpha}} :\overset{\triangle}{=} a_*, \tag{3.2}$$

so we obtain $g_a(t_a) > 0$ as $a < a_*$. Hence

$$t_a^2 g_a(t_a) = \frac{1}{2}t_a^2 - \frac{1}{4}C_* a^3 t_a - \frac{1}{6+2\alpha}S_{h,l}^{-(3+\alpha)}t_a^{6+2\alpha} > 0 \quad \text{as } a < a_*. \tag{3.3}$$

From the above deduction, we know that when $a < a_*$, the maximum value of $g_a(t)$ is positive. Clearly, $g_a(t) = 0$ has exactly two roots $0 < R_1 < R_2 < \infty$. Thus, we summarize the properties of $g_a(t)$ as follows:

$$\begin{gathered} \lim_{r\to 0^+} g_a(t) = -\infty \quad \text{and} \quad \lim_{r\to+\infty} g_a(t) = -\infty, \\ g_a(R_1) = g_a(R_2) = 0 \quad \text{and} \quad g_a(t_a) > 0, \\ g_a(t) \text{ is increasing on } (0, t_a) \text{ and decreasing on } (t_a, +\infty). \end{gathered}$$

For $a \in (0, a_*)$, from (3.2), we obtain

$$a^{\frac{12+6\alpha}{5+2\alpha}}\mathcal{K} < 1 \implies a^{\frac{3}{5+2\alpha}} < \big(\frac{1}{\mathcal{K}}\big)^{\frac{1}{4+2\alpha}} = \mathcal{K}^{-\frac{1}{4+2\alpha}}.$$

So

$$\begin{aligned}
R_1 < t_a &= \Big(\frac{C_* a^3(3+\alpha)}{2(4+2\alpha)} S_{h,l}^{3+\alpha}\Big)^{\frac{1}{5+2\alpha}}\\
&= C_*^{\frac{1}{5+2\alpha}} \cdot S_{h,l}^{\frac{3+\alpha}{5+2\alpha}} \Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{\frac{1}{5+2\alpha}} a^{\frac{3}{5+2\alpha}}\\
&< C_*^{\frac{1}{5+2\alpha}} S_{h,l}^{\frac{3+\alpha}{5+2\alpha}} \Big(\frac{3+\alpha}{2(4+\alpha)}\Big)^{\frac{1}{5+2\alpha}} C_*^{\frac{4+2\alpha}{5+2\alpha}\left(-\frac{1}{4+2\alpha}\right)}\\
&\quad \times S_{h,l}^{-\frac{3+\alpha}{5+2\alpha}\left(-\frac{1}{4+2\alpha}\right)} \Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{5+2\alpha}\left(-\frac{1}{4+2\alpha}\right)} \Big(\frac{5+2\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{4+2\alpha}}\\
&= S_{h,l}^{\frac{3+\alpha}{5+2\alpha}\left(1+\frac{1}{4+2\alpha}\right)} \Big(\frac{3+\alpha}{2(4+2\alpha)}\Big)^{\frac{1}{5+2\alpha}+\frac{1}{5+2\alpha}\frac{1}{4+2\alpha}} \Big(\frac{5+2\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{4+2\alpha}}\\
&= S_{h,l}^{\frac{3+\alpha}{2(2+\alpha)}} \Big(\frac{2(4+2\alpha)}{3+\alpha} \cdot \frac{5+2\alpha}{2(4+2\alpha)}\Big)^{-\frac{1}{4+2\alpha}}\\
&< S_{h,l}^{\frac{3+\alpha}{2(2+\alpha)}}
\end{aligned}$$

since

$$\frac{2(4+2\alpha)}{3+\alpha} \cdot \frac{5+2\alpha}{2(4+2\alpha)} = \frac{5+2\alpha}{3+\alpha} > 1.$$

Hence

$$R_1^2 < S_{h,l}^{\frac{3+\alpha}{2+\alpha}}. \tag{3.4}$$

Let $\tau:\mathbb{R}^+\to[0,1]$ be a nonincreasing and C^∞ function that satisfies

$$\tau(t) = \begin{cases} 1 & \text{if } t \leqslant R_1,\\ 0 & \text{if } t \geqslant R_2. \end{cases}$$

Now, we introduce the truncated functional

$$I_\tau(u) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{s}\int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q} dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{\tau(\|\nabla u\|_2)}{6+2\alpha}\int_{\mathbb{R}^3} \big(I_\alpha * |u|^{3+\alpha}\big)|u|^{3+\alpha} dx,$$

we can easily show that $I_\tau \in C^1(E,\mathbb{R})$.

For $u\in S(a)$, by the Sobolev inequality, Lemmas 2.1-2.3 and (2.1), it is easy to see that

$$\begin{aligned}
I_\tau(u) &\ge \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{4} C_* a^3 \|\nabla u\|_2 - \frac{2^q}{s} C_{q,s}^{\frac{2-q}{2}} \|\nabla u\|_2^{q+\frac{3s}{2}-3} \|u\|_2^{3-q-\frac{s}{2}}\\
&\quad - \frac{\tau(\|\nabla u\|_2)}{6+2\alpha} S_{h,l}^{-(3+\alpha)} \|\nabla u\|_2^{6+2\alpha}\\
&:= \widetilde{h}(\|\nabla u\|_2),
\end{aligned}$$

where

$$\widetilde{h}(t) = \frac{1}{2}t^2 - \frac{1}{4}C_* a^3 t - \frac{2^q}{s} C_{q,s}^{\frac{2-q}{2}} a^{3-q-\frac{s}{2}} t^{q+\frac{3s}{2}-3} - \frac{\tau(t)}{6+2\alpha} S_{h,l}^{-(3+\alpha)} t^{6+2\alpha}.$$

Setting $u_t(x) = t^{3/2}u(tx)$, we have $\|u_t\|_2 = \|u\|_2$. By (3.1), when t is small enough, we have

$$\begin{aligned}
I(u_t) &\le \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{t}{4}\int_{\mathbb{R}^3} \phi_u |u|^2 dx - \frac{t^{\frac{3s}{2}+q-3}}{s}\int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q} dx\\
&\quad - \frac{t^{3+3\alpha}}{6+2\alpha}\int_{\mathbb{R}^3} \big(I_\alpha * |u|^{3+\alpha}\big)|u|^{3+\alpha} dx < 0.
\end{aligned} \tag{3.5}$$

Lemma 3.1. *For $a\in(0,a_*)$, the functional I_τ has the following properties:*

(i) *I_τ is coercive and bounded from below on $S(a)$. Moreover, if $I_\tau(u)\le 0$, then $\|\nabla u\|_2 \le R_1$ and $I_\tau(u) = I(u)$;*

(ii) *I_τ satisfies the $(PS)_c$ condition on $S(a)\cap E$ for all $c<0$.*

Proof. For (i), it follows from condition (A1) that when $\|\nabla u\|_2 \to +\infty$, $\widetilde{h}(\|\nabla u\|_2)$ is dominated by $\|\nabla u\|_2^2$ yielding

$$\begin{aligned} I_\tau(u) &\geq \widetilde{h}(\|\nabla u\|_2) \\ &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{4}C_* a^3\|\nabla u\|_2 - \frac{2^q}{s}C_{q,s}^{\frac{2-q}{2}}\|\nabla u\|_2^{q+\frac{3s}{2}-3}\|u\|_2^{3-q-\frac{s}{2}} - \frac{\tau(\|\nabla u\|_2)}{6+2\alpha}S_{h,l}^{-(3+\alpha)}\|\nabla u\|_2^{6+2\alpha} \\ &\to +\infty \quad \text{as } \|\nabla u\|_2 \to +\infty, \end{aligned} \tag{3.6}$$

which shows coercivity. When $\|\nabla u\|_2 \in [0, R_1]$, $I_\tau(u) \geq \widetilde{h}(\|\nabla u\|_2) = h(\|\nabla u\|_2)$. So we can get that I_τ is coercive and bounded from below on $S(a)$. By contradiction, if $I_\tau(u) \leq 0$ but $\|\nabla u\|_2 > R_1$, then by the property of $\widetilde{h}(\|\nabla u\|_2)$, we would have $I_\tau(u) > 0$, which contradicts the assumption. So $\|\nabla u\|_2 \leq R_1$ and thus the truncation function $\tau(\|\nabla u\|_2) = 1$, leading to $I_\tau(u) = I(u)$.

For (ii), let $\{u_n\}$ be a $(PS)_c$ sequence of I_τ restricted to $S(a) \cap E$ with $c < 0$. Then by (i), we have $\|\nabla u_n\|_2 \leq R_1$ for large n, and $\{u_n\}$ is also a bounded $(PS)_c$ sequence of $I|_{S(a)\cap E}$ with $c < 0$, i.e.,

$$I(u_n) \to c < 0, \quad \big\|I'|_{S(a)\cap E}(u_n)\big\|_{E^*} \to 0 \quad \text{as } n \to \infty.$$

Therefore, up to a subsequence (still denoted as $\{u_n\}$), there exists $u \in E$ such that

$$u_n \rightharpoonup u \text{ in } E, \quad u_n \to u \text{ in } L^t(\mathbb{R}^3) \text{ for } 2 < t < 6, \quad u_n(x) \to u(x) \quad \text{a.e. on } \mathbb{R}^3.$$

Note that, from Hölder inequality and Lemma 2.1, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|u_n - u|^s}{|x|^q}dx &= \int_{\mathbb{R}^3} \frac{|u_n - u|^q}{|x|^q} \cdot |u_n - u|^{s-q}dx \\ &\leq \Big(\int_{\mathbb{R}^3} \frac{|u_n - u|^2}{|x|^2}dx\Big)^{q/2}\Big(\int_{\mathbb{R}^3} |u_n - u|^{\frac{2(s-q)}{2-q}}dx\Big)^{\frac{2-q}{2}} \\ &\leq \Big(4\int_{\mathbb{R}^3} |\nabla(u_n - u)|^2dx\Big)^{\frac{q}{2}}\Big(\int_{\mathbb{R}^3} |u_n - u|^{\frac{2(s-q)}{2-q}}dx\Big)^{\frac{2-q}{2}}, \end{aligned} \tag{3.7}$$

thus we can obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q}\,dx = \int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q}\,dx \tag{3.8}$$

since $\{u_n\}$ is bounded in E and $2 < \frac{2(s-q)}{2-q} < 6$ by the condition (A1).

Then, by [26, Proposition 5.12], there exists $\lambda_n \in \mathbb{R}$ satisfying

$$-\Delta u_n + \lambda_n u_n - \phi_{u_n}u_n - \frac{1}{|x|^q}|u_n|^{s-2}u_n - \big(I_\alpha * |u_n|^{3+\alpha}\big)|u_n|^{1+\alpha}u_n = o_n(1) \tag{3.9}$$

in E^*, where E^* is the dual space of E. Therefore, for $\varphi \in E$, one has

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla u_n \nabla\varphi dx + \lambda_n\int_{\mathbb{R}^3} u_n\varphi dx - \int_{\mathbb{R}^3} \phi_{u_n}u_n\varphi dx \\ &- \int_{\mathbb{R}^3} \frac{|u_n|^{s-2}}{|x|^q}u_n\varphi dx - \int_{\mathbb{R}^3} \big(I_\alpha * |u_n|^{3+\alpha}\big)|u_n|^{1+\alpha}u_n\varphi dx \\ &= o_n(1)\|\varphi\|. \end{aligned} \tag{3.10}$$

Consequently, lettin $\varphi = u_n$, we have

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u_n|^2dx + \lambda_n a^2 \\ &= \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2dx + \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q}dx + \int_{\mathbb{R}^3} \big(I_\alpha * |u_n|^{3+\alpha}\big)|u_n|^{3+\alpha}dx + o_n(1). \end{aligned} \tag{3.11}$$

By Lemmas 2.1-2.3, we have

$$\begin{gathered} \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2dx \leq C_* a^3\|\nabla u_n\|_2, \\ \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q}dx \leq 2^q C_{q,s}^{\frac{2-q}{2}} a^{3-q-\frac{s}{2}}\|\nabla u\|_2^{q+\frac{3s}{2}-3}, \end{gathered}$$

$$\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} dx \le S_{h,l}^{-(3+\alpha)} \|\nabla u_n\|_2^{6+2\alpha}.$$

Thus, the boundedness of $\{u_n\}$ yields that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Up to a subsequence, we may assume $\lambda_n \to \lambda_a \in \mathbb{R}$. By (3.10), we claim that

$$-\Delta u + \lambda_a u - \phi_u u - \frac{|u|^{s-2}u}{|x|^q} - \left(I_\alpha * |u|^{3+\alpha}\right) |u|^{1+\alpha} u = 0. \tag{3.12}$$

Indeed, for any $\varphi \in E$, from $u_n \rightharpoonup u$ in E, it follows that

$$\int_{\mathbb{R}^3} \nabla u_n \nabla \varphi \, dx \to \int_{\mathbb{R}^3} \nabla u \nabla \varphi \, dx \quad \text{as } n \to \infty. \tag{3.13}$$

Noting that $\lambda_n \to \lambda_a$ as $n \to \infty$, we easily obtain

$$\lambda_n \int_{\mathbb{R}^3} u_n \varphi \, dx \to \lambda_a \int_{\mathbb{R}^3} u \varphi \, dx \quad \text{as } n \to \infty. \tag{3.14}$$

Furthermore, since $\{|u_n|^{3+\alpha}\}$ is bounded in $L^{\frac{6}{3+\alpha}}(\mathbb{R}^3)$ and $|u_n(x)|^{3+\alpha} \to |u(x)|^{3+\alpha}$ almost everywhere in $\mathbb{R}^3$. Then, we have $|u_n(x)|^{3+\alpha} \to |u(x)|^{3+\alpha}$ in $L^{\frac{6}{3+\alpha}}(\mathbb{R}^3)$.

By the Hardy-Littlewood-Sobolev inequality, we have

$$\|I_\alpha * |u_n|^{3+\alpha}\|_{\frac{6}{3-\alpha}} \le C \||u_n|^{3+\alpha}\|_{\frac{6}{3+\alpha}} \le C_I,$$

where $C_I > 0$ is independent on n. Thus, up to subsequences, it follows that

$$I_\alpha * |u_n|^{3+\alpha} \rightharpoonup I_\alpha * |u|^{3+\alpha} \quad \text{in} \quad L^{\frac{6}{3-\alpha}}(\mathbb{R}^3).$$

Furthermore, since

$$\int_{\mathbb{R}^3} \left(|u_n|^{1+\alpha} u_n\right)^{\frac{6}{2+\alpha}} \, dx = \int_{\mathbb{R}^3} |u_n|^6 \, dx \le C,$$

as $n \to \infty$, it holds that

$$|u_n(x)|^{1+\alpha} u_n \rightharpoonup |u(x)|^{1+\alpha} u \quad \text{in } L^{\frac{6}{2+\alpha}}(\mathbb{R}^3).$$

By Hölder inequality, we obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[\left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n(x)|^{1+\alpha} u_n(x)\right]^{\frac{6}{5}} dx \\
&= \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right)^{\frac{6}{5}} \left[|u_n(x)|^{1+\alpha} u_n(x)\right]^{\frac{6}{5}} dx \\
&= \left(\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right)^{\frac{6}{3-\alpha}} dx\right)^{\frac{3-\alpha}{5}} \left(\int_{\mathbb{R}^3} \left(|u_n(x)|^{1+\alpha} u_n(x)\right)^{\frac{6}{2+\alpha}} dx\right)^{\frac{2+\alpha}{5}} \le C.
\end{aligned}$$

So, it follows that

$$I_\alpha * |u_n|^{3+\alpha} |u_n(x)|^{1+\alpha} u_n \rightharpoonup I_\alpha * |u|^{3+\alpha} |u(x)|^{1+\alpha} u \quad \text{in} \quad L^{\frac{6}{5}}(\mathbb{R}^3).$$

Thus,

$$\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{1+\alpha} u_n \varphi dx \to \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{3+\alpha}\right) |u|^{1+\alpha} u \varphi dx. \tag{3.15}$$

Moreover, it follows Lemma 2.4 (viii), we have

$$\int_{\mathbb{R}^3} \left(\phi_{u_n} u_n - \phi_u u\right) \varphi dx \to 0 \quad as \quad n \to \infty. \tag{3.16}$$

So, from (3.8) and (3.13)-(3.16), (3.12) is proved.

Next we will prove that $\|u_n\|_E^2 \to \|u\|_E^2$, and

$$\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} dx \to \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{3+\alpha}\right) |u|^{3+\alpha} dx. \tag{3.17}$$

In fact, since $\|\nabla u_n\|_2 \le R_1$ for n large enough, by Lemma 2.5, there exist two positive measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^3)$ such that

$$|\nabla u_n|^2 \rightharpoonup \mu\,, \quad \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} \rightharpoonup \nu \tag{3.18}$$

as $n \to \infty$. Setting the functional $\Psi(v) : E \to \mathbb{R}^3$ given by

$$\Psi(v) = \frac{1}{2}\int_{\mathbb{R}^3} v^2 dx.$$

It follows that $S(a) = \Psi^{-1}(\frac{a^2}{2})$. We divide the proof into three steps.

Step 1. We prove that $\mathcal{J}$ is empty. By the conservation of the mass, $\mathcal{J}$ is not infinite. If that $\mathcal{J}$ is nonempty and finite, then μ_j and ν_j, which come from Lemma 2.5 are positive measures.

Take $\varphi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\varphi \in [0,1]$, $\varphi \equiv 1$ in $B_{1/2}(0)$, $\varphi \equiv 0$ in $\mathbb{R}^3 \setminus B_1(0)$. For any $\rho > 0$, set

$$\varphi_\rho(x) := \varphi\Big(\frac{x-x_j}{\rho}\Big) = \begin{cases} 1 & \text{if } |x-x_j| \leqslant \rho/2, \\ 0 & \text{if } |x-x_j| \geqslant \rho. \end{cases}$$

By the boundedness of $\{u_n\}$ in E, $\{\varphi_\rho u_n\}$ is also bounded in E. Thus, one has

$$\begin{aligned} o_n(1) &= \langle I'(u_n), \varphi_\rho u_n\rangle \\ &= \int_{\mathbb{R}^3} \varphi_\rho |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} u_n \nabla\varphi_\rho \nabla u_n dx - \int_{\mathbb{R}^3} \phi_{u_n}|u_n|^2 \varphi_\rho dx \\ &\quad - \int_{\mathbb{R}^3} \frac{\varphi_\rho |u_n|^s}{|x|^q} dx - \int_{\mathbb{R}^3} \varphi_\rho \left(I_\alpha * |u_n|^{3+\alpha}\right)|u_n|^{3+\alpha} dx. \end{aligned} \tag{3.19}$$

By (3.8), (3.16) and the definition of φ_ρ, one has

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} \frac{\varphi_\rho|u_n|^s}{|x|^q}dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3}\frac{\varphi_\rho|u|^s}{|x|^q}dx = \lim_{\rho\to 0}\int_{|x-x_j|\leqslant\rho}\frac{\varphi_\rho|u|^s}{|x|^q}dx = 0, \tag{3.20}$$

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}\varphi_\rho\phi_{u_n}|u_n|^2 dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho\phi_u|u|^2dx = 0. \tag{3.21}$$

Applying Hölder' inequality with the absolute continuity of Lebesgue integration, we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\varphi_\rho dx = 0. \tag{3.22}$$

Then, it follows from (3.19)-(3.22), Lemma 2.5 and the definition of φ_ρ that

$$\begin{aligned} \nu_j &= \lim_{\rho\to 0}\Big\{\int_{\mathbb{R}^3}\Big(I_\alpha * |u|^{3+\alpha}|u|^{3+\alpha}\varphi_\rho + \sum_{j\in\mathcal{J}}\nu_j\delta_{x_j}\varphi_\rho\Big)dx\Big\} \\ &= \lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\nu = \lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}(I_\alpha * |u_n|^{3+\alpha})|u_n|^{3+\alpha}\varphi_\rho dx \\ &= \lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho dx \\ &= \lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\mu \\ &\geq \lim_{\rho\to 0}\Big\{\int_{\mathbb{R}^3}\Big(|\nabla u|^2\varphi_\rho + \sum_{j\in\mathcal{J}}\mu_j\delta_{x_j}\varphi_\rho\Big)dx\Big\} = \mu_j. \end{aligned} \tag{3.23}$$

So we can deduce that $\nu_j \geq \mu_j$. By (2.4), we obtain

$$\mu_j \geq S_{h,l}\nu_j^{\frac{1}{3+\alpha}},$$

so we can obtain that

$$\mu_j \geq S_{h,l}\nu_j^{\frac{1}{3+\alpha}} \geq S_{h,l}\mu_j^{\frac{1}{3+\alpha}}.$$

Consequently, one has $\mu_j \geq S_{h,l}\mu_j^{\frac{1}{3+\alpha}}$. Then

$$R_1^2 \geq \lim_{\rho\to 0}\lim_{n\to\infty}\|\nabla u_n\|_2^2 \geq \lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^3}|\nabla u_n|^2\varphi_\rho dx = \lim_{\rho\to 0}\int_{\mathbb{R}^3}\varphi_\rho d\mu \geq \mu_j \geq S_{h,l}^{\frac{3+\alpha}{2+\alpha}}, \tag{3.24}$$

which contradicts with (3.4), then $\mathcal{J}$ is empty.

Step 2. We prove that $\mu_\infty = \nu_\infty$, where μ_∞ and ν_∞ come from Lemma 2.6. Let $\psi \in C_0^\infty(\mathbb{R}^3)$ be a cut-off function with $\psi \in [0,1]$, $\psi \equiv 0$ in $B_{1/2}(0)$, $\psi \equiv 1$ in $\mathbb{R}^3 \setminus B_1(0)$. For any $R > 0$, set

$$\psi_R(x) := \psi\left(\frac{x}{R}\right) = \begin{cases} 0 & \text{if } |x| \le R/2, \\ 1 & \text{if } |x| \ge R. \end{cases}$$

Again by the boundedness of $\{u_n\}$ in E, we know that $\{\psi_R u_n\}$ is also bounded in E. Thus,

$$\begin{aligned} o_n(1) &= \langle I'(u_n), \psi_R u_n \rangle \\ &= \int_{\mathbb{R}^3} \psi_R |\nabla u_n|^2 dx + \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_R dx - \int_{\mathbb{R}^3} \psi_R \ \phi_{u_n} |u_n|^2 dx \\ &\quad - \int_{\mathbb{R}^3} \psi_R \frac{|u_n|^s}{|x|^q} dx - \int_{\mathbb{R}^3} \psi_R \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} dx. \end{aligned} \tag{3.25}$$

By the definition of ψ_R, we have

$$\int_{|x|>R} |\nabla u_n|^2 dx \leqslant \int_{\mathbb{R}^3} \psi_R |\nabla u_n|^2 dx = \int_{|x|>R/2} |\nabla u_n|^2 dx. \tag{3.26}$$

Thus, in view of Lemma 2.6, we obtain

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_R |\nabla u_n|^2 dx = \mu_\infty, \tag{3.27}$$

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_R \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} dx = \nu_\infty. \tag{3.28}$$

By (3.8) and the definition of ψ_ρ, we have

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_R \frac{|u_n|^s}{|x|^q} dx = \lim_{R\to\infty} \int_{\mathbb{R}^3} \psi_R \frac{|u|^s}{|x|^q} dx = \lim_{R\to\infty} \int_{|x|>R/2} \psi_R \frac{|u|^s}{|x|^q} dx = 0. \tag{3.29}$$

Similarly, we have

$$\lim_{R\to\infty} \lim_{n\to\infty} \int_{\mathbb{R}^3} \psi_R \phi_{u_n} |u_n|^2 dx = 0, \tag{3.30}$$

$$\lim_{R\to 0} \lim_{n\to\infty} \int_{\mathbb{R}^3} u_n \nabla u_n \nabla \psi_R dx = 0. \tag{3.31}$$

So, from (3.27)-(3.31), taking the limit as $n \to \infty$, and then the limit as $R \to \infty$ in (3.25), we obtain

$$\mu_\infty = \nu_\infty. \tag{3.32}$$

Step 3. We prove that $\mu_\infty = \nu_\infty = 0$. Suppose by contradiction that $\mu_\infty = \nu_\infty > 0$. By (2.7), we know

$$S_{h,l}^2 \nu_\infty^{\frac{2}{3+\alpha}} \le \mu_\infty \Big(\int_{\mathbb{R}^3} d\mu + \mu_\infty \Big).$$

By (2.5), we have

$$\nu_\infty \le S_{h,l}^{-(3+\alpha)} \mu_\infty^{\frac{3+\alpha}{2}} \Big(\int_{\mathbb{R}^3} d\mu + \mu_\infty \Big)^{\frac{3+\alpha}{2}} \le S_{h,l}^{-(3+\alpha)} \mu_\infty^{\frac{3+\alpha}{2}} \left(R_1^2\right)^{\frac{3+\alpha}{2}} = S_{h,l}^{-(3+\alpha)} \nu_\infty^{\frac{3+\alpha}{2}} \left(R_1^2\right)^{\frac{3+\alpha}{2}},$$

which implies

$$\nu_\infty^{1-\frac{3+\alpha}{2}} \le S_{h,l}^{-(3+\alpha)} \left(R_1^2\right)^{\frac{3+\alpha}{2}}.$$

Since $1 - \frac{3+\alpha}{2} < 0$, we can deduce that

$$\nu_\infty \ge S_{h,l}^{\frac{2(3+\alpha)}{1+\alpha}} \left(R_1^2\right)^{-\frac{3+\alpha}{1+\alpha}}.$$

Then

$$R_1^2 \ge \lim_{R\to\infty} \lim_{n\to\infty} \|\nabla u_n\|_2^2 \ge \mu_\infty = \nu_\infty \ge S_{h,l}^{\frac{2(3+\alpha)}{1+\alpha}} \left(R_1^2\right)^{-\frac{3+\alpha}{1+\alpha}},$$

which shows that

$$R_1^2 \ge S_{h,l}^{\frac{2(3+\alpha)}{2(2+\alpha)}} = S_{h,l}^{\frac{3+\alpha}{2+\alpha}},$$

which contradicts (3.4). Therefore, we prove that $\mu_\infty = \nu_\infty = 0$. Consequently, by Steps 1-3, we obtain

$$\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} dx \to \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{3+\alpha}\right) |u|^{3+\alpha} dx. \tag{3.33}$$

Next, we prove that $u \neq 0$. Indeed, if $u = 0$, from (3.8) and the concentration-compactness principle, we can show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q}\, dx = 0, \lim_{n\to\infty} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} dx = 0,$$

and by Lemma 2.4 (vii), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^2 dx = 0.$$

Hence, from (3.1), the definition of I_τ and when $\|\nabla u\|_2 \in [0, R_1]$, one has

$$\begin{aligned}
0 > c &= \lim_{n\to\infty} I_\tau(u_n) = \lim_{n\to\infty} I(u_n) \\
&= \lim_{n\to\infty} \Big[\frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{6+2\alpha} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^{3+\alpha})|u_n|^{3+\alpha}\, dx\Big] \\
&\geq \lim_{n\to\infty} \frac{1}{2}\|\nabla u_n\|_2^2 \geq 0,
\end{aligned}$$

which is absurd.

Taking into account (3.11), we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \left[\|\nabla u_n\|_2^2 + \lambda_n \|u_n\|_2^2\right] \\
&= \lim_{n\to\infty} \Big[\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^2 dx + \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q} dx + \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{3+\alpha}\right) |u_n|^{3+\alpha} dx\Big] \\
&= \int_{\mathbb{R}^3} \phi_u |u|^2 dx + \int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q} dx + \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{3+\alpha}\right) |u|^{3+\alpha} dx \\
&= \|u\|_E^2 + \lambda_a \|u\|_2^2.
\end{aligned} \tag{3.34}$$

Next, we claim that $\lambda_a > 0$.

$$\begin{aligned}
0 > c &= \lim_{n\to\infty} I_\tau(u_n) \\
&= \lim_{n\to\infty} I(u_n) \\
&= \lim_{n\to\infty} \Big(I(u_n) - \frac{1}{2}\langle I'(u_n) + \lambda_n \Psi'(u_n), u_n\rangle\Big) \\
&= \lim_{n\to\infty} \Big[\frac{1}{2}\|\nabla u_n\|_2^2 - \frac{1}{s} \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q}\, dx - \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^2\, dx \\
&\quad - \frac{1}{6+2\alpha} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^{3+\alpha})|u_n|^{3+\alpha}\, dx - \frac{1}{2}\Big(\|\nabla u_n\|_E^2 + \lambda_n a^2 \\
&\quad - \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^2\, dx - \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q}\, dx - \int_{\mathbb{R}^3} (I_\alpha * |u_n|^{3+\alpha})|u_n|^{3+\alpha}\, dx\Big)\Big] \\
&= \lim_{n\to\infty} \Big[-\frac{1}{2}\lambda_n a^2 + \Big(\frac{1}{2} - \frac{1}{s}\Big) \int_{\mathbb{R}^3} \frac{|u_n|^s}{|x|^q}\, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^2\, dx \\
&\quad + \frac{2+\alpha}{6+2\alpha} \int_{\mathbb{R}^3} (I_\alpha * |u_n|^{3+\alpha})|u_n|^{3+\alpha}\, dx\Big] \\
&= -\frac{1}{2}\lambda_a a^2 + \Big(\frac{1}{2} - \frac{1}{s}\Big) \int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q}\, dx + \frac{1}{4} \int_{\mathbb{R}^3} \phi_u |u|^2\, dx \\
&\quad + \frac{2+\alpha}{6+2\alpha} \int_{\mathbb{R}^3} (I_\alpha * |u|^{3+\alpha})|u|^{3+\alpha}\, dx,
\end{aligned}$$

so

$$\frac{1}{2}\lambda_a a^2 > \Big(\frac{1}{2}-\frac{1}{s}\Big)\int_{\mathbb{R}^3}\frac{|u|^s}{|x|^q}\,dx + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u|u|^2\,dx + \frac{2+\alpha}{6+2\alpha}\int_{\mathbb{R}^3}(I_\alpha * |u|^{3+\alpha})|u|^{3+\alpha}\,dx,$$

which indicates that $\lambda_a > 0$ since $s > 2$.

From (3.34) we obtain

$$\begin{aligned}\lambda_a\|u\|_2^2 &\leqslant \liminf_{n\to\infty}\big(\lambda_a\|u_n\|_2^2\big) \le \limsup_{n\to\infty}\big(\lambda_a\|u_n\|_2^2\big)\\ &\leqslant \limsup_{n\to\infty}\big(\lambda_a\|u_n\|_2^2\big) + \liminf_{n\to\infty}\big(\|\nabla u_n\|_2^2 - \|\nabla u\|_2^2\big)\\ &\le \limsup_{n\to\infty}\big(\|\nabla u_n\|_2^2 + \lambda_a\|u_n\|_2^2\big) - \|\nabla u\|_2^2 = \lambda_a\|u\|_2^2.\end{aligned}$$

So that

$$\lim_{n\to\infty}\big(\lambda_a\|u_n\|_2^2\big) = \lambda_a\|u\|_2^2,$$

that is $\lim_{n\to\infty}\|u_n\|_2^2 = \|u\|_2^2$. By (3.34), one has

$$\lim_{n\to\infty}\|\nabla u_n\|_2^2 = \|\nabla u\|_2^2.$$

Hence, $u_n \to u$ in E and $\|u\|_2^2 = a^2$. The proof is complete. $\square$

For $\varepsilon > 0$, we define the set

$$I_\tau^{-\varepsilon} = \{u \in E \cap S(a) : I_\tau(u) \leqslant -\varepsilon\} \subset E. \tag{3.35}$$

Since $I_\tau(u)$ is continuous and even on E , we see that $I_\tau^{-\varepsilon}$ is closed and symmetric. Then, we have the following lemma, which can be similarly proved as [3, Lemma 3.2].

Lemma 3.2. *Give $k \in \mathbb{N}$, there exists $\varepsilon_k := \varepsilon(k) > 0$ such that $\gamma(I_\tau^{-\varepsilon_k}) \geqslant k$.*

We introduce the set

$$\Sigma_k := \{D \subset E \cap S(a) : D \text{ is closed and symmetric}, \gamma(D) \geqslant k\}, \tag{3.36}$$

and define the minimax value

$$c_k := \inf_{D\in\Sigma_k}\sup_{u\in D} I_\tau(u) > -\infty \tag{3.37}$$

for all $k \in \mathbb{N}$.

To show Theorem 1.2, we set

$$K_c := \{u \in E \cap S(a) : I_\tau'(u) = 0, I_\tau(u) = c\}, \tag{3.38}$$

then, we can prove the following result.

Lemma 3.3. *For $a \in (0, a_*)$. If $c = c_k = c_{k+1} = \cdots = c_{k+r}$, we have $\gamma(K_c) \ge r + 1$. In particular, $I_\tau(u)$ has at least $r + 1$ nontrivial critical points.*

Proof. For a fixed $k \in \mathbb{N}$, by Lemma 3.2, there exists $\varepsilon_k := \varepsilon(k) > 0$ such that if $0 < \varepsilon \le \varepsilon_k$, we obtain $\gamma(I_\tau^{-\varepsilon_k}) \ge k$, and so $I_\tau^{-\varepsilon_k} \in \Sigma_k$. Moreover, one obtains

$$c_k \le \sup_{u\in I_\tau^{-\varepsilon_k}} I_\tau(u) = -\varepsilon_k < 0.$$

Assume that $0 > c = c_k = c_{k+1} = \cdots = c_{k+r}$, then by Lemma 3.1 (ii), $I_\tau(u)$ verifies the $(PS)_c$ condition at the level $c < 0$. Thus, K_c is a compact set. From [15, Theorem 2.1], we see that $I_\tau(u)|_{S(a)}$ has at least $r + 1$ nontrivial critical points. The proof is complete. $\square$

By Lemma 3.1, the critical points of $I_\tau(u)$ obtained in Lemma 3.3 are the critical points of I, which complete the first part of Theorem 1.2.

4. Existence of ground state solution

Define the truncated constraint set

$$\mathcal{B}_a := \{u \in S(a) : \|\nabla u\|_2 \leq R_1\}.$$

Furthermore, for $u \in \mathcal{B}_a$, we have $\tau(\|\nabla u\|_2) = 1$ and thus $I_\tau(u) = I(u)$.

Insprired by [24], we define

$$\zeta_a(x) = ae^{\frac{-\pi}{2}|x|^2}. \tag{4.1}$$

A direct calculation shows that

$$\|\zeta_a\|_2^2 = \int_{\mathbb{R}^3} |\zeta_a|^2\,dx = a^2 \int_{\mathbb{R}^3} e^{-\pi|x|^2}dx = a^2,$$

leading to $\zeta \in S(a)$. Furthermore, we have

$$\int_{\mathbb{R}^3} |\nabla\zeta_a|^2\,dx = a^2\pi^2 \int_{\mathbb{R}^3} |x|^2 e^{-\pi|x|^2}dx = a^2\pi^2\mathcal{W}_3 \int_0^{+\infty} e^{-\pi\rho^2}\rho^4 d\rho = \frac{a^2}{2}\mathcal{W}_3\pi^{-\frac{1}{2}}\Gamma\left(\frac{5}{2}\right).$$

Thus, we can deduce that

$$\begin{aligned}
\|\zeta_a\|_E^2 &= \int_{\mathbb{R}^3} |\zeta_a|^2\,dx + \int_{\mathbb{R}^3} |\nabla\zeta_a|^2\,dx \\
&= a^2 + \frac{a^2}{2}\mathcal{W}_3\pi^{-\frac{1}{2}}\Gamma\big(\frac{5}{2}\big) \\
&= a^2\Big(1 + \frac{1}{2}\mathcal{W}_3\pi^{-\frac{1}{2}}\Gamma\big(\frac{5}{2}\big)\Big).
\end{aligned}$$

For $a \in (0, a_*)$ sufficiently small, $\|\nabla\zeta_a\|_2 < R_1$, so $\zeta_a \in \mathcal{B}_a$. We can get that $\mathcal{B}_a$ is nonempty. Therefore, we can define

$$m_a := \inf_{u\in\mathcal{B}_a} I(u).$$

Lemma 4.1. *For $a \in (0, a_*)$. Let $\{u_n\} \subset \mathcal{B}_a$ with $\|\nabla u_n\|_2 \leq r_1 < R_1$ and $I_\tau(u_n) \to m_a$. Then there exists $\{\tilde{u}_n\} \subset \mathcal{B}_a$ such that $\|\tilde{u}_n - u_n\|_E \to 0$ and $\|I'_\tau|_{\mathcal{B}_a}(\tilde{u}_n)\|_{E^*} \to 0$.*

Proof. Since $\|\nabla u_n\|_2 \leq r_1 < R_1$, there exists $\rho > 0$ such that $B_n := \{v \in S(a) : \|v - u_n\|_E \leq \rho\} \subset \mathcal{B}_a^\circ$. By the Ekeland variational principle applied to I_τ on $(B_n, \|\cdot\|_E)$, there exists $\tilde{u}_n \in B_n$ satisfying $\|\tilde{u}_n - u_n\|_E \to 0$ and

$$I_\tau(v) \geq I_\tau(\tilde{u}_n) - o(1)\|v - \tilde{u}_n\|_E, \quad \forall v \in B_n,$$

which implies $\|I'_\tau|_{S(a)}(\tilde{u}_n)\|_{E^*} \to 0$. Since $\tilde{u}_n \in \mathcal{B}_a^\circ$, we have $I'_\tau|_{\mathcal{B}_a}(\tilde{u}_n) = I'_\tau|_{S(a)}(\tilde{u}_n)$. □

Lemma 4.2. *For $a \in (0, a_*)$, the system* (1.1) *has a couple ground state solution (u_a, λ_a).*

Proof. The proof proceeds in three steps.

Step 1: For any $u \in \mathcal{B}_a$, by Lemma 3.1 and the definition of $\widetilde{h}(\|\nabla u\|_2)$, we can deduce that

$$I(u) = I_\tau(u) \geq \tilde{h}(\|\nabla u\|_2) \geq \min_{t\in[0,R_1]} \tilde{h}(t) > -\infty.$$

Hence $m_a > -\infty$.

$$\begin{aligned}
m_a &\leqslant I(\zeta_a) \\
&= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\zeta_a|^2dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi_{\zeta_a}|\zeta_a|^2dx - \frac{1}{s}\int_{\mathbb{R}^3} \frac{|\zeta_a|^s}{|x|^q}dx - \frac{1}{6+2\alpha}\int_{\mathbb{R}^3} \left(I_\alpha * |\zeta_a|^{3+\alpha}\right)|\zeta_a|^{3+\alpha}\,dx \\
&< \frac{1}{2}\|\zeta_a\|_E^2 \\
&\leq a^2\Big(1 + \frac{1}{2}\mathcal{W}_3\pi^{-\frac{1}{2}}\Gamma\left(\frac{5}{2}\right)\Big) \to 0 \quad \text{as } a \to 0,
\end{aligned}$$

Thus $-\infty < m_a \leq I(\zeta_a) < 0$.

Step 2: Let $\{u_n\} \subset \mathcal{B}_a$ satisfy $I(u_n) \to m_a$. By definition, $\|u_n\|_2 = a$ and $\|\nabla u_n\|_2 \leq R_1$. So $\{u_n\}$ is bounded in E. By Lemma 4.1, we can replace the sequence $\{u_n\}$ with $\{\tilde{u}_n\}$ such that $I'(\tilde{u}_n) \to 0$ and $I(\tilde{u}_n) \to m_a$. We also get that $\{\tilde{u}_n\}$ is away from $\partial\mathcal{B}_a$. Otherwise, $0 > m_a \leftarrow I(\tilde{u}_n) \geq h(|\nabla \tilde{u}_n|_2) = 0$ since $|\nabla \tilde{u}_n|_2 = R_1$ for $\tilde{u}_n \in \partial\mathcal{B}_a$. It is a contradiction. So $I'_{S(a)}(\tilde{u}_n) = I'_{\mathcal{B}_a}(\tilde{u}_n) \to 0$. Up to subsequence, we assume that

$$\tilde{u}_n \rightharpoonup u_a \text{ in } E, \quad \tilde{u}_n \to u_a \text{ in } L^t(\mathbb{R}^3) \text{ for } 2 < t < 6, \quad \tilde{u}_n(x) \to u_a(x) \text{ a.e. on } \mathbb{R}^3.$$

So $\{\tilde{u}_n\}$ is a $(PS)_{m_a}$ sequence for $I_\tau = I$ restricted to $S(a)$. Arguing again as in the proof of Lemma 3.1, I_τ satisfies the $(PS)_{m_a}$ condition and $\tilde{u}_n \to u_a$ strongly in E and $\lambda_n \to \lambda_a$. So, we have $\|u_a\|_2 = \lim_{n\to\infty} \|\tilde{u}_n\|_2 = a$ and $\|\nabla u_a\|_2 \leq \liminf_{n\to\infty} \|\nabla \tilde{u}_n\|_2 \leq R_1$. Thus, $u_a \in \mathcal{B}_a$.

Step 3: We show (u_a, λ_a) is a normalized ground state solution. Let $v \in S(a)$ satisfy $I'_{S(a)}(v) = 0$ and $I_{S(a)}(v) < 0$ for $a \in (0, a_*)$, we obtain $\|\nabla v\|_2 \leq R_1$, thus $v \in \mathcal{B}_a$. Hence, $I(u_a) \leq I(v)$. From the arbitrariness of v, we know that (u_a, λ_a) is a normalized ground state solution to (1.1). □

Proof of Theorem 1.2. From Lemmas 3.1-3.3, we can get the existence of multiple solutions. Furthermore, from Lemma 4.2, (u_a, λ_a) is a couple ground state solution of the system (1.1).

Let (u_{a_n}, λ_{a_n}) be a couple sequence ground state solutions of the system (1.1). From Lemma 3.1, we know that $\|\nabla u_{a_n}\|_2 \leq R_1$ uniformly in a, n, then $u_{a_n} \rightharpoonup u_0$ for some $u_0 \in E$, similar arguing as in the proof of Lemma 3.1, we obtain $u_{a_n} \to u_0$ strongly in E and $\lambda_{a_n} \to \lambda_0$. Combing with $\|u_{a_n}\|_2 = a_n \to 0$ as $n \to \infty$, we have $u_{a_n} \to 0$ a.e. in $\mathbb{R}^3$, thus $\|u_{a_n}\|_E \to 0$ as $n \to \infty$. Hence $\|u_a\|_E \to 0$ as $a \to 0$. □

5. Orbital stability

In this section, we study the orbital stability of u_a obtained in Theorem 1.2. Firstly, we give the definition of orbital stability.

Definition 5.1. The set Σ_a is said to be orbitally stable if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any initial data ψ_0 satisfying

$$\inf_{u\in\Sigma_a} \|\psi_0 - u\|_E < \delta,$$

the corresponding solution $\psi(t)$ to (1.1) with initial data ψ_0 satisfies

$$\inf_{u\in\Sigma_a} \|\psi(t) - u\|_E < \varepsilon \quad \text{for all } t > 0.$$

Firstly, we consider that the solution ψ of (1.1) exists globally. We have

$$\begin{aligned} I_\tau(u) &\geq I_\tau(u) \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{4}C_* a^3 \|\nabla u\|_2 - \frac{1}{s} C \|\nabla u\|_2^{q+\frac{3s}{2}-3} \|u\|_2^{3-q-\frac{s}{2}} - \frac{\tau\left(\|\nabla u\|_2\right)}{6+2\alpha} S_{h,l}^{-(3+\alpha)} \|\nabla u\|_2^{6+2\alpha} \\ &:= \widetilde{h}(\|\nabla u\|_2), \end{aligned}$$

where

$$\widetilde{h}(t) = \frac{1}{2}t^2 - \frac{1}{4}C_* a^3 t - -\frac{1}{s} C t^{q+\frac{3s}{2}-3} a^{3-q-\frac{s}{2}} - \frac{\tau\left(t\right)}{6+2\alpha} S_{h,l}^{-(3+\alpha)} t^{6+2\alpha}. \tag{5.1}$$

Denoting by $\psi(t, \cdot)$ the solution to (1.1) with initial data u_0 and denoting by $[0, T_{\max})$ the maximal existence interval for $\psi(t, \cdot)$, we have classically that either $\psi(t, \cdot)$ is globally defined for positive times, or $\|\nabla \psi(t, \cdot)\|_2 = +\infty$ as $t \to T_{\max}$; see [22].

We define

$$\begin{aligned} \Omega : &= \left\{v \in S(a) : I_\tau|'_{S(a)}(v) = 0, I_\tau(v) = m_a\right\} \\ &= \left\{v \in S(a) : I|'_{S(a)}(v) = 0, I(v) = m_a, \|\nabla v\|_2^2 \leqslant R_1, I\left(R_1\right) = 0\right\}. \end{aligned}$$

Assuming that r_1 satisfies $\widetilde{h}\left(r_1\right) = m_a$, by using $m_a < 0$ and the definition of Ω, we have

$$r_1 < R_1, \quad \forall v \in \Omega.$$

Note that, since $\widetilde{h}$ is monotonically decreasing with respect to a, we can take $a_1 > a$ to satisfy $\widetilde{h}\,(R_1) = \frac{m_a}{2}$. There is $\delta > 0$ such that if $u_0 \in E$ and $\operatorname{dist}_E(u_0, \Omega) < \delta$. Then

$$\|u_0\|_2 \leqslant a_1, \quad \|\nabla u_0\|_2^2 \leqslant r_1 + \frac{R_1 - r_1}{2}, \quad I_\tau(u_0) \leqslant \frac{2}{3} m_a.$$

Let $a = \|u_0\|_2$. Note that $\|\psi(t,\cdot)\|_2 = \|u_0\|_2$ for all $t \in (0, T_{\max})$ by the conservation of the mass. If there is $\tilde{t} \in (0, T_{\max})$ such that $\|\nabla \psi(\tilde{t},\cdot)\|_2 = R_1$, then

$$I_\tau(\psi(\tilde{t},\cdot)) \geqslant \widetilde{h}\,(R_1) = \frac{m_a}{2},$$

which contradicts the conservation of the energy

$$I_\tau(\psi(\tilde{t},\cdot)) = I_\tau\,(u_0) \leqslant \frac{2}{3} m_a, \quad t \in (0, T_{\max})\,.$$

Thus,

$$\|\nabla \psi(t,\cdot)\|_2 < R_1, \quad t \in [0, T_{\max})\,,$$

which implies that $\psi(t,\cdot)$ is globally defined for positive times. We have obtained the solution ψ of (1.1) exists globally.

Assume by contradiction that there exist ε_0 and a solution sequence $\{\psi_n(t_n)\}_{n=1}^{\infty}$ of (1.1), which corresponds $\{t_n\}_{n=1}^{\infty}$, satisfies

$$\inf_{u \in \Omega} \|\psi_n\,(t_n) - u\|_E \geqslant \varepsilon_0. \tag{5.2}$$

Proof of Theorem 1.3. Assume by contradiction that there exist $\varepsilon_0 > 0$ and $\{\psi_n\,(t_n)\}$ satisfying (5.2). By conservation laws and a priori estimates: $\|\psi_n(t_n)\|_2^2 \to a$, $\|\nabla\psi_n(t_n)\|_2 \leq r_1 < R_1$, and $I_\tau(\psi_n(t_n)) \to m_a$. By Lemma 4.1, there exists $\{\tilde{u}_n\} \subset \mathcal{B}_a$ with $\|\tilde{u}_n - \psi_n(t_n)\|_E \to 0$ and $\|I'_\tau|_{\mathcal{B}_a}(\tilde{u}_n)\|_{E^*} \to 0$. So there exists $u \in \Omega$ such that $\|\tilde{u}_n - u\|_E \to 0$ from Theorem 1.1. Hence $\|\psi_n(t_n) - u\|_E \leq \|\psi_n(t_n) - \tilde{u}_n\|_E + \|\tilde{u}_n - u\|_E \to 0$, which contradicts (5.2). This completes the proof. $\square$

6. Asymptotic behavior

Let (u_d, ϕ^d) be the radial ground-state solution of (1.3)–(1.4) and the corresponding energy functional be

$$I_d(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{s}\int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q} dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi^d u^2 dx - \frac{1}{6+2\alpha}\int_{\mathbb{R}^3} \big(I_\alpha * |u|^{3+\alpha}\big)|u|^{3+\alpha} dx,$$

where $u \in S(a)$ and ϕ^d is the unique solution of $-\Delta\phi + d^2\Delta^2\phi = 4\pi u^2$ in $\mathbb{R}^3$. The energy functional corresponding to (1.5) is

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{s}\int_{\mathbb{R}^3} \frac{|u|^s}{|x|^q} dx - \frac{1}{4}\int_{\mathbb{R}^3} \phi^0 u^2 dx - \frac{1}{6+2\alpha}\int_{\mathbb{R}^3} \big(I_\alpha * |u|^{3+\alpha}\big)|u|^{3+\alpha} dx,$$

where $u \in S(a)$ and ϕ^0 is the unique solution of $-\Delta\phi = 4\pi u^2$ in $\mathbb{R}^3$.

Proof of Theorem 1.4. By Lemma 3.1 (i), we have $\|\nabla u_d\|_2 \leq R_1$, and then

$$\|u_d\|_E = \Big(\int_{\mathbb{R}^3} \Big(|\nabla u_d|^2 + |u_d|^2\Big)\,dx\Big)^{1/2} \leq \big(R_1^2 + a^2\big)^{1/2} < C,$$

which implies that $\{u_d\}$ is bounded in E, and up to a subsequence (still denoted by u_d),

$$u_d \rightharpoonup u_0 \quad \text{in } E, \quad u_d \to u_0 \quad \text{in } L^p(\mathbb{R}^3)\ \forall p \in (2,6), \quad u_d(x) \to u_0(x) \quad \text{a.e. on } \mathbb{R}^3.$$

For any test function $\varphi \in C_0^\infty(\mathbb{R}^3)$ we have

$$\begin{aligned} &\int_{\mathbb{R}^3} \nabla u_d \nabla \varphi dx + \lambda_d \int_{\mathbb{R}^3} u_d \varphi dx \\ &= \int_{\mathbb{R}^3} \phi^d u_d \varphi dx + \int_{\mathbb{R}^3} \frac{|u_d|^{s-2} u_d \varphi}{|x|^q} dx + \int_{\mathbb{R}^3} \big(I_\alpha * |u_d|^{3+\alpha}\big)|u_d|^{1+\alpha} u_d \varphi dx. \end{aligned} \tag{6.1}$$

Using the weak convergences, we can pass to the limit and obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla u_0 \nabla \varphi dx + \lambda_0 \int_{\mathbb{R}^3} u_0 \varphi dx \\
&= \int_{\mathbb{R}^3} \phi^0 u_0 \varphi dx + \int_{\mathbb{R}^3} \frac{|u_0|^{s-2}\varphi}{|x|^q} dx + \int_{\mathbb{R}^3} \big(I_\alpha * |u_0|^{3+\alpha}\big)|u_0|^{1+\alpha} u_0 \varphi dx,
\end{aligned} \tag{6.2}$$

where $\lambda_d \to \lambda_0$ (up to a subsequence) and $\lambda_0 > 0$ by the same argument as in Lemma 3.1.

From Lemma 2.7, there exists a subsequence (still denoted by d) and (u_0, ϕ^0) is a solution of the limit Schrödinger-Poisson system.

$$\phi^d \to \phi^0 \quad \text{in } D^{1,2}(\mathbb{R}^3) \quad \text{and} \quad d\Delta \phi^d \to 0 \quad \text{in } L^2(\mathbb{R}^3),$$

where ϕ^0 is the unique solution of $-\Delta \phi = 4\pi u_0^2$.

Next we prove that $u_d \to u_0$ in E. Taking $\varphi = u_d$ and $\varphi = u_0$ respectively in (6.1), we can obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla u_d \nabla (u_d - u_0) dx + \lambda_d \int_{\mathbb{R}^3} u_d (u_d - u_0) dx \\
&= \int_{\mathbb{R}^3} \phi^d u_d (u_d - u_0) dx + \int_{\mathbb{R}^3} \frac{|u_d|^{s-2}(u_d - u_0)}{|x|^q} dx \\
&\quad + \int_{\mathbb{R}^3} \big(I_\alpha * |u_d|^{3+\alpha}\big)|u_d|^{1+\alpha} u_d (u_d - u_0) dx.
\end{aligned} \tag{6.3}$$

Similarly, taking $\varphi = u_d$ and $\varphi = u_0$ respectively in (6.2), so

$$\begin{aligned}
&\int_{\mathbb{R}^3} \nabla u_0 \nabla (u_d - u_0) dx + \lambda_0 \int_{\mathbb{R}^3} u_0 (u_d - u_0) dx \\
&= \int_{\mathbb{R}^3} \phi^0 u_0 (u_d - u_0) dx + \int_{\mathbb{R}^3} \frac{|u_0|^{s-2}(u_d - u_0)}{|x|^q} dx + \int_{\mathbb{R}^3} \big(I_\alpha * |u_0|^{3+\alpha}\big)|u_0|^{1+\alpha} u_0 (u_d - u_0) dx.
\end{aligned}$$

By Lemma 2.4($viii$), we have

$$\int_{\mathbb{R}^3} (\phi^d u_d - \phi^0 u_0)(u_d - u_0) dx \to 0. \tag{6.4}$$

Similar as the proof of (3.8), we have

$$\int_{\mathbb{R}^3} \frac{|u_d - u_0|^s}{|x|^q} dx \to 0. \tag{6.5}$$

By (3.15), we obtain that

$$\int_{\mathbb{R}^3} \big[(I_\alpha * |u_d|^{3+\alpha})|u_d|^{1+\alpha} u_d - (I_\alpha * |u_0|^{3+\alpha})|u_0|^{1+\alpha} u_0\big](u_d - u_0) dx \to 0. \tag{6.6}$$

Then from (6.4)-(6.6), with $\lambda_d \to \lambda_0$, we have

$$\int_{\mathbb{R}^3} |\nabla (u_d - u_0)|^2 \, dx + \lambda_0 \int_{\mathbb{R}^3} |u_d - u_0|^2 \, dx = o(1).$$

Since $\lambda_0 > 0$,

$$u_d \to u_0 \quad \text{in } D^{1,2}(\mathbb{R}^3)$$

and

$$u_d \to u_0 \quad \text{in } L^2(\mathbb{R}^3).$$

Thus, $u_d \to u_0$ in E and (u_0, ϕ^0) solves (1.5).

By the lower semi-continuity of the norm and the convergence of the nonlinear terms we obtain

$$I_0(u_0) \le \liminf_{d \to 0} I_d(u_d).$$

On the other hand, for any $v \in S(a)$, $I_d(v) \to I_0(v)$ as $d \to 0$. Then

$$\limsup_{d \to 0} \inf_{v \in S(a)} I_d(v) \le \inf_{v \in S(a)} I_0(v),$$

so

$$I_0(u_0) \le \liminf_{d\to 0} I_d(u_d) \le \limsup_{d\to 0} I_d(u_d) \le I_0(v).$$

Hence $I_0(u_0) \le I_0(v)$ for all $v \in S(a)$. Taking the infimum over $v \in S(a)$ on the right-hand side yields

$$I_0(u_0) \le \inf_{v\in S(a)} I_0(v).$$

Since $u_0 \in S(a)$ (because $\|u_0\|_2 = a$), the reverse inequality

$$\inf_{v\in S(a)} I_0(v) \le I_0(u_0)$$

is trivial. Therefore,

$$I_0(u_0) = \inf_{v\in S(a)} I_0(v)$$

which means u_0 attains the least energy of the limit functional I_0 under the mass constraint, i.e. u_0 is a ground state of the limiting Schrödinger-Poisson system. Therefore $I_0(u_0) = \inf_{u\in S(a)} I_0(u)$, i.e. (u_0, ϕ^0) is a radial ground-state solution of the limit Schrödinger-Poisson system. This completes the proof. $\square$

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